ENUMERATION OF CONCAVE INTEGER PARTITIONS

JAN SNELLMAN AND MICHAEL PAULSEN

Abstract. An integer partition $\lambda \vdash n$ corresponds, via its Ferrers diagram, to an artinian monomial ideal $I \subset \mathbb{C}[x,y]$ with $\dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n$. If λ corresponds to an integrally closed ideal we call it *concave* . We study generating functions for the number of concave partitions, unrestricted or with at most r parts.

1. concave partitions

By an *integer partition* $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ we mean a weakly decreasing sequence of non-negative integers, all but finitely many of which are zero. The non-zero elements are called the *parts* of the partition. When writing a partition, we often will only write the parts; thus $(2,1,1,0,0,0,\dots)$ may be written as $(2,1,1)$.

We write $r = \langle \lambda \rangle$ for the number of parts of λ, and $n = |\lambda| = \sum_i \lambda_i$; equivalently, we write $\lambda \vdash n$ if $n = |\lambda|$. The set of all partitions is denoted by $\mathcal{P}$, and the set of partitions of n by $\mathcal{P}(n)$. We put $|\mathcal{P}(n)| = p(n)$. By subscripting any of the above with r we restrict to partitions with at most r parts.

We will use the fact that $\mathcal{P}$ forms a monoid under component-wise addition.

For an integer partition $\lambda \vdash n$ we define its *Ferrers diagram* $F(\lambda) = \left\{ (i,j) \in \mathbb{N}^2 \middle| i < \lambda_{j+1} \right\}$. In figure 1 the black dots comprise the Ferrers diagram of the partition $\mu = (4,4,2,2)$.

Then $F(\lambda)$ is a finite *order ideal* in the partially ordered set $(\mathbb{N}^2, \leq)$, where $(a,b) \leq (c,d)$ iff $a \leq c$ and $b \leq d$. In fact, integer partitions correspond precisely to finite order ideals in this poset.

The complement $I(\lambda) = \mathbb{N}^2 \setminus F(\lambda)$ is a monoid ideal in the additive monoid $\mathbb{N}^2$. Recall that for a monoid ideal I the *integral closure* $\bar{I}$ is

$$\left\{ \mathbf{a} \middle| \ell \mathbf{a} \in I \text{ for some } \ell \in \mathbb{Z}_+ \right\} \tag{1}$$

and that I is *integrally closed* iff it is equal to its integral closure.

Definition 1. The integer partition λ is *concave* iff $I(\lambda)$ is integrally closed. We denote by $\bar{\lambda}$ the unique partition such that $I(\bar{\lambda}) = \overline{I(\lambda)}$.

Now let R be the complex monoid ring of $\mathbb{N}^2$. We identify $\mathbb{N}^2$ with the set of commutative monomials in the variables x, y, so that $R \simeq \mathbb{C}[x,y]$. Then a monoid ideal $I \subset \mathbb{N}^2$ corresponds to the monomial ideal J in R generated by the monomials $\left\{ x^i y^j \middle| (i,j) \in I \right\}$. Furthermore, since the monoid ideals of the form $I(\lambda)$ are precisely those with finite complement to $\mathbb{N}^2$, those

1991 *Mathematics Subject Classification.* 05A17; 13B22.

Key words and phrases. Integer partitions, monomial ideals, integral closure.

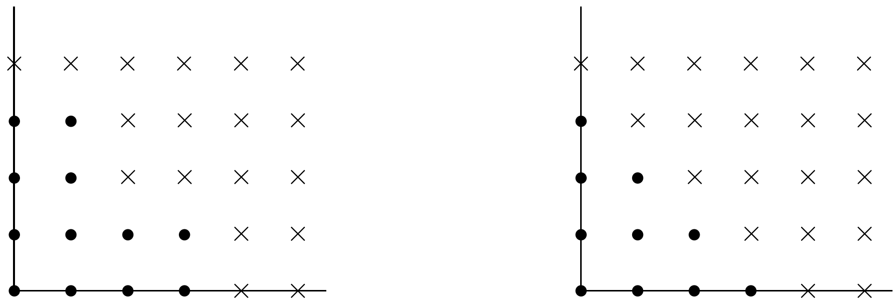

FIGURE 1. μ and $\bar{\mu}$

monoid ideals will correspond to monomial ideals $J \subset R$ such that R/J has a finite $\mathbb{C}$-vector space basis (consisting of images of those monomials not in J). By abuse of notation, such monomial ideals are called *artinian*, and the $\mathbb{C}$-vector space dimension of R/J is called the *colength* of J.

We get in this way a bijection between

(1) integer partitions of n,
(2) order ideals in $(\mathbb{N}^2, \leq)$ of cardinality n,
(3) monoid ideals in $\mathbb{N}^2$ whose complement has cardinality n, and
(4) monomial ideals in R of colength n.

Recall that if $\mathfrak{a}$ is an ideal in the commutative unitary ring S, then the *integral closure* $\bar{\mathfrak{a}}$ consists of all $u \in S$ that fulfill some equation of the form

$$s^n + b_1 s^{n-1} + \cdots + b_0, \qquad b_i \in \mathfrak{a}^i \tag{2}$$

Then $\mathfrak{a}$ is always contained in its integral closure, which is an ideal. The ideal $\mathfrak{a}$ is said to be *integrally closed* if it coincides with its integral closure.

For the special case $S = R$, we have that the integral closure of a monomial ideal is again a monomial ideal, and that the latter monomial ideal corresponds to the integral closure of the monoid ideal corresponding to the former monomial ideal. Hence, we have a bijection between

(1) concave integer partitions of n,
(2) integrally closed monoid ideals in $\mathbb{N}^2$ whose complements have cardinality n, and
(3) integrally closed monomial ideals in R of colength n.

Fröberg and Barucci [3] studied the growth of the number of ideals of colength n in certain rings, among them local noetherian rings of dimension 1. Studying the growth of the number of monomial ideals of colength n in R is, by the above, the same as studying the partition function $p(n)$. In this article, we will instead study the growth of the number of integrally closed monomial ideals in R, that is, the number of concave partitions of n.

2. Inequalities defining concave partitions

It is in general a hard problem to compute the integral closure of an ideal in a commutative ring. However, for monomial ideals in a polynomial ring, the following theorem, which can be found in e.g. [6], makes the problem feasible.

Theorem 2. *Let $I \subset \mathbb{N}^2$ be a monoid ideal, and regard $\mathbb{N}^2$ as a subset of $\mathbb{Q}^2$ in the natural way. Let $\mathrm{conv}_{\mathbb{Q}}(I)$ denote the convex hull of I inside $\mathbb{Q}^2$. Then the integral closure of I is given by*

$$\mathrm{conv}_{\mathbb{Q}}(I) \cap \mathbb{N}^2 \tag{3}$$

Example 3. The partition $\mu = (4,4,2,2)$ corresponds to the monoid ideal $((0,4),(2,2),(4,0))$, which has integral closure $((0,4),(1,3),(2,2),(3,1),(4,0))$. It follows that $\overline{\mu} = (4,3,2,1)$. In figure 1 we have drawn the lattice points belonging to $F(\mu)$ as dots, and the lattice points belonging to $I(\lambda)$ as crosses.

The above theorem gives the following characterization of concave partitions:

Lemma 4. *Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ be a partition. Then λ is concave iff for all positive integers $i < j < k$,*

$$\lambda_j < 1 + \lambda_i \frac{k-j}{k-i} + \lambda_k \frac{j-i}{k-i} \tag{4}$$

or, equivalently, if

$$\lambda_i(j-k) + \lambda_j(k-i) + \lambda_k(i-j) < k-i \tag{5}$$

3. Generating functions for super-concave partitions

We will enumerate concave partitions by considering another class of partitions which is more amenable to enumeration, yet is close to that of concave partitions.

Definition 5. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ be a partition. Then λ is *super-concave* iff for all positive integers $i < j < k$,

$$\lambda_i(j-k) + \lambda_j(k-i) + \lambda_k(j-i) \leq 0 \tag{6}$$

The reader should note that it is actually a *stronger* property to be super-concave than to be concave. Unlike the latter property, it is not necessarily preserved by conjugation: the partition (2) is super-concave, hence concave, but its conjugate $(1,1)$ is concave but not super-concave.

Theorem 6. *Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ be a partition, and let $\mu = (\mu_1, \mu_2, \mu_3, \dots)$ be its conjugate, so that $|\{\, j | \mu_j = i \,\}| = \lambda_i - \lambda_{i+1}$ for all i. Then the following are equivalent:*

(i) λ is super-concave,

(ii) for all positive ℓ,

$$-\lambda_\ell + 2\lambda_{\ell+1} - \lambda_{\ell+2} \leq 0 \tag{7}$$

(iii) for all positive ℓ,

$$\lambda_{\ell+1} - \lambda_\ell \geq \lambda_{\ell+2} - \lambda_{\ell+1} \tag{8}$$

(iv) $|\{\, k | \mu_k = i \,\}| \geq |\{\, k | \mu_k = j \,\}|$ whenever $i \leq j$.

Proof. (i) $\iff$ (ii): Let $\mathbf{e}_i$ be the vector with 1 in the i'th coordinate and zeros elsewhere, let $\mathbf{f}_j = -\mathbf{e}_j + 2\mathbf{e}_{j+1} - \mathbf{e}_{j+2}$, and let $\mathbf{t}_{i,j,k} = (j-k)\mathbf{e}_i + (k-i)\mathbf{e}_j + (j-i)\mathbf{e}_k$. Clearly, (6) is equivalent with $\mathbf{t}_{i,j,k} \cdot \lambda \leq 0$, and (7) is equivalent with $\mathbf{f}_j \cdot \lambda \leq 0$. We have that $\mathbf{f}_\ell = \mathbf{t}_{\ell,\ell+1,\ell+2}$. Conversely, we claim that $\mathbf{t}_{i,j,k}$ is a positive linear combination of different $\mathbf{f}_\ell$. From this claim, it follows that if λ fulfills (7) for all ℓ then λ is super-concave.

We can without loss of generality assume that $i = 1$. Then it is easy to verify that

$$\mathbf{t}_{1,j,k} = \sum_{\ell=1}^{j-2} \ell(k-j)\mathbf{f}_\ell + \sum_{\ell=j-1}^{k-2} \ell(j-1)(k-\ell-1)\mathbf{f}_\ell \tag{9}$$

(ii) $\iff$ (iii) $\iff$ (iv) : This is obvious. □

The *difference operator* Δ is defined on partitions by

$$\Delta(\lambda_1, \lambda_2, \lambda_3, \dots) = (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 - \lambda_4, \dots) \tag{10}$$

We get that the *second order difference operator* Δ^2 is given by

$$\begin{aligned} \Delta^2(\lambda_1, \lambda_2, \lambda_3, \dots) &= \Delta(\Delta(\lambda_1, \lambda_2, \lambda_3, \dots)) = \\ &= (\lambda_1 - 2\lambda_2 + \lambda_3, \lambda_2 - 2\lambda_3 + \lambda_4, \lambda_3 - 2\lambda_4 + \lambda_5, \dots) \end{aligned} \tag{11}$$

Corollary 7. *The super-concave partitions are precisely those with non-negative second differences.*

Definition 8. Let $p_{sc}(n)$ denote the number of super-concave partitions of n, and $p_{sc}(n, r)$ denote the number of super-concave partitions of n with at most r parts. Let similarly $p_c(n)$ and $p_c(n, r)$ denote the number of super-concave partitions of n, and the number of super-concave partitions of n with at most r parts, respectively. For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ let $\mathbf{x}^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots$, and define

$$\begin{aligned} PS(\mathbf{x}) &= \sum_{\lambda \text{ super-concave}} \mathbf{x}^\lambda \\ PS_r(x_1, \dots, x_r) &= PS(x_1, x_2, \dots, x_r, 0, 0, 0, \dots) = \sum_{\substack{\lambda \text{ super-concave} \\ \lambda_{r+1}=0}} \mathbf{x}^\lambda \\ PC(\mathbf{x}) &= \sum_{\lambda \text{ concave}} \mathbf{x}^\lambda \\ PC_r(x_1, \dots, x_r) &= PC(x_1, x_2, \dots, x_r, 0, 0, 0, \dots) = \sum_{\substack{\lambda \text{ concave} \\ \lambda_{r+1}=0}} \mathbf{x}^\lambda \end{aligned} \tag{12}$$

Partitions with non-negative second differences have been studied by Andrews [2], who proved that there are as many such partitions of n as there are partitions of n into triangular numbers.

Canfield et al [4] have studied partitions with non-negative m'th differences. Specialising their results to the case $m = 2$, we conclude:

Theorem 9. *Let n, r be denote positive integers.*

(i) *There is a bijection between partitions of n into triangular numbers and super-concave partitions.*

(ii) The multi-generating function for super-concave partitions is given by

$$PS(\mathbf{x}) = \frac{1}{\prod_{i=1}^{\infty}\left(1-\prod_{j=1}^{i} x_j^{1+i-j}\right)}$$
$$= 1 + x_1 + {x_1}^2 + {x_1}^3 + {x_1}^4 + {x_1}^2 x_2 + {x_1}^5 + {x_1}^4 x_2 + {x_1}^3 x_2 + \dots \tag{13}$$

(iii) The multi-generating function for super-concave partitions with at most r parts is given by

$$PS_r(x_1, x_2, \dots, x_r) = \frac{1}{\prod_{i=1}^{r}\left(1-\prod_{j=1}^{i} x_j^{1+i-j}\right)} \tag{14}$$

(iv) The generating function for super-concave partitions is

$$PS(t) = \sum_{n=0}^{\infty} p_{sc}(n) t^n = \prod_{i=1}^{\infty} \frac{1}{1-t^{\frac{i(i+1)}{2}}} \tag{15}$$

and the one for super-concave partitions with at most r parts is

$$PS_r(t) = \sum_{n=0}^{\infty} p_{sc}(n,r) t^n = \prod_{i=1}^{r} \frac{1}{1-t^{\frac{i(i+1)}{2}}} \tag{16}$$

(v) The proportion of super-concave partitions with at most r parts among all partitions with at most r parts is

$$\frac{r!}{\prod_{i=1}^{r} \frac{i(i+1)}{2}}. \tag{17}$$

(vi) As $n \to \infty$,

$$p_{sc}(n) \sim c n^{-3/2} \exp(3Cn^{1/3})$$
$$C = 2^{-1/3}\left[\zeta(3/2)\Gamma(3/2)\right]^{2/3}, \quad c = \frac{\sqrt{3}}{12}\left(\frac{C}{\pi}\right)^{3/2} \tag{18}$$

The sequence $\left(p_{sc}(n)\right)_{n=0}^{\infty}$ is identical to sequence A007294 in OEIS [8]. We have submitted the sequences $\left(p_{sc}(n,r)\right)_{n=0}^{\infty}$, for $r = 3, 4$, in OEIS [8], as A086159 and A086160. The sequence for $r = 2$ was already in the database, as A008620.

3.1. **Other apperances of super-concave partitions in the literature.**

The bijection between partitions into triangular numbers and partitions with non-negative second difference is mentioned in A007294 in OEIS [8], together with a reference to Andrews [2]. That sequence has been contributed by Mira Bernstein and Roland Bacher; we thank Philippe Flajolet for drawing our attention to it.

Gert Almkvist [1] gives an asymptotic analysis of $p_{sc}(n)$ which is finer than (18).

Another derivation of the generating functions above can found in a forthcoming paper "Partition Bijections, a Survey" [7] by Igor Pak. He observes that the set of super-concave partitions with at most r parts consists of the lattice points of the unimodular cone spanned by the vectors $v_0 = (1, \dots, 1)$ and $v_i = (i-1, i-2, \dots, 1, 0, 0, \dots)$ for $1 \le i \le r$.

Corteel and Savage [5] calculate rational generating functions for classes of partitions defined by linear homogeneous inequalities. This applies to super-concave partitions, but not directly to concave partitions, since the inequalities (5) defining them are inhomogeneous.

4. Generating functions for concave partitions

Theorem 10. *Let r be a positive integer. Then*

$$PC_r(x_1,\ldots,x_r) = \frac{Q_r(x_1,\ldots,x_r)}{\prod_{i=1}^{r}\left(1-\prod_{j=1}^{i} x_j^{1+i-j}\right)} \tag{19}$$

where $Q_r(x_1,\ldots,x_r)$ is a polynomial satisfying

(i) $Q_r(x_1,\ldots,x_r)$ has integer coefficients,
(ii) $Q_r(1,\ldots,1) = 1$,
(iii) all exponent vectors of the monomials that occur in Q_r are weakly decreasing, and
(iv) $Q_r(x_1,\ldots,x_r) = Q_{r+1}(x_1,\ldots,x_r,0)$.

Furthermore,

$$PC(\mathbf{x}) = \frac{Q(\mathbf{x})}{\prod_{i=1}^{\infty}\left(1-\prod_{j=1}^{i} x_j^{1+i-j}\right)} \tag{20}$$

where $Q(\mathbf{x})$ is a formal power series with the property that for each ℓ, $Q(x_1,\ldots,x_\ell,0,0,\ldots) = Q_\ell(x_1,\ldots,x_\ell)$; in other words,

$$Q = 1 + \sum_{i=1}^{\infty}(Q_i - Q_{i-1})$$

Proof. Let A be the matrix with r columns whose rows consists of all truncations of the vectors $\mathbf{t}_{i,j,k}$ introduced in the proof of Theorem 6, for $i<j<k$, $k<r+2$ For example, if $r=3$ and if we order the 3-subsets of $\{1,2,3,4<\}$ lexicographically we get that

$$A = \begin{pmatrix} -1 & 2 & -1 \\ -2 & 3 & 0 \\ -1 & 0 & 3 \\ 0 & -1 & 2 \end{pmatrix}$$

Then a super-concave partition with at most r parts corresponds to a solution to

$$A\mathbf{z} \le \mathbf{0}, \qquad \mathbf{z} \ge \mathbf{0} \tag{21}$$

whereas a concave partition with at most r parts corresponds to a solution to

$$A\mathbf{z} \le \mathbf{b}, \qquad \mathbf{z} \ge \mathbf{0} \tag{22}$$

where the entry of $\mathbf{b}$ which corresponds to the row of A indexed by (i,j,k) is $i-k$. It follows from a theorem in Stanley's "green book" [9] that the multi-generating functions of these two solution sets have the same denominator, and that their numerator evaluates to the same value after substituting 1 for each formal variable.

All monomials in

$$\prod_{i=1}^{r}\left(1-\prod_{j=1}^{i} x_j^{1+i-j}\right)$$

have weakly decreasing exponent vectors, hence this is also true for $PC_r(x_1,\dots,x_r)$.

The assertion about $PC(\mathbf{x})$ follows by passing to the limit. □

Our calculations indicate that

$$\begin{aligned} Q_1(\mathbf{x}) &= 1 \\ Q_2(\mathbf{x}) &= 1 + x_1x_2 - x_1^2x_2 \\ Q_3(\mathbf{x}) &= Q_2(\mathbf{x}) + x_3\left({x_1}^5{x_2}^3 - {x_1}^4{x_2}^3 - 2\,{x_1}^3{x_2}^2 + {x_1}^2{x_2}^2 + x_1x_2\right) \end{aligned} \tag{23}$$

Corollary 11. *(i) The generating function for concave partitions with at most r parts is given by*

$$PC_r(t) = \sum_{n=0}^{\infty} p_c(n,r)t^n = \frac{Q_r(t)}{\prod_{i=1}^{r}\left(1-t^{\frac{i(i+1)}{2}}\right)} \tag{24}$$

where $Q_r(1) = 1$, and the numerator has degree strictly smaller than $r^3/6 + r^2/2 + r/3$.

(ii) The proportion of concave partitions with at most r parts among all partitions with at most r parts is

$$\frac{r!}{\prod_{i=1}^{r}\frac{i(i+1)}{2}}. \tag{25}$$

Proof. The only thing which does not follow immediately from substituting $x_i = t$ in the previous theorem is the assertion about the degree of the numerator. From Stanley's "grey book" [10, Theorem 4.6.25] we have that the rational function $PC_r(t,\dots,t)$ is of degree < 0. The degree of the denominator is

$$\sum_{i=1}^{r}\frac{i(i+1)}{2} = \frac{r^3}{6} + \frac{r^2}{2} + \frac{r}{3}$$

so the result follows. □

We can therefore say with absolute certainty that the first $Q_r(t)$ are as follows:

$$\begin{aligned} Q_1(t) &= 1 \\ Q_2(t) &= 1 + t^2 - t^3 \\ Q_3(t) &= 1 + t^2 + t^5 - 2t^6 - t^8 + t^9 \\ Q_4(t) &= 1 + t^2 + t^4 + t^5 - t^6 - t^7 + 2t^9 - 2t^{10} - t^{11} - 2t^{12} + \\ &\quad + 2t^{13} - t^{14} - t^{15} + t^{16} + t^{17} + t^{18} - t^{19} \end{aligned} \tag{26}$$

Hence, we belive that

$$PC(t) = \frac{1 + t^2 + O(t^3)}{\prod_{i=1}^{\infty}\left(1-t^{\frac{i(i+1)}{2}}\right)} \tag{27}$$

We've calculated that

$$PC(t) = \sum_{n=0}^{\infty} p_c(n)t^n = 1 + t + 2t^2 + 3t^3 + 4t^4 + 7t^5 + 9t^6 + 11t^7 + \\ + 17t^8 + 23t^9 + 28t^{10} + 39t^{11} + 48t^{12} + 59t^{13} + 79t^{14} + \\ + 100t^{15} + 121t^{16} + 152t^{17} + 185t^{18} + 225t^{19} + 280t^{20} + O(t^{21}) \quad (28)$$

It seems likely that $\log p_c(n)$ grows as $n^{1/3}$ (i.e. approximately as fast as pseudo-convex partitions), but we can not prove this, since we have no estimates of the numerator in (27).

We have submitted $\left(p_c(n)\right)_{n=0}^{\infty}$ to the OEIS [8]; it is A084913. The sequences $\left(p_c(n,r)\right)_{n=0}^{\infty}$ are A086161, A086162, and A086163 for $r = 2, 3, 4$.

References

[1] Gert Almkvist. Asymptotics of various partitions. manuscript.

[2] George E. Andrews. MacMahon's partition analysis. II. Fundamental theorems. *Ann. Comb.*, 4(3-4):327–338, 2000. Conference on Combinatorics and Physics (Los Alamos, NM, 1998).

[3] Valentina Barucci and Ralf Fröberg. On the number of ideals of finite colength. In *Geometric and combinatorial aspects of commutative algebra (Messina, 1999)*, volume 217 of *Lecture Notes in Pure and Appl. Math.*, pages 11–19. Dekker, New York, 2001.

[4] Rod Canfield, Sylvie Corteel, and Pawel Hitczenko. Random partitions with non-negative r-th differences. *Adv. in Appl. Math.*, 27(2-3):298–317, 2001. Special issue in honor of Dominique Foata's 65th birthday (Philadelphia, PA, 2000).

[5] S. Corteel and C. D. Savage. Partitions and compositions defined by inequalities. math.CO/0309110.

[6] David Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer Verlag, 1995.

[7] Igor Pak. Partition bijections, a survey. *Ramanujan Journal*, to appear.

[8] Neil J. A. Sloane. The on-line encyclopedia of integer sequences. `http://www.research.att.com/~njas/sequences/index.html`.

[9] Richard P. Stanley. *Combinatorics and Commutative Algebra*, volume 41 of *Progress in Mathematics*. Birkhäuser, 2 edition, 1996.

[10] Richard P. Stanley. *Enumerative combinatorics. Vol. 1.* Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

Department of Mathematics, Stockholm University, SE-10691 Stockholm, Sweden

E-mail address: Jan.Snellman@math.su.se