

# A Combinatorial Interpretation of the Eigensequence for Composition

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## Abstract

The monic sequence that shifts left under convolution with itself is the Catalan numbers with 130+ combinatorial interpretations. Here we establish a combinatorial interpretation for the monic sequence that shifts left under *composition*: it counts permutations that contain a 3241 pattern only as part of a 35241 pattern. We give two recurrences, the first allowing relatively fast computation, the second similar to one for the Catalan numbers. Among the  $4 \times 4! = 96$  similarly restricted patterns involving 4 letters (such as 4231: a 431 pattern only occurs as part of a 4231), four different counting sequences arise: 64 give the Catalan numbers, 16 give the Bell numbers, 12 give sequence [A051295](#) in OEIS, and 4 give a new sequence with an explicit formula.

## 1 Introduction

The composition of two sequences  $(a_n)_{n \geq 1}$  and  $(b_n)_{n \geq 1}$  is  $(c_n)_{n \geq 1}$  defined by  $C(x) = A(B(x))$  where  $A, B, C$  are the respective generating functions,  $A(x) = \sum_{n \geq 1} a_n x^n$  and so on. A sequence is monic if its first term is 1. There is a unique monic sequence  $(b_n)_{n \geq 1}$  whose composition with itself is equal to its left shift,  $(b_2, b_3, \dots)$ . This sequence  $(b_n)_{n \geq 1}$  begins 1, 1, 2, 6, 23, 104, 531,  $\dots$  and is called an eigensequence for composition [\[1\]](#).

Consider a permutation on a set of positive integers as a list (or word of distinct letters). A subpermutation is a subword (letters in same order but not necessarily contiguous). Thus 253 is a subpermutation of 21534. A factor is a subpermutation in which the letters are contiguous. A standard permutation is one on an initial segment of the positive integers. The reduced form,  $\text{reduce}(\pi)$ , of a permutation  $\pi$  on an arbitrary set of positive integers is the standard permutation obtained by replacing its smallest entry by 1, next smallest by 2, and so on. Thus  $\text{reduce}(352) = 231$ . The length  $|\pi|$  of a permutation  $\pi$  is simply the number of letters in it. Given a permutation  $\pi$  and a standard permutation  $\rho$  of weakly smaller length—a “pattern”—a subpermutation of  $\pi$  whose reduced form is  $\rho$  is said to be an instance of the pattern  $\rho$  in  $\pi$ . Thus 352 forms a 231 pattern in 35124. A 321-avoiding permutation, for example, is one containing no instance of a 321 pattern. The number of 321-avoiding permutations is the Catalan number  $C_n$  [2].

Let  $\mathcal{A}_n$  denote the set of permutations on  $[n]$  in which the pattern 3241 only occurs as part of a 35241 pattern (35241OK permutations for short). This curious pattern restriction arises in connection with 2-stack-sortable permutations, which can be characterized as permutations that are both 35241OK and 2341-avoiding [3]. With  $a_n = |\mathcal{A}_n|$ , the first several terms of  $(a_n)_{n \geq 0}$  coincide with those of  $(b_n)_{n \geq 1}$  above. This suggests that  $a_n = b_{n+1}$  and the main objective of this paper is to prove bijectively that this is so (§3 and §4). In §2, we consider the structure of a 35241OK permutation and deduce two recurrences for  $a_n$ . A bonus section (§5) classifies all similarly restricted patterns involving 4 letters by their counting sequences.

## 2 Structure and Recurrences for 35241OK Permutations

Every permutation  $\pi$  on  $[n]$  has the form  $\pi = m_1 L_1 m_2 L_2 \dots m_r L_r$  where  $m_1 < m_2 < \dots < m_r = n$  are the left-to-right maxima (LRmax for short) of  $\pi$ . With this notation, the following characterization of 35241OK permutations is easy to verify.

**Theorem 1.** *A permutation  $\pi$  on  $[n]$  is 35241OK if and only if*

- (i)  $L_1 < L_2 < \dots < L_r$  in the sense that  $u \in L_i, v \in L_j$  with  $i < j$  implies  $u < v$ , and
- (ii) each  $L_i$  is 35241OK. □

Recall  $\mathcal{A}_n$  is the set of all 35241OK permutations on  $[n]$ . Let  $\mathcal{A}_{n,k}$  denote the subset of

$\mathcal{A}_n$  with first entry  $k$ ,  $\mathcal{C}_n$  the subset of  $\mathcal{A}_n$  with first two entries increasing ( $n \geq 2$ ,  $\mathcal{C}_1 := \mathcal{A}_1 = \{(1)\}$ ). Set  $a_n = |\mathcal{A}_n|$ ,  $a_{n,k} = |\mathcal{A}_{n,k}|$ ,  $c_n = |\mathcal{C}_n|$ .

**Theorem 2.**  $a_n$  is given by the recurrence relations  $a_0 = c_1 = 1$  and

$$\begin{aligned} (i) \quad a_n &= \sum_{i=0}^{n-1} a_i c_{n-i} & n \geq 1 \\ (ii) \quad c_n &= \sum_{i=0}^{n-1} i a_{n-1,i} & n \geq 2 \\ (iii) \quad a_{n,k} &= \begin{cases} \sum_{i=0}^{k-1} a_i \sum_{j=k-i}^{n-1-i} a_{n-1-i,j} & 1 \leq k \leq n-1 \\ a_{n-1} & k = n \end{cases} \end{aligned}$$

**Proof** (i) counts  $\mathcal{A}_n$  by  $i = |L_1|$  since  $m_1 L_1 K \rightarrow (\text{reduce}(L_1), \text{reduce}(m_1 K))$  is a bijection to  $\mathcal{A}_i \times \mathcal{C}_{n-i}$ . (ii) counts  $\mathcal{C}_n$  by second entry, say  $i+1$ , because  $L_1 = \emptyset$  and, with  $m_2 = i+1$ ,  $m_1 m_2 L_2 \dots m_r L_r \rightarrow (m_1, \text{reduce}(m_2 L_2 \dots m_r L_r))$  is a bijection to  $[i] \times \mathcal{A}_{n-1,i}$ . (iii) counts  $\mathcal{A}_{n,k}$  ( $1 \leq k \leq n-1$ ) by  $i = |L_1|$  and  $j =$  first entry of  $\text{reduce}(m_2 L_2 \dots)$  since  $j \geq k-i$ ,  $m_1 = k$  and  $m_1 L_1 m_2 L_2 \dots \rightarrow (L_1, \text{reduce}(m_2 L_2 \dots))$  is a bijection to  $\mathcal{A}_i \times \mathcal{A}_{n-1-i,j}$ .  $\square$

There is a more elegant (but less computationally efficient) recurrence involving a sum over compositions. Recall that a composition  $\mathbf{c}$  of  $n$  is a list  $\mathbf{c} = (c_1, c_2, \dots, c_r)$  of positive integers whose sum is  $n$  and if  $\mathbf{c} = (c_i)_{i=1}^r$  and  $\mathbf{d} = (d_i)_{i=1}^r$  are same-length compositions of  $n$ , then  $\mathbf{d}$  dominates  $\mathbf{c}$  ( $\mathbf{d} \succeq \mathbf{c}$ ) if  $d_1 + \dots + d_i \geq c_1 + \dots + c_i$  for  $i = 1, 2, \dots, r-1$  (of course, equality holds for  $i = r$ ). Let  $\mathcal{C}_n$  denote the set of all compositions of  $n$  (there are  $2^{n-1}$  of them). A permutation  $\pi = m_1 L_1 m_2 L_2 \dots m_r L_r$  (recall the  $m_i$  are the LRmax entries) determines two same-length compositions of  $[n]$ :  $\mathbf{c} = (c_i)_{i=1}^r$  with  $c_i = |m_i L_i| = 1 + |L_i|$  and  $\mathbf{d} = (d_i)_{i=1}^r$  with  $d_i = m_i - m_{i-1}$  ( $m_0 := 0$ ) and necessarily  $\mathbf{d} \succeq \mathbf{c}$  (or else a left-to-right max would occur among the  $L_i$ ). Summing over  $\mathbf{c}$ , Theorem 1 yields

**Theorem 3.**

$$a_n = \sum_{\mathbf{c} \in \mathcal{C}_n} |\{\mathbf{d} \in \mathcal{C}_n : \mathbf{d} \succeq \mathbf{c}\}| a_{c_1-1} a_{c_2-1} \dots a_{c_r-1} \quad n \geq 1$$

$\square$

Omitting the first factor in the summand, the recurrence  $a_0 = 1$ ,  $a_n = \sum_{\mathbf{c} \in \mathcal{C}_n} a_{c_1-1} a_{c_2-1} \dots a_{c_r-1}$  ( $n \geq 1$ ) is well known to generate the Catalan numbers (for example, count Dyck paths

by the locations of their returns to ground level).

### 3 Preparing for the Main Bijection

The generating function  $B(x) = \sum_{n \geq 1} b_n x^n$  for the “shifts left under composition” sequence of §1 is characterized by  $B(B(x)) = \frac{B(x)}{x} - 1$ . If  $(b_n)$  is the counting sequence for some species, say B-structures, then  $[x^n]B(B(x))$  is the number of pairs  $(X, \mathbf{Y})$  where  $X$  is a B-structure of unspecified size  $k$ ,  $1 \leq k \leq n$  and  $\mathbf{Y}$  is a  $k$ -list of B-structures of total size  $n$  (a  $k$ -list is simply a list with  $k$  entries). On the other hand,  $[x^n](\frac{B(x)}{x} - 1) = b_{n+1}$ . Hence, to show that  $b_{n+1} = a_n$ , our main objective, we need a bijection from the set  $\mathcal{A}_n$  of 35241OK permutations on  $[n]$  to pairs  $(\rho, \mathbf{v})$  where  $\rho \in \mathcal{A}_{k-1}$  for some  $1 \leq k \leq n$  and  $\mathbf{v}$  is a  $k$ -list of 35241OK permutations (possibly empty) of total length  $n - k$ .

Indeed, given  $\pi \in \mathcal{A}_n$ , the position of  $n$  in  $\pi$  determines  $k$  and the right factor of  $\pi$  starting at  $n$  determines  $\rho$  as follows. It is convenient to define the LIT (longest increasing terminal) entries of a permutation  $\pi$  on  $[n]$  to be  $k, k + 1, \dots, n$  where  $k$  is the smallest integer such that  $k, k + 1, \dots, n$  appear in that order in  $\pi$ . Note that the LIT entries form a terminal segment of the LRmax entries. For example, the LIT entries are double underlined and the remaining LRmax entries are single underlined in 214765893. The LIT entries of a permutation on arbitrary positive integers are defined analogously. Now decompose  $\pi$  as  $\sigma n \tau$ .

**Theorem 4.** *A permutation  $\sigma n \tau$  on  $[n]$  is 35241OK if and only if  $\sigma, \tau$  are both 35241OK and each entry of  $\sigma$  that exceeds the smallest entry of  $\tau$  is an LIT entry of  $\sigma$ .*

**Proof** ( $\Rightarrow$ ) Suppose  $b \in \sigma$  exceeds  $c \in \tau$  yet  $b$  is not an LIT entry of  $\sigma$ . Then either (i) there exists  $a \in \sigma$  preceding  $b$  with  $a > b$ , and  $abnc$  is an offending 3241 pattern, or (ii) there exist  $u, v$  following  $b$  in  $\sigma$  with  $u > v > b$  and this time  $uvnc$  is an offending 3241.

( $\Leftarrow$ ) routine. □

Set  $k = |n\tau|$ , the length of the right factor of  $\pi$  starting at  $n$ , and set  $\rho = \text{reduce}(\tau) \in \mathcal{A}_{k-1}$ , capturing the underlying permutation of  $\tau$ . The support of  $\tau$  can be captured by placing, for each  $c \in \text{support}(\tau)$ , an asterisk (or star) right before the smallest entry of  $\sigma$  that exceeds  $c$  (necessarily an LIT entry in view of Theorem 4) or after  $\max(\sigma)$  if there is no such entry. No information is lost if  $\sigma$  is then reduced (keeping the stars in place) to produce  $\sigma^*$  say, because only the LIT entries of  $\sigma$  will be affected and the stars

determine both the original LIT entries of  $\sigma$  and the support of  $\tau$ . For example,  $\pi = (2, 8, 3, 1, 11, 4, 6, 5, 13, 7, 15, 9, 10, 14, 12) \in \mathcal{A}_{15}$  gives  $k = 5$ ,  $\tau = (9, 10, 14, 12)$  and yields the pair  $\rho = (1, 2, 4, 3)$ ,  $\sigma^* = (2, 8, 3, 1, *, *, 9, 4, 6, 5, *, 10, *, 7)$ , and  $\pi$  can be recovered from  $\rho$  and  $\sigma^*$ . The length of  $\sigma^*$  (disregarding the stars) coincides with the total length  $n - k$  of the desired list of  $\underline{35241OK}$  permutations. This reduces the problem to giving a bijection from  $\underline{35241OK}$  permutations on  $[n]$  with  $k - 1$  stars distributed arbitrarily, just before LIT entries or just after the maximum entry  $n$ , to  $k$ -lists of (possibly empty)  $\underline{35241OK}$  permutations of total length  $n$ .

Let  $X_{n,k}$  denote the subset of the preceding “starred” permutations on  $[n]$  with (i) no stars immediately following or preceding the max  $n$ , and (ii) at most one star preceding each non-max LIT entry. In other words,  $X_{n,k}$  is the set of  $\underline{35241OK}$  permutations on  $[n]$  with  $k - 1$  distinct LIT entries other than  $n$  preceded (or marked) by a star. We now show that the following result suffices to construct the desired bijection.

**Theorem 5.** *With  $X_{n,k}$  as just defined, there is a bijection from  $X_{n,k}$  to the set of  $k$ -lists of nonempty  $\underline{35241OK}$  permutations of total length  $n$ .*

Consider a permutation with (possibly) multiple stars in the allowed locations, for example, with  $k = 8$  and hence 7 stars and with  $a, b$  denoting non-max LIT entries,

$$\dots *** a \dots * b \dots * n ** \dots$$

We will show how to represent this as an element of  $X_{n,j}$  for some  $j$  together with a bit sequence of  $j$  0s and  $k - j$  1s. The preceding Theorem then converts the element of  $X_{n,j}$  to a  $j$ -list of nonempty  $\underline{35241OK}$  permutations of total length  $n$ , while the 1s in the bit sequence specify the locations for empty permutations and we have the desired  $k$ -list of permutations of total length  $n$ .

First, collapse all contiguous stars to a single star and delete the stars (if any) surrounding  $n$ . The example yields

$$\dots *** a \dots * b \dots * n ** \dots \quad \rightarrow \quad \dots * a \dots * b \dots n \dots \quad \in X_{n,j}$$

with  $j = 3$  and  $j - 1 = 2$  stars. Second, replace each star immediately preceding a non-max LIT entry and  $n$  itself by 1, all other stars by 0, and then suppress the permutation entries to get a bit sequence  $j$  1s and  $k - j$  0s; the example yields

$$\dots *** a \dots * b \dots * n ** \dots \quad \rightarrow \quad (00 \overset{*a}{1} \overset{*b}{1} \overset{n}{0} 100).$$

Clearly, the original “multiple-starred” permutation can be uniquely recovered from the element of  $X_{n,j}$  and the bit sequence, and we are all done as soon as we establish Theorem 5. In fact, we will prove Theorem 5 with a specific type of bijection that permits a further reduction in the problem.

Define the LRmax factors of a permutation  $m_1L_1m_2L_2\dots m_rL_r$  to be  $m_1L_1, m_2L_2, \dots, m_rL_r$ . Then there is a refined form of Theorem 5 that goes as follows.

**Theorem 6.** *Let  $X_{n,k}$  denote the set of  $\underline{35241OK}$  permutations on  $[n]$  with  $k - 1$  of the non-max LIT entries marked. Let  $Y_{n,k}$  denote the set of  $k$ -lists of nonempty  $\underline{35241OK}$  permutations of total length  $n$ . There is a bijection from  $X_{n,k}$  to  $Y_{n,k}$ ,  $\pi \rightarrow \mathbf{v}$ , of the following type: it rearranges the LRmax factors of  $\pi$ , then splits them into a  $k$ -list of permutations and, finally, reduces each one to yield  $\mathbf{v}$ .*

It suffices to present this bijection for the special case where  $L_1, L_2, \dots, L_r$  are all increasing lists, equivalently, since  $L_1 < L_2 < \dots < L_r$  for a  $\underline{35241OK}$  permutation, the entire list  $L_1L_2\dots L_r$  is increasing. (In the general case, sort each  $L_i$  and then apply the special case bijection except, before reducing, replace each sorted  $L_i$  by the original list  $L_i$ .)

Note that permutations  $\pi = m_1L_1m_2L_2\dots m_rL_r$  for which the full list  $L_1L_2\dots L_r$  is increasing are precisely the 321-avoiding permutations (and automatically  $\underline{35241OK}$ ). In summary, we have shown in this section that the entire problem boils down to

**Theorem 7.** *There is a bijection of the following specific type from 321-avoiding permutations  $\pi$  on  $[n]$  with  $k - 1$  of the non-max LIT entries marked to  $k$ -lists  $\mathbf{v}$  of nonempty 321-avoiding permutations of total length  $n$ : it rearranges the LRmax factors of  $\pi$ , then splits them into a  $k$ -list of permutations and, finally, reduces each one to yield  $\mathbf{v}$ .*

## 4 The Main Bijection

We now prove Theorem 7 and use

$$\underline{3} \ 1 \ \underline{5} \ \underline{2} \ \underline{8} \ 4 \ 6 \ \underline{12} \ \underline{7} \ \underline{15} \ 9 \ \underline{17} \ 10 \ 11 \ \underline{20} \ \underline{25} \ \underline{26}^* \ 13 \ \underline{27} \ \underline{28}^* \ 14 \ \underline{29}^* \ 16 \ \underline{30} \ 18 \ 19 \ 21 \ 22 \ 23 \ 24$$

with  $n = 30$  and  $k = 4$  as a working example (as before, the LIT entries are double underlined and the other LRmax entries are single underlined). First, we split the permutation into panes each consisting of one or more LRmax factors. To do so, use a moving window

initially covering a right factor of the permutation and consisting of  $k$  panes starting, respectively, at the very first LIT entry and the first LIT entry following each starred LIT entry. (Since the window gradually moves left, dropping the last pane and sometimes acquiring a new first pane, it is drawn below the permutation for clarity.)



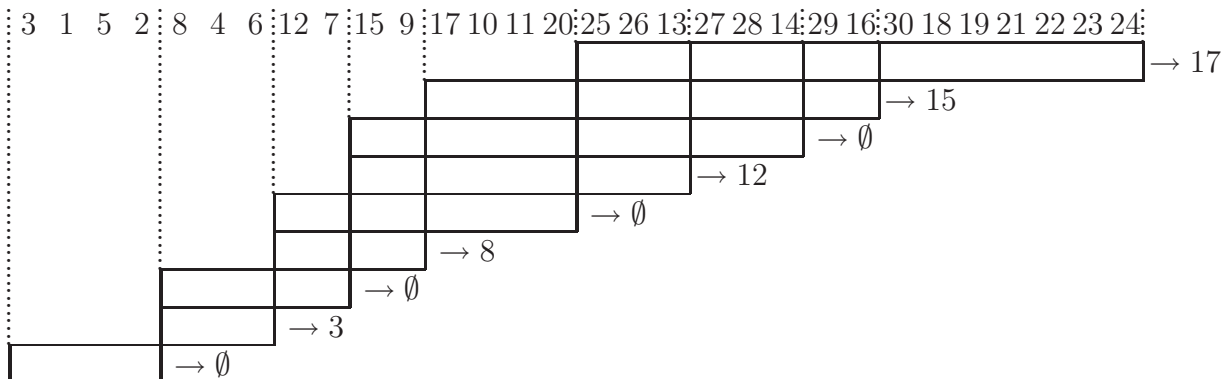
The Initial Window

Proceed as follows until the entire permutation is covered by nonoverlapping panes. (It's just possible that the initial window covers the entire permutation but this happens only for the identity permutation. In this case associate the symbol  $\emptyset$  with the window and proceed to the second step below.) Find the maximum non-LRmax entry in all but the last pane of the current window (the max of the empty set is considered to be  $-\infty$ ) and denote this number  $m$ . Look for LRmax entries not yet empaned that exceed  $m$ . There are 2 cases.

Case 1. There is no such LRmax entry. In this case associate  $\emptyset$  with the current window, and delete the last pane to obtain the next window.

Case 2. There exist LRmax entries not yet empaned that exceed  $m$ . In this case associate the smallest such entry,  $M$  say, with the current window and shift the window left by prepending a pane starting at  $M$  and deleting the last pane.

The resulting panes and the successive windows, each with an associated LRmax or  $\emptyset$  symbol, are shown.



The Moving Window

Second, rearrange the panes using a  $k$ -row array formed from a list consisting of (i) the windows' associated LRmax entries and  $\emptyset$ s, and (ii) the first entries of the  $k$  panes of the initial window:

$$\emptyset \ 3 \ \emptyset \ 8 \ \emptyset \ 12 \ \emptyset \ 15 \ 17 \ 25 \ 27 \ 29 \ 30$$

Insert the list entries right to left into array entries lower right to upper left moving up each column in turn EXCEPT, when a  $\emptyset$  is inserted, that row is “not accepting any more entries” and is henceforth skipped over. The resulting array is

$$\begin{array}{cccc} & \emptyset & 12 & 25 \\ & & \emptyset & 27 \\ \emptyset & 3 & 8 & 15 & 29 \\ & \emptyset & 17 & 30 \end{array}$$

Now form a list of  $k$  permutations by concatenating the panes initiated by the LRmax entries in each row:

$$12 \ 7 \ 25 \ 26 \ 13, \quad 27 \ 28 \ 14, \quad 3 \ 1 \ 5 \ 2 \ 8 \ 4 \ 6 \ 15 \ 9 \ 29 \ 16, \quad 17 \ 10 \ 11 \ 20 \ 30 \ 18 \ 19 \ 21 \ 22 \ 23 \ 24$$

Finally, reduce each one to get the desired  $k$ -list of 321-avoiding permutations:

$$2 \ 1 \ 4 \ 5 \ 3, \quad 2 \ 3 \ 1, \quad 3 \ 1 \ 5 \ 2 \ 7 \ 4 \ 6 \ 9 \ 8 \ 11 \ 10, \quad 3 \ 1 \ 2 \ 6 \ 11 \ 4 \ 5 \ 7 \ 8 \ 9 \ 10.$$

Note that LIT entries in these reduced permutations correspond to LIT entries in the original as do LRmax entries.

To reverse the process, suppose given a list  $(\pi_1, \pi_2, \dots, \pi_k)$  of  $k$  321-avoiding permutations of total length  $n$  and let us use the previous example with  $k = 4$  and  $n = 30$ . Double underline the LIT entries in each  $\pi_i$  and single underline the remaining LRmax entries. Form a pane from the first LIT entry to the end of each permutation.

$$\begin{array}{cccc} \pi_1 & \pi_2 & \pi_3 & \pi_4 \\ \underline{2} \ 1 \ \underline{\underline{4}} \ \underline{\underline{5}} \ 3, & \underline{\underline{2}} \ \underline{\underline{3}} \ 1, & \underline{3} \ 1 \ \underline{5} \ 2 \ \underline{7} \ 4 \ 6 \ \underline{9} \ 8 \ \underline{\underline{11}} \ \underline{\underline{10}}, & \underline{3} \ 1 \ 2 \ \underline{6} \ \underline{\underline{11}} \ 4 \ 5 \ 7 \ 8 \ 9 \ 10 \end{array}$$

### The Initial Panes

Place  $n, n - 1, n - 2, \dots$  below the LIT entries working right to left.



$$\underline{2} \ 1 \ \boxed{\begin{array}{|c|c|c|} \hline \underline{4} & \underline{5} & 3 \\ \hline 25 & 26 & \end{array}}, \ \boxed{\begin{array}{|c|c|} \hline \underline{2} & \underline{3} & 1 \\ \hline 27 & 28 & \end{array}}, \ \underline{3} \ 1 \ \underline{5} \ 2 \ \underline{7} \ 4 \ 6 \ \underline{9} \ 8 \ \boxed{\begin{array}{|c|c|} \hline \underline{11} & \underline{10} \\ \hline 29 & \end{array}}, \ \underline{3} \ 1 \ 2 \ \underline{6} \ \boxed{\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \underline{11} & 4 & 5 & 7 & 8 & 9 & 10 \\ \hline 30 & & & & & & & & & \end{array}}$$

Place LIT Entries

Set  $b =$  largest integer in  $[n]$  not yet placed; here  $b = 24$ . Start with the last permutation in the list. Take the largest entry that is blank below (here, 10), place  $b$  in the blank, decrement  $b$  and repeat as long as these largest entries are (i) empaned or (ii) underlined. Then add a pane that begins at the leftmost newly added entry unless it is already empaned.

$$\underline{2} \ 1 \ \boxed{\begin{array}{|c|c|c|} \hline \underline{4} & \underline{5} & 3 \\ \hline 25 & 26 & \end{array}}, \ \boxed{\begin{array}{|c|c|} \hline \underline{2} & \underline{3} & 1 \\ \hline 27 & 28 & \end{array}}, \ \underline{3} \ 1 \ \underline{5} \ 2 \ \underline{7} \ 4 \ 6 \ \underline{9} \ 8 \ \boxed{\begin{array}{|c|c|} \hline \underline{11} & \underline{10} \\ \hline 29 & \end{array}}, \ \boxed{\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \underline{3} & 1 & 2 & \underline{6} & \underline{11} & 4 & 5 & 7 & 8 & 9 & 10 \\ \hline 17 & & & 20 & 30 & 18 & 19 & 21 & 22 & 23 & 24 \end{array}}$$

Work Last Permutation

Decrement  $b$  (now  $b = 16$ ) and proceed to the next permutation to the left in the list (considering the last to be the left neighbor of the first). Repeat until  $b = 0$  and no blanks remain. The second step gives

$$\underline{2} \ 1 \ \boxed{\begin{array}{|c|c|c|} \hline \underline{4} & \underline{5} & 3 \\ \hline 25 & 26 & \end{array}}, \ \boxed{\begin{array}{|c|c|} \hline \underline{2} & \underline{3} & 1 \\ \hline 27 & 28 & \end{array}}, \ \underline{3} \ 1 \ \underline{5} \ 2 \ \underline{7} \ 4 \ 6 \ \boxed{\begin{array}{|c|c|} \hline \underline{9} & \underline{8} & \underline{11} & \underline{10} \\ \hline 15 & 29 & 16 & \end{array}}, \ \boxed{\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \underline{3} & 1 & 2 & \underline{6} & \underline{11} & 4 & 5 & 7 & 8 & 9 & 10 \\ \hline 17 & & & 20 & 30 & 18 & 19 & 21 & 22 & 23 & 24 \end{array}}$$

Work Penultimate Permutation

and the final result (omitting the no-longer-needed underlines) is

$$\boxed{\begin{array}{|c|c|c|} \hline 2 & 1 & 4 & 5 & 3 \\ \hline 12 & 7 & 25 & 26 & 13 \end{array}}, \ \boxed{\begin{array}{|c|c|} \hline 2 & 3 & 1 \\ \hline 27 & 28 & 14 \end{array}}, \ \boxed{\begin{array}{|c|c|c|c|c|c|} \hline 3 & 1 & 5 & 2 & 7 & 4 & 6 & 9 & 8 & 11 & 10 \\ \hline 3 & 1 & 5 & 2 & 8 & 4 & 6 & 15 & 9 & 29 & 16 \end{array}}, \ \boxed{\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 3 & 1 & 2 & 6 & 11 & 4 & 5 & 7 & 8 & 9 & 10 \\ \hline 17 & 10 & 11 & 20 & 30 & 18 & 19 & 21 & 22 & 23 & 24 \end{array}}$$

Final Result

The marked LIT entries are now retrieved as the max entry in the last pane of all but the last permutation, here 26, 28, 29. Arrange the panes in increasing order of first entries to retrieve the original starred permutation.

## 5 Underlined 4-Patterns

There are  $4 \times 4! = 96$  similarly restricted patterns involving 4 letters (let's call them underlined 4-patterns), for instance,  $\underline{4}231$ : a 431 pattern only occurs as part of a 4231. We say that a permutation meeting this pattern restriction satisfies  $\underline{4}231$  or is  $\underline{4}231$ OK. Four different counting sequences arise. We will show that 64 give the Catalan numbers, 16 give the Bell numbers, 12 give sequence [A051295](#) in OEIS, and 4 give the sequence with  $n$ th term  $= (n-1)! + \sum_{k=0}^{n-2} \sum_{\substack{i,j \geq 0 \\ i+j \leq k}} k^{\bar{i}} (n-2-k)^{\underline{j}}$  using rising/falling factorial notation.

The complement of a permutation  $\pi$  on  $[n]$  is  $n+1-\pi$  (entrywise). The complement, reverse and inverse of an underlined 4-pattern are defined in the obvious way so that a permutation satisfies a given underlined 4-pattern if and only if its complement (resp. reverse, resp. inverse) satisfy the complement (resp. reverse, resp. inverse) of the underlined 4-pattern. Under the action of the 8-element abelian group generated by the complement, reverse and inverse operations, the 96 underlined 4-patterns are partitioned into equivalence classes, the members of each one all having the same counting sequence.

In fact, 64 underlined 4-patterns entail avoidance of the associated 3-pattern altogether. For each of the 6 patterns of length 3, the permutations avoiding it are counted by the Catalan numbers [2]. Hence these 64 are counted by Catalan numbers [A000108](#). (Incidentally, they split into 5 equivalence classes of size 8 and 6 of size 4.) The remaining 32—the nontrivial underlined 4-patterns—split into 5 classes as shown in the Table (the top row contains convenient class representatives).

$3\underline{2}4\underline{1}$	$3\underline{1}4\underline{2}$	$\underline{1}342$	$\underline{1}324$	$3\underline{2}1\underline{4}$
$134\underline{2}$	$\underline{3}142$	$\underline{1}423$	$132\underline{4}$	$\underline{4}123$
$1\underline{4}23$	$314\underline{2}$	$231\underline{4}$	$423\underline{1}$	$\underline{1}432$
$231\underline{4}$	$3\underline{1}42$	$243\underline{1}$	$\underline{4}231$	$234\underline{1}$
$\underline{2}431$	$24\underline{1}3$	$312\underline{4}$		
$\underline{3}124$	$\underline{2}413$	$324\underline{1}$		
$4\underline{1}32$	$241\underline{3}$	$\underline{4}132$		
$421\underline{3}$	$2\underline{4}13$	$\underline{4}213$		
Bell	Bell	A051295	A051295	new

The 5 equivalence classes of nontrivial underlined 4-patterns and their counting sequences

It remains to verify the counting sequences.

**Bell Sequence** Consider two canonical ways to represent a set partition of  $[n]$ .

**Canonical Increasing** largest entry first in each block, rest of block increasing, blocks arranged in increasing order of first element, as in 412-6-735.

**Canonical Decreasing** each block decreasing, blocks arranged in increasing order of first element, as in 421-6-753.

**Theorem 8.** *Splitting a permutation on  $[n]$  into its LRmax factors  $m_1L_1, \dots, m_rL_r$  is a bijection from*

(i) *3241OK permutations on  $[n]$  to set partitions of  $[n]$  in canonical increasing form.*

(ii) *3142OK permutations on  $[n]$  to set partitions of  $[n]$  in canonical decreasing form.*

**Proof** (i) If  $\pi$  is 3241OK yet some  $L_i$  fails to be increasing, then there exist  $u$  preceding  $v$  in  $L_i$  with  $u > v$  and  $m_iuv$  is an offending 321 pattern. Conversely, if each  $L_i$  is increasing, then a 321 must have its “2” and “1” in different blocks and  $\pi$  is  $\pi$  is 3241OK.

(ii) If  $\pi$  is 3142OK yet some LRmax factor  $m_iL_i$  fails to be decreasing, then there exist  $u$  preceding  $v$  in  $L_i$  with  $u < v$  and  $m_iuv$  is an offending 312 pattern. Conversely, if the LRmax factors of  $\pi$  are all decreasing, then a 312 must have its “1” and “2” in different blocks and  $\pi$  is 3142OK.  $\square$

Since set partitions are counted by the Bell numbers [A000110](#), this verifies the first 2 columns in the Table.

**A051295 Sequence** Let us count 1324OK permutations on  $[n]$ . We claim each such permutation  $\pi$  has the form  $\pi_1 1 \pi_2$  where  $\pi_1$  is a 1324OK permutation on  $[n - k + 2, n]$  with  $k$  the position of 1 in  $\pi$ , and  $\pi_2$  is an arbitrary permutation on  $[2, n - k + 1]$ . To see the claim, observe that each such  $\pi$  is 1324OK. Conversely, if  $\pi$  is 1324OK, all entries preceding 1 in  $\pi$  necessarily exceed all entries following 1 or else there is a subpermutation  $u1v$  with  $u < v$  forming an offending 324 pattern. While  $\pi_1$  must be 1324OK, the entry 1 relieves  $\pi_2$  of any obligation, and the claim follows. This decomposition implies that the number  $u_n$  of 1324OK permutations on  $[n]$  satisfies  $u_n = \sum_{k=1}^n u_{k-1}(n - k)!$ , and hence  $(u_n)$  is sequence [A051295](#) in OEIS.

Next, we count  $\underline{1}342$ OK permutations on  $[n]$  by position  $k$  of 1. If 1 occurs in first position, the rest of the permutation is arbitrary. Otherwise, we claim the permutation has the form (with 1 in position  $k \geq 2$ )

$$a_1 \dots a_{k-1} 1 \pi[2, a_{k-1} - 1] \pi[a_{k-1} + 1, a_{k-2} - 1] \dots \pi[a_2 + 1, a_1 - 1] \pi[a_1 + 1, n]$$

where  $a_1 > a_2 > \dots > a_{k-1}$  and  $\pi[b, c]$  denotes an arbitrary permutation on the interval of integers  $[b, c]$  (understood to be the empty permutation if  $b > c$ ).

Such permutations are clearly  $\underline{1}342$ OK. Conversely, given a  $\underline{1}342$ OK permutation, the entries  $a_1, \dots, a_{k-1}$  preceding 1 are necessarily decreasing left to right for otherwise there is a subpermutation  $ab1$  with  $a < b$ . This is a 342 pattern with no hope of the required “1”. And for  $2 \leq i \leq k-1$ , all elements of  $[a_i + 1, a_{i-1} - 1]$  precede all elements of  $[a_{i-1} + 1, a_{i-2} - 1] \sqcup \dots \sqcup [a_1 + 1, n]$  else some  $b > a_{i-1}$  precedes some  $c < a_{i-1}$  in the permutation and  $a_{i-1}bc$  is an offending 342 pattern because all entries preceding  $a_{i-1}$  exceed  $a_{i-1}$ . The claim follows.

With  $u_{n,k}$  the number of  $\underline{1}342$ OK permutations on  $[n]$  for which 1 occurs in position  $k$ , this decomposition implies the generating function identity for each  $k \geq 1$

$$\sum_{n \geq 0} u_{n,k} x^n = x^k \left( \sum_{n \geq 0} n! x^n \right)^k$$

and so  $(u_{n,k})$  forms sequence [A084938](#) in OEIS with row sums [A051295](#). This verifies columns 3 and 4 of the Table.

The fact that  $\underline{1}324$  and  $\underline{1}342$  are Wilf-equivalent (that is, have the same counting sequence) despite being in different equivalence classes can be explained by a simple bijection from  $\underline{1}324$ OK permutations to  $\underline{1}342$ OK permutations. Factor a  $\underline{1}324$ OK permutation  $\pi$  as

$$m_1 L_1 m_2 L_2 \dots m_r L_r$$

where  $m_1, m_2, \dots$  are now the left-to-right *minima* of  $\pi$ . Then reassemble as

$$m_1 m_2 \dots m_r L_r L_{r-1} \dots L_1.$$

This works because  $L_i$  is a permutation on  $[m_i + 1, m_{i-1} - 1]$  for  $1 \leq i \leq r$  ( $m_0 := n + 1$ ) for otherwise there is an entry  $b$  following  $m_i$  in  $\pi$  with  $b > m_{i-1}$  and  $m_{i-1}m_i b$  is an offending 324 pattern.

**New Sequence** The  $321\underline{4}$ OK permutations on  $[n]$  can be counted directly by position of  $n$ . If  $n$  occurs in last position, the rest of the permutation is unrestricted— $(n-1)!$

choices. If  $n$  occurs in position  $n-1$  and the last entry is  $i$ , then  $i+1, i+2, \dots, n$  must occur in that order else  $i$  terminates an offending 321. Thus the permutation is determined by the positions of  $1, 2, \dots, i-1$ — $(n-2)^{i-1}$  choices where  $j^i = j(j-1)\dots(j-i+1)$  is the falling factorial.

Now suppose  $n$  occurs in position  $k \leq n-2$ . The subpermutation following  $n$  must be increasing and hence has the form  $1 \leq a < y_1 < y_2 < \dots < y_{n-k-2} < b < n$ . Fix  $a$  and  $b$  and let  $(x_i)_{i=1}^r = (a, b) \setminus (y_i)_{i=1}^{n-k-2}$  entailing  $r = (b-a-1) - (n-k-2) = b-t$  with  $t = a+n-k-1$ . The subpermutation preceding  $n$  has no restriction on the entries  $< a$  and placing these entries gives  $(k-1)^{a-1}$  choices. However, its entries  $> b$  must be increasing left to right (else there exist  $u > v > b$  with  $u$  preceding  $v$  and  $uvb$  is an offending 321) and must all lie to the right of the  $x_i$ s (else some  $u > b$  precedes some  $x_i$  and  $u > b > x_i > a$  implies  $ux_i a$  is an offending 321). Thus the subpermutation preceding  $n$  and involving entries  $> a$  yields only  $r!$  choices: arrange the  $x_i$ s. All told, the desired count is (with  $t := a+n-k-1$  and  $n \geq 1$ )

$$(n-1)! + \sum_{i=1}^{n-1} (n-2)^{i-1} + \sum_{k=1}^{n-2} \sum_{a=1}^k \sum_{b=t}^{n-1} (k-1)^{a-1} \binom{b-a-1}{n-k-2} (b-t)! =$$

$$(n-1)! + \sum_{k=0}^{n-2} \sum_{\substack{i,j \geq 0 \\ i+j \leq k}} k^{\bar{i}} (n-2-k)^{\underline{j}}.$$

This sequence begins  $(1, 1, 2, 5, 15, 55, 248, 1357, 8809, 66323, 568238, \dots)_{n \geq 0}$  and is entry-wise  $\geq$  the next largest counting sequence **A051295**  $= (1, 1, 2, 5, 15, 54, 235, 1237, \dots)$ ; both seem to be  $\sim (n-1)!$  asymptotically.  $\square$

A pattern restriction similar to one of our underlined 4-patterns arises in a recent paper [4] on “Patience Sorting” a deck of cards into piles, namely, the restriction that a 342 pattern in which the “4” and “2” are contiguous occurs only as part of a 3142 pattern. It is almost immediate, however, that the permutations satisfying this seemingly weaker restriction coincide with our 3142OK permutations (and hence are counted by the Bell numbers).

## References

- [1] Mira Bernstein and N. J. A. Sloane, Some canonical sequences of integers, *Linear Algebra and its Applications*, Vol. 226-228 (1995), pp. 57–72; errata Vol. 320 (2000),

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- [2] Richard P. Stanley, *Enumerative Combinatorics Vol. 2*, Cambridge University Press, 1999. Exercise 6.19 and related material on Catalan numbers are available online at <http://www-math.mit.edu/~rstan/ec/> .
- [3] S. Dulucq, S. Gire, and O. Guibert, A combinatorial proof of J. West's conjecture, *Discrete Math.* 187 (1998) 71–96.
- [4] Alexander Burstein, Combinatorics of Patience Sorting Piles, [arXiv math.CO/0506358](https://arxiv.org/abs/math/0506358), 18 Jun 2005, 19 pp.