ON ASYMPTOTIC CONSTANTS RELATED TO PRODUCTS OF BERNOULLI NUMBERS AND FACTORIALS

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ABSTRACT. We discuss the asymptotic expansions of certain products of Bernoulli numbers and factorials, e.g.,

$$\prod_{\nu=1}^{n} |B_{2\nu}| \quad \text{and} \quad \prod_{\nu=1}^{n} (k\nu)!^{\,\nu^r} \quad \text{as} \quad n \to \infty$$

for integers $k \geq 1$ and $r \geq 0$. Our main interest is to determine exact expressions, in terms of known constants, for the asymptotic constants of these expansions and to show some relations among them.

1. Introduction

Let B_n be the nth Bernoulli number. These numbers are defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \qquad |z| < 2\pi,$$

where $B_n = 0$ for odd $n > 1$. The Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{\nu=1}^{\infty} \nu^{-s} = \prod_{p} (1 - p^{-s})^{-1}, \quad s \in \mathbb{C}, \ \operatorname{Re} s > 1. \tag{1.1}$$

By Euler's formula we have for even positive integers n that

$$\zeta(n) = -\frac{1}{2} \frac{(2\pi i)^n}{n!} B_n. \tag{1.2}$$

Products of Bernoulli numbers occur in certain contexts in number theory. For example, the Minkowski–Siegel mass formula states for positive integers n with $8 \mid n$ that

$$M(n) = \frac{|B_k|}{2k} \prod_{\nu=1}^{k-1} \frac{|B_{2\nu}|}{4\nu}, \quad n = 2k,$$

which describes the mass of the genus of even unimodular positive definite $n \times n$ matrices, for details see [12, p. 252]. We introduce the following constants which we shall need further on.

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Lemma 1.1. *There exist the constants*

$$\mathcal{C}_1 = \prod_{\nu=2}^{\infty} \zeta(\nu) = 2.2948565916\ldots,$$

$$\mathcal{C}_2 = \prod_{\nu=1}^{\infty} \zeta(2\nu) = 1.8210174514\ldots,$$

$$\mathcal{C}_3 = \prod_{\nu=1}^{\infty} \zeta(2\nu+1) = 1.2602057107\ldots.$$

Proof. We have $\log(1+x) < x$ for real $x > 0$. Then

$$\log \prod_{\nu=1}^{\infty} \zeta(2\nu) = \sum_{\nu=1}^{\infty} \log \zeta(2\nu) < \sum_{\nu=1}^{\infty} (\zeta(2\nu) - 1) = \frac{3}{4}. \tag{1.3}$$

The last sum of (1.3) is well known and follows by rearranging in geometric series, since we have absolute convergence. We then obtain that $\pi^2/6 < \mathcal{C}_2 < e^{3/4}$, $\zeta(3) < \mathcal{C}_3 < \mathcal{C}_2$, and $\mathcal{C}_1 = \mathcal{C}_2 \mathcal{C}_3$. □

To compute the infinite products above within a given precision, one can use the following arguments. A standard estimate for the partial sum of $\zeta(s)$ is given by

$$\zeta(s) - \sum_{\nu=1}^{N} \nu^{-s} < \frac{N^{1-s}}{s-1}, \quad s \in \mathbb{R},\ s > 1.$$

This follows by comparing the sum of ν^{-s} and the integral of x^{-s} in the interval (N, ∞). Now, one can estimate the number N depending on s and the needed precision. However, we use a computer algebra system, that computes $\zeta(s)$ to a given precision with already accelerated built-in algorithms. Since $\zeta(s) \to 1$ monotonically as $s \to \infty$, we next have to determine a finite product that suffices the precision. From above, we obtain

$$\zeta(s) - 1 < 2^{-s} \left(1 + \frac{2}{s-1} \right), \quad s \in \mathbb{R},\ s > 1. \tag{1.4}$$

According to (1.3) and (1.4), we then get an estimate for the remainder of the infinite product by

$$\log \prod_{\nu > N'} \zeta(\nu) < 2^{-N' + \varepsilon}$$

where we can take $\varepsilon = 3/N'$; the choice of ε follows by $2^x \geq 1 + x \log 2$ and (1.4).

We give the following example where the constant $\mathcal{C}_1$ plays an important role; see Finch [8]. Let $a(n)$ be the number of non-isomorphic abelian groups of order n. The constant $\mathcal{C}_1$ equals the average of the numbers $a(n)$ by taking the limit. Thus, we have

$$\mathcal{C}_1 = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a(n).$$

By definition the constant $\mathcal{C}_2$ is connected with values of the Riemann zeta function on the positive real axis. Moreover, this constant is also connected with values of the Dedekind eta function

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{\nu=1}^{\infty} (1 - e^{2\pi i \nu \tau}), \quad \tau \in \mathbb{C},\ \operatorname{Im} \tau > 0,$$

on the upper imaginary axis.

Lemma 1.2. *The constant* $\mathcal{C}_2$ *is given by*

$$1/\mathcal{C}_2 = \prod_p p^{\frac{1}{12}}\ \eta\left(i \frac{\log p}{\pi} \right)$$

where the product runs over all primes.

Proof. By Lemma 1.1 and the Euler product (1.1) of $\zeta(s)$, we obtain

$$\mathcal{C}_2 = \prod_{\nu=1}^{\infty} \prod_p (1 - p^{-2\nu})^{-1} = \prod_p \prod_{\nu=1}^{\infty} (1 - p^{-2\nu})^{-1}$$

where we can change the order of the products because of absolute convergence. Rewriting $p^{-2\nu} = e^{2\pi i \nu \tau}$ with $\tau = i \log p \,/\pi$ yields the result. □

We used Mathematica [17] to compute all numerical values in this paper. The values were checked again by increasing the needed precision to 10 more digits.

2. Preliminaries

We use the notation $f \sim g$ for real-valued functions when $\lim_{x\to\infty} f(x)/g(x) = 1$. As usual, $O(\cdot)$ denotes Landau's symbol. We write $\log f$ for $\log(f(x))$.

Definition 2.1. Define the linear function spaces

$$\Omega_n = \operatorname*{span}_{0 \le \nu \le n} \{x^\nu,\, x^\nu \log x\}, \quad n \ge 0,$$

over $\mathbb{R}$ where $f \in \Omega_n$ is a function $f : \mathbb{R}^+ \to \mathbb{R}$. Let

$$\Omega_\infty = \bigcup_{n \ge 0} \Omega_n.$$

Define the linear map $\psi : \Omega_\infty \to \mathbb{R}$ which gives the constant term of any $f \in \Omega_\infty$. For the class of functions

$$F(x) = f(x) + O(x^{-\delta}), \quad f \in \Omega_n, \quad n \ge 0, \quad \delta > 0, \tag{2.1}$$

define the linear operator $[\cdot] : C(\mathbb{R}^+; \mathbb{R}) \to \Omega_\infty$ such that $[F] = f$ and $[F] \in \Omega_n$. Then $\psi([F])$ is defined to be the asymptotic constant of F.

We shall examine functions $h : \mathbb{N} \to \mathbb{R}$ which grow exponentially; in particular these functions are represented by certain products. Our problem is to find an asymptotic function $\tilde{h} : \mathbb{R}^+ \to \mathbb{R}$ where $h \sim \tilde{h}$. If $F = \log \tilde{h}$ satisfies (2.1), then we have $[\log \tilde{h}] \in \Omega_n$ for a suitable n and we identify $[\log \tilde{h}] = [\log h] \in \Omega_n$ in that case.

Lemma 2.2. *Let $f \in \Omega_n$ where*

$$f(x) = \sum_{\nu=0}^{n} (\alpha_\nu \, x^\nu + \beta_\nu \, x^\nu \log x)$$

with coefficients $\alpha_\nu, \beta_\nu \in \mathbb{R}$. Let $g(x) = f(\lambda x)$ with a fixed $\lambda \in \mathbb{R}^+$. Then $g \in \Omega_n$ and $\psi(g) = \psi(f) + \beta_0 \log \lambda$.

Proof. Since $g(x) = f(\lambda x)$ we obtain

$$g(x) = \sum_{\nu=0}^{n} (\alpha_\nu \, (\lambda x)^\nu + \beta_\nu \, (\lambda x)^\nu (\log \lambda + \log x)).$$

This shows that $g \in \Omega_n$. The constant terms are α_0 and $\beta_0 \log \lambda$, thus $\psi(g) = \psi(f) + \beta_0 \log \lambda$. □

Definition 2.3. For a function $f : \mathbb{R}^+ \to \mathbb{R}$ we introduce the notation

$$f(x) = \sum_{\nu \geq 1}{}' \, f_\nu(x)$$

with functions $f_\nu : \mathbb{R}^+ \to \mathbb{R}$ in case f has a divergent series expansion such that

$$f(x) = \sum_{\nu=1}^{m-1} f_\nu(x) + \theta_m(x) f_m(x), \quad \theta_m(x) \in (0,1), \quad m \geq N_f,$$

where N_f is a suitable constant depending on f.

Next we need some well known facts which we state without proof, cf. [10].

Proposition 2.4. *Let*

$$H_0 = 0, \qquad H_n = \sum_{\nu=1}^{n} \frac{1}{\nu}, \quad n \geq 1,$$

be the nth harmonic number. These numbers satisfy $H_n = \gamma + \log n + O(n^{-1})$ for $n \geq 1$, where $\gamma = 0.5772156649...$ is Euler's constant.

Proposition 2.5 (Stirling's series)**.** *The Gamma function $\Gamma(x)$ has the divergent series expansion*

$$\log \Gamma(x+1) = \frac{1}{2} \log(2\pi) + \left(x + \frac{1}{2} \right) \log x - x + \sum_{\nu \geq 1}{}' \, \frac{B_{2\nu}}{2\nu(2\nu-1)} \, x^{-(2\nu-1)}, \; x > 0.$$

Remark 2.6. When evaluating the divergent series given above, we have to choose a suitable index m such that

$$\sum_{\nu \geq 1}{}' \, \frac{B_{2\nu}}{2\nu(2\nu-1)} \, x^{-(2\nu-1)} = \sum_{\nu=1}^{m-1} \frac{B_{2\nu}}{2\nu(2\nu-1)} \, x^{-(2\nu-1)} + \theta_m(x) R_m(x)$$

and the remainder $|\theta_m(x) R_m(x)|$ is as small as possible. Since $\theta_m(x) \in (0,1)$ is not effectively computable in general, we have to use $|R_m(x)|$ instead as an error bound. Schäfke and Finsterer [15], among others, showed that the so-called Lindelöf error bound $L = 1$ for the estimate $L \geq \theta_m(x)$ is best possible for positive real x.

Proposition 2.7. *If $\alpha \in \mathbb{R}$ with $0 \leq \alpha < 1$, then*

$$\prod_{\nu=1}^{n} (\nu - \alpha) = \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)} \sim \frac{\sqrt{2\pi}}{\Gamma(1-\alpha)} \left(\frac{n}{e}\right)^n n^{\frac{1}{2}-\alpha} \quad as \quad n \to \infty.$$

Proposition 2.8 (Euler)**.** *Let $\Gamma(x)$ be the Gamma function. Then*

$$\prod_{\nu=1}^{n-1} \Gamma\left(\frac{\nu}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}.$$

Proposition 2.9 (Glaisher [9], Kinkelin [11])**.** *Asymptotically, we have*

$$\prod_{\nu=1}^{n} \nu^\nu \sim \mathcal{A}\, n^{\frac{1}{2}n(n+1)+\frac{1}{12}}\, e^{-\frac{n^2}{4}} \quad as \quad n \to \infty$$

where $\mathcal{A} = 1.2824271291...$ is the Glaisher–Kinkelin constant, which is given by

$$\log \mathcal{A} = \frac{1}{12} - \zeta'(-1) = \frac{\gamma}{12} + \frac{1}{12}\log(2\pi) - \frac{\zeta'(2)}{2\pi^2}.$$

Numerous digits of the decimal expansion of the Glaisher–Kinkelin constant $\mathcal{A}$ are recorded as sequence A074962 in OEIS [16].

3. Products of factorials

In this section we consider products of factorials and determine their asymptotic expansions and constants. For these asymptotic constants we derive a divergent series representation as well as a closed formula.

Theorem 3.1. *Let k be a positive integer. Asymptotically, we have*

$$\prod_{\nu=1}^{n} (k\nu)! \sim \mathcal{F}_k\, \mathcal{A}^k\, (2\pi)^{\frac{1}{4}} \left(\frac{k\,n}{e^{3/2}}\right)^{\frac{k}{2}n(n+1)} \left(2\pi k e^{k/2-1}\, n\right)^{\frac{n}{2}} n^{\frac{1}{4}+\frac{k}{12}+\frac{1}{12k}} \quad as \quad n \to \infty$$

with certain constants $\mathcal{F}_k$ which satisfy

$$\log \mathcal{F}_k = \frac{\gamma}{12k} + \sum_{j\geq 2}{}' \frac{B_{2j}\, \zeta(2j-1)}{2j(2j-1)\, k^{2j-1}}.$$

Moreover, the constants have the asymptotic behavior that

$$\lim_{k\to\infty} \mathcal{F}_k = 1, \quad \lim_{k\to\infty} \mathcal{F}_k^k = e^{\gamma/12}, \quad and \quad \prod_{k=1}^{n} \mathcal{F}_k \sim \mathcal{F}_\infty\, n^{\gamma/12} \quad as \quad n \to \infty$$

with

$$\log \mathcal{F}_\infty = \frac{\gamma^2}{12} + \sum_{j\geq 2}{}' \frac{B_{2j}\, \zeta(2j-1)^2}{2j(2j-1)}.$$

Theorem 3.2. *If k is a positive integer, then*

$$\log \mathcal{F}_k = -\left(k + \frac{1}{k}\right) \log \mathcal{A} + \frac{1}{12k} - \frac{1}{12k} \log k + \frac{k}{4} \log(2\pi) - \sum_{\nu=1}^{k-1} \frac{\nu}{k} \log \Gamma\left(\frac{\nu}{k}\right).$$

We will prove Theorem 3.2 later, since we shall need several preparations.

Proof of Theorem 3.1. Let $k \geq 1$ be fixed. By Stirling's approximation, see Proposition 2.5, we have

$$\log(k\nu)! = \frac{1}{2} \log(2\pi) + \left(k\nu + \frac{1}{2}\right) \log(k\nu) - k\nu + f(k\nu) \tag{3.1}$$

where we can write the remaining divergent sum as

$$f(k\nu) = \frac{1}{12k\nu} + \sum_{j \geq 2}{}' \frac{B_{2j}}{2j(2j-1)\,(k\nu)^{2j-1}}.$$

Define $S(n) = 1 + \cdots + n = n(n+1)/2$. By summation we obtain

$$\sum_{\nu=1}^{n} \log(k\nu)! = \frac{n}{2} \log(2\pi k) + \frac{1}{2} \log n! - kS(n) + kS(n) \log k + k \sum_{\nu=1}^{n} \nu \log \nu + \sum_{\nu=1}^{n} f(k\nu).$$

The term $\frac{1}{2} \log n!$ is evaluated again by (3.1). Proposition 2.9 provides that

$$k \sum_{\nu=1}^{n} \nu \log \nu = k \log \mathcal{A} + kS(n) \log n + \frac{k}{12} \log n - \frac{k}{2}\left(S(n) - \frac{n}{2}\right) + O(n^{-\delta})$$

with some $\delta > 0$. Since $\lim_{n\to\infty} H_n - \log n = \gamma$, we asymptotically obtain for the remaining sum that

$$\lim_{n\to\infty} \left(\sum_{\nu=1}^{n} f(k\nu) - \frac{1}{12k} \log n\right) = \frac{\gamma}{12k} + \sum_{j \geq 2}{}' \frac{B_{2j}\,\zeta(2j-1)}{2j(2j-1)\,k^{2j-1}} =: \log \mathcal{F}_k. \tag{3.2}$$

Here we have used the following arguments. We choose a fixed index $m > 2$ for the remainder of the divergent sum. Then

$$\lim_{n\to\infty} \sum_{\nu=1}^{n} \theta_m(k\nu) \frac{B_{2m}}{2m(2m-1)\,(k\nu)^{2m-1}} = \eta_m \frac{B_{2m}\,\zeta(2m-1)}{2m(2m-1)\,k^{2m-1}} \tag{3.3}$$

with some $\eta_m \in (0,1)$, since $\theta_m(k\nu) \in (0,1)$ for all $\nu \geq 1$. Thus, we can write (3.2) as an asymptotic series again. Collecting all terms, we finally get the asymptotic formula

$$\begin{aligned}\sum_{\nu=1}^{n} \log(k\nu)! = \log \mathcal{F}_k + k \log \mathcal{A} + \frac{1}{4}\log(2\pi) + \; kS(n)\left(-\frac{3}{2} + \log(kn)\right) \\ + \frac{n}{2}\left(\log(2\pi k) + \frac{k}{2} - 1 + \log n\right) \\ + \left(\frac{1}{4} + \frac{k}{12} + \frac{1}{12k}\right)\log n + O(n^{-\delta'})\end{aligned}$$

with some $\delta' > 0$. Note that the exact value of δ' does not play a role here. Now, let k be an arbitrary positive integer. From (3.2) we deduce that

$$\log \mathcal{F}_k = \frac{\gamma}{12k} + O(k^{-3}) \qquad \text{and} \qquad k \log \mathcal{F}_k = \frac{\gamma}{12} + O(k^{-2}). \tag{3.4}$$

The summation of (3.2) yields

$$\sum_{k=1}^{n} \log \mathcal{F}_k = \frac{\gamma}{12} H_n + \sum_{k=1}^{n} \sum_{j \geq 2}{}' \frac{B_{2j}\, \zeta(2j-1)}{2j(2j-1)\, k^{2j-1}}. \tag{3.5}$$

Similar to (3.2) and (3.3), we can write again:

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} \log \mathcal{F}_k - \frac{\gamma}{12}\log n\right) = \frac{\gamma^2}{12} + \sum_{j\geq 2}{}' \frac{B_{2j}\, \zeta(2j-1)^2}{2j(2j-1)} =: \log \mathcal{F}_\infty. \tag{3.6}$$

□

The case $k = 1$ of Theorem 3.1 is related to the so-called Barnes G-function, cf. [2]. Now we shall determine exact expressions for the constants $\mathcal{F}_k$. For $k \geq 2$ this is more complicated.

Lemma 3.3. *We have* $\mathcal{F}_1 = (2\pi)^{\frac{1}{4}}\, e^{\frac{1}{12}} / \mathcal{A}^2$.

Proof. Writing down the product of $n!$ repeatedly in $n+1$ rows, one observes by counting in rows and columns that

$$n!^{n+1} = \prod_{\nu=1}^{n} \nu! \prod_{\nu=1}^{n} \nu^{\nu}. \tag{3.7}$$

From Proposition 2.5 we have

$$(n+1)\log n! = \frac{n+1}{2}\log(2\pi) - n(n+1) + (n+1)\left(n + \frac{1}{2}\right)\log n + \frac{1}{12} + O(n^{-1}).$$

Comparing the asymptotic constants of both sides of (3.7) when $n \to \infty$, we obtain

$$(2\pi)^{\frac{1}{2}}\, e^{\frac{1}{12}} = \mathcal{F}_1\, \mathcal{A}\, (2\pi)^{\frac{1}{4}} \cdot \mathcal{A}$$

where the right side follows by Theorem 3.1 and Proposition 2.9. □

Proposition 3.4. *Let k, l be integers with $k \geq 1$. Define*

$$F_{k,l}(n) := \prod_{\nu=1}^{n} (k\nu - l)! \quad \text{for} \quad 0 \leq l < k.$$

Then $[\log F_{k,l}] \in \Omega_2$ and $F_{k,0}(n) \cdots F_{k,k-1}(n) = F_{1,0}(kn)$. Moreover

$$F_{k,l}(n)/F_{k,l+1}(n) = k^n \prod_{\nu=1}^{n} \left(\nu - \frac{l}{k} \right) \quad \text{for} \quad 0 \leq l < k-1$$

and $[\log(F_{k,l}/F_{k,l+1})] = [\log F_{k,l}] - [\log F_{k,l+1}] \in \Omega_1$.

Proof. We deduce the proposed products from $(k\nu - l)!/(k\nu - (l+1))! = k\nu - l$ and

$$\prod_{\nu=1}^{n} (k\nu)!(k\nu - 1)! \cdots (k\nu - (k-1))! = \prod_{\nu=1}^{kn} \nu!. \tag{3.8}$$

Proposition 2.7 shows that $[\log(F_{k,l}/F_{k,l+1})] \in \Omega_1$. Since the operator $[\cdot]$ is linear, it follows that

$$[\log(F_{k,l}/F_{k,l+1})] = [\log F_{k,l} - \log F_{k,l+1}] = [\log F_{k,l}] - [\log F_{k,l+1}] \in \Omega_1. \tag{3.9}$$

From Theorem 3.1 we have $[\log F_{k,0}] \in \Omega_2$. By induction on l and using (3.9) we derive that $[\log F_{k,l}] \in \Omega_2$ for $0 < l < k$. □

Lemma 3.5. *Let k be an integer with $k \geq 2$. Define the $k \times k$ matrix*

$$M_k := \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

where all other entries are zero. Then $\det M_k = k$ and the matrix inverse is given by $M_k^{-1} = \frac{1}{k} \widetilde{M_k}$ with

$$\widetilde{M_k} = \begin{pmatrix} k-1 & k-2 & k-3 & \cdots & 2 & 1 & 1 \\ -1 & k-2 & k-3 & \cdots & 2 & 1 & 1 \\ -1 & -2 & k-3 & \cdots & 2 & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ -1 & -2 & -3 & \cdots & 2 & 1 & 1 \\ -1 & -2 & -3 & \cdots & -(k-2) & 1 & 1 \\ -1 & -2 & -3 & \cdots & -(k-2) & -(k-1) & 1 \end{pmatrix}.$$

Proof. We have $\det M_2 = 2$. Let $k \geq 3$. We recursively deduce by the Laplacian determinant expansion by minors on the first column that

$$\det M_k = (-1)^{1+1} \det M_{k-1} + (-1)^{1+k} \det T_{k-1}$$

where the latter matrix T_{k-1} is a lower triangular matrix having -1 in its diagonal. Therefore

$$\det M_k = \det M_{k-1} + (-1)^{1+k} \cdot (-1)^{k-1} = k - 1 + 1 = k$$

by induction on k. Let I_k be the $k \times k$ identity matrix. The equation $M_k \cdot \widetilde{M_k} = k\, I_k$ is easily verified by direct calculation, since M_k has a simple form. □

Proof of Theorem 3.2. The case $k = 1$ agrees with Lemma 3.3. For now, let $k \geq 2$. We use the relations between the functions $F_{k,l}$, resp. $\log F_{k,l}$, given in Proposition 3.4. Since $[\log F_{k,l}] \in \Omega_2$, we can work in Ω_2. The matrix M_k defined in Lemma 3.5 mainly describes the relations given in (3.8) and (3.9). Furthermore we can reduce our equations to $\mathbb{R}$ by applying the linear map ψ, since we are only interested in the asymptotic constants. We obtain the linear system of equations

$$M_k \cdot x = b\,, \qquad x, b \in \mathbb{R}^k,$$

where

$$x = (\psi([\log F_{k,0}]), \ldots, \psi([\log F_{k,k-1}]))^T$$

and $b = (b_1, \ldots, b_k)^T$ with

$$b_{l+1} = \psi([\log(F_{k,l}/F_{k,l+1})]) = \frac{1}{2}\log(2\pi) - \log\Gamma\left(1 - \frac{l}{k}\right) \quad \text{for} \quad l = 0, \ldots, k-2$$

using Proposition 2.7. The last element b_k is given by Theorem 3.1, Lemma 3.3, and Lemma 2.2:

$$\begin{aligned} b_k = \psi([\log(F_{1,0}(kn))]) &= \frac{1}{4}\log(2\pi) + \log\mathcal{F}_1 + \log\mathcal{A} + \frac{5}{12}\log k \\ &= \frac{1}{2}\log(2\pi) - \log\mathcal{A} + \frac{1}{12} + \frac{5}{12}\log k. \end{aligned}$$

By Lemma 3.5 we can solve the linear system directly with

$$x = \frac{1}{k}\,\widetilde{M_k} \cdot b.$$

The first row yields

$$x_1 = \frac{1}{k}\, b_k + \frac{1}{k}\sum_{\nu=1}^{k-1}(k - \nu)\, b_\nu.$$

On the other side, we have

$$x_1 = \psi([\log F_{k,0}]) = \log\mathcal{F}_k + \frac{1}{4}\log(2\pi) + k\log\mathcal{A}.$$

This provides

$$\begin{aligned} \log\mathcal{F}_k = &-\left(k + \frac{1}{k}\right)\log\mathcal{A} + \left(\frac{k}{4} + \frac{1}{2k} - \frac{1}{2}\right)\log(2\pi) \\ &+ \frac{5}{12k}\log k + \frac{1}{12k} - \sum_{\nu=2}^{k-1}\frac{\nu - 1}{k}\log\Gamma\left(\frac{\nu}{k}\right) \end{aligned} \tag{3.10}$$

after some rearranging of terms. By Euler's formula, see Proposition 2.8, we have

$$\frac{1}{k}\sum_{\nu=1}^{k-1}\log\Gamma\Big(\frac{\nu}{k}\Big)=\Big(\frac{1}{2}-\frac{1}{2k}\Big)\log(2\pi)-\frac{1}{2k}\log k. \tag{3.11}$$

Finally, substituting (3.11) into (3.10) yields the result. □

Remark 3.6. Although the formula for $\mathcal{F}_k$ has an elegant short form, one might also use (3.10) instead, since this formula omits the value $\Gamma(1/k)$. Thus we easily obtain the value of $\mathcal{F}_2$ from (3.10) at once: $\mathcal{F}_2=(2\pi)^{\frac{1}{4}}\,2^{\frac{5}{24}}\,e^{\frac{1}{24}}/\mathcal{A}^{\frac{5}{2}}$.

Corollary 3.7. *Asymptotically, we have*

$$\prod_{\nu=1}^{n-1}\Gamma\Big(\frac{\nu}{n}\Big)^{\nu}\sim\frac{e^{\frac{1-\gamma}{12}}}{\mathcal{A}}\left(\frac{(2\pi)^{\frac{1}{4}}}{\mathcal{A}}\right)^{n^2}\Big/\,n^{\frac{1}{12}}\quad \text{as}\quad n\to\infty$$

with the constants $e^{\frac{1-\gamma}{12}}/\mathcal{A}=0.8077340270...$ *and* $(2\pi)^{\frac{1}{4}}/\mathcal{A}=1.2345601953....$

Proof. On the one hand, we have by (3.4) that

$$n\log\mathcal{F}_n=\frac{\gamma}{12}+O(n^{-2}).$$

On the other hand, Theorem 3.2 provides that

$$n\log\mathcal{F}_n=-\left(n^2+1\right)\log\mathcal{A}+\frac{1}{12}-\frac{1}{12}\log n+\frac{n^2}{4}\log(2\pi)-\sum_{\nu=1}^{n-1}\nu\log\Gamma\Big(\frac{\nu}{n}\Big).$$

Combining both formulas easily gives the result. □

Since we have derived exact expressions for the constants $\mathcal{F}_k$, we can improve the calculation of $\mathcal{F}_\infty$. The divergent sum of $\mathcal{F}_\infty$, given in Theorem 3.1, is not suitable to determine a value within a given precision, but we can use this sum in a modified way. Note that we cannot use the limit formula

$$\log\mathcal{F}_\infty=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\log\mathcal{F}_k-\frac{\gamma}{12}\log n\right)$$

without a very extensive calculation, because the sequence $\gamma_n=H_n-\log n$ converges too slowly. Moreover, the computation of $\mathcal{F}_k$ involves the computation of the values $\Gamma(\nu/k)$. This becomes more difficult for larger k.

Proposition 3.8. *Let m,n be positive integers. Assume that $m>2$ and the constants $\mathcal{F}_k$ are given by exact expressions for $k=1,\dots,n$. Define the computable values $\eta_k\in(0,1)$ implicitly by*

$$\log\mathcal{F}_k=\frac{\gamma}{12k}+\sum_{j=2}^{m-1}\frac{B_{2j}\,\zeta(2j-1)}{2j(2j-1)\,k^{2j-1}}+\eta_k\frac{B_{2m}\,\zeta(2m-1)}{2m(2m-1)\,k^{2m-1}}.$$

Then

$$\log \mathcal{F}_\infty = \frac{\gamma^2}{12} + \sum_{j=2}^{m-1} \frac{B_{2j}\,\zeta(2j-1)^2}{2j(2j-1)} + \theta_{n,m}\,\frac{B_{2m}\,\zeta(2m-1)^2}{2m(2m-1)}$$

with $\theta_{n,m} \in (\theta_{n,m}^{\min}, \theta_{n,m}^{\max}) \subset (0,1)$ *where*

$$\theta_{n,m}^{\min} = \zeta(2m-1)^{-1} \sum_{k=1}^{n} \frac{\eta_k}{k^{2m-1}}, \qquad \theta_{n,m}^{\max} = 1 + \zeta(2m-1)^{-1} \sum_{k=1}^{n} \frac{\eta_k - 1}{k^{2m-1}}.$$

The error bound for the remainder of the divergent sum of $\log \mathcal{F}_\infty$ *is given by*

$$\theta_{n,m}^{\mathrm{err}} = \left(1 - \zeta(2m-1)^{-1} \sum_{k=1}^{n} \frac{1}{k^{2m-1}}\right) \frac{|B_{2m}|\,\zeta(2m-1)^2}{2m(2m-1)}.$$

Proof. Let $n \geq 1$ and $m > 2$ be fixed integers. The divergent sums for $\log \mathcal{F}_k$ and $\log \mathcal{F}_\infty$ are given by Theorem 3.1. Since we require exact expressions for $\mathcal{F}_k$, we can compute the values η_k for $k = 1, \ldots, n$. We define

$$\eta_{m,k} = \eta'_{m,k} = \eta_k \quad \text{for} \quad k = 1, \ldots, n$$

and

$$\eta_{m,k} = 0, \quad \eta'_{m,k} = 1 \quad \text{for} \quad k > n.$$

We use (3.5) and (3.6) to derive the bounds:

$$\theta_{n,m}^{\min} = \zeta(2m-1)^{-1} \sum_{k=1}^{\infty} \frac{\eta_{m,k}}{k^{2m-1}} \; < \; \theta_{n,m} \; < \; \zeta(2m-1)^{-1} \sum_{k=1}^{\infty} \frac{\eta'_{m,k}}{k^{2m-1}} = \theta_{n,m}^{\max}.$$

We obtain the suggested formulas for $\theta_{n,m}^{\min}$ and $\theta_{n,m}^{\max}$ by evaluating the sums with $\eta_{m,k} = 0$, resp. $\eta'_{m,k} = 1$, for $k > n$. The error bound is given by the difference of the absolute values of the minimal and maximal remainder. Therefore

$$\theta_{n,m}^{\mathrm{err}} = (\theta_{n,m}^{\max} - \theta_{n,m}^{\min}) R = \left(1 - \zeta(2m-1)^{-1} \sum_{k=1}^{n} \frac{1}{k^{2m-1}}\right) R$$

with $R = |B_{2m}|\zeta(2m-1)^2/2m(2m-1)$. ∎

Result 3.9. Exact expressions for $\mathcal{F}_k$:

$$\mathcal{F}_1 = (2\pi)^{\frac{1}{4}}\, e^{\frac{1}{12}} / \mathcal{A}^2, \qquad \mathcal{F}_2 = (2\pi)^{\frac{1}{4}}\, 2^{\frac{5}{24}}\, e^{\frac{1}{24}} / \mathcal{A}^{\frac{5}{2}},$$

$$\mathcal{F}_3 = (2\pi)^{\frac{5}{12}}\, 3^{\frac{5}{36}}\, e^{\frac{1}{36}} / \mathcal{A}^{\frac{10}{3}}\, \Gamma\!\left(\tfrac{2}{3}\right)^{\frac{1}{3}}, \quad \mathcal{F}_4 = (2\pi)^{\frac{1}{2}}\, 2^{\frac{1}{3}}\, e^{\frac{1}{48}} / \mathcal{A}^{\frac{17}{4}}\, \Gamma\!\left(\tfrac{3}{4}\right)^{\frac{1}{2}}.$$

We have computed the constants $\mathcal{F}_k$ by their exact expression. Moreover, we have determined the index m of the smallest remainder of their asymptotic divergent series and the resulting error bound given by Theorem 3.1.

Constant	Value	m	Error bound
$\mathcal{F}_1$	1.04633506677050318098...	4	$6.000 \cdot 10^{-4}$
$\mathcal{F}_2$	1.02393741163711840157...	7	$7.826 \cdot 10^{-7}$
$\mathcal{F}_3$	1.01604053706462099128...	10	$1.198 \cdot 10^{-9}$
$\mathcal{F}_4$	1.01204589802394464624...	13	$1.948 \cdot 10^{-12}$
$\mathcal{F}_5$	1.00963997283647705086...	16	$3.272 \cdot 10^{-15}$
$\mathcal{F}_6$	1.00803362724207326544...	20	$5.552 \cdot 10^{-18}$

The weak interval of $\mathcal{F}_\infty$ is given by Theorem 3.1. The second value is derived by Proposition 3.8 with parameters $m = 17$ and $n = 7$. Thus, exact expressions of $\mathcal{F}_1, \dots, \mathcal{F}_7$ are needed to compute $\mathcal{F}_\infty$ within the given precision.

Constant	Value / Interval	m	Error bound
$\mathcal{F}_\infty$	(1.02428, 1.02491)	4	$6.050 \cdot 10^{-4}$
$\mathcal{F}_\infty$	1.02460688265559721480...	17	$6.321 \cdot 10^{-22}$

4. Products of Bernoulli numbers

Using results of the previous sections, we are now able to consider several products of Bernoulli numbers and to derive their asymptotic expansions and constants.

Theorem 4.1. *Asymptotically, we have*

$$\prod_{\nu=1}^{n} |B_{2\nu}| \;\sim\; \mathcal{B}_1 \left(\frac{n}{\pi e^{3/2}}\right)^{n(n+1)} (16\pi n)^{\frac{n}{2}}\, n^{\frac{11}{24}} \quad \text{as} \quad n \to \infty,$$

$$\prod_{\nu=1}^{n} \frac{|B_{2\nu}|}{2\nu} \;\sim\; \mathcal{B}_2 \left(\frac{n}{\pi e^{3/2}}\right)^{n^2} \left(\frac{4n}{\pi e}\right)^{\frac{n}{2}} \Big/ n^{\frac{1}{24}} \quad \text{as} \quad n \to \infty$$

with the constants

$$\begin{aligned} \mathcal{B}_1 &= \mathcal{C}_2 \mathcal{F}_2 \mathcal{A}^2 (2\pi)^{\frac14} &&= \mathcal{C}_2\, (2\pi)^{\frac12}\, 2^{\frac{5}{24}}\, e^{\frac{1}{24}} / \mathcal{A}^{\frac12}, \\ \mathcal{B}_2 &= \mathcal{C}_2 \mathcal{F}_2 \mathcal{A}^2 / (2\pi)^{\frac14} &&= \mathcal{C}_2\, 2^{\frac{5}{24}}\, e^{\frac{1}{24}} / \mathcal{A}^{\frac12}. \end{aligned}$$

Proof. By Euler's formula (1.2) for $\zeta(2\nu)$ and Lemma 1.1 we obtain

$$\prod_{\nu=1}^{n} |B_{2\nu}| \;\sim\; \mathcal{C}_2 \prod_{\nu=1}^{n} \frac{2 \cdot (2\nu)!}{(2\pi)^{2\nu}} \;\sim\; \mathcal{C}_2\, 2^n (2\pi)^{-n(n+1)} \prod_{\nu=1}^{n} (2\nu)! \quad \text{as} \quad n \to \infty.$$

Theorem 3.1 states for $k = 2$ that

$$\prod_{\nu=1}^{n} (2\nu)! \;\sim\; \mathcal{F}_2\, \mathcal{A}^2\, (2\pi)^{\frac14} \left(\frac{2n}{e^{3/2}}\right)^{n(n+1)} (4\pi n)^{\frac{n}{2}}\, n^{\frac{11}{24}} \quad \text{as} \quad n \to \infty.$$

The expression for $\mathcal{F}_2$ is given in Remark 3.6. Combining both asymptotic formulas above gives the first suggested formula. It remains to evaluate the following product:

$$\prod_{\nu=1}^{n} (2\nu) = 2^n\, n! \;\sim\; (2\pi)^{\frac12} \left(\frac{2n}{e}\right)^{n} n^{\frac12} \quad \text{as} \quad n \to \infty.$$

After some rearranging of terms we then obtain the second suggested formula. □

Remark 4.2. Milnor and Husemoller [14, pp. 49–50] give the following asymptotic formula without proof:

$$\prod_{\nu=1}^{n} |B_{2\nu}| \;\sim\; \mathcal{B}'\, n!\, 2^{n+1}\, F(2n+1) \quad \text{as} \quad n \to \infty \tag{4.1}$$

where

$$F(n) = \left(\frac{n}{2\pi e^{3/2}}\right)^{\frac{n^2}{4}} \left(\frac{8\pi e}{n}\right)^{\frac{n}{4}} \Big/ \, n^{\frac{1}{24}} \tag{4.2}$$

and $\mathcal{B}' \approx 0.705$ is a certain constant. This constant is related to the constant $\mathcal{B}_2$.

Proposition 4.3. *The constant* $\mathcal{B}'$ *is given by*

$$\mathcal{B}' = 2^{\frac{1}{24}}\, 2^{-\frac{3}{2}}\, \mathcal{B}_2 = \mathcal{C}_2\, e^{\frac{1}{24}} / 2^{\frac{5}{4}} \mathcal{A}^{\frac{1}{2}} = 0.7048648734\ldots.$$

Proof. By Theorem 4.1 we have

$$\prod_{\nu=1}^{n} \frac{|B_{2\nu}|}{2\nu} \;\sim\; \mathcal{B}_2\, G(n) \quad \text{as} \quad n \to \infty \tag{4.3}$$

with

$$G(n) = \left(\frac{n}{\pi e^{3/2}}\right)^{n^2} \left(\frac{4n}{\pi e}\right)^{\frac{n}{2}} \Big/ \, n^{\frac{1}{24}}.$$

We observe that (4.1) and (4.3) are equivalent so that

$$2\,\mathcal{B}' F(2n+1) \;\sim\; \mathcal{B}_2\, G(n) \quad \text{as} \quad n \to \infty.$$

We rewrite (4.2) in the suitable form

$$F(2n+1) = \left(\frac{n+\frac{1}{2}}{\pi e^{3/2}}\right)^{n^2+n+\frac{1}{4}} \left(\frac{4\pi e}{n+\frac{1}{2}}\right)^{\frac{n}{2}+\frac{1}{4}} \Big/ \, 2^{\frac{1}{24}} \left(n+\frac{1}{2}\right)^{\frac{1}{24}}.$$

Hence, we easily deduce that

$$G(n)/F(2n+1) = \left(1+\frac{1}{2n}\right)^{-n^2-\frac{n}{2}+\frac{1}{24}} e^{\frac{n}{2}}\, 2^{\frac{1}{24}} \left(\frac{e^{1/2}}{4}\right)^{\frac{1}{4}}.$$

It is well known that

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{n} = e^{x} \qquad \text{and} \qquad \lim_{n\to\infty} e^{-xn} \left(1+\frac{x}{n}\right)^{n^2} = e^{-\frac{x^2}{2}}.$$

Evaluating the asymptotic terms, we get

$$2\,\mathcal{B}'/\mathcal{B}_2 \;\sim\; G(n)/F(2n+1) \;\sim\; e^{\frac{1}{8}}\, e^{-\frac{1}{4}}\, 2^{\frac{1}{24}}\, e^{\frac{1}{8}}\, 2^{-\frac{1}{2}} \quad \text{as} \quad n \to \infty,$$

which finally yields $\mathcal{B}' = 2^{\frac{1}{24}}\, 2^{-\frac{3}{2}}\, \mathcal{B}_2$. □

Theorem 4.4. *The Minkowski–Siegel mass formula asymptotically states for positive integers n with $4 \mid n$ that*

$$M(2n) = \frac{|B_n|}{2n} \prod_{\nu=1}^{n-1} \frac{|B_{2\nu}|}{4\nu} \sim \mathcal{B}_3 \left(\frac{n}{\pi e^{3/2}} \right)^{n^2} \Big/ \left(\frac{4n}{\pi e} \right)^{\frac{n}{2}} n^{\frac{1}{24}} \quad \textit{as} \quad n \to \infty$$

with $\mathcal{B}_3 = \sqrt{2}\, \mathcal{B}_2$.

Proof. Let n always be even. By Proposition 2.5 and (1.2) we have

$$2^{-n} \left| \frac{B_n/n}{B_{2n}/2n} \right| = 2 \frac{\zeta(n)}{\zeta(2n)} \pi^n \frac{n!}{(2n)!} \sim \sqrt{2} \left(\frac{4n}{\pi e} \right)^{-n} \quad \text{as} \quad n \to \infty,$$

since $\zeta(n)/\zeta(2n) \sim 1$ and

$$\log \left(\frac{n!}{(2n)!} \right) \sim n - n \log n - \left(2n + \frac{1}{2} \right) \log 2 \quad \text{as} \quad n \to \infty.$$

We finally use Theorem 4.1 and (4.3) to obtain

$$M(2n) = 2^{-n} \left| \frac{B_n/n}{B_{2n}/2n} \right| \prod_{\nu=1}^{n} \frac{|B_{2\nu}|}{2\nu} \sim \sqrt{2}\, \mathcal{B}_2 \left(\frac{4n}{\pi e} \right)^{-n} G(n) \quad \text{as} \quad n \to \infty,$$

which gives the result. □

Result 4.5. The constants $\mathcal{B}'$, $\mathcal{B}_\nu$ ($\nu = 1, 2, 3$) mainly depend on the constant $\mathcal{C}_2$ and the Glaisher–Kinkelin constant $\mathcal{A}$.

Constant	Expression	Value
$\mathcal{A}$		1.28242712910062263687...
$\mathcal{C}_2$		1.82101745149929239040...
$\mathcal{B}_1$	$\mathcal{C}_2 (2\pi)^{\frac{1}{2}}\, 2^{\frac{5}{24}}\, e^{\frac{1}{24}} / \mathcal{A}^{\frac{1}{2}}$	4.85509664652226751252...
$\mathcal{B}_2$	$\mathcal{C}_2 2^{\frac{5}{24}}\, e^{\frac{1}{24}} / \mathcal{A}^{\frac{1}{2}}$	1.93690332773294192068...
$\mathcal{B}_3$	$\mathcal{C}_2 2^{\frac{17}{24}}\, e^{\frac{1}{24}} / \mathcal{A}^{\frac{1}{2}}$	2.73919495508550621998...
$\mathcal{B}'$	$\mathcal{C}_2\, e^{\frac{1}{24}} / 2^{\frac{5}{4}} \mathcal{A}^{\frac{1}{2}}$	0.70486487346802031057...

5. Generalizations

In this section we derive a generalization of Theorem 3.1. The results show the structure of the constants $\mathcal{F}_k$ and the generalized constants $\mathcal{F}_{r,k}$, which we shall define later, in a wider context. For simplification we introduce the following definitions which arise from the Euler-Maclaurin summation formula.

The sum of consecutive integer powers is given by the well known formula

$$\sum_{\nu=0}^{n-1} \nu^r = \frac{B_{r+1}(n) - B_{r+1}}{r+1} = \sum_{j=0}^{r} \binom{r}{j} B_{r-j} \frac{n^{j+1}}{j+1}, \quad r \geq 0,$$

where $B_m(x)$ is the mth Bernoulli polynomial. Now, the Bernoulli number $B_1 = -\frac{1}{2}$ is responsible for omitting the last power n^r in the summation above. Because we further need the summation up to n^r, we change the sign of B_1 in the sum as follows:

$$S_r(n) = \sum_{\nu=1}^{n} \nu^r = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} B_{r-j} \frac{n^{j+1}}{j+1}, \quad r \geq 0.$$

This modification also coincides with

$$\zeta(-n) = (-1)^{n+1} \frac{B_{n+1}}{n+1}$$

for nonnegative integers n. We define the extended sum

$$S_r(n; f(\diamond)) = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} B_{r-j} \frac{n^{j+1} f(j+1)}{j+1}, \quad r \geq 0,$$

where the symbol $\diamond$ is replaced by the index $j+1$ in the sum. Note that S_r is linear in the second parameter, i.e.,

$$S_r(n; \alpha + \beta f(\diamond)) = \alpha S_r(n) + \beta S_r(n; f(\diamond)).$$

Finally we define

$$D_k(x) = \sum_{j \geq 1}{}' \widehat{B}_{2j,k}\, x^{-(2j-1)} \quad \text{where} \quad \widehat{B}_{m,k} = \frac{B_m}{m(m-1)k^{m-1}}.$$

Theorem 5.1. *Let r be a nonnegative integer. Then*

$$\prod_{\nu=1}^{n} \nu^{\nu^r} \sim \mathcal{A}_r\, Q_r(n) \quad \textit{as} \quad n \to \infty,$$

where $\mathcal{A}_r$ is the generalized Glaisher–Kinkelin constant defined by

$$\log \mathcal{A}_r = -\zeta(-r)\, H_r - \zeta'(-r).$$

Moreover, $\log Q_r \in \Omega_{r+1}$ *with*

$$\log Q_r(n) = (S_r(n) - \zeta(-r)) \log n + S_r(n; H_r - H_\diamond).$$

Proof. This formula and the constants easily follow from a more general formula for real $r > -1$ given in [10, 9.28, p. 595] and after some rearranging of terms. □

Remark 5.2. The case $r = 0$ reduces to Stirling's approximation of $n!$ with $\mathcal{A}_0 = \sqrt{2\pi}$. The case $r = 1$ gives the usual Glaisher–Kinkelin constant $\mathcal{A}_1 = \mathcal{A}$. The expression $S_r(n; H_r - H_\diamond)$ does not depend on the definition of B_1, since the term with B_1 is cancelled in the sum. Graham, Knuth, and Patashnik [10, 9.28, p. 595] notice that the constant $-\zeta'(-r)$ has been determined in a book of de Bruijn [7, §3.7] in 1970. The theorem above has a long history. In 1894 Alexeiewsky [3] gave the identity

$$\prod_{\nu=1}^{n} \nu^{\nu^r} = \exp\left(\zeta'(-r, n+1) - \zeta'(-r)\right)$$

where $\zeta'(s,a)$ is the partial derivative of the Hurwitz zeta function with respect to the first variable. Between 1903 and 1913, Ramanujan recorded in his notebooks [5, Entry 27, pp. 273–276] (the first part was published and edited by Berndt [5] in 1985) an asymptotic expansion for real $r > -1$ and an analytic expression for the constant $C_r = -\zeta'(-r)$. However, Ramanujan only derived closed expressions for C_0 and C_{2r} $(r \geq 1)$ in terms of $\zeta(2r+1)$; see (5.9) below. In 1933 Bendersky [4] showed that there exist certain constants $\mathcal{A}_r$. Since 1980, several others have investigated the asymptotic formula, including MacLeod [13], Choudhury [6], and Adamchik [1, 2].

Theorem 5.3. *Let k, r be integers with $k \geq 1$ and $r \geq 0$. Then*

$$\prod_{\nu=1}^{n} (k\nu)!^{\nu^r} \sim \mathcal{F}_{r,k}\, \mathcal{A}_r^{\frac{1}{2}}\, \mathcal{A}_{r+1}^{k}\, P_{r,k}(n)\, Q_r(n)^{\frac{1}{2}}\, Q_{r+1}(n)^k \quad \text{as} \quad n \to \infty.$$

The constants $\mathcal{F}_{r,k}$ and functions $P_{r,k}$ satisfy that $\lim_{k\to\infty} \mathcal{F}_{r,k} = 1$ and $\log P_{r,k} \in \Omega_{r+2}$ where

$$\begin{aligned}\log P_{r,k}(n) &= \frac{1}{2} S_r(n) \log(2\pi k) + k\, S_{r+1}(n) \log(k/e) \\ &\quad + \widehat{B}_{r+2,k} \log n + \sum_{j=1}^{\lfloor \frac{r+1}{2} \rfloor} \widehat{B}_{2j,k}\, S_{r+1-2j}(n).\end{aligned}$$

The constants $\mathcal{A}_r$ and functions Q_r are defined as in Theorem 5.1.

The determination of exact expressions for the constants $\mathcal{F}_{r,k}$ seems to be a very complicated and extensive task in the case $r > 0$. The next theorem gives a partial result for $k = 1$ and $r \geq 0$.

Theorem 5.4. *Let r be a nonnegative integer. Then*

$$\log \mathcal{F}_{r,1} = \frac{1}{2} \log \mathcal{A}_r - \log \mathcal{A}_{r+1} + S_r(1; \widehat{B}_{1+\diamond,1} - \log \mathcal{A}_\diamond).$$

Case $r = 0$:

$$\log \mathcal{F}_{r,1} = \frac{1}{12} + \frac{1}{2} \log \mathcal{A}_0 - 2 \log \mathcal{A}_1.$$

Case $r > 0$:

$$\log \mathcal{F}_{r,1} = \alpha_{r,0} + \sum_{j=1}^{r+1} \alpha_{r,j} \log \mathcal{A}_j$$

where

$$\alpha_{r,j} = \begin{cases} \frac{B_{r+1}}{2r(r+1)}, & r \not\equiv j \pmod 2, \quad j = 0; \\ \sum\limits_{j=0}^{r} \binom{r}{j} \frac{B_{r-j}\, B_{j+2}}{(j+1)^2(j+2)}, & r \equiv j \pmod 2, \quad j = 0; \\ -\delta_{r+1,j} - \binom{r+1}{j} \frac{B_{r+1-j}}{r+1}, & r \not\equiv j \pmod 2, \quad j > 0; \\ 0, & r \equiv j \pmod 2, \quad j > 0 \end{cases}$$

and $\delta_{i,j}$ is Kronecker's delta.

Proof of Theorem 5.3. Let k and r be fixed. We extend the proof of Theorem 3.1. From (3.1) we have

$$\log(k\nu)! = \frac{1}{2}\log(2\pi k) + k\nu\log\left(\frac{k}{e}\right) + \left(k\nu + \frac{1}{2}\right)\log\nu + D_k(\nu). \tag{5.1}$$

The summation yields

$$\sum_{\nu=1}^{n} \nu^r \log(k\nu)! = F_1(n) + F_2(n) + F_3(n)$$

where

$$F_1(n) = \frac{1}{2}S_r(n)\log(2\pi k) + kS_{r+1}(n)\log(k/e),$$
$$F_2(n) = k\sum_{\nu=1}^{n}\nu^{r+1}\log\nu + \frac{1}{2}\sum_{\nu=1}^{n}\nu^r\log\nu,$$
$$F_3(n) = \sum_{\nu=1}^{n}\nu^r D_k(\nu).$$

Theorem 5.1 provides

$$F_2(n) = k\left(\log\mathcal{A}_{r+1} + \log Q_{r+1}(n)\right) + \frac{1}{2}\left(\log\mathcal{A}_r + \log Q_r(n)\right) + O(n^{-\delta})$$

with some $\delta > 0$. Let $R = \lfloor\frac{r+1}{2}\rfloor$. By definition we have

$$x^r D_k(x) = \sum_{j=1}^{R}\widehat{B}_{2j,k}\, x^{r+1-2j} + {\sum_{j>R}}'\, \widehat{B}_{2j,k}\, x^{r+1-2j} =: E_1(x) + E_2(x).$$

Therewith we obtain that

$$F_3(n) = \sum_{j=1}^{R}\widehat{B}_{2j,k}\, S_{r+1-2j}(n) + \sum_{\nu=1}^{n}E_2(\nu).$$

For the second sum above we consider two cases. We use similar arguments which we have applied to (3.2) and (3.3). If r is odd, then

$$\lim_{n\to\infty}\sum_{\nu=1}^{n}E_2(\nu) = {\sum_{j>R}}'\, \widehat{B}_{2j,k}\,\zeta(2j-(r+1)). \tag{5.2}$$

Note that $\widehat{B}_{r+2,k} = 0$ in that case. If r is even, then we have to take care of the term ν^{-1}. This gives

$$\lim_{n\to\infty}\left(\sum_{\nu=1}^{n}E_2(\nu) - \widehat{B}_{r+2,k}\log n\right) = \gamma\,\widehat{B}_{r+2,k} + {\sum_{j>R+1}}'\, \widehat{B}_{2j,k}\,\zeta(2j-(r+1)). \tag{5.3}$$

The right hand side of (5.2), resp. (5.3), defines the constant $\log \mathcal{F}_{r,k}$. Finally we have to collect all results for F_1, F_2, and F_3. This gives the constants and the function $P_{r,k}$. It remains to show that $\lim_{k\to\infty} \log \mathcal{F}_{r,k} = 0$. This follows by $\widehat{B}_{2j,k} \to 0$ as $k \to \infty$. □

The following lemma gives a generalization of Equation (3.7) in Lemma 3.3. After that we can give a proof of Theorem 5.4.

Lemma 5.5. *Let n, r be integers with $n \geq 1$ and $r \geq 0$. Then*

$$n!^{S_r(n)} \prod_{\nu=1}^{n} \nu^{\nu^r} = \prod_{\nu=1}^{n} \nu!^{\nu^r} \prod_{\nu=1}^{n} \nu^{S_r(\nu)}. \tag{5.4}$$

Proof. We regard the following enumeration scheme which can be easily extended to n rows and n columns:

$$\begin{array}{ccc} 1^{1^r} & \boxed{2^{1^r}} & \boxed{3^{1^r}} \\ 1^{2^r} & 2^{2^r} & \boxed{3^{2^r}} \\ 1^{3^r} & 2^{3^r} & 3^{3^r} \end{array}$$

The product of all elements, resp. non-framed elements, in the νth row equals $n!^{\nu^r}$, resp. $\nu!^{\nu^r}$. The product of the framed elements in the νth column equals $\nu^{S_r(\nu-1)}$. Thus

$$n!^{S_r(n)} = \prod_{\nu=1}^{n} \nu!^{\nu^r} \prod_{\nu=1}^{n} \nu^{S_r(\nu)-\nu^r}.$$ □

Proof of Theorem 5.4. Let $r \geq 0$. We take the logarithm of (5.4) to obtain

$$F_1(n) + F_2(n) = F_3(n) + F_4(n) \tag{5.5}$$

where

$$F_1(n) = S_r(n) \log n!, \quad F_2(n) = \sum_{\nu=1}^{n} \nu^r \log \nu,$$

$$F_3(n) = \sum_{\nu=1}^{n} \nu^r \log \nu!, \quad F_4(n) = \sum_{\nu=1}^{n} S_r(\nu) \log \nu.$$

Next we consider the asymptotic expansions $\tilde{F}_j$ of the functions F_j ($j = 1, \ldots, 4$) when $n \to \infty$. We further reduce the functions $\tilde{F}_j$ via the maps

$$C(\mathbb{R}^+; \mathbb{R}) \xrightarrow{[\,]} \Omega_\infty \xrightarrow{\psi} \mathbb{R}$$

to the constant terms which are the asymptotic constants of $[\tilde{F}_j]$ in Ω_∞. Consequently (5.5) turns into

$$\psi([\tilde{F}_1]) + \psi([\tilde{F}_2]) = \psi([\tilde{F}_3]) + \psi([\tilde{F}_4]). \tag{5.6}$$

We know from Theorem 5.1 and Theorem 5.3 that

$$\psi([\tilde{F}_2]) = \log \mathcal{A}_r \quad \text{and} \quad \psi([\tilde{F}_3]) = \log \mathcal{F}_{r,1} + \frac{1}{2} \log \mathcal{A}_r + \log \mathcal{A}_{r+1}.$$

For $\tilde{F}_4$ we derive the expression

$$\psi([\tilde{F}_4]) = S_r(1; \log \mathcal{A}_\diamond), \tag{5.7}$$

since each term $s_j \nu^j$ in $S_r(\nu)$ produces the term $s_j \log \mathcal{A}_j$. It remains to evaluate $\tilde{F}_1$. According to (5.1) we have

$$\log n! = \frac{1}{2}\log(2\pi) - n + \left(n + \frac{1}{2}\right)\log n + D_1(n) =: E(n) + D_1(n).$$

Thus

$$\tilde{F}_1(x) = S_r(x)E(x) + S_r(x)D_1(x).$$

Since $S_r E \in \Omega_\infty$ has no constant term, we deduce that

$$\psi([\tilde{F}_1]) = \psi([S_r D_1]) = S_r(1; \widehat{B}_{1+\diamond,1}).$$

The latter equation is similarly derived as (5.7), whereas we regard the constant terms of the product of the polynomial S_r and the Laurent series D_1. From (5.6) we finally obtain

$$\log \mathcal{F}_{r,1} = \frac{1}{2}\log \mathcal{A}_r - \log \mathcal{A}_{r+1} + S_r(1; \widehat{B}_{1+\diamond,1} - \log \mathcal{A}_\diamond).$$

Now, we shall evaluate the expression above. For $r = 0$ we get

$$\log \mathcal{F}_{0,1} = \frac{1}{12} + \frac{1}{2}\log \mathcal{A}_0 - 2\log \mathcal{A}_1,$$

since

$$S_0(1; \widehat{B}_{1+\diamond,1} - \log \mathcal{A}_\diamond) = \widehat{B}_{2,1} - \log \mathcal{A}_1 = \frac{1}{12} - \log \mathcal{A}_1.$$

For now, let $r > 0$. We may represent $\log \mathcal{F}_{r,1}$ in terms of $\log \mathcal{A}_j$ as follows:

$$\log \mathcal{F}_{r,1} = \alpha_{r,0} + \sum_{j=1}^{r+1} \alpha_{r,j} \, \log \mathcal{A}_j.$$

The term $\alpha_{r,0}$ is given by

$$\alpha_{r,0} = S_r(1; \widehat{B}_{1+\diamond,1}) = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} B_{r-j} \frac{\widehat{B}_{j+2,1}}{j+1}$$

where the sum runs over even j, since $\widehat{B}_{j+2,1} = 0$ for odd j. If r is odd, then the sum simplifies to the term $B_{r+1}/2r(r+1)$. Otherwise we derive for even r that

$$\alpha_{r,0} = \sum_{j=0}^{r} \binom{r}{j} \frac{B_{r-j}\, B_{j+2}}{(j+1)^2(j+2)}.$$

It remains to determine the coefficients $\alpha_{r,j}$ for $r+1 \geq j \geq 1$. Since $\frac{1}{2}x^r - x^{r+1} - S_r(x)$ is an odd, resp. even, polynomial for even, resp. odd, $r > 0$, this property transfers in a similar way to $\frac{1}{2}\log\mathcal{A}_r - \log\mathcal{A}_{r+1} - S_r(1;\log\mathcal{A}_\diamond)$, such that $\alpha_{r,j}=0$ when $2 \mid r-j$. Otherwise we get

$$\alpha_{r,j} = -\binom{r}{j-1}\frac{B_{r-(j-1)}}{j} - \delta_{r+1,j} = -\binom{r+1}{j}\frac{B_{r+1-j}}{r+1} - \delta_{r+1,j} \tag{5.8}$$

for $2 \nmid r-j$, where the term $-\log\mathcal{A}_{r+1}$ is represented by $-\delta_{r+1,j}$. □

Corollary 5.6. *Let r be an odd positive integer. Then*

$$\begin{aligned}\log\mathcal{F}_{r,1} &= -\frac{r!}{(2\pi i)^{r+1}}\left(\frac{\zeta(r+1)}{r} + \sum_{j=1}^{\frac{r-1}{2}}\zeta(r+1-2j)\zeta(2j+1) - \frac{(r+2)\zeta(r+2)}{2}\right)\\ &= (-1)^{\frac{r-1}{2}}\frac{r!}{2}\left(\frac{|B_{r+1}|}{r(r+1)!} + \sum_{j=1}^{\frac{r-1}{2}}\frac{|B_{r+1-2j}|\,\zeta(2j+1)}{(r+1-2j)!\,(2\pi)^{2j}} - \frac{(r+2)\zeta(r+2)}{(2\pi)^{r+1}}\right).\end{aligned}$$

Proof. As a consequence of the functional equation of $\zeta(s)$ and its derivative, we have for even positive integers n, cf. [5, p. 276], that

$$\log\mathcal{A}_n = -\zeta'(-n) = -\frac{1}{2}\frac{n!}{(2\pi i)^n}\zeta(n+1) \tag{5.9}$$

where the left hand side of (5.9) follows by definition. Theorem 5.4 provides

$$\log\mathcal{F}_{r,1} = \frac{B_{r+1}}{2r(r+1)} + \sum_{j=1}^{\frac{r+1}{2}}\alpha_{r,2j}\,\log\mathcal{A}_{2j}.$$

Combining (5.8) and (5.9) gives the second equation above. By Euler's formula (1.2) we finally derive the first equation. □

Remark 5.7. For the sake of completeness, we give an analogue of (5.9) for odd integers. From the logarithmic derivatives of $\Gamma(s)$ and the functional equation of $\zeta(s)$, see [5, pp. 183, 276], it follows for even positive integers n, that

$$\log\mathcal{A}_{n-1} = \frac{B_n}{n}H_{n-1} - \zeta'(1-n) = \frac{B_n}{n}(\gamma + \log(2\pi)) + 2\frac{(n-1)!}{(2\pi i)^n}\zeta'(n)$$

where

$$\zeta'(n) = -\sum_{\nu=2}^{\infty}\log(\nu)\,\nu^{-n}.$$

However, MATHEMATICA is able to compute values of ζ' for positive and negative argument values to any given precision.

Result 5.8. Exact expressions for $\mathcal{F}_{r,1}$ in terms of $\mathcal{A}_j$:

Constant	Expression	Value
$\mathcal{F}_{0,1}$	$e^{\frac{1}{12}}\mathcal{A}_0^{\frac{1}{2}}\mathcal{A}_1^{-2}$	1.04633506677050318098...
$\mathcal{F}_{1,1}$	$e^{\frac{1}{24}}\mathcal{A}_2^{-\frac{3}{2}}$	0.99600199446870605433...
$\mathcal{F}_{2,1}$	$e^{\frac{7}{540}}\mathcal{A}_1^{-\frac{1}{6}}\mathcal{A}_3^{-\frac{4}{3}}$	0.99904614418135586848...
$\mathcal{F}_{3,1}$	$e^{-\frac{1}{720}}\mathcal{A}_2^{-\frac{1}{4}}\mathcal{A}_4^{-\frac{5}{4}}$	1.00097924030236153773...
$\mathcal{F}_{4,1}$	$e^{-\frac{67}{18900}}\mathcal{A}_1^{\frac{1}{30}}\mathcal{A}_3^{-\frac{1}{3}}\mathcal{A}_5^{-\frac{6}{5}}$	1.00007169725554110099...
$\mathcal{F}_{5,1}$	$e^{\frac{1}{2520}}\mathcal{A}_2^{\frac{1}{12}}\mathcal{A}_4^{-\frac{5}{12}}\mathcal{A}_6^{-\frac{7}{6}}$	0.99937792615674804266...

Exact expressions for $\mathcal{F}_{r,1}$ in terms of $\zeta(2j+1)$:

$$\mathcal{F}_{1,1} = \exp\left(\frac{1}{24} - \frac{3\zeta(3)}{8\pi^2}\right),$$
$$\mathcal{F}_{3,1} = \exp\left(-\frac{1}{720} - \frac{\zeta(3)}{16\pi^2} + \frac{15\zeta(5)}{16\pi^4}\right),$$
$$\mathcal{F}_{5,1} = \exp\left(\frac{1}{2520} + \frac{\zeta(3)}{48\pi^2} + \frac{5\zeta(5)}{16\pi^4} - \frac{105\zeta(7)}{16\pi^6}\right).$$

For the first 15 constants $\mathcal{F}_{r,1}$ $(r = 0, \ldots, 14)$ we find that

$$\max_{0\le r\le 14} |\mathcal{F}_{r,1} - 1| < 0.05,$$

but, e.g., $\mathcal{F}_{19,1} \approx 371.61$ and $\mathcal{F}_{20,1} \approx 1.16 \cdot 10^{-7}$.

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