THE JOY OF SET

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Abstract. The card game called *Set* inspires combinatorial problems related to triplets of points in $\mathbb{Z}_3^k$, p_1, p_2, p_3, such that $p_1 + p_2 + p_3 = (0^k)$.

Résumé. Le jeu des cartes qui s'appelle *Set* inspire les questions combinatoires qui sont liés aux triplés des points de l'espace $\mathbb{Z}_3^k$, p_1, p_2, p_3, tel que $p_1 + p_2 + p_3 = (0^k)$.

1. Introduction - The Game of Set

The card game of *Set* was invented by a population geneticist by the name of Marsha Jean Falco in 1974. She was studying the condition of epilepsy in German Shepherds and began representing genetic data on the dogs by drawing symbols on cards and then searching for patterns in the data. After realizing the potential as a challenging puzzle, with some encouragement from friends and family she developed and marketed the card game. It is now available in many gaming stores in the U. S. and worldwide.

A deck of *Set* has $3^4 = 81$ different cards. Each card has a white background with either one, two, or three forms that are either an ellipse, a diamond or a squiggle (on any given card all of the shapes are the same). The forms on the card are drawn in either red, green or purple ink, and each of the shapes are either filled solid, shaded with lines, or just an outline. In other words, there are four distinguishing properties on each of these cards: shape, number, color and shading. For each of these properties, a card can take on one of three different values giving 3^4 different possibilities. Some examples of cards are shown in Figure 1.

The game is played by placing 12 cards from the deck face up on a table. The players scan the upright cards to locate a collection of three such that each of the four values for the properties on their faces are either all the same or all different. A collection of 3 cards with this property is called a 'set.' The first player to locate a 'set' removes it from the board and three more cards are dealt face up. The player with the most number of sets after all of the cards have been dealt and all sets removed wins the game.

Figure 1. An example of three cards from a Set deck that form a set.

With 12 cards on the table it is possible that no sets exist in the upright ones. After all players agree that there isn't a set on the table, three more cards are dealt. If again no sets are showing, three more are dealt until a set appears. The cards are not replaced if there are already twelve on the table.

A shareware computer version of the card game is distributed by the company SET Enterprises, Inc. and may be downloaded for free from their web site. The freeware part of the program deals cards that have only three properties. It also includes a demo version of the full computer game, but the demo version is rigged (i.e. the cards are not chosen at random from the deck). Both the simplified and demo versions are much easier than the original game, however the version available for purchase has more options than is possible with just the deck of cards.

There are two obvious generalizations to this game. The first would be to add more properties (where for a deck with k properties there are 3^k possible cards). The second would be to vary the number of values that these properties could take on (that is, if each property takes on p different values and there are k properties then there are p^k different cards).

After playing a few rounds of the game, several questions came to my mind about this game. The first was "What is the largest number of cards that one can expose that does not include a set?" Although it is easy to experiment with fewer than 4 properties, it is difficult to generalize the results with more.

This very question had been answered for a deck with k properties where k is 4 or less by some recreational mathematicians and their results are available on the *Set* web site. They had found by a computer exhaustive search that in a deck with 4 properties, it is possible to place 20 cards without finding a set. With a deck that has 3 properties, it is possible to have 9 cards that do not contain a set; with 2 properties, 4 cards; with 1 property, 2 cards.

The first attack to try to answer this problem is to look at Sloane's On-line Encyclopedia of Integer Sequences [4]. The entry 2, 4, 9, 20 returns a maximum number of responses, however none of them are suggestive of the problem that is considered here. Four values is just not enough in this case to determine if the sequence had been considered before in a different situation.

2. The computer version- Advanced Set

My interest was in trying to answer this and several other questions that this game inspires. Having a way of playing the game with more than 4 properties was the motivation to try to learn Java well enough to write the computer program that will play Set with larger decks of cards.

The computer program presented here has the possibility of playing the game with k different properties where k is between 1 and 7. This version of the game not only varies the shape, color, number and shading of the objects on the cards, but also the size, left/right/center orientation, and background color. The game played with a deck with 4 properties is challenging, while the game played with 5, 6 and 7 properties ranges from hard to nearly impossible.

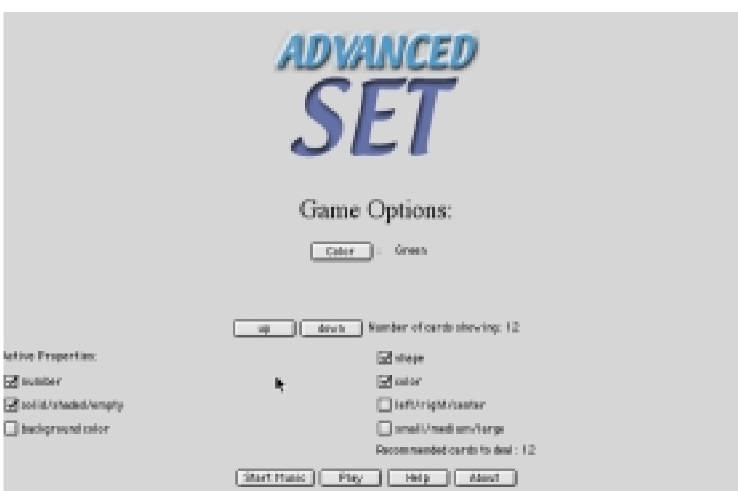


Figure 2. Opening window of Advanced Set

The opening screen is shown in Figure 2, it contains the controls necessary to set the options for the game. The background color is one of four different values. Below that, there are a couple of buttons that set the base number of cards to be dealt, this can be between 3 and 36. Below that there are seven different checkboxes, the first four will be checked already when the program starts as they correspond to the original game of set. For a deck with 3 properties the game will recommend that 9 cards are dealt; with 4, 12 cards; with 5, 16 cards; with 6, 24 cards; and with 7, 32 cards. These recommendations were chosen by experiment. About 1 in every 10 times that the recommended cards are dealt there will not be a set.

The first button below the checkboxes starts or stops the theme music. This music may not start to play immediately if it takes a while to load it over the internet. It was borrowed from a recording of 'The Flying Lizards.' It only plays during the appearance of the title screen and not during the game itself. The second button starts the game, while the last two buttons provide more information about the program.

Figure 3 shows the game board being played with a green background, 12 cards dealt, with 4 different properties (in this case shape, shading, left/right/center orientation and number) in the deck. Clicking on a card will select (or unselect) it. When three cards are selected the computer will decide if they are a set and remove them if they are. There is a button that activates sounds so that a positive noise will be made if a set is found, a negative one if a the triple turns out not to be a set.

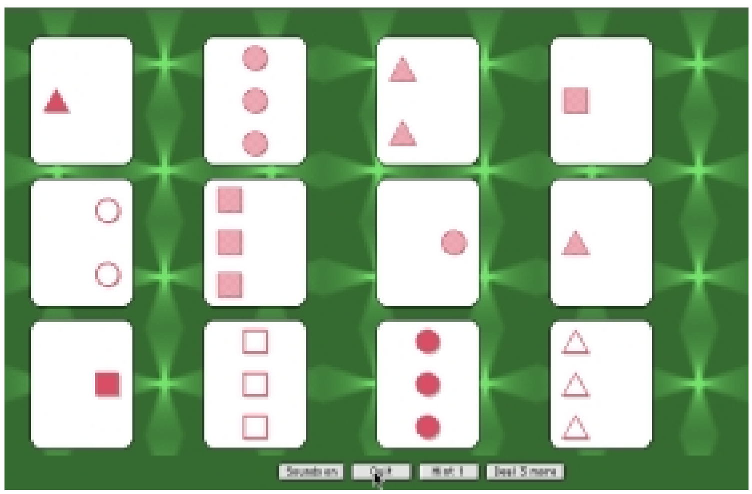


Figure 3. The playing field of Advanced Set

There is a button that will give up to three hints. The first hint will be whether a set exists in the cards that are showing (the message will be either 'Triple exists' or 'No triple exists'). The second hint will be the position of one card that is part of a set. The third and last hint will be the position of a second card that is in a triple.

If no set exists, the only way to continue is to deal three more cards. If a set does exist then the button that deals three cards will be disabled.

The quit button will immediately end the game and return to the main screen.

3. Some Mathematics of Set

The card game suggests many combinatorial questions both enumerative and algebraic that seem to be fairly difficult to solve but are very elementary to state. This could be a wide open area of research as none of the questions listed below seem to be trivial. By listing a few problems of this sort I hope to generate some interest in the mathematical questions that this game inspires. Perhaps they are equivalent to well known combinatorial problems and this game will provide another way of looking at them.

Consider the set of tuples of length k such that the entries are 0, 1 or 2. We represent the space of such tuples by the symbol $\mathbb{Z}_3^k$ considering them as the ring of k-tuples of integers mod 3. For two such elements in this space $A, B \in \mathbb{Z}_3^k$, we set $A + B := C \in \mathbb{Z}_3^k$ where $C_i = A_i + B_i \ mod\ 3$ for $1 \leq i \leq k$. Also for $a \in \mathbb{Z}_3$ and $A \in \mathbb{Z}_3^k$, we define $aA := B \in \mathbb{Z}_3^k$ where $B_i = aA_i \ mod\ 3$.

We can identify a deck of *Set* cards with the space $\mathbb{Z}_3^k$ by giving an order to the properties and then assigning 0, 1 or 2 to each of the values that the properties can take on thereby associating to each card a tuple of k values with entries in $\mathbb{Z}_3$.

Definition 1. *A* set *is a collection of three points* $\{A, B, C\} \subset \mathbb{Z}_3^k$ *such that* $A + B + C = (0^k)$ *and* $A \neq B \neq C$.

This definition agrees with the meaning of a set from the card game since it may easily be checked that the sum of A_i, B_i and C_i is $0\ mod\ 3$ if and only if either all values are the same or all are different. The first question that we consider is, 'How many sets are there in a deck with k properties?' It is an elementary counting problem to solve it, and it is about the only combinatorial question we consider here that we currently have a solution to.

Proposition 2. *The number of different sets such that r properties all have different values (with $1 \leq r \leq k$) and $k - r$ properties all have the same values is*

$$\binom{k}{r} 3^{k-r} 6^{r-1}$$

The total number of sets in a deck with k properties is the sum of these numbers which is $3^{k-1}(3^k - 1)/2$.

The integers from 1 to 3^k may be identified with these tuples by associating each integer with its base 3 expansion. The isomorphism may be given explicitly by the map that sends $A \in \mathbb{Z}_3^k$ to

$$\phi(A) = 1 + \sum_{i=1}^{k} A_i 3^{i-1}. \tag{1}$$

This map can be used to specify a faithful action of the symmetric group S_{3^k} on $\mathbb{Z}_3^k$ by composing it with $\sigma \in S_{3^k}$ and defining $\sigma A = \phi^{-1} \sigma \phi(A)$.

Definition 3. *The subgroup $\mathcal{I}_k \subset S_{3^k}$ is defined to be*

$$\mathcal{I}_k = \{\sigma \in S_{3^k} \mid \textit{if } \{A, B, C\} \textit{ is a set then } \{\sigma A, \sigma B, \sigma C\} \textit{ is also a set } \} \tag{2}$$

We note that if $a + b + c = 0\ mod\ 3$ then $\tau_a + \tau_b + \tau_c = 0\ mod\ 3$ for any permutation $\tau \in S_3$ (where we have identified $\tau_0 = \tau_3$). Hence $(\tau^{(1)}, \ldots, \tau^{(k)}) \in S_3^k$ where

$$(\tau^{(1)}, \ldots, \tau^{(k)}) A = (\tau^{(1)}_{A_1}, \ldots, \tau^{(k)}_{A_k}) \tag{3}$$

is in $\mathcal{I}_k$. As well we see that for $\alpha \in S_k$, the action of α on elements $A \in \mathbb{Z}_3^k$ by

$$\alpha[A] = (A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_k}) \tag{4}$$

is well defined. Clearly then, the group $S_3 \ltimes S_k = \{(\tau^{(1)}, \ldots, \tau^{(k)})\alpha | \tau^{(i)} \in S_3, \alpha \in S_k\}$ is a subgroup of $\mathcal{I}_k$. However, it is not clear that elements of $S_3 \ltimes S_k$ are the only elements of this group.

Considering $\mathbb{Z}_3^k$ as a module over the field $\mathbb{Z}_3$, what is the breakdown of the space into irreducible representations of S_k? As this is a module over a finite field the representation theory is not as well understood, but it should be possible to determine it for this case.

Open Problem 1. *Determine the order of the group $\mathcal{I}_k$.*

Having a more direct description of this group would perhaps give some insight into some of the other combinatorial problems we will mention here. There is an elementary way of at least arriving at a tighter bound on the maximum size of $\mathcal{I}_k$. We show the following proposition.

Proposition 4. *If $\sigma \in \mathcal{I}_k$ and $A, B \in \mathbb{Z}_3^k$, then*

$$\sigma(2A) = 2\sigma(A) + 2\sigma(0^k)$$
$$\sigma(A+B) = \sigma(A) + \sigma(B) + 2\sigma(0^k)$$
$$\sigma(A) = \sigma(0^k) + \sum_{i=1}^{k} A_i\sigma(E^{(i)}) + 2A_i\sigma(0^k)$$

where for $1 \le i \le k$, $E^{(i)} = (0^{i-1}10^{k-i})$.

Proof. For any $A \in \mathbb{Z}_3^k$, we have that $A + 2A + (0^k) = (0^k)$. Hence, for $\sigma \in \mathcal{I}_k$, $\sigma A + \sigma(2A) + \sigma(0^k) = (0^k)$. Thus $\sigma A = 2\sigma(2A) + 2\sigma(0^k)$. Using this relation, since $(A+B) + 2A + 2B = \sigma(A+B) + \sigma(2A) + \sigma(2B) = (0^k)$, we may show the second relation. Applying these two relations on the equation $A = \sum_{i=1}^{k} A_i E^{(i)}$ shows the third part of the proposition. □

The third relation in the previous proposition demonstrates that by specifying the value of $\sigma E^{(i)}$ and $\sigma(0^k)$ determines σ for every other element in $\mathbb{Z}_3^k$. This demonstrates the following corollary.

Corollary 5.

$$3!^k k! \le |\mathcal{I}_k| \le (3^k)_{k+1} \tag{5}$$

where $(a)_k = a(a-1)\cdots(a-k+1)$.

Consider the question mentioned in the first section. What is the maximum number of cards that one can have that does not contain a set.

Open Problem 2. *What is the largest collection of points in $\mathbb{Z}_3^k$ such that no subset of size 3 forms a set?*

A subset of points in $\mathbb{Z}_3^k$ that does not contain a set will be called a *no-set*. We already know that for $k = 1, 2, 3, 4$ the answer to this open problem is $2, 4, 9, 20$. These were computed by a computer program, and an example of one of each size is listed at the web page for the game of *Set* [1].

The calculation of the largest no-set is definitely suggestive of the type of problem considered in Ramsey theory. The motivating problem within Ramsey theory is the calculation of the following numbers.

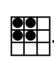
Definition 6. *Let $r(p,t,n)$ be the smallest integer such that there is a set X of order $r(p,t,n)$ where for every coloring of the p element subsets of X with t different colors, there is a subset $Y \subseteq X$ of order n such that all of the p-element subsets of Y are the same color.*

To be slightly more precise, a coloring of a set U with t different colors is a map from U to the set $\{1, \ldots, t\}$.

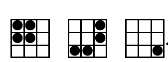
The existence of these integers $r(p,t,n)$ was proven in a theorem of Ramsey in 1930. The situation that we are considering here seems similar, but there is something different going on here and the exact connection with the Ramsey numbers is not obvious.

If $k = 1$, then of course $\{(0), (1)\}$ is a no-set.

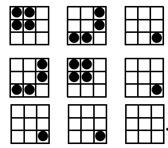
If $k = 2$, then $\{(00), (01), (10), (11)\}$ is a no-set of size 4. This may be represented graphically by drawing a 3×3 array and placing a dot in an entry if the coordinates of the point are in the set.

.

If $k = 3$, then $\{(000), (010), (100), (110), (121), (211), (021), (201), (222)\}$ is a no-set of size 9. This may be represented by the $3 \times 3 \times 3$ array:

.

If $k = 4$, then $\{(0000), (0100), (1000), (1100), (1210), (2110), (0210), (2010), (2220), (0011), (0111), (1011), (1111), (1201), (2101), (0201), (2001), (2221), (2202), (2212)\}$ is a no-set of size 20. This may be represented by the $3 \times 3 \times 3 \times 3$ array:



Perhaps there is a way of constructing a single no-set of $\mathbb{Z}_3^k$ that has maximal size. For the case of $k = 5$, if the pattern that appears in the configuration for $k = 4$ is repeated in a manner similar to the generalization from $k = 3$ to $k = 4$, it is possible to construct collection of 40 points that do not contain a set. It is not clear if this no-set is maximal. It would be interesting to know if the orbit any particular no-set of size n under the group $\mathcal{I}_k$ is the set of all no-sets of size n.

Open Problem 3. *What is the number of no-sets of size n?*

We would also like to be able to answer questions on the other end of the spectrum, namely, how is it possible to maximize the number of sets that appear in a collection of cards?

Open Problem 4. *What is the maximum number of sets that can be found in a collection of n points?*

This problem was considered by some recreational mathematicians, Tom Magliery and Judd Weeks [2], who found a collection of 12 cards from the standard Set deck with 14 different sets. Their solution was found by choosing 9 cards that all have two properties the same (these contain 12 different sets) and it is possible to choose the last two cards so that they form two additional sets. It is not known whether 14 is maximal or not.

In a related problem, we would like to be able to answer the following question.

Open Problem 5. *How many subsets of points from $\mathbb{Z}_3^k$ of size n are there that contain exactly ℓ sets?*

There are several generalizations of these problems if we allow the properties to have more than 3 different values. Instead of considering the space of $\mathbb{Z}_3^k$ we may look at $\mathbb{Z}_p^k$, but then there may be several different notions of what should be the definition of a set.

The first would be the definition that generalizes the notion of set as it was introduced from the game, that is, a collection of p cards is a set if all values taken on for any one property are either all the same or all different. Another definition of a set could be subset of p k-tuples in $\mathbb{Z}_p^k$ where the sum of the points is (0^k). Yet another possible definition is a collection of p points in $\mathbb{Z}_p^k$ that are all co-linear. In the case when p is 3, all of these definitions coincide.

Acknowledgments

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