PROVE OR DISPROVE
100 CONJECTURES FROM THE OEIS

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Abstract. Presented here are over one hundred conjectures ranging from easy to difficult, from many mathematical fields. I also briefly summarize methods and tools that have led to this collection.

Dedicated to all contributors to the OEIS,
on occasion of its 100,000th entry.

The On-Line Encyclopedia of Integer Sequences [OEIS] is a database containing the start terms of nearly 100,000 sequences (as of Autumn 2004), together with formulæ and references. It was reviewed by Sloane in 2003 [Slo03]. In the first section of this work, I describe how such a database can assist with finding of conjectures, and the tools necessary to this end; section 2 then lists over one hundred so found propositions from many fields that have been checked numerically to some degree but await proof. Finally, I conclude in section 3, giving a webpage that follows the status of the assertions. An appendix provides links to corresponding OEIS entries.

1. Finding connections

With a string of numbers representing an integer sequence, a conjecture appears already whenever a possible formula for a prefix of the sequence is found, or when a transformation is discovered that maps from one prefix to that of another sequence. The OEIS consists just of such prefixes. While the online interface to the OEIS allows searching for simple patterns in the database, its 'superseeker' eMail service is able to find formulæ for several types of sequences directly, by computation, or indirectly, by applying transformations and comparing with the entries. However, this way, it is not possible to do bulk searches of the whole or parts of the database.

To help with systematic work, Neil Sloane, the creator/editor/maintainer of the OEIS, offers a file where only the numbers are collected, and this file served as input to several computer programs I wrote over two years of work as associate editor. The methods can be divided according to program usage and intention:

- using a C program, I extended the database's 10^5 sequences with their first and second differences, with subsequent lexicographical sort and visual inspection using a simple scrolling program (like Unix `less`). This method sounds awkward but is not very much so since close matches show clearly through all noise. It yielded several conjectures, among many 'trivial' identities and a few hundred duplicates.
- for frequent offline lookups, I built a further extended database by applying many transformations (bisections, pairwise sums, part-gcd, odd part, to state the more non-obvious) to a core set of OEIS sequences. The resulting file with 370MB size was searched with Unix `grep` when needed and using several different strategies,

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and this yielded a good part of the conjectures of section 2, not to speak of another lot of identities that were included for reference in the database.

- special scans of the OEIS numbers, like that of Plouffe[Plo92], searching for sequences of specific kind, most notably for C–finite sequences, using my own implementation of `guessgf`[BP92] in Pari[GP], and for *bifurcative** sequences. It has to be noted that scanning for specific types of sequences is nontrivial and cannot be fully automatized—even in the case of C–finiteness—, one still has to check for false positives. From the scans, I have only included those conjectures that seemed the most surprising to me; everything else served as immediate improvement of the information contained in the OEIS.

Comparing the quantities of output from the methods, simple transformations resulted in the most hits, I also had the impression that C–finiteness is quite common: about 12 per cent of sequences have the property. On the other hand, to squeeze out more identities from the database, ever more specialized transformations/scans would be necessary, with less and less results *per method*. There is definitely a kind of fractality to it, perhaps reflecting expertise structure of OEIS submitters.

2. The conjectures

From the very start, Neil Sloane had encouraged all kind of integer sequences to be submitted. The decision has lead to a statistically representative snapshot of sequences from all mathematical fields where integers occur. A similar mixture is visible in my following conjectures which deal with power series expansions, number theory, and enumerative combinatorics, as well as additive and combinatorial number theory and, due to my special interest, bifurcative and other nonlinear sequences. This is also roughly the order in which the statements are presented. I refrained from cluttering the presentation with the respective A–numbers, also to encourage independent recalculation.

Please note that, throughout the section, I use a set of commonly known abbreviations that are listed in Table 1 and some of whom were introduced by Graham et al.[Gra94].

2.1. **Easy start: special functions, binomials, and more.**

$$(1-4x)^{3/2} = 1 - 6x + \sum_{n>1} \frac{12(2n-4)!}{n!(n-2)!} x^n. \tag{1}$$

$$(1+x^2C(x)^2)C(x)^2 = \sum_{n\geq 0} \frac{6n(2n)!}{n!(n+1)!(n+2)} x^n, \quad C(x) = \frac{1-\sqrt{1-4x}}{2x}. \tag{2}$$

Please show how to get the *reduced* numerator/denominator in:

$$(1-x)^{1/4} = \sum_{n\geq 0} \frac{\prod_{k=1}^{n}(5-4k)2^{v_2(k)}/k}{2^{3n-e_1(n)}} x^n. \tag{3}$$

$$[x^n]\, P_{n+2}(x) = \frac{1}{2^{n+2}}(n+1)\binom{2n+2}{n+1}, \quad (P_n \text{ the Legendre polynomials}). \tag{4}$$

$$\sum_{k=0}^{n} -v_2\big([x^{2k}]\, P_{2n}(x)\big) = 2n^2 + 2n - 2\sum_{i=0}^{n} e_1(i). \tag{5}$$

$$B(2n, \tfrac{1}{2}) = \frac{a}{b}, B(2n, \tfrac{1}{4}) = \frac{c}{d} \quad \Longrightarrow \quad a = c, \quad B(n,x) \text{ Bernoulli polynomials}. \tag{6}$$

*formerly called divide-and-conquer.

p	a prime
$[n \text{ even}]$	1 if n even, 0 otherwise
$\#$	number of, cardinality ...
(a, b)	greatest common divisor of a, b
$\texttt{lcm}\,(a, b)$	least common multiple of a, b
$\lfloor x \rfloor$	floor, greatest integer $\leq x$
$\lceil x \rceil$	ceiling, smallest integer $\geq x$
$a \mid b$	a divides b
$\{n \mid A\}$	the set of all n with property A
$d(n)$	$\tau(n), \sigma_0(n)$, number of divisors of n
$\phi(n)$	Euler totient/phi function
$\sigma(n)$	$\sigma_1(n)$, sum of divisors of n
F_n	n-th Fibonacci number, with $F_0 = 0$
$P(x)$	a polynomial in x
$[x^n]\,P(x)$	n-th coefficient of polynomial or power series P
$\left[\frac{x^n}{n!}\right] f(x)$	n-th coefficient of Taylor expansion of f
$\lg x$	base-2 logarithm of x
$v_2(n)$	dyadic valuation of n, exponent of 2 in n
$e_1(n)$	number of ones in binary representation of n
Res	resultant

TABLE 1. Symbols and abbreviations.

$$\arctan(\tanh x \tan x) = \sum_{n\geq 0} (-1)^n 2^{6n+2} (2^{4n+2} - 1) \frac{B_{4n+2}}{2n+1} \cdot \frac{x^{4n+2}}{(4n+2)!}. \tag{7}$$

$$\tau(2^n) = [x^{2^n}]\, x \prod_{k\geq 1} (1 - x^k)^{24} = [x^n] \frac{1}{2048x^2 + 24x + 1}. \tag{8}$$

$$7 \mid \left[\frac{x^{6k+4}}{(6k+4)!}\right] \frac{1}{2 - \cosh(x)}. \tag{9}$$

$$4 \mid \left[\frac{x^{6k+4}}{(6k+4)!}\right] \exp(\cos x - 1), \quad 11 \mid \left[\frac{x^{10k}}{(10k)!}\right] \exp(\cos x - 1) \ldots \tag{10}$$

$$\mathrm{Res}\,(x^n - 1, 4x^2 - 1) = 4^n - 2^n - (-2)^n + (-1)^n. \tag{11}$$

$$\left\{n \,\middle|\, n^2 - 1 \mid \tbinom{2n}{n}\right\} \setminus \{2\} \subset \left\{n \,\middle|\, n!(n-1)! \mid 2(2n-3)!\right\}. \tag{12}$$

$$\sum_{k=0}^{n} (k+1) \sum_{l=0}^{k} 2^l \binom{k}{l} \binom{n-k}{l} = [x^n]\, \frac{1-x}{(1-2x-x^2)^2}. \tag{13}$$

$$\sum_{k=0}^{n+1} (k+1) \left[\binom{2n+1}{k} - \binom{2n+1}{k-1}\right] = \frac{n+2}{2} \binom{2n+2}{n+1} - 4^n. \tag{14}$$

$$a(n) = \sum_{k=0}^{n} \left[\binom{n}{k} \mod 2\right] \cdot 2^k \quad \Longrightarrow \quad a_{2n+1} = 3a_{2n}. \tag{15}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor} D^k \binom{n}{2k+1} = [x^n]\frac{x}{1-2x+(1-D)x^2}. \tag{16}$$

$$\left(\binom{2n}{n},\binom{3n}{n},\ldots,\binom{(n-1)n}{n}\right)=1 \iff n+1=\sum_i p_i^{e_i} \;\wedge\; m=\max_i p_i^{e_i} \;\wedge\; \frac{n+1}{m}>m. \tag{17}$$

$$\{a_n \mid \text{Least term in period of cont. frac. of } \sqrt{a_n}=20\}=100n^2+n. \tag{18}$$

Define $\mathrm{PCF}(n)$ the period of the continued fraction expansion for $\sqrt{n}$. Then

$$\mathrm{PCF}\,(n)=\mathrm{PCF}\,(n+1)\equiv 1 \bmod 2 \implies n\equiv 1 \bmod 24. \tag{19}$$

(20) The largest term in the periodic part of the cont. frac. of $\sqrt{3^n+1}$ is $2\cdot\lfloor(\sqrt{3})^n\rfloor$.

The numerators of the continued fraction convergents to $\sqrt{27}$ are

$$[x^n]\,\frac{5+26x+5x^2-x^3}{1-52x^2+x^4}. \tag{21}$$

2.2. **Classical number theory.**

$$n=5^i11^j \implies n|\sum_{k=1}^{10}k^n. \tag{22}$$

$$n+1\,|\,d(n!^n). \tag{23}$$

Group the natural numbers such that the product of the terms of the n-th group is divisible by $n!$. Let a_n the first term of the n-th group. Then

$$a_n=\left\lfloor\frac{(n-1)^2}{2}+1\right\rfloor. \tag{24}$$

$$\#\{\text{cubic residues mod } 8^n\}=\frac{4\cdot 8^n+3}{7}. \tag{25}$$

$$\mathrm{lcm}\,(3n+1,3n+2,3n+3)=\tfrac{3}{4}(9n^3+18n^2+11n+2)(3+(-1)^n). \tag{26}$$

(26a) Let $a_n=\prod_{k=1}^{n}\mathrm{lcm}\,(k,n-k+1)$. Then $a_n=n^2(n-1)!^2$ for n even, $n+1$ prime. Also, if n is odd and >3, $2(n+1)a_n$ is a perfect square, the root of which has the factor $\frac{1}{2}n(n-1)((n-1)/2)!$.

$$\left\{\min(x)\;\middle|\; p\,|\,px-x-1\right\}=p-1, \quad p=\text{prime}\,(n). \tag{27}$$

$$\emptyset=\{n \mid \forall\, i,j,\ 0<i<j<n,\ n>3:\ 2^n+2^i+2^j+1 \text{ is composite}\}. \tag{28}$$

$$p\equiv 5 \bmod 12 \iff \emptyset=\left\{x \;\middle|\; x^4\equiv 9 \bmod p\right\}. \tag{29}$$

$$d(n)=d(n+1)=\cdots=d(n+6) \implies n\equiv 5 \bmod 16. \tag{30}$$

$$\frac{\sigma(2n)}{\sigma(n)} = \frac{4 \cdot 2^{v_2(n)} - 1}{2 \cdot 2^{v_2(n)} - 1}. \tag{31}$$

$$\forall k > 0: \quad \emptyset \neq \left\{x \mid \phi(x) = 2^k p\right\} \iff 2p+1 \text{ prime}. \tag{32}$$

(33)
$$\left\{\text{composite } n \mid \phi(n+12) = \phi(n)+12 \wedge \sigma(n+12) = \sigma(n)+12\right\} \implies n \equiv 64 \bmod 72.$$

$$\left\{n \mid \sigma(d(n^3)) = d(\sigma(n^2))\right\} \implies n \equiv 1 \bmod 24. \tag{34}$$

$$\left\{n \mid \sigma(n) = 2u(n)\right\} \implies n \equiv 108 \bmod 216, \quad u(n) = \sum_{\substack{d \mid n \\ (d,n/d)=1}} d. \tag{35}$$

$$\left\{n \mid t(n) = t(t(n)-n)\right\} = \left\{n \mid n = 5 \cdot 2^k \vee n = 7 \cdot 2^k, k > 0\right\}, t(n) = |\phi(n) - n|. \tag{36}$$

$$\left\{n \mid \phi(n^2+1) = n\phi(n+1)\right\} = \{8\} \cup \left\{n \mid n^2+1 = \text{ prime } \wedge n+1 = \text{ prime }\right\}. \tag{37}$$

$$\left\{n \mid |n - 2d(n) - 2\phi(n) - 2| = 2\right\} = \{2, 72\} \cup \left\{16p \mid p > 2\right\}. \tag{38}$$

$$\{\text{Local maxima of } \sigma(n)\} \subset \left\{m \mid m = \sigma(l) \wedge l = \text{local maximum of } d(n)\right\}. \tag{39}$$

$$F_n \bmod 9 \text{ has period } 24. \tag{40}$$

$$(2^p - 1, F_p) > 1 \implies (2^{kp} - 1, F_{kp}) > 1. \tag{41}$$

$$(2^p - 1, F_p) > 1 \implies 8p \mid (2^p - 1, F_p) - 1. \tag{42}$$

$$(2^p - 1, F_p) > 1 \wedge p \not\equiv 1 \bmod 10 \implies \frac{(2^p - 1, F_p) - 1}{8p} > 1. \tag{43}$$

$$\#\{k \mid F_k \mid F_n\} = d(n) - [n \text{ even}]. \tag{44}$$

$$k \text{ squarefree} \wedge \mathbb{Q}(\sqrt{-k}) \text{ has class number } n \implies \max k \equiv 19 \bmod 24. \tag{45}$$

Define $\varsigma(n)$ the smallest prime factor of n. Let a_n the least number such that the number of numbers $k \leq a_n$ with $k > \varsigma(k)^n$ exceeds the number of numbers with $k \leq \varsigma(k)^n$. Then

$$a_n = 3^n + 3 \cdot 2^n + 6. \tag{46}$$

2.3. **Additive/combinatorial number theory.**

$$\#\Big\{m \mid m = [q^i] \prod_{k=1}^{n} \sum_{j=0}^{k} q^j \Big\} = [x^n] \frac{x(1-x+2x^2-2x^3+x^4)}{(1+x^2)(1-x)^3}. \tag{47}$$

$$\#\Big\{m \mid m = [q^i] \prod_{k=1}^{n} 1 + \sum_{j=0}^{k} q^{2j+1} \ \wedge\ n > 6\Big\} = n^2 - 3. \tag{48}$$

Let $A_{n\geq 0}$ the number of distinct entries in the $n \times n$ multiplication table, i.e., the number of distinct products ij with $1 \leq i, j \leq n$, then A_n goes $\{0, 1, 3, 6, 9, 14, \dots\}$. Let further B the set of composite numbers > 9 that are not equal to the product of their *aliquot divisors* (divisors of n without n). Then

$$A_n - A_{n-1} = \frac{n + \text{ its smallest divisor} > 1}{2}, \text{ if } n \notin B. \tag{49}$$

Define the sequence $\langle a_n \rangle$ as $a_1 = 2$, $a_2 = 7$, and a_n the smallest number which is uniquely $a_j + a_k$, $j < k$. The sequence starts $\{2, 7, 9, 11, 13, 15, 16, 17, 19, 21, 25 \dots\}$. Then

$$a_n - a_{n-1} \text{ has period } 26. \tag{50}$$

Let A the set of numbers whose cubes can be partitioned into two nonzero squares, and B the set of numbers that are the sum of two nonzero squares. Then

$$A = B. \tag{51}$$

2.3.1. *Sum-free sequences.* The sequence with start values $\{a_1, a_2, \dots, a_s\}$ and further values $a_{n>s}$ satisfying "a_n is the smallest number $> a_{n-1}$ not of the form $a_i + a_j + a_k$ for $1 \leq i < j < k \leq n$". Then

$$\text{Start with } \{1, 2, 3\}: \quad a_{n+6} - a_{n+5} = a_{n+1} - a_n, \quad \text{for } n > 6. \tag{52}$$

$$\text{Start with } \{0, 1, 2, 3\}: \quad a_n = 1 \ \vee\ a_n \equiv 2, 3 \bmod 8. \tag{53}$$

$$\text{Start with } \{1, 2, 4\}: \quad a_n = \frac{26n - 125 - 11 \cdot (-1)^n}{4}, \quad \text{for } n > 14. \tag{54}$$

$$\text{Start with } \{1, 3, 4\}: \quad a_{n+8} - a_{n+7} = a_{n+1} - a_n, \quad \text{for } n > 7. \tag{55}$$

$$\text{Start with } \{0, 1, 3, 4\}: \quad a_n = 1 \ \vee\ a_n \equiv 3, 4 \bmod 10. \tag{56}$$

$$\text{Start with } \{1, 3, 5\}: \quad a_{n+5} - a_{n+4} = a_{n+1} - a_n, \quad \text{for } n > 6. \tag{57}$$

Let $a_1 = 1, a_2 = 2$ and a_n the smallest number not of form a_i, $a_i + a_{n-1}$, or $|a_i - a_{n-1}|$. Then

$$a_n = 3n + 3 - 2 \cdot 2^{\lfloor \lg(n+2) \rfloor}, \quad \text{for } n > 2. \tag{58}$$

Let $a_n = n$ for $n < 4$ and a_n the least integer $> a_{n-1}$ not of form $2a_i + a_j$, $1 \leq i < j < n$. Then

$$a_n = 4n - 10 - (n \bmod 2), \quad \text{for } n > 3. \tag{59}$$

2.3.2. *Progression-free sequences.* The sequence with start values $\{a_1, a_2, \ldots, a_s\}$ and further values $a_{n>s}$ satisfying "a_n is the smallest number $> a_{n-1}$ that builds no three-term arithmetic progression with any a_k, $1 \le k < n$" is well–defined. This peculiar definition was chosen to allow for any start values. We abbreviate such a sequence with start values $a, b, \ldots$ as $A_3(a, b, \ldots, n)$ and present several conjectures for them.

$$A_3(1,3,n) = A_3(1,2,n) + [n \text{ is even}] = \frac{3-n}{2} + 2\lfloor n/2 \rfloor + \frac{1}{2}\sum_{k=1}^{n-1} 3^{v_2(k)}. \tag{60}$$

$$A_3(1,4,n) = A_3(1,3,n) + [n \text{ is even}] + [\lceil n/2 \rceil \text{ is even}]. \tag{61}$$

(62)

$$A_3(1,7,n) = b_n + \sum_{k=1}^{n-1} \frac{3^{v_2(n)}+1}{2}, \text{ where } b_n = \{1,6,5,6,2,6,5,7,\ldots\}_{n\ge 1} \text{ has period } 8.$$

$$A_3(1,10,n) = b_n + \sum_{k=1}^{n-1} \frac{3^{v_2(n)}+1}{2}, \tag{63}$$

$$\text{where } b_n = \{1,9,8,9,5,10,10,10,\ldots\}_{n\ge 1} \text{ has period } 8.$$

$$A_3(1,19,n) = b_n + \sum_{k=1}^{n-1} \frac{3^{v_2(n)}+1}{2}, \tag{64}$$

$$\text{where } b_n = \{1,18,17,18,14,18,17,19,5,18,17,18,14,19,19,19\ldots\}_{n\ge 1} \text{ has period } 16.$$

$$\text{In general,} \quad A_3(1,m,n) = b_n + \sum_{k=1}^{n-1} \frac{3^{v_2(n)}+1}{2}, \quad m = 1+3^k \ \vee\ 1+2\cdot 3^k, k \ge 0, \tag{65}$$

$$\text{where } b_n \text{ has period } P \le 2^{\lfloor \frac{k+3}{2} \rfloor}.$$

$$A_3(1,2,3,n+2) = 1 + 2^{\lfloor \lg n \rfloor} + \sum_{k=1}^{n} \frac{3^{v_2(n)}+1}{2}. \tag{66}$$

The numbers n such that the n-th row of Pascal's triangle contains an arithmetic progression are

$$n = 19 \ \vee\ n = \tfrac{1}{8}\left[2k^2 + 22k + 37 + (2k+3)(-1)^k\right], k > 0. \tag{67}$$

2.4. **Enumerative combinatorics.**

The number of $n \times n$ invertible binary matrices A such that $A + I$ is invertible is

$$2^{\frac{n(n-1)}{2}} a_n, \quad \text{with } a_n = \langle a_0 = 1, a_n = (2^n - 1)a_{n-1} + (-1)^n \rangle. \tag{68}$$

Consider flips between the d-dimensional tilings of the unary zonotope $Z(D, d)$. Here the codimension $D - d$ is equal to 3 and d varies. Then the number of flips is

$$a(d) = (d^2 + 11d + 24)2^{d-1}. \tag{69}$$

The sequence a_n shifts left twice under binomial transform and is described by both $\langle a_0 = a_1 = 1, a_n = \sum_{k=0}^{n-2} \binom{n-2}{k} a_k \rangle$ and (formally)

$$\sum_{n\ge 0} a_n x^n = \sum_{k\ge 0} \frac{x^{2k}}{(1-kx)(1-x-kx)\prod_{m=0}^{k-1}(1-mx)^2}. \tag{70}$$

The number of self-avoiding closed walks, starting and ending at the origin, of length $2n$ in the strip $\{0, 1, 2\} \times \mathbb{Z}$ is

$$\frac{1}{125}\Big\{(315n-168)2^{n-2}+(-1)^{\lfloor n/2\rfloor}\big[55\lfloor n/2\rfloor+78-(135\lfloor n/2\rfloor+36)(-1)^n\big]\Big\}, n>1. \tag{71}$$

The number of non-palindromic reversible strings with n beads of 4 colors is

$$\begin{cases} \frac{1}{2}4^n - \frac{1}{2}2^n, & n \text{ even}, \\ \frac{1}{2}4^n - 2^n, & n \text{ odd}. \end{cases} \tag{72}$$

The number of non-palindromic reversible strings with $n-1$ beads of 2 colors (4 beads are black) is

$$\begin{cases} \frac{1}{48}(n^4 - 10n^3 + 32n^3 - 38n + 15), & n \text{ odd}, \\ \frac{1}{48}(n^4 - 10n^3 + 32n^3 - 32n), & n \text{ even}. \end{cases} \tag{73}$$

The number of non-palindromic reversible strings with n black beads and $n-1$ white beads is

$$\begin{cases} \dfrac{1}{4}\left[\dbinom{2n}{n} - \dbinom{n}{n/2}\right], & n \text{ even}, \\ \dfrac{1}{2}\left[\dbinom{2n}{n} - \dbinom{2n-1}{n-1} - \dbinom{n-1}{(n-1)/2}\right], & n \text{ odd}. \end{cases} \tag{74}$$

The number of necklaces of n beads of 2 colors (6 of them black) is

$$[x^n]\,\frac{x^6(1-x+x^2+4x^3+2x^4+3x^6+x^7+x^8)}{(1-x)^4(1+x)^2(1-x^3)(1-x^6)}. \tag{75}$$

The number of edges in the 9-partite Turan graph of order n is

$$[x^n]\,\frac{x}{(1-x)^2}\left[\frac{1}{1-x} - \frac{1}{1-x^9}\right]. \tag{76}$$

The number of binary strings of length n that can be reduced to null by repeatedly removing an entire run of two or more consecutive identical digits is

$$2^n - 2nF_{n-2} - (-1)^n - 1. \tag{77}$$

The number of nonempty subsets of $\{1, 2, \ldots, n\}$ in which exactly $1/2$ of the elements are $\leq (n-1)/2$ is

$$\binom{n}{\lfloor (n-1)/2 \rfloor} - 1. \tag{78}$$

The number of level permutations of $2n-1$ is

$$\frac{(2n-1)!}{2^{2n-2}}\binom{2n-2}{n-1}. \tag{79}$$

The number of rooted trees with n nodes and 3 leaves is

$$\tfrac{1}{288}\big(6n^4 - 40n^3 + 108n^2 - 120n - 41 + 9(-1)^n + 32[(n \bmod 3) + (n+1 \bmod 3)]\big). \tag{80}$$

Let a_n the number of $(2 \times n)$ binary arrays with a path of adjacent 1's from the upper left corner to anywhere in right hand column. Then

$$a_{n+2} = 2P_n + 5P_{n+1}, \quad P_n = \text{Pell numbers}. \tag{81}$$

The number of strings over $\mathbb{Z}_3$ of length n with trace 0 and subtrace 1 is

$$[x^n]\,\frac{x(-6x^4+6x^3)}{(1-3x)(1+3x^2)(1-3x+3x^2)}. \tag{82}$$

The number of strings of length n over GF(4) with trace 0 and subtrace 0 is

$$[x^n]\,\frac{x(-26x^3+13x^2-5x+1)}{(1-2x)(1-4x)(1+4x^2)}. \tag{83}$$

The number of symmetric ways to lace a shoe that has n pairs of eyelets, such that each eyelet has at least one direct connection to the opposite side, is

$$\sum_{k=0}^{n} k!\binom{n}{k} F_{k+2}. \tag{84}$$

The number of minimax trees with n nodes is 2^n times the number of labelled ordered partitions of a $2n$-set into odd parts, that is,

$$2^n\left[\frac{x^{2n}}{(2n)!}\right]\frac{1}{1-\sinh x}. \tag{85}$$

The number of unlabeled alternating octopi with n black nodes and k white nodes has the g.f.

$$\sum_{k,n\geq 1}\frac{\phi(k)}{k}\log\left(\frac{(1-x^n y^n)^2}{1-x^n y^n(3+x^n+y^n)}\right).$$

The conjecture is now that the number of those octopi with n black nodes and n white nodes (the diagonal of the above array) is

$$-2+3\sum_{d|n}\frac{\phi(n/d)\binom{2d}{d}}{2n}. \tag{86}$$

Finally, the number of dimer tilings of the graph $S_k\times P_{2n}$ (S_k the star graph on k nodes, P_n the path with length n) is

$$[x^n]\,\frac{1-x}{1-(k+1)x+x^2}. \tag{87}$$

2.5. **Nonlinear recurrences and other sequences.**

(88)
$$a_n=[x^n]\,\frac{-4x^5+x^4+x^3-3x^2-2x+6}{(1-x)(1-x-x^2-x^5)} \iff \left\langle a_0=6, a_1=10, a_n=\left\lfloor\frac{a_{n-1}^2}{a_{n-2}}+\frac{1}{2}\right\rfloor\right\rangle.$$

$$a_n=[x^n]\,\frac{-x^5+x^4-x^3+x^2-2x+3}{(1-x)(1-2x-x^3-x^5)} \iff \left\langle a_0=3, a_1=7, a_n=\left\lfloor\frac{a_{n-1}^2}{a_{n-2}}\right\rfloor\right\rangle. \tag{89}$$

(90)
$$a_n=[x^n]\,\frac{-3x^5+2x^4+x^3-x^2-2x+4}{(1-x)(1-2x-x^2-2x^5)} \iff \left\langle a_0=4, a_1=10, a_n=\left\lfloor\frac{a_{n-1}^2}{a_{n-2}}\right\rfloor\right\rangle.$$

$$a_n=[x^n]\,\frac{2x^3+x^2-4x+5}{-x^4+2x^2-3x+1} \iff \left\langle a_0=5, a_1=11, a_n=\left\lfloor\frac{a_{n-1}^2}{a_{n-2}}+\frac{1}{2}\right\rfloor\right\rangle. \tag{91}$$

(92)
$$a_n=[x^n]\,\frac{3x^5+2x^4+x^3+4x^2-x+6}{-x^6-x^3+x^2-2x+1} \iff \left\langle a_0=6, a_1=11, a_n=\left\lfloor\frac{a_{n-1}^2}{a_{n-2}}+\frac{1}{2}\right\rfloor\right\rangle.$$

$$\begin{aligned}&\langle a_0=a_1=1, a_n=(|n-1-a_{n-1}| \bmod n-1)+(|n-1-a_{n-2}| \bmod n)\rangle\\ &\qquad\qquad\Longrightarrow\quad a_{n+3}=a_n,\quad n>6.\end{aligned} \tag{93}$$

$$\langle a_0=x, a_{n+1}=a_n(a_n+1)\rangle \implies [x^{2^n-3}]\,a_k=\frac{2^{3k+2}-2^k}{3}. \tag{94}$$

$$\text{(95)} \quad \left\langle \nu_0 = \nu_1 = 1,\ \nu_n = \nu_{n-1} + 3\nu_{n-2} \sum_{i=0}^{n-2} q^i \right\rangle \implies [q^1]\,\nu_n = [x^n]\frac{x^3(9x+3)}{(1-3x-3x^2)^2}.$$

$$\text{(96)} \quad a_n = \sum_{k=1}^{\infty} \left\lfloor 2\Big(\frac{\sqrt{5}+1}{2}\Big)^{n-k} \right\rfloor \iff \sum_{n\geq 0} a_n x^n = \frac{x(x^5+x^4-4x^2+3)}{(1-x)(1-x^2)(1-x-x^2)}.$$

$$\text{(97)} \quad \left\langle a_0 = 1, a_{n+1} = \left\lfloor \frac{a_n}{\sqrt{5}-2} \right\rfloor \right\rangle \iff \sum_{n\geq 0} a_n x^n = \frac{1-x-x^2}{(1-x)(1-4x-x^2)}.$$

Let the sequence $\langle a_n \rangle$ be defined such that $a_1 = C$ and $a_{n+1} =$ the smallest difference > 1 between d and p/d for any divisor d of the partial product $p = \prod_{k=1}^{n} a_k$ of the sequence. Then $a_n = \{19, 18, 29, 27, 9, \dots\}$ for $C = 19$, and $a_n = \{21, 4, 5, 13, 8, 2, \dots\}$ for $C = 21$ and

$$\text{(98)} \quad a_n = 3^k, C = 19, n > 3 \quad \text{and} \quad a_n = 2^m, C = 21, n > 4.$$

2.6. **Binary representation, k-regular and bifurcative sequences.**

$$\text{(99)} \quad \#\big\{m \,\big|\, m = v_2\tbinom{n}{j},\ 0 \leq j \leq n\big\} = \lfloor \lg(n+1) \rfloor + 1 - v_2(n+1).$$

$$\text{(100)} \quad e_1(m) \equiv 0 \bmod 2 \iff m \in \langle a_0 = 0,\ a_{2n} = a_n + 2n,\ a_{2n+1} = -a_n + 6n + 3\rangle.$$

$$\text{(101)} \quad e_1(m) \equiv 1 \bmod 2 \iff m \in \langle a_0 = 1,\ a_{2n} = a_n + 2n,\ a_{2n+1} = -a_n + 6n + 3\rangle.$$

$$\text{(102)} \quad a_n = n + [x^n]\frac{x}{1-x}\sum_{k\geq 0} 2^k x^{3\cdot 2^k} \iff a_n \text{ in binary does not begin 100.}$$

$$\text{(103)} \quad \langle a_0 = 0, a_1 = 1, a_{2n} = a_n, a_{2n+1} = a_{n+1} - a_n\rangle \ \wedge\ a_{3k} = 0 \iff \text{no adjacent 1s in binary of } k.$$

$$\text{(104)} \quad \langle a_1 = 3, a_{2n} = 4a_n - 2n, a_{2n+1} = 4a_n - 2n + 2^{\lfloor \lg(4n+2) \rfloor}\rangle \iff \text{binary of } a_n \text{ is binary of } n \text{ twice juxtaposed.}$$

$$\text{(105)} \quad a_n = n \ \mathtt{XOR}\ (n+m) \iff a_n = [x^n]\frac{P(x)}{(1-x)^2 \prod_{k\geq 0} 1 + x^{2^{e_k}}}, \quad \sum_{k\geq 0} 2^{e_k} = m.$$

$$\text{(106)} \quad \langle a_0 = a_1 = 0, a_{4n} = 2a_{2n}, a_{4n+2} = 2a_{2n+1} + 1, a_{4n+1} = 2a_{2n} + 1, a_{4n+3} = 2a_{2n+1}\rangle \iff a_n = \{\text{Replace each pair of adjacent bits of } n \text{ by their mod 2 sum}\}.$$

$$\text{(107)} \quad a_n = \sum_{k=1}^{n-1} k \ \mathtt{AND}\,(n-k) \iff \langle a_0 = a_1 = 0, a_{2n} = 2a_{n-1} + 2a_n + n, a_{2n+1} = 4a_n\rangle.$$

(108)

$$a_n = \sum_{k=1}^{n-1} k \ \mathtt{XOR}\,(n-k) \iff \langle a_0 = a_1 = 0, a_{2n} = 2a_{n-1} + 2a_n + 4n - 4, a_{2n+1} = 4a_n + 6n\rangle.$$

(109)
$$a_n = \sum_{k=1}^{n-1} k\,\texttt{OR}\,(n-k) \iff \langle a_0 = a_1 = 0, a_{2n} = 2a_{n-1}+2a_n+5n-4, a_{2n+1} = 4a_n+6n\rangle.$$

(110) $$a_n = \#\{(i,j)\,|\,0 \le i,j < n \wedge i\,\texttt{AND}\,j > 0\} \iff \langle a_0 = a_1 = 0, a_{2n} = 3a_n + n^2, a_{2n+1} = a_n + 2a_{n+1} + n^2 - 1\rangle.$$

(111) $$n = \sum_{k\ge 0} 2^k e_k \;\wedge\; a_n = \sum_{k\ge 0}(-1)^k e_k \;\wedge\; |a_n| = 3 \implies n \in \left\{m \,\middle|\, m = 3k \wedge k = 3i \wedge e_1(k) \equiv 1 \bmod 2\right\}.$$

(112) $$n = \sum_{k\ge 0} 2^k e_k \;\wedge\; a_n = \sum_{k\ge 0}(-1)^k e_k \;\wedge\; a_n = 0 \implies n = 3m \;\wedge\; m \notin \left\{k \,\middle|\, k = 3i \wedge e_1(k) \equiv 1 \bmod 2\right\}.$$

(113)
$$\langle a_0 = 0, a_{2n} = 1 - a_n, a_{2n+1} = -a_n\rangle \wedge a_{3k} = 0 \implies k \text{ in base-4 contains only } -1, 0, 1.$$

$$\max \sum_{j=0}^{n} [x^j] \sum_{k\ge 0} \frac{x^{2^k}}{1 + x^{2^k} + x^{2^{k+1}}} = \lfloor \log_4 n \rfloor + 1. \tag{114}$$

Define the sequence a_n by $a_1 = 1$ and $a_n = M_n + m_n$, where $M_n = \max_{1\le i<n}(a_i + a_{n-i})$, and $m_n = \min_{1\le i<n}(a_i + a_{n-i})$. Let further b_n the number of partitions of $2n$ into powers of 2 (number of *binary partitions*). Then

$$m_n = \tfrac{3}{2} b_{n-1} - 1, \quad M_n = n + \sum_{k=1}^{n-1} m_n, \quad a_n = M_{n+1} - 1. \tag{115}$$

Let a_n defined as the limit in the infinite of the sequence b_n, with $b_1 = 1$, $b_2 = n$, and $b_{n+2} = \lceil \frac{1}{2}(b_n + b_{n+1}) \rceil$, and c_n the number of ones in the base-(-2)-representation of n. Then

$$c_n = 3a_{n+1} - 2n - 3 \quad \text{and} \quad a_{n+1} - a_n = \langle d_{4n} = 0, d_{2n+1} = 1, d_{4n+2} = d_{n+1}\rangle. \tag{116}$$

Let a_n the number of subwords of length n in the word generated by $\texttt{a} \mapsto \texttt{aab}$, $\texttt{b} \mapsto \texttt{b}$. Then

$$\sum_{n\ge 0} a_n x^n = 1 + \frac{1}{1-x} + \frac{1}{(1-x)^2}\left(\frac{1}{1-x} - \sum_{k\ge 1} x^{2^k+k-1}\right). \tag{117}$$

3. Conclusions

Working over two years with the OEIS showed me that simple computer programs suffice for many tasks; where I had to write programs myself, it was not visible that the task could be fully automatized—mathematics is essentially human. The work as editor was rewarding not only in itself but also in that it yielded a huge collection of conjectures as byproduct. However, I expect further gains in that regard as becoming ever more difficult as scans and transformations have to become more specialized and complex.

I do not intend to work on proving the majority of propositions presented here but I provide below a webpage giving the status of work done on them. Given their number, it is quite possible that a few are already in the literature. I hope the reader excuses my not researching these ones: it is very difficult nowadays to access pay–only journals from outside university.

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