Divide-and-conquer generating functions

Part I

Elementary Sequences

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Divide-and-conquer functions satisfy equations in $F(z), F(z^2), F(z^4)\dots$. Their generated sequences are mainly used in computer science, and they were analyzed pragmatically, that is, now and then a sequence was picked out for scrutiny. By giving several classes of ordinary generating functions together with recurrences, we hope to help with the analysis of many such sequences, and try to classify a part of the divide-and-conquer sequence zoo.

Nowadays, it is routine to work within the equivalence of linear recurrences and rational o.g.f.s., or even broader, with hypergeometrical functions and binomial identities. Mysteries, however, remain with many sequences generated by recurrences of other type. Empirical studies played the main part of this work where we focus on recurrences of the form

$$a_{0\text{ or }1} = c, \quad a_{2n} = f(a_n, a_{n+i_1}, \dots, n), \quad a_{2n+1} = g(a_n, a_{n+j_1}, \dots, n), \tag{0.1}$$

and their power series generating functions.

In the first section, we introduce the reader to a class of functions that can generate said sequences. Then, several g.f.s are given together with corresponding recurrences and, partly, informal proofs. The last two sections list example sequences and some further questions.

1. Introduction and Definitions

The first wide survey of the sequences generated by recurrences of form (0.1) was done by Allouche and Shallit[AS1] who managed to give a broad definition of such sequences by the notion of 2-regularity. As many of the k-regular sequences are interesting to computer scientists, but k-regularity seems too fuzzy a concept, it was necessary to find more detailed properties that can be used to both distinguish between sequences and make them more amenable to analysis. One fruitful attempt in this regard is the fitting term of *divide-and-conquer* by Dumas who revamped[D1] an old concept by Mahler—Dumas contemplates[D2] the following definitions.

A formal series $f(z)$ is called **Mahlerian** if it satisfies an equation

$$c_0(z)f(z) + c_1(z)f(z^2) + \cdots + c_N(z)f(z^{2^N}) = 0 \tag{1.1}$$

where the $c_k(z)$ are polynomials, not all zero. Likewise, Mahlerian sequences satisfy a recurrence of the form

$$\sum_\ell c_{0,\ell}f_{n-\ell} + \sum_\ell c_{1,\ell}f_{(n-\ell)/2} + \cdots + \sum_\ell c_{N,\ell}f_{(n-\ell)/2^\ell} = 0, \tag{1.2}$$

where the sums are finite and the coefficients $c_{k,\ell}$ are not all zero.

A series $f(z)$ is of the **divide-and-conquer** (DC) type if it satisfies an equation with a right member

$$c_0(z)f(z) + c_1(z)f(z^2) + \cdots + c_N(z)f(z^{2^N}) = b(z) \tag{1.3}$$

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in which $b(z)$ is a formal series and $c_k(z)$ are polynomials not all zero. A sequence is of the divide-and-conquer type if its generating series is of the divide-and-conquer type.

Examples. The sequence $\nu(n) = e_1(n)$, which counts ones in the binary representation of n, is of DC type since its g.f. is

$$F(z) = \sum_{i\geq 0} e_1(i)z^i = \frac{1}{1-z}\sum_{k\geq 0}\frac{z^{2^k}}{1+z^{2^k}} \quad = \quad \frac{1}{1-z}\left(\frac{z}{1+z}+\frac{z^2}{1+z^2}+\cdots\right),$$

$$F(z^2) = \frac{1}{1-z^2}\left(\frac{z^2}{1+z^2}+\frac{z^4}{1+z^4}+\cdots\right) = \frac{1}{1-z^2}\left((1-z)F(z)-\frac{z}{1+z}\right),$$

$$\Longrightarrow \qquad (1-z^2)F(z^2)-(1-z)F(z) = -\frac{z}{1+z}.$$

The Thue-Morse sequence on $\{1,-1\}$, $t(n)$, is not only of DC type but also Mahlerian:

$$T(z) = \sum_{i\geq 0} t(i)z^i = \prod_{k\geq 0} 1-z^{2^k} = (1-z)(1-z^2)(1-z^4)\cdots,$$

$$T(z^2) = (1-z^2)(1-z^4)\cdots \quad = \quad T(z)/(1-z),$$

$$\Longrightarrow \qquad T(z^2)-T(z)/(1-z) = 0. \qquad \blacklozenge$$

We will call generating functions and the associated sequences **of elementary divide-and-conquer type**, if

- the g.f. can be written as sum or product of rational functions and infinite sums or products in the unknown (here z),
- where z appears in the sum/product only as a rational function of z^{2^k} (where k is the sum/product index starting with 0).
- The index k may appear in the product/sum additionally as exponent of an integer factor, and we allow only integer coefficients to the monomials.

The previous examples are both elementary in this regard.

That the above defined functions are of DC type is easily seen as they can be transformed into functional equations of DC type by singling out the first term of the sum/product, as performed in the examples. Not all functions of DC type seem to be elementary, however, for example, we were not able to find an elementary form of $F(z) = \sum 2^{e_0(i)}z^i$, where $e_0(n)$ counts the number of zeros in the binary representation of n. On the other hand, $F(z)$ is of DC type since $F(z^2) = (1+F(z))/(z+2)$.

In the next section, we show that elementary DC functions admit simple recurrences, and that many of the well known sequences concerning the binary representation of n are generated by them. This will likely allow to handle whole classes of sequences of such type analytically — Prodinger has shown[Pr] that applying the Mellin transform to power series generating functions, in order to find asymptotic behaviour, yields useful results. Even if this turns out to be difficult in the end, the classification of these sequences according to their ordinary g.f.s will give identities and clarify the sequence zoo.

2. MAIN THEOREM

Where the theorems in this section are given unproven, there is overwhelming evidence for them from our empirical studies — usually, generated sequences were compared with recursions up to index 100 and above. We can now only encourage the reader to find more formal proofs. We also provide in the next section tables listing the well known sequences falling into the given categories.

Theorem. *Let α, c, d be integers. The following generating functions of elementary divide-and-conquer type have coefficients satisfying the recurrences*

$$(2.1)\quad F(z) = \sum_{i=0}^{\infty} a_i z^i = \sum_{k=0}^{\infty} \frac{c^k z^{2^k}}{1 - z^{2^k}} \implies \begin{pmatrix} a_{2n} = c a_n + 1 \\ a_{2n+1} = 1 \end{pmatrix}, \quad |c| > 0,$$

$$(2.2)\quad \sum \frac{c^k z^{2^k}}{1 - z^{2^{k+1}}} \implies \begin{pmatrix} a_{2n} = c a_n \\ a_{2n+1} = 1 \end{pmatrix}, \quad |c| > 0,$$

$$(2.3)\quad \prod 1 + c z^{2^k} \implies \begin{pmatrix} a_1 = 1 \\ a_{2n} = a_n \\ a_{2n+1} = c a_n \end{pmatrix}, \quad |c| > 0,$$

$$(2.4)\quad \frac{1}{1-z} \sum \frac{\alpha^k (d z^{2^k} + c z^{2^{k+1}})}{1 + z^{2^k}} \implies \begin{pmatrix} a_0 = 0 \\ a_{2n} = \alpha \cdot a_n + c \\ a_{2n+1} = \alpha \cdot a_n + d \end{pmatrix}, \quad |\alpha| > 0.$$

Let $c_1, c_2, \ldots, c_D$ be integers, then also

$$(2.5)\quad \prod \left(1 + c z^{2^k} + \sum_{i=1}^{D} c_i z^{2^{k+1} i} \right) \implies \begin{pmatrix} a_{n<0} = 0,\ a_0 = 1 \\ a_{2n} = a_n + \sum_{i=1}^{D} c_i a_{n-i} \\ a_{2n+1} = c a_n \end{pmatrix},$$

$$(2.6)\quad \sum \frac{1}{1 - \sum_{i=1}^{D} c_i z^{2^k i}} \implies \begin{pmatrix} a_{2n} = a_n + b_{2n} \\ a_{2n+1} = b_{2n+1} \end{pmatrix},$$

where b_n is a sequence satisfying the linear recurrence $b_n = \sum_{i=1}^{D} c_i b_{n-i}$.

First, the infinite sums/products are not really; it suffices in practice to compute the first $\lceil \log_2 n \rceil$ terms because any exponent of z will then be greater than the index of the coefficient we want to get. This fact can be applied in the proofs, as well. A symbolic calculator program like PARI helps with visualizing what subsequences are generated.

To prove (2.1), see that $\frac{1}{1-z^e}$ generates $a_n = [\,e \text{ divides } n\,]$, and any factor z^e shifts the sequence to the right by e places. Therefore,

$$\sum v_2(i) z^i = \sum_{k \geq 1} \frac{z^{2^k}}{1 - z^{2^k}}, \quad \text{where } v_2(n) = -1 + \sum [\,2^k \text{ divides } n\,],$$

the exponent of the highest power of 2 dividing n. Consequently, the g.f. in (2.1) adds c^k if 2^k divides n. We then have

$$a_n = \sum_{k=0}^{v_2(n)} c^k = \begin{cases} 1 + c \sum_{k=0}^{v_2(n)-1} c^k, & \text{for even } n; \\ 1, & \text{for odd } n. \end{cases}$$

Similar observations apply to (2.2).

Although (2.3) is a special case of (2.5), it deserved to be mentioned, as there is a simplified form for the coefficients: $a_n = c^{e_1(n)}$. It is obvious that every exponent $\leq 2^k$ is generated by the first k terms of the product, and that, every time a one appears in the binary representation of n, the coefficient of z^n is multiplied with c. Finally, the recurrence is an automaton that has the same behaviour, and the recurrence for $e_1(n)$ shows it too, just with addition instead of multiplication.

The principle of a machinery that does one thing when we have a one bit, and the other with a zero bit, is exploited to the fullest with (2.4), where we have a multiplication with α for any bit, followed by addition of c or d, respectively. But this means also that any recurrence/g.f. of the form (2.4) is a linear combination of two functions, where we have for the case $\alpha = 1$

$$\sum e_1(i)z^i = \frac{1}{1-z}\sum_{k\geq 0}\frac{z^{2^k}}{1+z^{2^k}}, \qquad \sum e_0(i)z^i = \frac{1}{1-z}\sum_{k\geq 0}\frac{z^{2^{k+1}}}{1+z^{2^k}}.$$

We will list the function pairs more completely in the examples section. □

As said, (2.5) and (2.6) are given as well supported conjectures. It might be possible to construct from them o.g.f.s of several classes of DC type sequences where only the recurrences are known.

For another generalization of functions dependent on the binary representation, the reader is referred to Dumas who discusses[D2] 2-rational sequences that are defined using linear algebra. One defines a **2-rational sequence** u_n by a linear representation. Such a representation consists of a row matrix λ, a column matrix γ, and two square matrices A_0 and A_1 whose sizes are respectively $1 \times N$, $N \times 1$, and $N \times N$ for a certain integer N. If an integer n has as binary expansion

$$n = (n_\ell \dots n_0)_2,$$

the value of the sequence for n is

$$u_n = \lambda A_{n_\ell} \cdots A_{n_0} \gamma. \tag{2.7}$$

3. Example Sequences

The use of Neil Sloane's *Online Encyclopedia of Integer Sequences*[OEIS] was invaluable in finding the given sequences, and we provide their A-numbers from the database for further references.

Type	Parameter(s)	Name/Description	OEIS/References
(2.1)	$c = 1$	2^{a_n} divides $2n, v_2(n)+1$	A001511, [AS1]
	$c = 2$	n XOR $n-1$	A038712
	$c = -1$	first Feigenbaum symbolic seq.	A035263, [K]
(2.2)	$c = 2$	highest power of 2 dividing n	A006519
(2.3)	$c = 2$	Gould's seq., $2^{e_1(n)}$	A001316, [AS1]
	$c = 3$	$3^{e_1(n)}$	(A048883), [PT]
	$c = -1$	Thue-Morse seq. on $\{1, -1\}$	[AS2]

Type	Parameter(s)	Name/Description	OEIS/References
(2.4)	$c = 0, d = 1$	ones-counting seq., $e_1(n), \nu(n)$	A000120, [AS1]
$\alpha = 1$	$c = 1, d = 0$	$e_0(n)$	A023416
	$c = 1, d = 1$	binary length	A070939
	$c = 1, d = -1$	$e_0(n) - e_1(n)$	A037861
	$c = 2, d = 1$	a stopping problem	(A061313)
$\alpha = 2$	$c = 0, d = 1$	natural numbers, n	A000027
	$c = 1, d = 0$	interchange 0s and 1s	A035327, [AS3]
	$c = 1, d = 1$	a_{n-1} OR n	A003817
$\alpha = -1$	$c = 0, d = 1$	alternating bit sum for n	A065359, [AS3]
	$c = 1, d = 0$		A083905
	$c = 1, d = 1$		A030300
$\alpha = 3$	$c = 0, d = 1$	ternary(n) contains no 2	A005836, [AS1]
$\alpha = 4$	$c = 0, d = 1$	Moser–de Bruijn sequence	A000695, [AS1]
(2.5)	$c = 1, c_1 = 1$	Stern-Brocot (Carlitz) seq.	A002487,[AS1][AS3]
	$c = 1, c_1 = -1$	a fractal sequence	A005590, [AS1]
	$c = 3, c_1 = 2$	odd entries in Pascal(1 . . . n)	A006046

4. Questions

What is $\sum 2^{e_0(i)} z^i$? And what function generates Per Nørgård's infinity sequence, defined as $\{a_0 = 0, a_{2n} = -a_n, a_{2n+1} = a_n + 1\}$? Can the sequences in this work expressed in 2-rational form?

References

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[AS2] J.-P. Allouche and J. O. Shallit, *The Ubiquitous Prouhet-Thue-Morse Sequence*, in C. Ding. T. Helleseth and H. Niederreiter, eds., Sequences and Their Applications: Proceedings of SETA '98, Springer-Verlag, 1999, pp. 1-16. `http://www.math.uwaterloo.ca/~shallit/Papers/ubiq.ps`

[AS3] J.-P. Allouche, J. Shallit, *The Ring of k-regular Sequences, II* (preprint), `http://www.lri.fr/~allouche/kreg2.ps`

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