The Mathematical Knight

Noam D. Elkies
Richard P. Stanley

Introduction

Much has been said of the affinity between mathematics and chess: two domains of human thought where very limited sets of rules yield inexhaustible depths, challenges, frustrations and beauty. Both fields support a venerable and burgeoning technical literature and attract much more than their share of child prodigies. For all that, the intersection of the two domains is not large. While chess and mathematics may favor similar mindsets, there are few places where a chess player or analyst can benefit from a specific mathematical idea, such as the symmetry of the board and of most pieces' moves (see for instance [24]) or the combinatorial game theory of Berlekamp, Conway, and Guy (as in [4]). Still, when mathematics does find applications in chess, striking and instructive results often arise.

This two-part article shows several such applications that feature the knight and its characteristic $(2, 1)$ leap. It is based on portions of a book tentatively entitled *Chess and Mathematics*, currently in preparation by the two authors of this article, that will cover all aspects of the interactions between chess and mathematics. Mathematically, the choice of $(2, 1)$ and of the 8×8 board may seem to be a special case of no particular interest, and indeed we shall on occasion indicate variations and generalizations involving other leap parameters and board sizes. But long experience points to the standard knight's move and chessboard size as felicitous choices not only for the game of chess but also for puzzles and problems involving the board and pieces, including several of our examples.

This first part concentrates on puzzles such as the knight's tour. Many of these are clearly mathematical problems in a very thin disguise (for instance, a closed knight's tour is a Hamiltonian circuit on a certain graph $\mathcal{G}$), and can be solved or at least better understood using the terminology and techniques of combinatorics. We also relate a few of these ideas with practical endgame technique (see Diagrams 1ff., 10, 11). The second part shows some remarkable chess problems featuring the knight or knights. Most "practical" chess players have little patience for the art of chess problems, which has evolved a long way from its origins in instructive exercises. But the same formal concerns that may deter the over-the-board player give some problems a particular appeal to mathematicians. For instance, we will exhibit a position, constructed by P. O'Shea and published in 1989, where White, with only king and knight, has just one way to force mate in 48 (the current record). We also show the longest known legal game of chess that is determined completely by its last move (discovered by Rösler in 1994) — which happens to be checkmate by promotion to a knight.

Algebraic notation.
We assume that the reader is familiar with the rules of chess, but require very little knowledge of chess strategy. (The reader who knows, or is willing to accept as intuitively obvious, that king and queen win against king or even king and knight if there is no immediate draw, will have no difficulty following the analysis.) The reader will, however, have to follow the notation for chess moves, either by visualizing the moves on the diagram or by setting up the position on the board. Several notation systems have been used; the most common one nowadays, and the one we use here, is "algebraic notation", so called because of the coordinate system used to name the squares of the board. In the remaining paragraphs of this introductory section we outline this notation system. Readers already fluent in algebraic notation may safely skip ahead to Section 1.

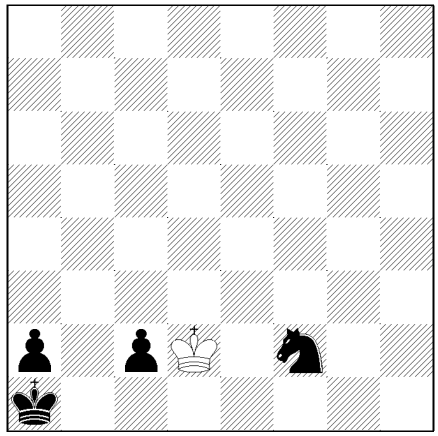
Each square on the 8×8 board is uniquely determined by its row and column, called "rank" and "file" respectively. The ranks are numbered from 1 to 8, the files named by letters a through h. In the initial array, ranks 1 and 2 are occupied by White's pieces and pawns, ranks 8 and 7 by Black's, both queens are on the d-file, and both kings on the e-file. Thus, viewed from White's side of the board (as are all the diagrams in this article), the ranks are numbered from bottom to top, the files from left to right. We name a square by its column followed by the row; for instance, the White king in Diagram 1 below is at d2. Each of the six kinds of chessmen is referred to by a single letter, usually its initial: K, Q, R, B, P are king, queen, rook, bishop, and pawn (often lower-case p is seen for pawn). We cannot use the initial letter for the knight because K is already the king, so we use its phonetic initial, N for kNight. For instance, Diagram 1 can be described as: White Kd2, Black Ka1, Nf2, Pa2, Pc2. To notate a chess move we name the piece and its destination square, interpolating "×" if the move is a capture. For pawn moves the P is usually suppressed; for pawn captures, it is replaced by the pawn's file. Thus in Diagram 11, Black's pawn moves are notated a2 and a×b2 rather than Pa2 and P×b2. We follow a move by "+" if it gives check, and by "!" or "?" if we regard it as particularly strong or weak. In some cases "!" is used to indicate a thematic move, i.e., a move that is essential to the "theme" or main point of the problem. As an aid to following the analysis, moves are numbered consecutively, from the start of the game or from the diagram. For instance, we shall begin the discussion of Diagram 1 by considering the possibility "1.K×c2 Nd3!". Here "1" indicates that these are White's and Black's first moves from the diagram; "K×c2" means that the White king captures the unit on c2; and "Nd3!" means that the Black knight moves to the unoccupied square d3, and that this is regarded as a strong move (the point here being that Black prevents 2.Kc1 even at the cost of letting White capture the knight). When analysis begins with a Black move, we use "..." to represent the previous White move; thus "1 ... Nd3!" is the same first Black move.

A few further refinements are needed to subsume promotion and castling, and to ensure that every move is uniquely specified by its notation. For instance, if Black were to move first in Diagram 1 and promoted his c2-pawn to a queen (giving check), we would write this as 1 ... c1Q+, or more likely 1 ... c1Q+?, because we shall see that after 2.K×c1 White can draw. Short and long castling are notated 0-0 and 0-0-0 respectively. If the piece and destination square do not specify the move uniquely, we also give the departure square's file, rank, or both. An extreme example: Starting from Diagram 9, "Nb1" uniquely specifies a move of the c3 knight. But to move it to d5 we would write "Ncd5" (because other knights on the b- and f-files could also reach d5); to a4, "N3a4" (not "Nca4" because of the knight on c5); and to e4, "Nc3e4" (why?).

1 A chess endgame

We begin by analyzing a relatively simple chess position (Diagram 1 below). This may look like an endgame from actual play, but is a composed position — an "endgame study" — created (by NDE) to bring the key point into sharper focus.

Diagram 1



White to move

White, reduced to bare king, can do no better than draw, and even that with difficulty: Black will surely win if either pawn safely promotes to a queen. A natural try is 1.K×c2, eliminating one pawn and imprisoning two of Black's remaining three men in the corner. But 1 … Nd3! breaks the blockade (Diagram 2a). Black threatens nothing but controls the key square c1. The rules of chess do not allow White to pass the move; unable to go to c1, the king must move elsewhere and release Black's men. After 2.K×d3 (or any other move) Kb1 followed by 3 …a1Q, Black wins easily.

Diagram 2a

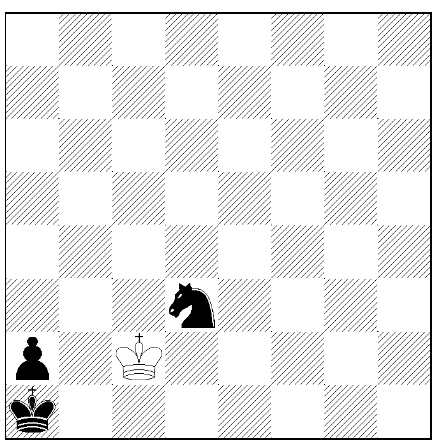
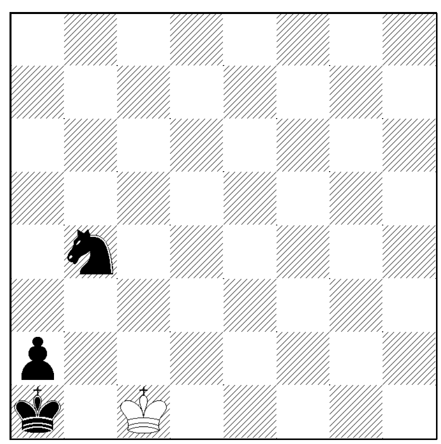


Diagram 2b



Returning to Diagram 1, let us try instead 1.Kc1! This still locks in the Black Ka1 and Pa2, and prepares to capture the Pc2 next move, for instance 1 … Nd3+ 2.K×c2, arriving at Diagram 2a with Black to move. White has in effect succeeded in passing the move to Black by taking a detour from d2 to c2. Now it is Black who cannot pass, and any move restores the White king's access to c1. For instance, play may continue 2 … Nb4+ 3.Kc1, reaching Diagram 2b. Black is still bottled up. If it were White to move in Diagram 2b, White would have to release Black with Kd1 or Kd2 and lose; but again Black must move and allow White back to c2, for instance 3 … Nd3+ 4.Kc2 and we are back at Diagram 2a.

So White does draw — at least if Black obligingly shuttles the knight between d3 and b4 to match the White king's oscillations between c1 and c2. But what if Black tries to improve on this? While the king is limited to those two squares, the knight can roam over almost the entire board. For instance, from Diagram 2a Black might bring the knight to the far corner in m moves, reaching a position such as Diagram 3a, and then back to d3 in n moves. If $m + n$ is odd, then Black will win since it will be White's turn to move. Instead of d3, Black can aim for b3 or e2, which also control c1; but each of these is two knight moves away from d3, so we get an equivalent parity condition. Alternatively, Black might try to reach b4 from d3 in an *even* number of moves, to reach Diagram 2b with White to move; and again Black could aim for another square that controls c2. But each of these squares is one or three knight moves away from d3, so again would yield a closed path of odd length through d3.

Can Black thus pass the move back to White? For that matter, what should White do in Diagram 3b? Does either Kc1 or K×c2 draw, or is White lost regardless of this choice?

Diagram 3a

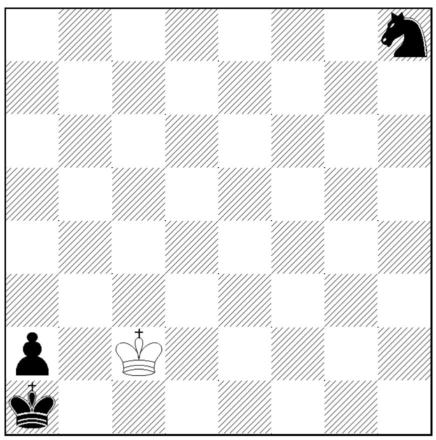
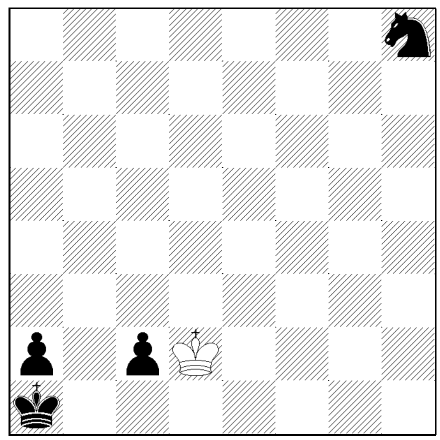


Diagram 3b



White to move

The outcome of Diagram 2a thus hinges on the answer to the following problem in graph theory:

Let $\mathcal{G} = \mathcal{G}_{8,8}$ be the graph whose vertices are the 64 squares of the 8×8 chessboard and whose edges are the pairs of squares joined by a knight's move. Does $\mathcal{G}$ have a cycle of odd length through d3?

Likewise White's initial move in Diagram 3b and the outcome of this endgame comes down to the related question concerning the same graph $\mathcal{G}$:

What are the possible parities of lengths of paths on $\mathcal{G}$ from h8 to c1 or c2?

The answers result from the following basic properties of $\mathcal{G}$:

Lemma. *(i) The graph $\mathcal{G}$ is connected. (ii) The graph is bipartite, the two parts comprising the 32 light squares and 32 dark squares of the chessboard.*

Proof: Part (i) is just the familiar fact that a knight can get from any square on the chessboard to any other square. Part (ii) amounts to the observation that every knight move connects a light and a dark square.

Corollaries. 1) There are no knight cycles of odd length on the chessboard. 2) Two squares of the same color are connected by knight-move paths of even length but not of odd length; two square of opposite color are connected by knight-move paths of odd length but not of even length.

We thus answer our chess questions: White draws both Diagram 1 and Diagram 3b by starting with Kc1. More generally, for any initial position of the Black knight, White chooses between c1 and c2 by moving to the square of the same color as the one occupied by the knight.

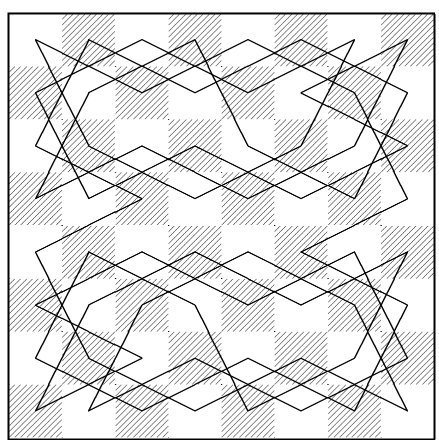
Remark. Our analysis would reach the same conclusions if the Black pawn on c2 were removed from Diagrams 1 and 3b; we included this superfluous pawn only as bait to make the wrong choice of c2 more tempting.

Puzzle 1. For which rectangular boards (if any) does part (i) or (ii) of the Lemma fail? That is, which $\mathcal{G}_{m,n}$ are not connected, or not bipartite? (All puzzles and all diagrams not explicated in the text have solutions at the end of this article.)

Knight's tours and the Thirty-Two Knights

The graph $\mathcal{G}$ arises often in problems and puzzles involving knights. For instance, the perennial knight's tour puzzle asks in effect for a Hamiltonian path on $\mathcal{G}$; a "re-entrant" or "closed" knight's tour is just a Hamiltonian circuit. The existence of such tours is classical — even Euler spent some time constructing them, finding among others the following elegant centrally symmetric tour (from [9, p. 191]):

Diagram 4

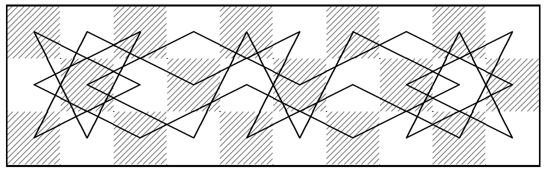


a closed knight's tour constructed by Euler

The extensive literature on knight's tours includes many examples, which, when numbered along the path from 1 to 64, yield semi-magic squares (all row and column sums equal 260), sometimes with further "magic" properties, but it is not yet known whether a fully magic knight's tour (one with major diagonals as well as rows and columns summing to 260), either open or closed, can exist.

More generally, we may ask for Hamiltonian circuits on $\mathcal{G}_{m,n}$ for other m, n; that is, for closed knight's tours on other rectangular chessboards. A necessary condition is that $\mathcal{G}_{m,n}$ be a connected graph with an even number of vertices. Hence we must have $2|mn$ and both m, n at least 3 (cf. Puzzle 1). But not all $\mathcal{G}_{m,n}$ satisfying this condition admit Hamiltonian circuits. For instance, one easily checks that $\mathcal{G}_{3,4}$ is not Hamiltonian. Nor are $\mathcal{G}_{3,6}$ and $\mathcal{G}_{3,8}$, but $\mathcal{G}_{3,10}$ has a Hamiltonian circuit, as does $G_{3,n}$ for each even $n > 10$. For instance, the next diagram shows a closed knight's tour on the 3×10 board:

Diagram 5



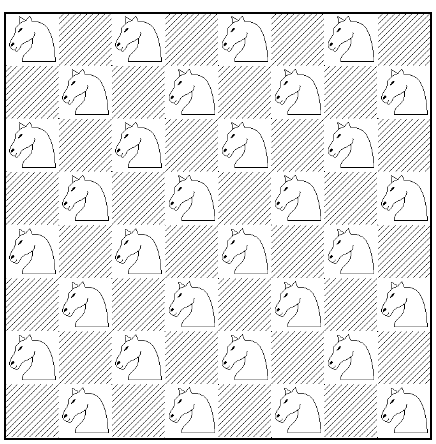
a closed knight's tour on the 3×10 board

There are sixteen such tours (ignoring the board symmetries). More generally, enumerating the closed knight's tours on a $3 \times (8 + 2n)$ board yields a sequence 16, 176, 1536, 15424, ... satisfying a constant linear recursion of degree 21 that was obtained independently by Knuth and NDE in April, 1994. See [23, Sequence A070030]. In 1997, Brendan McKay first computed that there are 13267364410532 (more than 1.3×10^{13}) closed knight's tours on the 8×8 board ([19]; see also [23, Sequence A001230],[26]).

We return now from enumeration to existence. After $\mathcal{G}_{3,n}$ the next case is $\mathcal{G}_{4,n}$. This is trickier: the reader might try to construct a closed knight's tour on a 4×11 board, or to prove that none exists. We answer this question later.

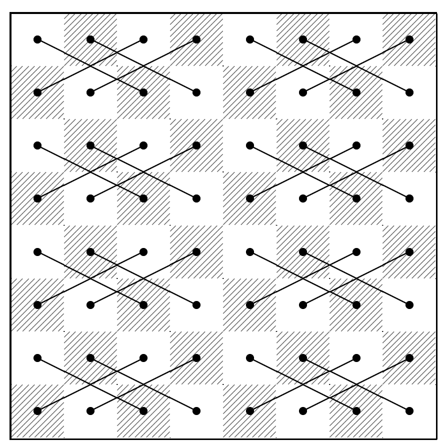
What of maximal cliques and cocliques on $\mathcal{G}$? A clique is just a collection of pairwise defending (or attacking) knights. Clearly there can be no more than two knights, again because $\mathcal{G}$ is bipartite: two squares of the same color cannot be a knight's move apart, and any set of more at least three squares must include two of the same color. Cocliques are more interesting: how many pairwise *non*attacking knights can the chessboard accommodate?[1] We follow Golomb ([21], via M. Gardner [9, p. 193]). Again the fact that $\mathcal{G}$ is bipartite suggests the answer (Diagram 6):

Diagram 6



32 mutually nonattacking knights

Diagram 7



A one-factor in $\mathcal{G}$

It is not hard to see that we cannot do better: the 64 squares may be partitioned into 32 pairs each related by a knight move, and then at most one square from each pair can be

[1] Burt Hochberg jokes (in [11, p. 5], concerning the analogous problem for queens) that the answer is 64, all White pieces or all Black: pieces of the same color cannot attack each other! Of course this joke, and similar jokes such as crowding several pieces on a single square, are extraneous to our analysis.

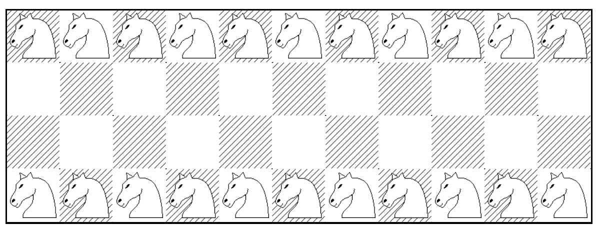
used. See Diagram 7. This is Patenaude's solution in [21]. Such a pairing of $\mathcal{G}$ is called a "one-factor" in graph theory. Similar one-factors exist on all $\mathcal{G}_{m,n}$ when $2|mn$ and m, n both exceed 2; they can be used to show that in general a knight coclique on an $m \times n$ board has size at most $mn/2$ for such m, n.

Puzzle 2. What happens if m, n are both odd, or if $m \leq 2$ or $n \leq 2$?

Are Diagram 6 and its complement the only maximal cocliques? Yes, but this is harder to show. One elegant proof, given by Greenberg in [21], invokes the existence of a closed knight's tour, such as Euler's Diagram 4. In general, on a circuit of length $2M$ the only sets of M pairwise nonadjacent vertices are the set of even-numbered vertices and the set of odd-numbered ones on the circuit. Here $M = 32$, and the knight's tour in effect embeds that circuit into $\mathcal{G}$, so *a fortiori* there can be at most two cocliques of size M on $\mathcal{G}$ — and we have already found them both!

Of course this proof applies equally to any board with a closed knight's tour: on any such board the light- and dark-squared subsets are the only maximal cocliques. Conversely, a board for which there are further maximal cocliques cannot support a closed knight's tour. For example, any $4 \times n$ board has a mixed-color maximal coclique, as illustrated for $n = 11$ in the next diagram:

Diagram 8



a third maximal knight coclique on the 4×11 board

This yields possibly the cleanest proof that *there is no closed knight's tour on a* $4 \times n$ *board for any* n. (According to Jelliss [14], this fact was known to Euler and first proved by C. Flye Sainte-Marie in 1877; Jelliss attributes the above clean proof to Louis Posa.)

Warning: the existence of a closed knight's tour is a sufficient but not necessary condition for the existence of only two maximal knight cocliques. It is known that an $m \times n$ board supports a closed tour if and only if its area mn is an even integer > 24 and neither m nor n is 1, 2, or 4. In particular, as noted above there are no closed knight's tours on the 3×6 and 3×8 boards, though as it happens on each of these boards the only maximal knight cocliques are the two obvious monochromatic ones.

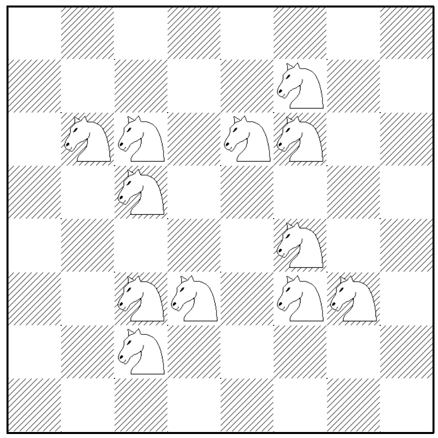
More about $\mathcal{G}$: Domination number, girth, and the knight metric

Another classic puzzle asks: how many knights does it take to either occupy or defend every square on the board? In graph theory parlance this asks for the "domination number" of $\mathcal{G}$.[2]

[2] This terminology is not entirely foreign to the chess literature: A piece is said to be "dominated" when it can move to many squares but will be lost on any of them. (The meaning of "many" in this definition is not precise because domination is an artistic concept, not a mathematical one.) The introduction of this term into the chess lexicon is attributed to Henri Rinck ([12, p. 93], [16, p. 151]). The task of constructing economical domination positions, where a few chessmen cover many squares, has a pronounced combinatorial flavor; the great composer of endgame studies G.M. Kasparyan devoted an entire book to the subject, *Domination in 2545 Endgame Studies*, Progress Publishers, Moscow, 1980.

For the standard 8×8 board, the following symmetrical solution with 12 knights has long been known:

Diagram 9



All unoccupied squares controlled

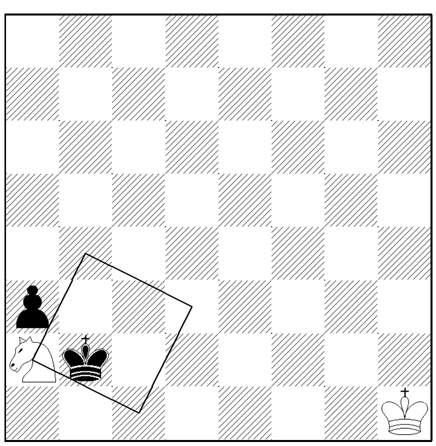
Puzzle 3. Prove that this solution is unique up to reflection.

The knight domination number for chessboards of arbitrary size is not known, not even asymptotically. See [9, Ch.14] for results known at the time for square boards of order up to 15, most dating back to 1918 [1, Vol.2, p. 359]. If we ask instead that every square, occupied or not, be defended, then the 8×8 chessboard requires 14 knights. On an $m \times n$ board, at least $mn/8$ knights are needed since a knight defends at most 8 squares.

Puzzle 4. Prove that $mn/8 + O(m + n)$ knights suffice. HINT: treat the light and dark squares separately.

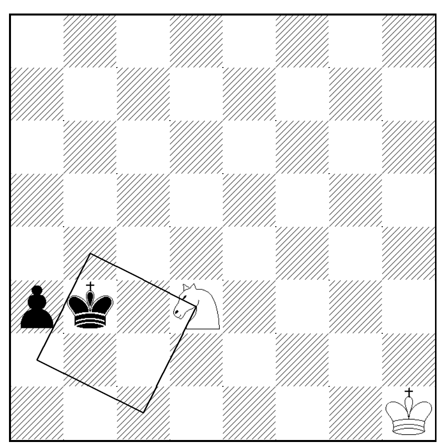
We already noted that $\mathcal{G}$, being bipartite, has no cycles of odd length. (We also encountered the non-existence of 3-cycles as "$\mathcal{G}$ has no cliques of size 3".) Thus the girth (minimal cycle length) of $\mathcal{G}$ is at least 4. In fact the girth is exactly 4, as shown for instance in Diagram 10.

Diagram 10



White to move draws

Diagram 10a



After 2 Nd3!

This square cycle is important to endgame theory: a White knight traveling on the cycle can prevent the promotion of the Black pawn on a3 supported by its king. To draw this position White must either block the pawn or capture it, even at the cost of the knight. The point is seen after 1.Nb4 Kb3 2.Nd3! (reaching Diagram 10a) a2 3.Nc1+!, "forking" king and pawn and giving White time for 4.N×a2 and a draw. On other Black moves from Diagram 10a White resumes control of a2 with 3.Nc1 or 3.Nb4; for instance 2 ... Kc2 3.Nb4+ or 2 ... Kc3 3.Nc1 Kb2 (else Na2+) 4.Nd3+! etc. Note that the White king was not needed. [3]

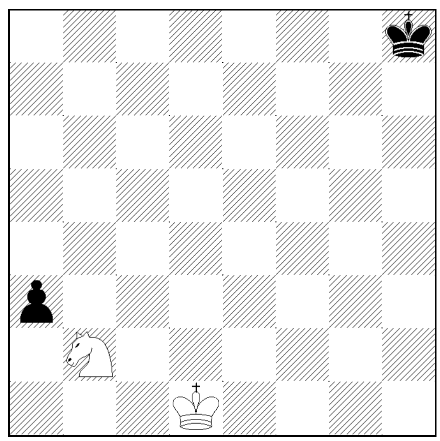
Puzzle 5. Construct a position where this Nd5 resource is White's only way to draw.

Warning: this puzzle is hard, and requires considerably more chess background than anything else in this article. The construction requires some delicacy: is not enough to simply stalemate the White king, since then White can play 2.Na2 with impunity; on the other hand if the White king is put in Zugzwang (so that it has some legal moves, but all of them lose), then the direct 1 ... a2 2.N×a2 K×a2 wins for Black.

Even more important for the practical chessplayer is the distance function on $\mathcal{G}$, which encodes the number of moves a knight needs to get from any square to any other. The diameter (maximal distance) on $\mathcal{G}$ is 6, which is attained only by diagonally opposite corners. This is to be expected, but shorter distances bring some surprises. The following table shows the distance from each vertex of $\mathcal{G}$ to a corner square:

5	4	5	4	5	4	5	6
4	3	4	3	4	5	4	5
3	4	3	4	3	4	5	4
2	3	2	3	4	3	4	5
3	2	3	2	3	4	3	4
2	1	4	3	2	3	4	5
3	4*	1	2	3	4	3	4
♘	3	2	3	2	3	4	5

Diagram 11



White loses

The starred entry is due to the board edges: a knight can travel from any square to any diagonally adjacent square in two moves except when one of them is a corner square. But the other irregularities of the table at short distances do not depend on edge effects. Anywhere on the board, it takes the otherwise agile knight three moves to reach an orthogonally adjacent square, and four moves to travel two squares diagonally. This peculiarity must be absorbed by any chessplayer who would learn to play with or against knights. One consequence, known to endgame theory, is Diagram 11, which exploits both the generic irregularity and the special corner case. Even with White to move, this position is a win for Black, who will play ... a2 and ... a1Q. One might expect that the knight is close enough to stop this, but in fact it would take it three moves to reach a2 and four to reach a1, in each case one too many. In

[3] Note to more advanced chessplayers: it might seem that the knight does need a bit of help after 1.Nb4 Kb1!?, when either 2.Na2? or 2.Nd3? loses (in the latter case to 2 ... a2) but Black has no threat so White can simply make a random ("waiting") king move. But this is not necessary, as White could also draw by thinking (and playing) out of the a2-b4-d3-c1-a2 box: 1.Nb4 Kb1 2.Nd5! If now 2 ... a2 then 3.Nc3+ is a new drawing fork, and otherwise White plays 3.Nb4 and resumes the square dance.

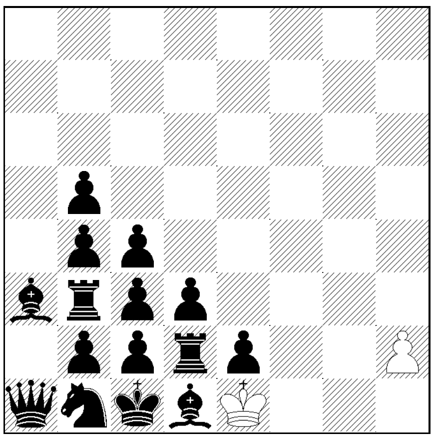
fact this knight helps Black by blocking the White king's approach to a1!

Puzzle 6. Determine the knight distance from $(0,0)$ to (m,n) on an infinite board as a function of the integers m,n.

Further puzzles

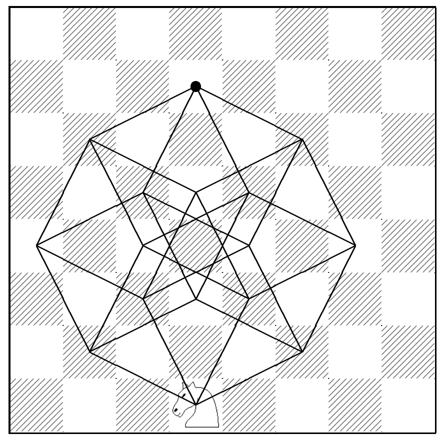
We conclude the first part with several more puzzles that exploit or extend our discussion:

Diagram 12



White to play and mate as quickly as possible

Diagram 13



the 4! shortest knight paths from d1 to d7

Puzzle 7. How does White play in Diagram 12 to force checkmate as quickly as possible against any Black defense?

Yes, it's White who wins, despite having only king and pawn against 15 Black men. But these men are almost paralyzed, with only the queen able to move in its corner prison. White must keep it that way: if he ever moves his king, Black will sacrifice his e2-pawn by promoting it, bring the Black army to life and soon overwhelm White. So White must move only the pawn, and the piece that it will promote to. That's good enough for a draw, but how to actually win?

Puzzle 8. (See Diagram 13.) There are exactly $24 = 4!$ paths that a knight on d1 can take to reach d7 in four moves; plotting these paths on the chessboard yields a beautiful projection of (the 1-skeleton of) the 4-dimensional hypercube! Explain.

Puzzle 9. We saw that there is an essentally unique maximal configuration of 32 mutually non-defending knights on the 8×8 board.
i) Suppose we allow each knight to be defended at most once. How many more knights can the board then accommodate?
ii) Now suppose we require each knight to be defended *exactly* once. What is the largest number of knights on the 8×8 board satisfying this constraint, and what are all the maximal configurations?

Puzzle 10. A "camel" is a $(3,1)$ leaper, that is, an unorthodox chess piece that moves from (x,y) to one of the squares $(x \pm 3, y \pm 1)$ or $(x \pm 1, y \pm 3)$. (A knight is a $(2,1)$ leaper.) Since there are eight such squares, it takes at least $mn/8$ camels to defend every square, occupied or not, on an $m \times n$ board. Are $mn/8 + O(m+n)$ sufficient, as in Puzzle 4?

Synthetic games

The remainder of this article will be devoted to composed chess problems featuring knights. A *synthetic game* [13] is a chess game composed (rather than played) in order to achieve some objective, usually in a minimal number of moves. Ideally the solution should be unique, but this is very rare. Failing this, we can hope for an "almost unique" solution, e.g., one where the final position is unique though not the move order. For instance, the shortest game ending in checkmate by a knight is 3.0 moves: 1.e3 Nc6 2.Ne2 Nd4 3.g3 Nf3 mate. White can vary the order of his moves and can play e4 and/or g4 instead of e3 and g3. The Black knight has two paths to f3. The biggest flaw, however, is that White could play c3/c4 instead of g3/g4, and Black could mate at d3. At least all 72 solutions share the central feature that White incarcerates his king at its home square. A better synthetic game involving a knight is the following.

Puzzle 11. Construct a game of chess in which Black checkmates White on Black's fifth move by promoting a pawn to a knight.

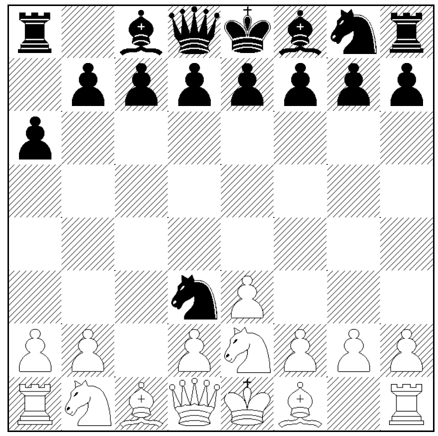
Proof games

A very successful variation of synthetic games that allows unique solutions are *proof games*, for which the length n of the game and the final position P are specified. In order for the condition (P, n) to be considered a sound problem, there should be a *unique* game in n moves ending in P. (Sometimes there will be more than one solution, but they should be related in some thematic way. Here we will only consider conditions (P, n) that are uniquely realizable, with the exception of Diagram 17.)

The earliest proof games were composed by the famous "Puzzle King" Sam Loyd in the 1890's but did not have unique solutions; the earliest sound (by today's standards) proof game seems to have been composed by T. R. Dawson in 1913. Although some interesting proof games were composed in subsequent years, the vast potential of the subject was not suspected until the fantastic pioneering efforts of Michel Caillaud in the early 1980's. A close to complete collection of all proof games published up to 1991 (around 160 problems) appears in [28].

Let us consider some proof games related to knights. We mentioned above that the shortest game ending in mate by knight has length 3.0 moves. None of the 72 solutions yield proof games with unique solutions, i.e., every terminal position has more than one way of reaching it in 3.0 moves. It is therefore natural to ask for the least number n (either an integer or half-integer) for which there exists a *uniquely realizable* game of chess in n moves ending with checkmate by knight, i.e., given the final position, there is a unique game that reaches it in n moves. Such a game was found independently by the two authors of this article in 1996 for $n = 4.0$, which is surely the minimum. The final position is shown in Diagram 14.

Diagram 14



Position after Black's 4th move. How did the game go?

Five other proof game problems involving knights are the following. The minimum known number of moves for achieving the game is given in parentheses. (We repeat that the game must be uniquely realizable from the number of moves and final position.)

Puzzle 12. Construct a proof game without any captures that ends with mate by a knight (4.5).

Puzzle 13. Construct a proof game ending with mate by a knight making a capture (5.5)

Puzzle 14. Construct a proof game ending with mate by a pawn promoting to a knight (5.5).

Puzzle 15. Construct a proof game ending with mate by a pawn promoting to a knight without a capture on the mating move (6.0).

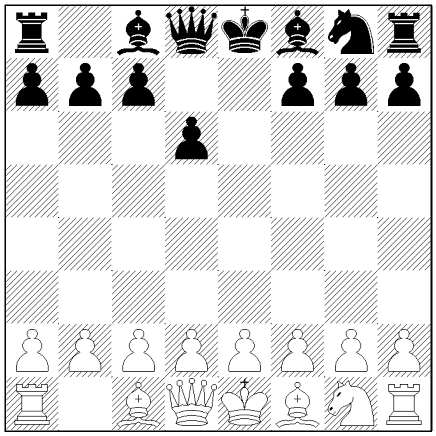
Puzzle 16. Construct a proof game ending with mate by a pawn promoting to a knight with no captures by the mating side throughout the game (7.0).

There is a remarkable variant of Puzzle 14. Rather than having the game determined by its final position and number of moves, it is instead completely determined by its last move (including the move number)! This is the longest known game with this property.

Puzzle 14′. Construct a game of chess with last move 6.g×f8N mate.

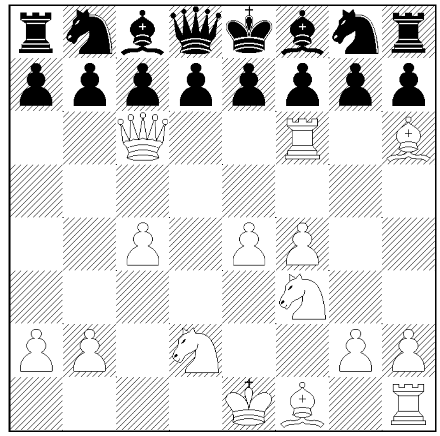
The above proof games focused on achieving some objective in the minimum number of moves. Many other proof games in which knights play a key role have been composed, of which we give a sample of five problems. Diagrams 15, 16, and 17 feature "impostors"—some piece(s) are not what they seem. The first of these (Diagram 15) is a classic problem that is one of the earliest of all proof games, while Diagram 16 is considerably more challenging. Diagram 17 features a different kind of impostor. Note that it has two solutions; it is remarkable how each solution has a different impostor. The complex and difficult Diagram 18 illustrates the *Frolkin theme*: the multiple capture of promoted pieces. Diagram 19 shows, in the words of Wilts and Frolkin [28, p. 53], that "the seemingly indisputable fact that a knight cannot lose a tempo is not quite unambiguous."

Diagram 15



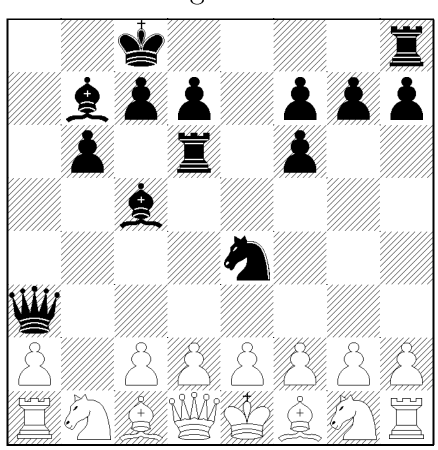
After Black's 4th. How did the game go?

Diagram 16



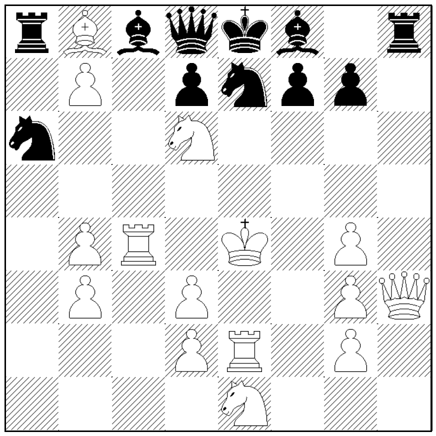
After Black's 12th. How did the game go?

Diagram 17



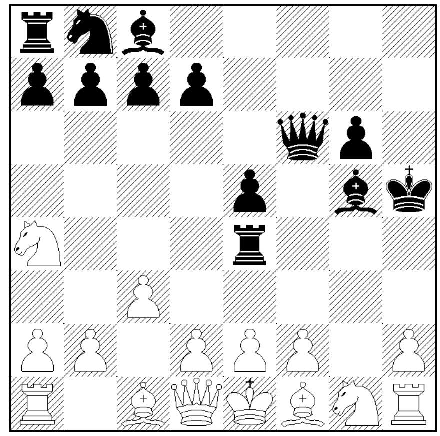
After White's 13th. How did the game go? Two solutions!

Diagram 18



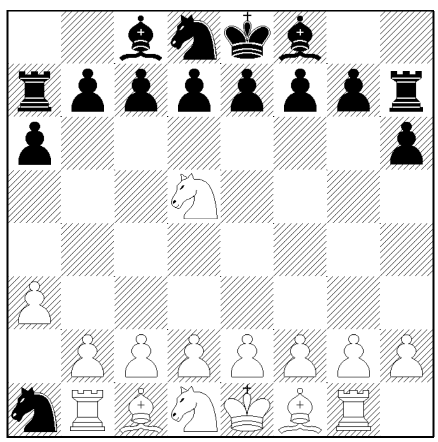
After White's 27th. How did the game go?

Diagram 19



After Black's 10th move. How did the game go?

Diagram 20



Mate in one

Retrograde analysis

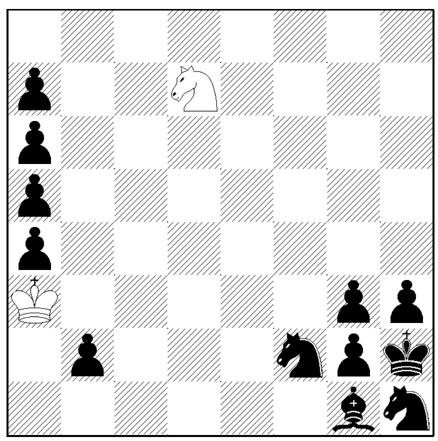
In retrograde analysis problems (called retro problems for short), it is necessary to deduce information from the current position concerning the prior history of the game. It is only assumed that the prior play is legal; no assumption is made that the play is "sensible." Proof games are a special class of retro problems. We will give only one illustration here of a retro problem that is not a proof game. It is based on considerations of parity, a common theme whenever knights are involved. Diagram 20 is a *mate in one*. A chess problem with this stipulation almost invariably involves an element of retrograde analysis, such as determining who has the move.[4]

Length records

[4] In a problem with the stipulation "Mate in n," it is assumed that White moves first unless it can be proved that Black has the move in order for the position to be legal.

Here one tries to construct a position that maximizes the number of moves which must elapse before a certain objective is satisfied. The most obvious and most-studied objective is checkmate. In other words, how large can n be in a problem with the objective "mate in n" (i.e., White to play and checkmate Black in n moves)? Chess problem standards demand that the solution should be unique if at all possible. It is too much to expect, especially for long-range problems, that White has a unique response to *every* Black move in order for White to achieve his objective. In other words, it is possible for Black to defend poorly and allow White to achieve his objective in more than one way, or even achieve it earlier than specified. The correct uniqueness condition is that the problem should be *dual-free*, which means that Black has at least one method of defending which forces each White move uniquely if White is to achieve his objective. The objective of checkmate can be combined with other conditions, such as White having only one unit besides his king. The ingenious Diagram 21 shows the current record for a "knight minimal," i.e., White's only unit besides his king is a knight. For other length records, as well as many other tasks and records, see [20].

Diagram 21



Mate in 48

Paradox

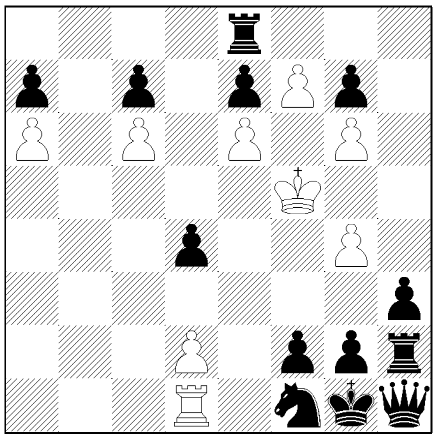
The term "paradox" has several meanings in both mathematics and ordinary discourse. We will regard a feature of a chess problem (or chess game) as paradoxical if it is seemingly opposed to common sense. For instance, common sense tells us that a material advantage is beneficial in winning a chess game or mating quickly. Thus *sacrifice* in an orthodox chess problem (i.e., a direct mate or study) is paradoxical. Of course it is just this paradoxical element that explains the appeal of a sacrifice. Another common paradoxical theme is underpromotion. Why not promote to the strongest possible piece, namely, the queen? This theme is related to that of sacrifice, since in each case the player is forgoing material. To be sure, underpromotion to knight in order to win, draw, or checkmate quickly is not so surprising (and has even occurred a fair number of times in games) since a knight can make moves forbidden to a queen. Tim Krabbé thus remarks in [15] that knighting hardly counts as a true "underpromotion."[5] Nevertheless, knight promotions can be used for surprising purposes that heighten the paradoxical effect.

Diagram 22 shows four knight sacrifices, all promoted pawns, with a total of five promotions

[5] More paradoxical are underpromotions to rooks and bishops, but we will not be concerned with them here.

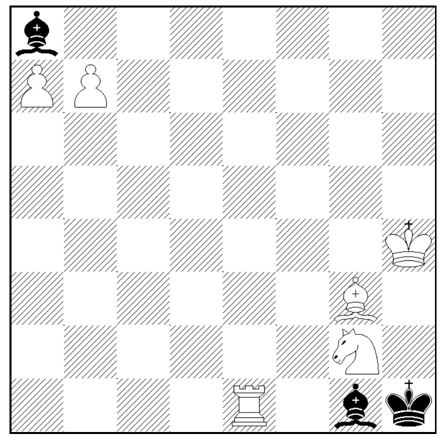
to knight. Diagram 23 shows a celebrated problem composed by Sam Loyd where a pawn promotes to a knight that threatens no pieces or checks and is hopelessly out of play. For some interesting comments by Loyd on this problem, see [27, p. 403].

Diagram 22



White to play and win

Diagram 23



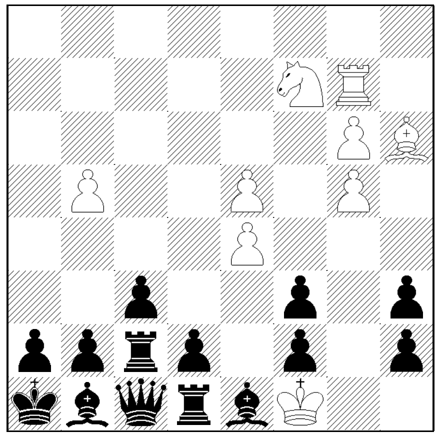
Mate in 3

Note that the impostors of Figures 15–17 may also be regarded as paradoxical, since we're trying to reach the position as quickly as possible, and it seems a waste of time to move knights into the original square(s) of other knights. Similarly the time-wasting 5.h×g8N 6.Nh6 7.N×f7 of Diagram 19 seems paradoxical—why not save a move by 5.h×g8B and 6.B×f7+?

Helpmate

In a *helpmate in n moves*, Black moves first and *cooperates* with White so that White mates Black on White's nth move. If the number of solutions of a helpmate is not specified, then there should be a unique solution. For a long time it was thought impossible to construct a sound helpmate with the theme of Diagram 24, featuring knight promotions. Note that the first obstacle to overcome is the avoidance of checkmating White or stalemating Black. The composer of this brilliant problem, Gabor Cseh, was tragically killed in an accident in 2001 at the age of 26.

Diagram 24

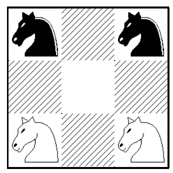


Helpmate in 10

Piece shuffle

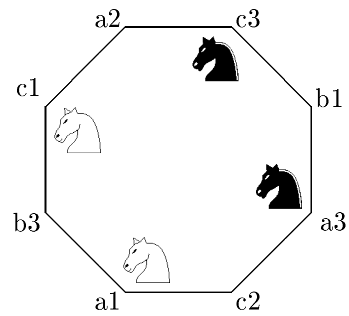
In *piece shuffles* or *permutation tasks*, a rearrangement of pieces is to be achieved in a minimum number of moves, sometimes subject to special conditions. They may be regarded as special cases of "moving counter problems" such as given in [2, pp. 769–777] or [3, pp. 58–68]. A classic example involving knights, going back to Guarini in 1512, is shown in Diagram 25. The knights are to exchange places in the minimum number of moves. (Each White knight ends up where a Black knight begins, and *vice versa*.) The systematic method for doing such problems, first enunciated by Dudeney [3, solution to #341] and called the method of "buttons and strings," is to form a graph whose vertices are the squares of the board, with an edge between two vertices if the problem piece (here a knight) can move from one vertex to the other. For Diagram 25 the graph is just an eight-cycle (with an irrelevant isolated vertex corresponding to the center square of the board). See Diagram 26. This representation of the problem makes it quite easy to see that the minimum number of moves is sixteen (eight by each color), achieved for instance by cyclically moving each knight four steps clockwise around the eight-cycle. If a White knight is added at b1 and a Black knight at b3, then somewhat paradoxically the minimum number of moves is reduced to eight! A variation of the stipulation of Diagram 25 is the following problem, whose solution is a bit tricky and essentially unique.

Diagram 25



Exchange the knights
in a minimum number of moves

Diagram 26



The graph corresponding
to Diagram 25

Puzzle 17 In Diagram 25 exchange the knights in a minimum number of move sequences, where a "move sequence" is an unlimited number of consecutive moves by the same knight.

For some more sophisticated problems similar to Diagram 25, see [10, pp. 114–124]. The most interesting piece shuffle problems connected with the game of chess (though not focusing on knights) are due to G. Foster [5, 6, 7, 8], created with the help of his computer program WOMBAT (Work Out Matrix By Algorithmic Techniques).

Puzzle answers, hints, and solutions

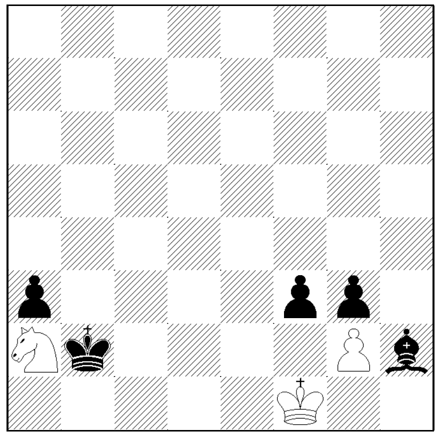
1 The graph $\mathcal{G}_{m,n}$ is connected for $m = n = 1$ (only one vertex) and not connected for $m = n = 3$ (the central square is an isolated vertex). With those two exceptions, $\mathcal{G}_{m,n}$ is connected if and only if $m > 2$ and $n > 2$. Every $G_{m,n}$ is bipartite, except $G_{1,1}$ (empty parts not allowed); each non-connected graph $G_{m,n}$ is bipartite in several ways except for $\mathcal{G}_{1,2} = \mathcal{G}_{2,1}$.

2 If $m = 1$ or $n = 1$ then $\mathcal{G}_{m,n}$ is disconnected, so the maximal coclique is the set of all mn vertices. The graph $\mathcal{G}_{2,n}$ (or $\mathcal{G}_{n,2}$) decomposes into two paths of length $\lfloor n/2 \rfloor$ and two of length $\lceil n/2 \rceil$. It thus has a one-factor if an only if $4|n$, and otherwise has cocliques of size $> n$; the maximal coclique size is $n + \delta$ where $\delta \in \{0, 1, 2\}$ and $n \equiv \pm\delta \bmod 4$. If m and n are odd integers greater than 1 then the maximal coclique size of $\mathcal{G}_{m,n}$ is $(mn+1)/2$, attained by placing a knight on each square of the same parity as a corner square of an $m \times n$ board. One can prove that this is maximal by deleting one of these squares and constructing a one-factor on the remaining $mn - 1$ vertices of $\mathcal{G}_{m,n}$.

3 Each of the four 2×2 corner subboards requires at least three knights, and no single knight may occupy or defend squares in two different subboards. Hence at least $4 \cdot 3 = 12$ knights are needed. For three knights to cover the $\{\mathrm{a1}, \mathrm{b1}, \mathrm{a2}, \mathrm{b2}\}$ subboard, one of them must be on c3; likewise f3, f6, c6 must be occupied if 12 knights are to suffice. It is now easy to verify that Diagram 9 and its reflection are the only ways to place the remaining 8 knights so as to cover the entire chessboard.

4 ([3, #319, p. 127]) On an infinite chessboard, each square of odd parity is a knight-move away from exactly one of the squares with coordinates $(2x, 2y)$ with $x \equiv y \bmod 4$. Intersecting this lattice with an $m \times n$ chessboard yields $mn/16 + O(m + n)$ knights that cover all odd squares at distance at least 3 from the nearest edge. Thus an extra $O(m + n)$ knights defend all the odd squares on the board. The same construction for the even squares yields a total of $mn/8 + O(m + n)$.

Diagram 27



White to move draws

5 One such position is Diagram 27 above. Once the a-pawn is gone, the position is a theoretical draw whether Black plays f×g2+ (Black can do no better than stalemate against K×g2, Kh1, Kg2 etc.) or f2 (ditto after Ke2, Kf1, etc.), or lets White play g×f3 and Kg2 and then jettison the f-pawn to reach the same draw that follows f×g2+. But as long as Black's a-pawn is on the board, White can move only the knight since g×f3 would liberate Black's bishop which could then force White's knight away (for instance 1.Nb4 Kb1 2.g×f3? g2+! 3.K×g2 Bd6 4.Nd5 Kb2) and safely promote the a-pawn. Black's pawn on f3 could also be on h3 with the same effect.

6 The distance is an integer, congruent to $m+n$ mod 2, that equals or exceeds each of $|m|/2$, $|n|/2$, and $(|m|+|n|)/3$. It is the smallest such integer except when in the cases already noted of $(m,n) = (0,\pm 1)$, $(\pm 1, 0)$, or $(\pm 2, \pm 2)$, when the distance exceeds the above lower bound by 2.

7 (adapted from Gorgiev) To win, White must promote the pawn to a knight, capture the pawns on b5 and c4, and then mate with N×b3 when the Black queen is on a1. Thus N×b3 must be an odd-numbered move. Therefore 1.h4, 2.h5, 3.h6, 4.h7, 5.h8N does not work because all knight paths from h8 to b3 have odd length. Since the knight cannot "lose the move", the pawn must do so on its initial move: 1.h3!, followed by 6.h8N!, 7.Nf7, 8.Nd6, 9.N×b5, 10.Nd6, 11.N×c4, 12.Na5. At this point the Black queen is on a2, having made 11 moves from the initial position; whence the conclusion: 12 ... Qa1 13.N×b3 mate. (We omitted from Gorgiev's original problem the initial move 1.Kf2×Ne1 Qa2-a1, which only served to give Black his entire army in the initial position and thus maximize the material disparity; and moved a Black pawn from c5 to b5 to make the solution unique, at some cost in strategic interest.)

8 Recall that a knight's move joins squares differing by one of the eight vectors $(\pm 1, \pm 2)$ or $(\pm 2, \pm 1)$, and check that to get some four of those to add to $(0, 6)$ we must use the four vectors with a positive ordinate in some order. Thus, to reach d7 from d1 (or, more generally, to travel six squares north with no obstruction from the edges of the board) in four moves, the knight must move once in each of its four north-going directions. Therefore a path corresponds to a permutation of the four vectors $(\pm 1, 2)$ and $(\pm 2, 1)$. The number of paths is thus $4! = 24$, and drawing them all yields the image of the 4-cube under a projection taking the unit vectors to $(\pm 1, 2)$ and $(\pm 2, 1)$. Instead of d1 and d7 we could also draw the

24 paths from a4 to g4 in four moves to get the same picture. Not b2 and f6, though: besides the 24 paths of Diagram 13 there are other four-move journeys, for instance b2-d3-f4-h5-f6.

9 (i) The maximum is still 32 (though there are many more configurations that attain this maximum). To show this, it is enough to prove that at most 8 knights can fit on a 4×4 board if each is to be defended at most once. This in turn can be seen by decomposing $\mathcal{G}_{4,4}$ as a union of four 4-cycles (Diagram 28), and noting that only two knights can fit on each 4-cycle.

(ii) Once again, the maximum is 32, this time with a new configuration (Diagram 29) unique up to reflection! (But note that this configuration has a cyclic group of 4 symmetries, unlike the elementary abelian 2-group of symmetries of the maximal coclique (Diagram 6).) That this is maximal follows from the first part of this puzzle. For uniqueness, our proof is too long to reproduce here in full; it proceeds as follows. In any 32-knight configuration, each of the four 4×4 corner subboards must contain 8 knights, two on each of its four 4-cycles. We analyze cases to show that it is impossible for two knights in different subboards to defend each other. We then show that Diagram 29 and its reflection are the only ways to fit four 8-knight configurations into an 8×8 board under this constraint.

10 Yes, $mn/8+O(m+n)$ camels suffice. The camel always stays on squares of the same color. The squares of one color may be regarded on a chessboard in its own right, tilted 45° and magnified by a factor of $\sqrt{2}$ — in other words, multiplied by the complex number $1+i$. On this board, the camel's move amounts to the ordinary knight's move since $3+i = (2-i)(1+i)$. We can thus adapt our solution of Puzzle 4. Explicitly, on an infinite chessboard each square with both coordinates odd is a camel's move away from exactly one square of the form $(4x, 8y)$. Thus camels at $(4x + a, 8x + b)$ $(a, b \in \{0, 1\})$ cover the entire board without duplication, and the intersection of this configuration with an $m \times n$ board covers all but $O(m + n)$ of its squares.

Diagram 28

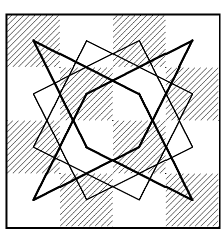
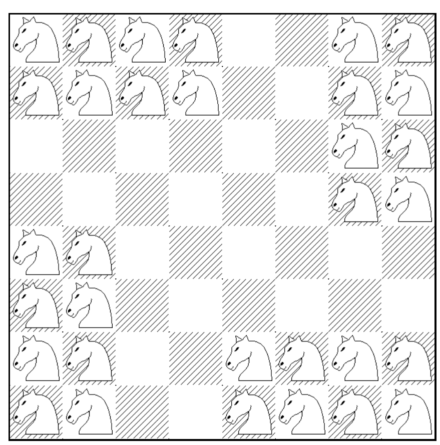


Diagram 29



Solution of Puzzle 8(ii)

11 1.d3 e5 2.Kd2 e4 3.Kc3 e×d3 4.b3 d×e2 5.Kb2 e×d1N mate. White can play d4 instead of d3 (so Black plays e×d4) and can vary his move order, but the final position is believed to be unique. This game first appeared in [17].

12 (G. Forslund, Retros Mailing List, June 1996) 1.e3 f5 2.Qf3 Kf7 3.Bc4+ Kf6 4.Qc6+ Ke5 5.Nf3 mate.

13 (G. Wicklund, Retros Mailing List, October 1996) 1.Nf3 e6 2.Ne5 Ne7 3.N×d7 e5 4.N×f8 Bd7 5.Ne6 Rf8 6.N×g7 mate.

14 (P. Rössler, *Problemkiste*, August 1994 (version)) 1.h4 d5 2.h5 Nd7 3.h6 Ndf6 4.h×g7 Kd7 5.Rh6 Ne8 6.g×f8N mate.

15 (G. Donati, Retros Mailing List, June 1996) 1.h4 g6 2.Rh3 g5 3.Re3 g×h4 4.f3 h3 5.Kf2 h2 6.Qe1 h1N mate.

16 (O. Heimo, Retros Mailing List, June 1996) 1.d4 e5 2.d×e5 d5 3.Qd4 Be6 4.Qb6 d4 5.Kd2 d3 6.Kc3 d2 7.a3 d1N mate.

14′ See solution to Puzzle 14.

17 a1-c2, c1-b3-a1, c3-a2-c1-b3, a3-b1-c3-a2-c1, c2-a3-b1-c3, a1-c2-a3, b3-c1. Seven move sequences.

Diagram solutions

Diagram 14. (N. Elkies, R. Stanley, 1996) 1.c4 Na6 2.c5 N×c5 3.e3 a6 4.Ne2 Nd3 mate.

Diagram 15. (G. Schweig, *Tukon*, 1938) 1.Nc3 d6 2.Nd5 Nd7 3.N×e7 Ndf6 4.N×g8 N×g8. The impostor is the knight at g8, which actually started out at b8.

Diagram 16. (U. Heinonen, *The Problemist* 1991) 1.c4 Nf6 2.Qa4 Ne4 3.Qc6 N×d2 4.e4 Nb3 5.Bh6 Na6! 6.Nd2 Nb4 7.Rc1 Nd5 8.Rc3 Nf6 9.Rf3 Ng8 10.Rf6 Nc5 11.f4 Na6 12.Ngf3 Nb8. Here both Black knights are impostors, as they have exchanged places! For a detailed analysis of this problem, see [16, pp. 207–209].

Diagram 17 (D. Pronkin, *Die Schwalbe*, 1985, 1st prize) 1.b4 Nf6 2.Bb2 Ne4 3.Bf6 e×f6 4.b5 Qe7 5.b6 Qa3 6.b×a7 Bc5 7.a×b8B Ra6 8.Ba7 Rd6 9.Bb6 Kd8 10.Ba5 b6 11.Bc3 Bb7 12.Bb2 Kc8 13.Bc1.

1.Nc3 Nf6 2.Nd5 Ne4 3.Nf6+ e×f6 4.b4 Qe7 5.b5 Qa3 6.b6 Bc5 7.b×a7 b6 8.a×b8N Bb7 9.Na6 0-0-0 10.Nb4 Rde8 11.Nd5 Re6 12.Nc3 Rd6 13.Nb1. This problem illustrates the *Phoenix theme*: a piece leaves its original square to be sacrificed somewhere else, then a pawn promotes to exactly the same piece which returns to the original square to replace the sacrificed piece. In the first solution the bishop at c1 is phoenix, while in the second it is the knight at b1! As if this weren't spectacular enough, Black castles in the second solution but not the first.

Diagram 18. (M. Caillaud, *Thèmes-64*, 1982, 1st prize) 1.a4 c5 2.a5 c4 3.a6 c3 4.a×b7 a5 5.Ra4 Na6 6.Rc4 a4 7.b4 a3 8.Bb2 a2 9.Na3 a1N! 10.Nb5 Nb3 11.c×b3 c2 12.Be5 c1N! 13.Bb8 Nd3+ 14.e×d3 e5 15.Qg4 e4 16.Ke2 e3 17.Kf3 e2 18.Ke4 e×f1N! 19.Nf3 Ng3+ 20.h×g3 h5 21.Re1 h4 22.Re2 h3 23.Ne1 h2 24.Qh3 h1N! 25.g4 Ng3+ 26.f×g3 Ne7 27.Nd6 mate. An amazing four promotions by Black to knight, all captured!

Diagram 19. (A. Frolkin, *Shortest Proof Games*, 1991) 1.g4 e5 2.g5 Be7 3.g6 Bg5 4.g×h7 Qf6 5.h×g8N! Rh4 6.Nh6 Re4 7.N×f7 K×f7 8.Nc3 Kg6 9.Na4 Kh5 10.c3 g6. If 5.h×g8B? Rh4 6.B×f7+ K×f7 7.Nc3 Re4 8.Na4 Kg6 9.c3 Kh5, then White must disturb his position before 10... g6. A knight is able to "lose a tempo" by taking two moves to get from g8 to f7, while a bishop must take one or at least three moves.

Diagram 20. (V. A. Korolikov, *Schach*, 1957) White's knights are on squares of the same color and hence have made an odd number of moves in all. Each White rook and the White king have made an even number of moves, and White has made one pawn move. No other

White unit (i.e., the queen and bishops) have moved. Hence White has made an even number of moves in all. Similarly Black has made an odd number of moves. Since White moved first it is currently Black's move, so Black mates in one with 1...N×c2 mate.

Diagram 21. (P. O'Shea, *The Problemist*, 1989, 1st prize) 1.Ne5 b1N+ (the only defense to 2.Nf3 mate) 2.Ka2 Nd2 3.Ka1 Nb3+ 4.Kb1 Nd2+ 5.Ka2. If Black moves either knight then checkmate is immediate, so 5... a3 is forced. Now White and Black repeat the maneuver Ka1, Nb3+, Kb1, Nd2+, Ka2 (any pawn moves by Black would just hasten the end): 8.Ka2 a4 11.Ka2 a5 14.Ka2 a6. Then 15.Ka1 Nb3+ 16.Kb1 a2+ 17.K×a2 Nd2. This manuever gets repeated until all Black's a-pawns are captured: 44.K×a2 Nd2 45.Ka1 Nb3+ 46.Kb1 Nd2+ 47.Ka2. Finally Black must allow 48.Nf3 mate or 48.Ng4 mate!

Diagram 22. (H. M. Lommer, *Szachy*, 1965) White cannot allow Black's rook at e8 to stay on the board, but how does White prevent Black from being stalemated without releasing the sleeping units in the h1 corner? 1.f×e8N d3 2.Nf6 (not 2.Nd6? e×d6, and stalemate cannot be prevented without releasing the h1 corner) g×f6 (capturing with the other pawn merely hastens the end) 3.g5 f×g5 4.g7 g4 5.g8N g3 6.Nf6 e×f6 7.Kg6 f5 8.e7 f4 9.e8N f3 10.Nd6 c×d6 11.c7 d5 12.c8N d4 13.Nb6 a×b6 14.a7 b5 15.a8N and wins, as White can play 19 N×f3 mate just after 18... b1Q. For the history of this problem, see [25, pp. xxi–xxii].

Diagram 23. (S. Loyd, *Holyoke Transcript*, 1876) 1.b×a8N! K×g2 2.Nb6, followed by 3.a8Q (or B) mate. Note that a knight is needed to prevent 2... B×a7. A queen or bishop promotion at move one would be stalemate, and a rook promotion leads nowhere. Normally a key move of capturing a piece is considered a serious flaw since it reduces Black's strength. Here, however, the capture seems to accomplish nothing so it is acceptable. Loyd himself says "[i]f the capture seems a hopeless move ...then it is obviously well concealed, and the most difficult key-move that could be selected" [18, p. 156]. For further problems by Loyd featuring distant knight promotion, see [27, pp. 402–403].

Diagram 24. (G. Cseh, *StrateGems*, 2000, 1st prize) 1.h1N! Nd6 2.h2 Nf5 3.Ng3+ N×g3 4.h1N! Ne2 5.f×e2+ Kg2! (not 5... K×e2?, since Black's tenth move would then check White) 6.f1N! Rc7 7.Bg3 R×c3 8.B×e5 R×c2 9.Bg7 R×c1 10.b×c1N! B×g7 mate. Four promotions to knight by Black.

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