Experimental Mathematics: Insight from Computation

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Prepared for
Joint Meetings

Simon Fraser University

San Antonio, Jan 13–16, 1999

URL: www.cecm.sfu.ca/preprints
URL ~jborwein/talks.html
URL: ~borwein/pi_cover.html

ABSTRACT.

The crucial role of high performance computing is now acknowledged throughout the physical, biological and engineering sciences: whether scientific computing, visualization, simulation or data-mining.

The emergence of powerful mathematical computing environments, the growing availability of fast, seriously parallel computers and the pervasive presence of the internet allow for (pure) mathematicians to partake of the same tools.

The unique features of our discipline make this both more problematic and more challenging.

◇ That said, many of the greatest computational benefits to mathematics are accessible through low-end "electronic blackboard" versions of computing.

• Through a handful of examples, embracing both high-end and low-end, I intend to illustrate the opportunities and issues with which I am personally familiar.

◇ Many involve

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

and its friends.

One Reference.

J. Borwein & R. Corless, "Emerging Tools for Experimental Mathematics," *MAA Monthly*, in press.

◇ Available as CECM Preprint 98:110.

◇ And I *may be* rash enough* to give my own perspective on what needs to be done to bring (symbolic) computing into the 21st century – that demands effectively using the parallel high performance systems of the next decade.

*At any rate verbally!

Hadamard "The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it."

◇ in E. Borel, *Lecons sur la theorie des fonctions*, 1928

Milnor "If I can give an abstract proof of something, I'm reasonably happy. But if I can get a concrete, computational proof and actually produce numbers I'm much happier."

"I'm rather an addict of doing things on the computer, because that gives you an explicit criterion of what's going on. I have a visual way of thinking, and I'm happy if I can see a picture of what I'm working with."

Comments & Caveats

Goal: **INSIGHT**
demands speed
$\equiv$ parallelism for:

† verification ('')
† validations; proofs *and* refutations
† "monster barring"
† what is "easy" changes

• HPC and HPN blur; merging disciplines and collaborators
• parallelism ≡ more space, speed & stuff
• exact ≡ hybrid ≡ symbolic '+' numeric
• for analysis, algebra, geometry & topology

Four Experiments

• 1. **Kantian** example: generating "the classical non-Euclidean geometries (hyperbolic, elliptic) by replacing Euclid's axiom of parallels (or something equivalent to it) with alternative forms."

• 2. The **Baconian** experiment is a contrived as opposed to a natural happening, it "is the consequence of 'trying things out' or even of merely messing about."

• 3. **Aristotelian** demonstrations: "apply electrodes to a frog's sciatic nerve, and lo, the leg kicks; always precede the presentation of the dog's dinner with the ringing of a bell, and lo, the bell alone will soon make the dog dribble."

• 4. The most important is **<u>Galilean</u>**: "a critical experiment – one that discriminates between possibilities and, in doing so, either gives us confidence in the view we are taking or makes us think it in need of correction."

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• From Peter Medawar's *Advice to a Young Scientist*

• And now the examples . . .

◇ "You shall soon gain the attention of those who count." (A recent fortune cookie)

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'METHODOLOGY'

• **1.** (*High Precision*) computation of object(s).
• **2.** *Pattern Recognition* of *real numbers* (Inverse Calculator) or *sequences* (GFun, Encylopedia).
• **3.** Extensive use of 'Integer Relation Methods': *PSLQ & LLL*.

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COMMENTS

• Exclusion bounds are especially useful.
• Great testbed for "experimental math" *.
• Proofs are often out of reach – understanding is not.

*Inverse Calculator space limits: 10Mb in 1985 to 10Gb today.

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Relation Detection

• A vector $(x_1, x_2, \cdots, x_n)$ of reals *possesses an integer relation* if there are integers a_i not all zero with $\boxed{0 =}$

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

Problem: Find a_i if such exist. If not, obtain lower bounds on the size of possible a_i.

• $(n = 2)$ *Euclid's algorithm* gives solution.

• $(n \geq 3)$ Euler, Jacobi, Poincare, Minkowski, Perron, others sought method.

• General algorithm in 1977 by Ferguson & Forcade. Since '77: LLL, HJLS, PSOS, PSLQ ('91).

Algebraic Numbers

If α is computed to high precision apply LLL to $(1, \alpha, \alpha^2, \cdots, \alpha^{n-1})$.

• Solution integers a_i are coefficients of a polynomial satisfied by α.

• If no relation is found, exclusion bounds are obtained.

Apéry application

• Thanks to *Apéry* it is well known that

$$\zeta(2) = 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}}$$

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}}$$

$$\zeta(4) = \frac{36}{17} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}}$$

◇ These results suggest

$$Z_5 = \zeta(5) / \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^5 \binom{2k}{k}}$$

might be a simple rational or algebraic.

• **PSLQ result**: If Z_5 satisfies a polynomial of degree ≤ 25 the Euclidean norm of coefficients exceeds 2×10^{37}.

A. First Tastes

A.1: Online glimpses at *Empirical Mathematics* and some pictures.

1983: Identifying

$$\sqrt{2}-1, \frac{\sqrt{3}-1}{2}, \sqrt{\frac{\sqrt{5}-1}{2}}$$

numerically was hard; now 'minpoly' or the "Inverse Calculator".

A.2: "modular machine".

1988-96: a cubic ${}_2F_1$. Consider a, b, c given respectively by

$$\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$

$$\sum_{m,n=-\infty}^{\infty} \omega^{n-m} q^{n^2+mn+m^2},$$

$$\sum_{m,n}^{\infty} q^{(n+\frac{1}{3})^2+(n+\frac{1}{3})(m+\frac{1}{3})+(m+\frac{1}{3})^2}.$$

$(\omega = e^{2\pi i/3})$

◇ These three functions lie on the *Fermat curve*

$$a^3 = b^3 + c^3$$

and one has the lovely *cubic* parameterization of a ${}_2F_1$ hypergeometric function:

$$F\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{c^3}{a^3}\right) = a \quad (1)$$

• In the modular world enough terms of a power series or a product are a *PROOF*.

• By **1996** (1) could be ‘discovered’ and automated in Salvy et als’ GFUN Maple package.

Coworkers: Bailey, P. Borwein, Garvan, Lisoněk, Macdonald

A.3: **MRI, 1997**. Estimating high-resolution spectra from short data records makes *maximum entropy reconstruction* particularly attractive in multi-dimensional NMR where short records are unavoidable.

The *Hoch and Stern information measure*, or *neg-entropy*, is defined in complex $n-$space by

$$H(X) = \sum_{j=1}^{n} h(X_j/b),$$

where h is convex and given (for scaling factor b) by: $h(z) \triangleq$

$$\boxed{|z| \ln\left(|z| + \sqrt{1+|z|^2}\right) - \sqrt{1+|z|^2}}$$

for quantum reasons.

• *Fenchel-Legendre conjugate*

$f^*(y) := \sup_x xy - f(x)$

• Our *symbolic convex analysis* package said:

$$h^*(z) = \cosh(|z|)$$

$\diamond$ Compare the *Shannon entropy*:

$(z \ln z - z)^* = \exp(z)$

$\diamond$ I'd never have tried by hand!

• Efficient *dual algorithms* now may be constructed.

Coworkers: Marechal, Naugler, $\cdots$, Fee

A.4: **Minpoly.** Consider matrices A, B, C, M:

$$A_{kj} := (-1)^{k+1}\binom{2n-j}{2n-k},$$

$$B_{kj} := (-1)^{k+1}\binom{2n-j}{k-1},$$

$$C_{kj} := (-1)^{k+1}\binom{j-1}{k-1}$$

$(k, j = 1, \ldots, n)$ and

$$\boxed{M := A + B - C}.$$

• In earlier work on *Euler Sums* we needed to prove M invertible: indeed $\boxed{M^{-1} = \frac{M+I}{2}}$.

• Key is discovering

$$A^2 = C^2 = I \qquad (2)$$
$$B^2 = CA, AC = B.$$

• Easy combinatorics once isolated.

• It follows that $B^3 = BCA = AA = I$, and that the group generated by A,B and C is S_3.

• One now easily shows

$$\boxed{M^2 + M = 2I}$$

as formal algebra.

• Truth is I started with instances of

'minpoly(M,x)'

and then emboldened

'minpoly(B,x)'

in Maple ...!

• Random matrices have full degree *minimal polynomials*.

B. π

• Different forms & uses of parallelism

B.1: (*Quartic algorithm*) Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}}$$

$$a_{k+1} = a_k(1 + y_{k+1})^4 - 2^{2k+3} \times y_{k+1}(1 + y_{k+1} + y_{k+1}^2)$$

Then a_k converges *quartically* to $1/\pi$.

• Used since 1986, with Salamin-Brent scheme, by Kanada (Tokyo).

• In 1997, Kanada computed over 51 billion digits on a Hitachi supercomputer (18 iterations, 25 hrs on 2^{10} cpu's); present world record.

⋄ 50 billionth decimal digit of π *or* $\frac{1}{\pi}$ is 04$\underline{2}$! And 0123456789 has finally appeared (Brouwer).

• Garvan and I (1995) found genuine η-based m-th order approximations to π.

• Discovery used enormous amounts of computer algebra (e.g. 'Andrews' $\Pi \Rightarrow \Sigma$ algorithm').

◇ A *nonic* (ninth-order) algorithm follows:

Set $a_0 = 1/3$, $r_0 = (\sqrt{3}-1)/2$, $s_0 = (1-r_0^3)^{1/3}$. Iterate

$$
\begin{aligned}
t &= 1 + 2r_k \\
u &= [9r_k(1 + r_k + r_k^2)]^{1/3} \\
v &= t^2 + tu + u^2 \\
m &= \frac{27(1 + s_k + s_k^2)}{v} \\
a_{k+1} &= ma_k + 3^{2k-1}(1-m) \\
s_{k+1} &= \frac{(1-r_k)^3}{(t+2u)v} \\
r_{k+1} &= (1-s_k^3)^{1/3}
\end{aligned}
$$

Then $1/a_k$ converges *nonically* to π.

• Higher order schemes slower than quartic.

• Kanada's estimate of time to run same FFT algorithm on serial machine: *"infinite"*.

Coworkers: Bailey, P. Borwein, Garvan, Kanada, Lisoněk

B.2: (*'Pentium farming' for binary digits.*) Bailey, P. Borwein and Plouffe (1996) discovered a series for π (and some other *polylogarithmic constants*) which allows one to compute hex–digits of π *without* computing prior digits.

• The algorithm needs very little memory, does not need multiple precision and the running time grows only slightly faster than linearly in the order of the digit being computed.

• The key, discovered using 'PSLQ', is:

$$\pi = \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \times \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$$

• Knowing an algorithm would follow they spent several months hunting for such a formula.

• Easy to prove in Mathematica, Maple or by hand.

"Reverse Mathematical Engineering"

• (Sept 97) Fabrice Bellard (INRIA) used a variant formula to compute 152 binary digits of π, starting at the *trillionth position* (10^{12}). This took 12 days on 20 workstations working in parallel over the Internet.

• (Aug 98) Colin Percival (SFU, age 17) finished a similar "embarassingly parallel" computation of *five trillionth digit* (using 25 machines at about 10 times the speed). In *Hex*:

$\underline{0}7E45733CC790B5B5979$

• More at *projects/pihex/*

• In a series of inspired computations using *polylogarithmic ladders* Broadhurst has since found – and proved – similar identities for $\zeta(3)$, $\zeta(5)$ and Catalan's constant G^*.

Coworkers: BBP, Bellard, Broadhurst, Percival.

*Why G was missed?

B.3: Broadhurst's binary formula for

$$G := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

$(= \mathrm{L}_{-4}(2))$

$$\begin{aligned} = 3 \sum_{k=0}^{\infty} \frac{1}{2 \cdot 16^k} \Big\{ & \frac{1}{(8k+1)^2} - \frac{1}{(8k+2)^2} \\ + & \frac{1}{2(8k+3)^2} - \frac{1}{2^2(8k+5)^2} \\ + & \frac{1}{2^2(8k+6)^2} - \frac{1}{2^3(8k+7)^2} \Big\} \\ + & \end{aligned}$$

$$\begin{aligned} -2 \sum_{k=0}^{\infty} \frac{1}{8 \cdot 16^{3k}} \Big\{ & \frac{1}{(8k+1)^2} + \frac{1}{2(8k+2)^2} \\ + & \frac{1}{2^3(8k+3)^2} - \frac{1}{2^6(8k+5)^2} \\ - & \frac{1}{2^7(8k+6)^2} - \frac{1}{2^9(8k+7)^2} \Big\} \end{aligned}$$

• He also gives some constants with ternary expansions.

C. Normal families

• High–level languages or computational speed?

• A family of primes $\mathcal{P}$ is *normal* if it contains no primes p, q such that p divides $q-1$.

◇ *Giuga's conjecture* ('51) is that

$$\sum_{k=1}^{n-1} k^{n-1} \equiv n-1 \pmod{n}$$

if and only if n is prime.

† *Agoh's Conjecture* ('95) is equivalent:

$$nB_{n-1} \equiv -1 \pmod{n} \Leftrightarrow$$

n is prime.

◇ *Lehmer's conjecture* ('32) is that

$$\phi(n) \mid n-1$$

if and only if n is prime. ("hard as odd perfect")

• For all three of these conjectures the set of prime factors of any counterexample n is a normal family.

• We exploited this property aggressively in our (Pari/Maple) computations.

• Lehmer's conjecture had been variously verified for up to 13 prime factors of n. We extended and unified this to 14 prime factors.

◇ We also examined the related condition

$$\boxed{\phi(n) \mid n+1}$$

known to have 8 solutions with up to 6 prime factors (Lehmer) ($2, F_0, \cdots, F_4$ and a rogue pair: 4919055 and 6992962672132095.)

• We extended this to 7 prime factors.

• But the next Lehmer cases (15 and 8) were way too large. The *curse of exponentiality*!

• Any counterexample to the Giuga conjecture must be a *Carmichael number*

$$\boxed{(p-1) \mid (n/p-1)}$$

* and an **odd** *Giuga number*: n squarefree and

$$\boxed{\sum_{p|n} 1/p - \prod_{p|n} 1/p \in Z}.$$

*$p \mid n$ and p prime

For example

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}-\frac{1}{30}=1.$$

◇ The largest we have found has 8 primes:

554079914617070801288578559178

$= 2 \times 3 \times 11$

$\times 2331 \times 47059$

$\times 2259696349 \times 110725121051.$

• We tried to 'use up' the only known branch-bound algorithm. 30 lines of Maple became 2 months in C++ which crashed in Tokyo; but confirming n has more than $13,800$ digits.

Coworkers: D. Borwein, P. Borwein, Girgensohn, Wong and Wayne State Undergraduates

D. Disjoint Genera

Theorem. There are at most 19 integers not of the form of $xy+yz+xz$ with $x, y, z \geq 1$. The only non-square-free are 4 and 18. The first 16 square-free are $1, 2, 6, 10, 22, 30, 42, 58, 70, 78,$ 102 $130, 190, 210, 330, 462$.

• If the 19th exists, it is greater than 10^{11} which the *Generalized Riemann Hypothesis* (GRH) excludes.

• The Matlab road to proof & the hazards of *Sloane's Encyclopedia*

Coworker: Choi

E. Khintchine

E.1: The celebrated *Khintchine constants* K_0, (K_{-1}) – the limiting geometric (harmonic) mean of the elements of *almost all* simple continued fractions – have efficient reworkings as *Riemann zeta* series.

• The rational ζ series we used: $\boxed{\log K_0 \log 2}$

$$= \sum_{n=1}^{\infty} \frac{\zeta(2n)-1}{n} \times \qquad (3)$$

$$\left(1 - \frac{1}{2} + \frac{1}{3} - \ldots + \frac{1}{2n-1}\right)$$

accelerated and with "recycling" evaluations of $\{\zeta(2s)\}$, gave us K_0 to thousands of digits. Here $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$.

• 7,350 digits suggest K_0's c.f. obeys its own prediction.

• A related challenge is to find natural constants that provably behave normally – in analogy to the *Champernowne* number

.0123456789101112 $\cdots$

E.2: Computing $\zeta(N)$

(K_{-1} needs $\zeta(2n+1)$.

• $\boxed{B_{2N} \cong \zeta(2N)}$ can be effectively computed (i) in parallel by *multi-section* methods*; or (ii) by FFT–enhanced *Newton (recycling) methods* on the series $\frac{\sinh}{\cosh}$.

*These have space advantages even as serial algorithms and work for *poly-exp* functions (Kevin Hare)

Ramanujan $\boxed{\zeta(2N+1)}$.

We chose identities of Ramanujan et al.:

$$\zeta(4N+3) = -2 \sum_{k\geq 1} \frac{1}{k^{4N+3}\left(e^{2\pi k}-1\right)}$$

$$+\frac{2}{\pi}\left\{\frac{4N+7}{4}\zeta(4N+4) - \sum_{k=1}^{N} \zeta(4k)\zeta(4N+4-4k)\right\}$$

$$\zeta(4N+1) = -\frac{2}{N} \sum_{k\geq 1} \frac{(\pi k+N)e^{2\pi k}-N}{k^{4N+1}(e^{2\pi k}-1)^2} +$$

$$+\tfrac{1}{2N\pi}\Big\{(2N+1)\zeta(4N+2)$$

$$+\textstyle\sum_{k=1}^{2N}(-1)^k 2k\zeta(2k)\zeta(4N+2-2k)\Big\}.$$

• Only a finite set of $\zeta(2N)$ values is required and the full precision value e^π is reused throughout.

◇ The number e^π is the easiest number to fast compute elliptically. One "differentiates" $e^{-s\pi}$ to obtain π (AGM).

• For $\zeta(4N+1)$ I've lately decoded "nicer" series from a few PSLQ cases of Plouffe. It is equivalent to:

$$\left\{2-(-4)^{-N}\right\}\sum_{k=1}^{\infty}\frac{\coth(k\pi)}{k^{4N+1}}$$

$$-(-4)^{-2N}\sum_{k=1}^{\infty}\frac{\tanh(k\pi)}{k^{4N+1}}$$

$$=Q_N\times\pi^{4N+1} \qquad (4)$$

with $Q_N :=$ in (4) an explicit rational:

$$\sum_{k=0}^{2N+1}\frac{B_{4N+2-2k}B_{2k}}{(4N+2-2k)!(2k)!}$$

$$\times$$

$$\left\{(-1)^{\binom{k}{2}}(-4)^N 2^k+(-4)^k\right\}$$

on substituting

$$\tanh(x)=1-\frac{2}{\exp(2x)+1}$$

and

$$\coth(x)=1+\frac{2}{\exp(2x)-1}$$

and solving for

$$\zeta(4N+1).$$

• Thus,

$$\zeta(5) = \frac{1}{294}\pi^5 +$$

$$\frac{2}{35}\sum_{k=1}^{\infty}\frac{1}{(1+e^{2k\pi})k^5}$$

$$+\frac{72}{35}\sum_{k=1}^{\infty}\frac{1}{(1-e^{2k\pi})k^5}.$$

• Will we ever be able to identify universal formulae like (4) automatically? My solution was highly human assisted.

Coworkers: Bailey, Crandall, Hare, Plouffe.

F. Apéry and Riemann

• Parallelizable "bootstrapped" discovery.

• Bradley and I (1997) considered the well-known formula for $\zeta(3)$, used by Apéry in his irrationality proof.

• No exact analogues exist for $\zeta(5), \zeta(7), \ldots$

• In other words, any relatively prime integers p and q such that

$$\zeta(5) \stackrel{?}{=} \frac{p}{q}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5\binom{2k}{k}}$$

have q astronomically large (as "lattice basis" techniques show). But $\cdots$

• PSLQ yields in polylogarithms:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5\binom{2k}{k}} - 2\zeta(5) = 80 \sum_{n>0} \left(\frac{1}{(2n)^5} - \frac{L}{(2n)^4}\right) \rho^{2n} - \frac{4}{3}L^5 + \frac{8}{3}L^3\zeta(2) + 4L^2\zeta(3)$$

where $L := \log(\rho)$ and $\rho := (\sqrt{5}-1)/2$; with similar formulae for A_4, A_6.

• A less known formula for $\zeta(5)$ due to Koecher suggested generalizations for $\zeta(7), \zeta(9), \ldots$. Again the coefficients were found by integer relation algorithms. *Bootstrapping* the earlier pattern kept the search space of manageable size.

• For example, and simpler than Koecher:

$$\zeta(7) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^7\binom{2k}{k}} + \frac{25}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^4}$$

• We were able – by finding integer relations for $n = 1, 2, \ldots, 10$ – to encapsulate the formulae for $\zeta(4n+3)$ in a single conjectured generating function, (entirely *ex machina):*

• For any complex z,

$$\sum_{n=1}^{\infty} \zeta(4n+3) z^{4n} = \sum_{k=1}^{\infty} \frac{1}{k^3(1 - z^4/k^4)} \qquad (5)$$

$$= \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k} (1 - z^4/k^4)} \times \prod_{m=1}^{k-1} \frac{1 + 4z^4/m^4}{1 - z^4/m^4}.$$

◇ Quite unexpectedly!

◇ The first ten cases show (5) has the form

$$\frac{5}{2} \sum_{k \geq 1} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \frac{P_k(z)}{(1 - z^4/k^4)}$$

for undetermined P_k; with abundant data to compute

$$P_k(z) = \prod_{m=1}^{k-1} \frac{1 + 4z^4/m^4}{1 - z^4/m^4}$$

• We proved many reformulations of (5), including a finite sum:

$$\sum_{k=1}^{n} \frac{2n^2}{k^2} \frac{\prod_{i=1}^{n-1}(4k^4 + i^4)}{\prod_{i=1,\, i \neq k}^{n}(k^4 - i^4)} = \binom{2n}{n} \qquad (6)$$

◇ Found via Gosper's (W-Z type) algorithm after a mistake in an electronic Petrie dish.

◇ This identity* was recently proved by Almkvist and Granville, thus finishing the proof of (5) and giving a rapidly converging series for any $\zeta(4n+3)$ where n is positive integer.

*Erdos, when shown this shortly before his death, rushed off. Twenty minutes later he returned saying he did not know how to prove it but if proven it would have implications for Apéry's result.

Question. What can one derive for

$$\zeta(4N+1)?$$

There are *too many* identities.

Coworkers: Bradley, Almkvist, Fee, Granville

G. Euler to Zagier •

Very large scale: mixing fields/tool/interfaces (Reduce, C++, Fortran, Pari, Snap) etc.

• *Euler sums* or *MZVs* ("multiple zeta values") are a wonderful generalization of the classical ζ.

• For natural $i_1, i_2, \ldots, i_k$

$$\zeta(i_1, i_2, \ldots, i_k) := \sum_{n_1 > n_2 > \dot{n}_k > 0} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}} \qquad (7)$$

⋄ Thus $\zeta(a)$ is as before and $\zeta(a, b)$

$$= \sum_{n=1}^{\infty} \frac{1 + \frac{1}{2^b} + \cdot + \frac{1}{(n-1)^b}}{n^a}$$

• The integer k is the sum's *depth* and $i_1 + i_2 + \cdots + i_k$ is its *weight*.

• Definition (7) clearly extends to alternating sums. MZVs have recently found interesting interpretations in high energy physics, knot theory, combinatorics ...

• MZVs satisfy many striking identities, of which

$$\zeta(2,1) = \zeta(3)$$

$$4\zeta(3,1) = \zeta(4)$$

are among the simplest.

◇ Euler himself found and partially proved theorems on reducibility of depth 2 to depth 1 ζ's ($\zeta(6,2)$ is 'irreducible').

◇ Our work on integral representations of MZVs led to a fast *Hölder convolution* algorithm for very high precision evaluations (projects/ EZFace/) thus allowing thorough examination via integer relation algorithms.

• This leads to amazing identities and important dimensional (reducibility) conjectures.

• Our simplest dimensional conjectures are surely beyond present proof techniques. Does $\zeta(5), G \in Q$?

Drinfeld(91)-Deligne

Conjecture. "The graded Lie algebra of *Grothendieck & Teichmuller* has no more than one generator in odd degrees, and no generators in even degrees."

... sits as a special case

• We detail an example motivated by an identity conjectured by Zagier and proved by Broadhurst: $\zeta(\{3,1\}_n) =$

$$\frac{1}{2n+1}\zeta(\{2\}_{2n}) \quad (8)$$

$$(= \frac{2\pi^{4n}}{(4n+2)!})$$

where $\{s\}_n$ is the string s repeated n times.

• In all known "non-decomposable" identities for Euler sums, all ζ-terms have the same weight.

• This is of great importance for guided integer relation searches, as it dramatically reduces the size of the search space.

• Broadhurst & Lisoněk then used Bailey's fast version of PSLQ to search for Zagier generalizations. They found that "cycles"

$$Z(m_1, m_2, \ldots, m_{2n+1}) := \zeta(\{2\}_{m_1}, 3, \{2\}_{m_2}, 1, \{2\}_{m_3}, 3, \ldots, 1, \{2\}_{m_{2n+1}}).$$

participate in many such identities.

Checking PSLQ input vectors from all Z values of fixed weight ($2K$, say) along with the value $\zeta(\{2\}_K)$ detected many identities, from which general patterns were "obvious".

• This led to a *conjecture* (among many):

$$\sum_{i=0}^{2n} Z(C^i S) \stackrel{?}{=} \zeta(\{2\}_{M+2n}) \qquad (9)$$

for S a string of $2n+1$ numbers summing to M, and $C^i S$ the cyclic shift of S by i places.

• My favourite is

$$8^n \zeta(\{-2,1\}_n) \stackrel{?}{=} \zeta(\{2,1\}_n).$$

Can n=2 be proven symbolically?

◇ Zagier's identity is the case of (9) with entries of S zero.

• The symmetry in (9) highlighted that Zagier-type identities have serious combinatorial content. For $M = 0, 1$ we could reduce (9) to evaluation of combinatorial sums; and thence to truly combinatorial proofs.*

*For $M \geq 2$ we have no proofs, but very strong evidence.

Kuhn "neither proof nor error is at issue. The transfer of allegiance from paradigm to paradigm is a conversion experience that cannot be forced."

Coworkers: Bailey, D. Borwein, Bradley, Broadhurst, Girgensohn, Lisonek, Yazdani