RESTRICTED DUMONT PERMUTATIONS, DYCK PATHS, AND NONCROSSING PARTITIONS

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Abstract. We complete the enumeration of Dumont permutations of the second kind avoiding a pattern of length 4 which is itself a Dumont permutation of the second kind. We also consider some combinatorial statistics on Dumont permutations avoiding certain patterns of length 3 and 4 and give a natural bijection between 3142-avoiding Dumont permutations of the second kind and noncrossing partitions that uses cycle decomposition, as well as bijections between 132-, 231- and 321-avoiding Dumont permutations and Dyck paths. Finally, we enumerate Dumont permutations of the first kind simultaneously avoiding certain pairs of 4-letter patterns and another pattern of arbitrary length.

1. Preliminaries

The main goal of this paper is to give analogues of enumerative results on certain classes of permutations characterized by pattern-avoidance in the symmetric group $\mathfrak{S}_n$. In the set of Dumont permutations (see below) we identify classes of restricted Dumont permutations with enumerative properties analogous to results on permutations. More precisely, we study the number of Dumont permutations of length $2n$ avoiding either a 3-letter pattern or a 4-letter pattern. We also give direct bijections between equinumerous sets of restricted Dumont permutations of length $2n$ and other objects such as restricted permutations of length n, Dyck paths of semilength n, or noncrossing partitions of $[n]=\{1,2\dots,n\}$.

1.1. **Patterns.** Let $\sigma\in\mathfrak{S}_n$ and $\tau\in\mathfrak{S}_k$ be two permutations. We say that τ *occurs* in σ, or $\sigma\in\mathfrak{S}_n$ *contains* τ, if σ has a subsequence $(\sigma(i_1),\dots,\sigma(i_k))$, $1\le i_1<\dots<i_k\le n$, that is order-isomorphic to τ (in other words, for any j_1 and j_2, $\sigma(i_{j_1})\le\sigma(i_{j_2})$ if and only if $\tau(j_1)\le\tau(j_2)$). Such a subsequence is called an *occurrence* (or an *instance*) of τ in σ. In this context, the permutation τ is called a *pattern*. If τ does not occur in σ, we say that σ *avoids* τ, or is τ-*avoiding*. We denote by $\mathfrak{S}_n(\tau)$ the set of permutations in $\mathfrak{S}_n$ avoiding a pattern τ. If T is a set of patterns, then $\mathfrak{S}_n(T)=\cap_{\tau\in T}\mathfrak{S}_n(\tau)$, i.e. $\mathfrak{S}_n(T)$ is the set of permutations in $\mathfrak{S}_n$ avoiding all patterns in T.

The first results in the extensive body of research on permutations avoiding a 3-letter pattern are due to Knuth [9], but the intensive study of patterns in permutations began with Simion and Schmidt [16] who considered permutations and involutions avoiding each set T of 3-letter patterns. One of the most frequently considered problems is the enumeration of $\mathfrak{S}_n(\tau)$ and $\mathfrak{S}_n(T)$ for various patterns τ and sets of patterns T. The inventory of cardinalities of $|\mathfrak{S}_n(T)|$ for $T\subseteq\mathfrak{S}_3$ is given in [16], and a similar inventory for $|\mathfrak{S}_n(\tau_1,\tau_2)|$, where $\tau_1\in\mathfrak{S}_3$ and $\tau_2\in\mathfrak{S}_4$ is given in [23]. Some results on $|\mathfrak{S}_n(\tau_1,\tau_2)|$ for $\tau_1,\tau_2\in\mathfrak{S}_4$ are obtained in [22]. The exact formula for $|\mathfrak{S}_n(1234)|$ and the generating function for $|\mathfrak{S}_n(12\dots k)|$ are

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found in [7]. Bóna [2] has found the exact value of $|\mathfrak{S}_n(1342)| = |\mathfrak{S}_n(1423)|$, and Stankova [18, 19] showed that $|\mathfrak{S}_n(3142)| = |\mathfrak{S}_n(1342)|$. For a survey of results on pattern avoidance, see [1, 8].

Another problem is finding equinumerously avoided (sets of) patterns, i.e. sets T_1 and T_2 such that $|\mathfrak{S}_n(T_1)| = |\mathfrak{S}_n(T_2)|$ for any $n \geq 0$. Such (sets of) patterns are called *Wilf-equivalent* and said to belong to the same *Wilf class*. There are eight symmetry operations on $\mathfrak{S}_n$ that map every pattern onto a Wilf-equivalent pattern, including:

- *reversal* r: $r(\tau)(j) = \tau(n+1-j)$, i.e. $r(\tau)$ is τ read right-to-left.
- *complement* c: $c(\tau)(j) = n+1-\tau(j)$, i.e. $c(\tau)$ is τ read upside down.
- $r \circ c = c \circ r$: $r \circ c(\tau)(j) = n+1-\tau(n+1-j)$, i.e. $r \circ c(\tau)$ is τ read right-to-left upside down.
- *inverse* i: $i(\tau) = \tau^{-1}$.

The set of patterns $\langle r, c, i\rangle(\tau) = \{\tau, r(\tau), c(\tau), r(c(\tau)), \tau^{-1}, r(\tau^{-1}), c(\tau^{-1}), r(c(\tau^{-1}))\}$ is called the *symmetry class* of τ.

Sometimes we will represent a permutation $\pi \in \mathfrak{S}_n$ by placing dots on an $n \times n$ board. For each $i = 1, \ldots, n$, we will place a dot with abscissa i and ordinate $\pi(i)$ (the origin of the board is at the bottom-left corner).

1.2. **Dumont permutations.** In this paper we answer some of the above problems in the case of Dumont permutations. A *Dumont permutation of the first kind* is a permutation $\pi \in \mathfrak{S}_{2n}$ where each even entry is followed by a descent and each odd entry is followed by an ascent or ends the string. In other words, for every $i = 1, 2, \ldots, 2n$,

$$\pi(i) \text{ is even} \implies i < 2n \text{ and } \pi(i) > \pi(i+1),$$
$$\pi(i) \text{ is odd} \implies \pi(i) < \pi(i+1) \text{ or } i = 2n.$$

A *Dumont permutation of the second kind* is a permutation $\pi \in \mathfrak{S}_{2n}$ where all entries at even positions are deficiencies and all entries at odd positions are fixed points or excedances. In other words, for every $i = 1, 2, \ldots, n$,

$$\pi(2i) < 2i,$$
$$\pi(2i-1) \geq 2i-1.$$

We denote the set of Dumont permutations of the first (resp. second) kind of length $2n$ by $\mathfrak{D}^1_{2n}$ (resp. $\mathfrak{D}^2_{2n}$). For example, $\mathfrak{D}^1_2 = \mathfrak{D}^2_2 = \{21\}$, $\mathfrak{D}^1_4 = \{2143, 3421, 4213\}$, $\mathfrak{D}^2_4 = \{2143, 3142, 4132\}$. We also define $\mathfrak{D}^1$*-Wilf-equivalence* and $\mathfrak{D}^2$*-Wilf-equivalence* similarly to the Wilf-equivalence on $\mathfrak{S}_n$. Dumont [4] showed that

$$|\mathfrak{D}^1_{2n}| = |\mathfrak{D}^2_{2n}| = G_{2n+2} = 2(1-2^{2n+2})B_{2n+2},$$

where G_n is the nth Genocchi number, a multiple of the Bernoulli number B_n. Lists of Dumont permutations $\mathfrak{D}^1_{2n}$ and $\mathfrak{D}^2_{2n}$ for $n \leq 4$ as well as some basic information and references for Genocchi numbers and Dumont permutations may be obtained in [15] and [17, A001469]. The exponential generating functions for the unsigned and signed Genocchi numbers are as follows:

$$\sum_{n=1}^{\infty} G_{2n}\frac{x^{2n}}{(2n)!} = x\tan\frac{x}{2}, \qquad \sum_{n=1}^{\infty} (-1)^n G_{2n}\frac{x^{2n}}{(2n)!} = \frac{2x}{e^x+1} - x = -x\tanh\frac{x}{2}.$$

Some cardinalities of sets of restricted Dumont permutations of length $2n$ parallel those of restricted permutations of length n. For example, the following results were obtained in [3, 11]:

- $|\mathfrak{D}^1_{2n}(\tau)| = C_n$ for $\tau \in \{132, 231, 312\}$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.
- $|\mathfrak{D}^2_{2n}(321)| = C_n$.
- $|\mathfrak{D}^1_{2n}(213)| = C_{n-1}$, so the operations r, c and $r \circ c$ do not necessarily produce $\mathfrak{D}^1$-Wilf-equivalent patterns.
- $|\mathfrak{D}^2_{2n}(231)| = 2^{n-1}$, while $|\mathfrak{D}^2_{2n}(312)| = 1$ and $|\mathfrak{D}^2_{2n}(132)| = |\mathfrak{D}^2_{2n}(213)| = 0$ for $n \geq 3$, so r, c and $r \circ c$ do not necessarily produce $\mathfrak{D}^2$-Wilf-equivalent patterns either.
- $|\mathfrak{D}^2_{2n}(3142)| = C_n$.
- $|\mathfrak{D}^1_{2n}(1342, 1423)| = |\mathfrak{D}^1_{2n}(2341, 2413)| = |\mathfrak{D}^1_{2n}(1342, 2413)| = s_{n+1}$, the $(n + 1)$-st little Schröder number [17, A001003], given by $s_1 = 1$, $s_{n+1} = -s_n + 2\sum_{k=1}^{n} s_k s_{n-k}$ $(n \geq 2)$.
- $|\mathfrak{D}^1_{2n}(2413, 3142)| = C(2; n)$, the generalized Catalan number (see [17, A064062]).

Note that the these results parallel some enumerative avoidance results in $\mathfrak{S}_n$, where the same or similar cardinalities are obtained:

- $|\mathfrak{S}_n(\tau)| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number, for any $\tau \in \mathfrak{S}_3$.
- $|\mathfrak{S}_n(123, 213)| = |\mathfrak{S}_n(132, 231)| = 2^{n-1}$.
- $|\mathfrak{S}_n(3142, 2413)| = |\mathfrak{S}_n(4132, 4231)| = |\mathfrak{S}_n(2431, 4231)| = r_{n-1}$, the $(n - 1)$-st large Schröder number [17, A006318], given by $r_0 = 1$, $r_n = r_{n-1} + \sum_{j=0}^{n-1} r_k r_{n-k}$, or alternatively, by $r_n = 2s_{n+1}$.

In this paper, we establish several enumerative and bijective results on restricted Dumont permutations.

In Section 2 we give direct bijections between $\mathfrak{D}^1_{2n}(132)$, $\mathfrak{D}^1_{2n}(231)$, $\mathfrak{D}^2_{2n}(321)$ and the class of Dyck paths of semilength n (paths from $(0, 0)$ to $(2n, 0)$ with steps $\mathbf{u} = (1, 1)$ and $\mathbf{d} = (1, -1)$ that never go below the x-axis). This allows us to consider some permutation statistics, such as length of the longest increasing (or decreasing) subsequence, and study their distribution on the sets $\mathfrak{D}^1_{2n}(132)$, $\mathfrak{D}^1_{2n}(231)$ and $\mathfrak{D}^2_{2n}(321)$.

In Section 3, we consider Dumont permutations of the second kind avoiding patterns in $\mathfrak{D}^2_4$. Note that [3] showed that $|\mathfrak{D}^2_{2n}(3142)| = C_n$ using block decomposition (see [12]), which is very surprising given that it is by far a more difficult task to count all permutations avoiding a single 4-letter pattern (e.g., see [2, 7, 18, 19, 21]).

Furthermore, we prove that $\mathfrak{D}^2_{2n}(4132) = \mathfrak{D}^2_{2n}(321)$ and, thus, $|\mathfrak{D}^2_{2n}(4132)| = C_n$. $\mathfrak{D}^2$-Wilf-equivalence of patterns of different lengths is another striking difference between restricted Dumont permutations and restricted permutations in general.

Refining the result $|\mathfrak{D}^2_{2n}(3142)| = C_n$ in [3], we consider some combinatorial statistics on $\mathfrak{D}^2_{2n}(3142)$ such as the number of fixed points and 2-cycles, and give a natural bijection between permutations in $\mathfrak{D}^2_{2n}(3142)$ with k fixed points and the set $NC(n, n-k)$ of noncrossing partitions of $[n]$ into $n-k$ parts that uses cycle decomposition. This is yet another surprising difference since pattern avoidance on permutations so far has not been shown to be related to their cycle decomposition in any natural way.

Finally, we prove that $|\mathfrak{D}^2_{2n}(2143)| = a_n a_{n+1}$, where $a_{2m} = \frac{1}{2m+1}\binom{3m}{m}$ and $a_{2m+1} = \frac{1}{2m+1}\binom{3m+1}{m+1}$. This allows us to relate 2143-avoiding Dumont permutations of the second kind with pairs of northeast lattice paths from $(0,0)$ to $(2n,n)$ and $(2n+1,n)$ that do not get above the line $y=x/2$.

Thus, we complete the enumeration problem of $\mathfrak{D}^2_{2n}(\tau)$ for all $\tau\in\mathfrak{D}^2_4$. Unfortunately, the same problem for Dumont permutations of the first kind (i.e. enumeration of permutations in $\mathfrak{D}^1_{2n}(\tau)$ avoiding a pattern in $\tau\in\mathfrak{D}^1_4=\{2143,3421,4213\}$ appears much harder to solve, and all cases remain unsolved. We do know, however, that no two patterns in $\mathfrak{D}^1_4$ are $\mathfrak{D}^1$-Wilf-equivalent [3]. On the other hand, avoidance of pairs of 4-letter patterns yields nice results [3].

τ	$\lvert\mathfrak{D}^1_{2n}(\tau)\rvert$	Reference
123	Open	
132	C_n	[11, Th. 2.2]
213	C_{n-1}	[3, Th. 2.1]
231	C_n	[11, Th. 4.3]
312	C_n	[11, Th. 4.3]
321	1	[3, Page 6]
2143	Open	
3421	Open	
4213	Open	
(1342, 1423)	s_{n+1}	[3, Th. 3.4]
(2341, 2413)	s_{n+1}	[3, Th. 3.5]
(1342, 2413)	s_{n+1}	[3, Th. 3.6]
(2341, 1423)	$b_n=3b_{n-1}+2b_{n-2}$	[3, Th. 3.7]
(1342, 4213)	2^{n-1}	[3, Th. 3.9]
(2413, 3142)	$C(2;n)$	[3, Th. 3.11]

τ	$\lvert\mathfrak{D}^2_{2n}(\tau)\rvert$	Reference
123	Open	
132	0	Obvious
213	0	Obvious
231	2^{n-1}	[3, Th. 2.2]
312	1	[3, Page 6]
321	C_n	[11, Th. 4.3]
3142	C_n	[3, Th. 3.1]
4132	C_n	Theorem 3.4
2143	a_na_{n+1}	Theorem 3.5

Table 1. Some avoidance results for Dumont permutations

Most known avoidance results are given in Table 1. Here $a_{2m}=\frac{1}{2m+1}\binom{3m}{m}$ and $a_{2m+1}=\frac{1}{2m+1}\binom{3m+1}{m+1}$ as defined earlier, $C(2;n)=\sum_{m=0}^{n-1}\frac{n-m}{n}\binom{n-1+m}{m}2^m$, and $b_0=1$, $b_1=1$, $b_2=3$.

2. Dumont permutations avoiding a single 3-letter pattern

In this section we consider some permutation statistics and study their distribution on certain classes of restricted Dumont permutations. We focus on the sets $\mathfrak{D}^1_{2n}(132)$, $\mathfrak{D}^1_{2n}(231)$ and $\mathfrak{D}^2_{2n}(321)$, whose cardinality is given by the Catalan numbers, as shown in [3, 11]. We construct direct bijections between these sets and the class of Dyck paths of semilength n, which we denote $\mathcal{D}_n$.

2.1. 132-avoiding Dumont permutations of the first kind.

In this section we present a bijection f_1 between $\mathfrak{D}^1_{2n}(132)$ and $S_n(132)$, which will allow us to enumerate 132-avoiding Dumont permutations of the first kind with respect to the length of the longest increasing subsequences. The bijection is defined as follows. Let $\pi=\pi_1\pi_2\cdots\pi_{2n}\in\mathfrak{D}^1_{2n}(132)$. First delete all the even entries of π. Next, replace each of the remaining entries π_i by $(\pi_i+1)/2$. Note that we only obtain integer numbers since the π_i that were not erased are odd. Clearly, since π was 132-avoiding, the sequence $f_1(\pi)$ that we obtain is a 132-avoiding permutation, that is, $f_1(\pi)\in\mathfrak{S}_n(132)$. For example, if $\pi=64357821$, then deleting the even entries we get 3571, so $f_1(\pi)=2341$.

To see that f_1 is indeed a bijection, we now describe the inverse map. Let $\sigma \in \mathfrak{S}_n(132)$. First replace each entry σ_i with $\sigma'_i := 2\sigma_i - 1$. Now, for every i from 1 to n, proceed according to one of the two following cases. If $\sigma'_i > \sigma'_{i+1}$, insert $\sigma'_i + 1$ immediately to the right of σ'_i. Otherwise (that is, $\sigma'_i < \sigma'_{i+1}$ or σ'_{i+1} is not defined), insert $\sigma'_i + 1$ immediately to the right of the rightmost element to the left of σ'_i that is bigger than σ'_i, or to the beginning of the sequence if such element does not exist. To see that $f_1^{-1}(\sigma) \in \mathfrak{D}^1_{2n}(132)$, note that every even entry $\sigma'_i + 1$ is inserted immediately to the right of either a smaller odd entry or a larger even entry, or at the beginning of the sequence, and it is always followed by a smaller entry. Also, after inserting the even entries, each odd entry σ'_i is followed by an ascent. For example, if $\sigma = 546231$, after the first step we get $(9, 7, 11, 3, 5, 1)$, so $f_1^{-1}(\sigma) = (9, 10, 8, 7, 11, 12, 4, 3, 5, 6, 2, 1)$.

Recall Krattenthaler's bijection between 132-avoiding permutations and Dyck paths [10]. We denote it $\varphi : \mathfrak{S}_n(132) \to \mathcal{D}_n$, and it can be defined as follows. Given a permutation $\pi \in \mathfrak{S}_n(132)$ represented as an $n \times n$ board, where for each entry $\pi(i)$ there is a dot in the i-th column from the left and row $\pi(i)$ from the bottom, consider a lattice path from $(n, 0)$ to $(0, n)$ not above the antidiagonal $y = n - x$ that leaves all dots to the right and stays as close to the antidiagonal as possible. Then $\varphi(\pi)$ is the Dyck path obtained from this path by reading an **u** every time the path goes west and a **d** every time it goes north. Composing f_1 with the bijection φ we obtain a bijection $\varphi \circ f_1 : \mathfrak{D}^1_{2n}(132) \to \mathcal{D}_n$.

Again through φ, the set $\mathfrak{S}_{2n}(132)$ is in bijection with $\mathcal{D}_{2n}$. Considering $\mathfrak{D}^1_{2n}(132)$ as a subset of $\mathfrak{S}_{2n}(132)$, we observe that $g_1 := \varphi \circ f_1^{-1} \circ \varphi^{-1}$ is an injective map from $\mathcal{D}_n$ to $\mathcal{D}_{2n}$. Here is a way to describe it directly only in terms of Dyck paths. Recall that a valley in a Dyck path is an occurrence of **du**, and that a tunnel is a horizontal segment whose interior is below the path and whose endpoints are lattice points belonging to the path (see [5, 6] for more precise definitions). Let $D \in \mathcal{D}_n$. For each valley in D, consider the tunnel whose left endpoint is at the bottom of the valley. Mark the up-step and the down-step that delimit this tunnel. Now, replace each unmarked down-step **d** with **dud**. Replace each marked up-step **u** with **uu**, and each marked **d** with **dd**. The path that we obtain after these operations is precisely $g_1(D) \in \mathcal{D}_{2n}$.

To justify the last claim, observe first that a permutation $\pi \in \mathfrak{D}^1_{2n}(132)$ can be decomposed uniquely either as $\pi = (\tau' + |\tau|, 2n - 1, 2n, \tau)$ or as $\pi = (2n, \tau, 2n - 1)$, where τ, τ' are again 132-avoiding Dumont permutations of the first kind, and $|\tau|$ denotes the size of τ. When applying φ to $\pi \in \mathfrak{D}^1_{2n}(132)$, the first decomposition translates into a Dyck path of the form $C = A\mathbf{uu}B\mathbf{dd}$, and the second decomposition gives a path $C = \mathbf{u}A\mathbf{dud}$, where A and B are Dyck paths. When the map f_1 is applied to π, even entries are deleted, so the first decomposition becomes $f_1(\pi) = (f_1(\tau') + |f_1(\tau)|, n, f_1(\tau))$, and the second $f_1(\pi) = (f_1(\tau), n)$. The translation of this operation in terms of Dyck paths is that the map $g_1^{-1} = \varphi \circ f_1 \circ \varphi^{-1}$ transforms the first decomposition into $g_1^{-1}(C) = g_1^{-1}(A)\mathbf{u}g_1^{-1}(B)\mathbf{d}$ and the second into $g_1^{-1}(C) = \mathbf{u}g_1^{-1}(A)\mathbf{d}$. The description of g_1 in the previous paragraph just reverses this construction. Through the map φ, each entry of the permutation has an associated tunnel in the path (as described in [5]). The construction describing g_1 creates tunnels that correspond to the even elements of $f_1^{-1}(\varphi^{-1}(D))$.

For example, if $D = \mathbf{uduududd}$, then underlining the marked steps we get $\mathbf{ud}\underline{\mathbf{u}}\mathbf{ud}\underline{\mathbf{udd}}$, so $g_1(D) = \mathbf{ududuuududuudddd}$.

Denote by $\text{lis}(\pi)$ (resp. $\text{lds}(\pi)$) the length of the longest increasing (resp. decreasing) subsequence of π. Using the above bijections we obtain the following result.

Theorem 2.1. *Let $L_k(z) := \sum_{n\geq 0} |\{\pi \in \mathfrak{D}^1_{2n}(132) : \text{lis}(\pi) \leq k)\}|\, z^n$ be the generating function for $\{132, 12\cdots(k+1)\}$-avoiding Dumont permutations of the first kind. Then we have the recurrence*

$$L_k(z) = 1 + \frac{zL_{k-1}(z)}{1 - zL_{k-2}(z)},$$

with $L_{-1}(z) = 0$ and $L_0(z) = 1$.

Proof. As shown in [10], the length of the longest increasing subsequence of a permutation $\pi \in \mathfrak{S}_{2n}(132)$ corresponds to the height of the path $\varphi(\pi) \in \mathcal{D}_{2n}$. Next we describe the statistic, which we denote λ, on the set of Dyck paths $\mathcal{D}_n$ that, under the injection $g_1 : \mathcal{D}_n \hookrightarrow \mathcal{D}_{2n}$, corresponds to the height in $\mathcal{D}_{2n}$. Let $D \in \mathcal{D}_n$. For each peak p of D, define $\lambda(p)$ to be the height of p plus the number of tunnels below p whose left endpoint is at a valley of D. Now let $\lambda(D) := \max_p\{\lambda(p)\}$ where p ranges over all the peaks of D. From the description of g_1 it follows that for any $D \in \mathcal{D}_n$, $\text{height}(g_1(D)) = \lambda(D)$. Thus, enumerating permutations in $\mathfrak{D}^1_{2n}(132)$ according to the parameter lis is equivalent to enumerating paths in $\mathcal{D}_n$ according to the parameter λ. More precisely, $L_k(z) = \sum_{D\in\mathcal{D}:\lambda(D)\leq k} z^{|D|}$. To find an equation for L_k, we use that every nonempty Dyck path D can be uniquely decomposed as $D = A\mathbf{u}B\mathbf{d}$, where $A, B \in \mathcal{D}$. We obtain that

$$L_k(z) = 1 + zL_{k-1}(z) + z(L_k(z) - 1)L_{k-2}(z),$$

where the term $zL_{k-1}(z)$ corresponds to the case where A is empty (for then $\lambda(\mathbf{u}B\mathbf{d}) = \lambda(B) + 1$, and $z(L_k(z) - 1)L_{k-2}(z)$ to the case there A is not empty. From this we obtain the recurrence

$$L_k(z) = 1 + \frac{zL_{k-1}(z)}{1 - zL_{k-2}(z)},$$

where $L_{-1}(z) = 0$ and $L_0(z) = 1$ by definition. □

It also follows from the definition of φ that the length of the longest decreasing subsequence of $\pi \in \mathfrak{S}_{2n}(132)$ corresponds to the number of peaks of the path $\varphi(\pi) \in \mathcal{D}_{2n}$. Looking at the description of g_1, we see that a peak is created in $g_1(D)$ for each unmarked down-step of d. The number of marked down-steps is the number of valleys of D. Therefore, if $D \in \mathcal{D}_n$, we have that the number of peaks of $g_1(D)$ is $\text{peaks}(g_1(D)) = \text{peaks}(D)+n-\text{valleys}(D) = n+1$. Hence, we have that for every $\pi \in \mathfrak{D}^1_{2n}(132)$, $\text{lds}(\pi) = n + 1$.

2.2. 231**-avoiding Dumont permutations of the first kind.** As we did in the case of 132-avoiding Dumont permutations, we can give the following bijection f_2 between $\mathfrak{D}^1_{2n}(231)$ and $\mathfrak{S}_n(231)$. Let $\pi \in \mathfrak{D}^1_{2n}(231)$. First delete all the odd entries of π. Next, replace each of the remaining entries π_i by $\pi_i/2$. Note that we only obtain integer entries since the remaining π_i were even. Compare this to the analogous transformation described in Section 3.1 for Dumont permutations of the second kind. Clearly the sequence $f_2(\pi)$ that we obtain is a 231-avoiding permutation (since so was π), that is, $f_2(\pi) \in \mathfrak{S}_n(231)$. For example, if $\pi = (2, 1, 10, 8, 4, 3, 6, 5, 7, 9)$, then deleting the odd entries we get $(2, 10, 8, 4, 6)$, so $f_2(\pi) = 15423$.

To see that f_2 is indeed a bijection, we define the inverse map as follows. Let $\sigma \in \mathfrak{S}_n(231)$. First replace each entry k with $2k$. Now, for every i from 1 to $n-1$, insert $2i-1$ immediately to the left of the first entry to the right of $2i$ that is bigger than $2i$ (if such an entry does not exist, insert $2i-1$ at the end of the sequence). For example, if $\sigma = 7215346$, after the first step we get $(14, 4, 2, 10, 6, 8, 12)$, so $f_2^{-1}(\sigma) = (14, 4, 2, 1, 3, 10, 6, 5, 8, 7, 9, 12, 11, 13)$.

Consider now the bijection $\varphi^R : \mathfrak{S}_n(231) \longrightarrow \mathcal{D}_n$ that is obtained by composing φ defined above with the reversal operation that sends $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathfrak{S}_n(231)$ to $\pi^R = \pi_n \cdots \pi_2\pi_1 \in \mathfrak{S}_n(132)$.

Through φ^R, the set $\mathfrak{S}_{2n}(231)$ is in bijection with $\mathcal{D}_{2n}$, so we can identify $\mathfrak{D}^1_{2n}(231)$ with a subset of $\mathcal{D}_{2n}$. The map $g_2 := \varphi^R \circ f_2^{-1} \circ (\varphi^R)^{-1}$ is an injection from $\mathcal{D}_n$ to $\mathcal{D}_{2n}$. Here is a way to describe it directly only in terms of Dyck paths. Given $D \in \mathcal{D}_n$, all we have to do is replace each down-step **d** of D with **udd**. The path that we obtain is precisely $g_2(D) \in \mathcal{D}_{2n}$. For example, if $D =$ **uduuududdd** (this example corresponds to the same σ given above), then $g_2(D) =$ **uuddduuuudduuddudddudd**. Given $g_2(D)$, one can easily recover D by replacing every **udd** by **d**.

Some properties of φ trivially translate to properties of φ^R. In particular, the length of the longest increasing subsequence of a 231-avoiding permutation π equals the number of peaks of $\varphi^R(\pi)$, and the length of the longest decreasing subsequence of π is precisely the height of $\varphi^R(\pi)$.

It follows from the description of g_2 in terms of Dyck paths that for any $D \in \mathcal{D}_n$, $g_2(D)$ has exactly n peaks (one for each down-step of D). Therefore, for any $\pi \in \mathfrak{D}^1_{2n}(231)$, the number of right-to-left minima of π is $\mathrm{rlm}(\pi) = n$. In fact it is not hard to see directly from the definition of 231-avoiding Dumont permutations that the right-to-left minima of $\pi \in \mathfrak{D}^1_{2n}(231)$ are precisely its odd entries, which necessarily form an increasing subsequence.

Also from the description of g_2 we see that $\mathrm{height}(g_2(D)) = \mathrm{height}(D) + 1$. In terms of permutations, this says that if $\pi \in \mathfrak{S}_n(231)$, then $\mathrm{lds}(f_2(\pi)) = \mathrm{lds}(\pi) + 1$. This allows us to enumerate 231-avoiding Dumont permutations with respect to the statistic lds. Indeed, $|\{\pi \in \mathfrak{D}^1_{2n}(231) : \mathrm{lds}(\pi) = k\}| = |\{D \in \mathcal{D}_n : \mathrm{height}(D) = k - 1\}|$.

2.3. **321-avoiding Dumont permutations of the second kind.** Let us first notice that a permutation $\pi \in \mathfrak{D}^2_{2n}(321)$ cannot have any fixed points. Indeed, assume that $\pi_i = i$ and let $\pi = \sigma i \tau$. Since π is 321-avoiding, it follows that σ is a permutation of $\{1, 2, \ldots, i-1\}$ and τ is a permutation of $\{i+1, i+2, \ldots, n\}$. Since $\pi \in \mathfrak{D}^2_{2n}$, i must be odd, but then the first element of τ is in an even position, and it is either a fixed point or an excedance, which contradicts the definition of Dumont permutations of the second kind.

It is known (see e.g. [14]) that a permutation is 321-avoiding if and only if both the subsequence determined by its excedances and the one determined by the remaining elements are increasing. It follows that a permutation in $\mathfrak{D}^2_{2n}(321)$ is uniquely determined by the values of its excedances. Another consequence is that if $\pi \in \mathfrak{D}^2_{2n}(321)$, then $\mathrm{lis}(\pi) = n$.

We can give a bijection between $\mathfrak{D}^2_{2n}(321)$ and $\mathcal{D}_n$. We define it in two parts. For the first part, we use the bijection ψ between $\mathfrak{S}_n(321)$ and $\mathcal{D}_n$ that was defined in [5], and which is closely related to the bijection between $\mathfrak{S}_n(123)$ and $\mathcal{D}_n$ given in [10]. Given $\pi \in \mathfrak{S}_n(321)$, consider again the $n \times n$ board with a dot in the i-th column from the left and row $\pi(i)$ from the bottom, for each i. Take the path with *north* and *east* steps that goes from $(0, 0)$ to the (n, n), leaving all the dots to the right, and staying always as close to the diagonal as possible. Then $\psi(\pi)$ is the Dyck path obtained from this path by reading an up-step every time the path goes north and a down-step every time it goes east.

If we apply ψ to a permutation $\pi \in \mathfrak{D}^2_{2n}(321)$ we get a Dyck path $\psi(\pi) \in \mathcal{D}_{2n}$. The second part of our bijection is just the map g_2^{-1} defined above, which consists in replacing every occurrence of **udd** with a **d**. It is not hard to check that $\pi \mapsto g_2^{-1}(\psi(\pi))$ is a bijection from $\mathfrak{D}^2_{2n}(321)$ to $\mathcal{D}_n$. For example, for $\pi = (3,1,5,2,6,4,9,7,10,8)$, we have that $\psi(\pi) =$ **uuudduuddudduuuddudd**, and $g_2^{-1}(\psi(\pi)) =$ **uududduudd**.

3. Dumont permutations avoiding a single 4-letter pattern

In this section we will determine the structure of permutations in $\mathfrak{D}^2_{2n}(\tau)$ and find the cardinality $|\mathfrak{D}^2_{2n}(\tau)|$ for each $\tau \in \mathfrak{D}^2_4 = \{2143, 3142, 4132\}$.

It was shown in [3] that $|\mathfrak{D}^2_{2n}(3142)| = C_n$. In Section 3.1, we refine this result with respect to the number of fixed points and 2-cycles in permutations in $\mathfrak{D}^2_{2n}(3142)$ and use cycle decomposition to give a natural bijection between permutations in $\mathfrak{D}^2_{2n}(3142)$ with k fixed points and the set $NC(n, n-k)$ of noncrossing partitions of $[n]$ into $n-k$ parts.

In Section 3.2, we prove that $\mathfrak{D}^2_{2n}(4132) = \mathfrak{D}^2_{2n}(321)$ and, thus, $|\mathfrak{D}^2_{2n}(4132)| = C_n$.

Finally, in Section 3.3 we prove that $|\mathfrak{D}^2_{2n}(2143)| = a_n a_{n+1}$, where $a_{2m} = \frac{1}{2m+1}\binom{3m}{m}$ and $a_{2m+1} = \frac{1}{2m+1}\binom{3m+1}{m+1}$. Thus, we can relate permutations in $\mathfrak{D}^2_{2n}(2143)$ and pairs of northeast lattice paths from $(0,0)$ to $(n, \lfloor \frac{n}{2} \rfloor)$ and $(n+1, \lfloor \frac{n+1}{2} \rfloor)$ that stay on or below $y = x/2$.

This completes the enumeration problem of $\mathfrak{D}^2_{2n}(\tau)$ for $\tau \in \mathfrak{D}^2_4$.

3.1. **Avoiding 3142.**

It was shown in [3] that $|\mathfrak{D}^2_{2n}(3142)| = C_n$; moreover, the permutations $\pi \in \mathfrak{D}^2_{2n}(3142)$ can be recursively described as follows:

$$\pi = (2k, 1, r \circ c(\pi') + 1, \pi'' + 2k), \tag{3.1}$$

where $\pi' \in \mathfrak{D}^2_{2k-2}(3142)$ and $\pi'' \in \mathfrak{D}^2_{2n-2k}(3142)$ (see Figure 1). From this block decomposition, it is easy to see that the subsequence of odd integers in π is increasing. Moreover, the odd entries are exactly those on the main diagonal and the first subdiagonal (i.e. those i for which $\pi(i) = i$ or $\pi(i) = i - 1$).

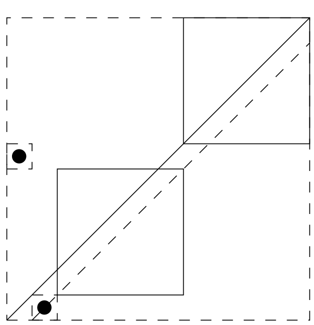

Figure 1. The block decomposition of a permutation in $D^2_{2n}(3142)$.

In Sections 3.1.1 and 3.1.2 we use the above decomposition to derive two bijections from $\mathfrak{D}^2_{2n}(3142)$ to sets of cardinality C_n.

3.1.1. *Subsequence of even entries.* The first bijection is $\phi : \mathfrak{D}^2_{2n}(3142) \to E_n \subset S_n$, where

$$E_n = \left\{(1/2)\pi_{ev} \mid \pi \in \mathfrak{D}^2_{2n}(3142)\right\},$$

and π_{ev} (resp. π_{ov}) is the subsequence of even (resp. odd) values in π. (Here $\frac{1}{2}\pi_{ev}$ is the permutation obtained by dividing all entries in π_{ev} by 2; in other words, if $\sigma = \frac{1}{2}\pi_{ev}$, then $\sigma(i) = \pi_{ev}(i)/2$ for all $i \in [n]$.) Define $\phi(\pi) = \frac{1}{2}\pi_{ev}$ for each $\pi \in \mathfrak{D}^2_{2n}(3142)$.

Permutations in E_n have a block decomposition similar to those in $\mathfrak{D}^2_{2n}(3142)$, namely,

$$\sigma \in E_n \iff \sigma = (k, r \circ c(\sigma'), k + \sigma'') \text{ for some } \sigma' \in E_{k-1} \text{ and } \sigma'' \in E_{n-k}.$$

The inverse $\phi^{-1} : E_n \to \mathfrak{D}^2_{2n}(3142)$ is easy to describe. Let $\sigma \in E_n$. Then $\pi = \phi^{-1}(\sigma)$ is obtained as follows: let $\pi_{ev} = 2\sigma$ (i.e. $\pi_{ev}(i) = 2\sigma(i)$ for all $i \in [n]$), then for each $i \in [n]$ insert $2i - 1$ immediately before $2\sigma(i)$ if $\sigma(i) < i$ or immediately after $2\sigma(i)$ if $\sigma(i) \geq i$. For instance, if $\sigma = 3124 \in E_4$, then $\pi_{ev} = 6248$ and $\pi = 61\,32\,54\,87 \in \mathfrak{D}^2_8(3142)$.

It is not difficult to show that E_n consists of exactly those permutations that, written in cyclic form, correspond to noncrossing partitions of $[n]$ by replacing pairs of parentheses with slashes. We remark that E_n is also the set of permutations whose tableaux (see [20]) have a single 1 in each column.

Theorem 3.1. *For a permutation ρ, define*

$$\operatorname{fix}(\rho) = |\{i \mid \rho(i) = i\}|, \qquad \operatorname{exc}(\rho) = |\{i \mid \rho(i) > i\}|,$$
$$\operatorname{fix}_{-1}(\rho) = |\{i \mid \rho(i) = i - 1\}|, \qquad \operatorname{def}(\rho) = |\{i \mid \rho(i) < i\}|.$$

Then for any $\pi \in \mathfrak{D}^2_{2n}(3142)$ and $\sigma = \phi(\pi) \in E_n$, we have

$$\operatorname{fix}(\pi) + \operatorname{fix}_{-1}(\pi) = n, \tag{3.2}$$
$$\operatorname{fix}(\pi) = \operatorname{def}(\sigma), \tag{3.3}$$
$$\operatorname{fix}_{-1}(\pi) = \operatorname{exc}(\sigma) + \operatorname{fix}(\sigma), \tag{3.4}$$
$$\operatorname{fix}(\sigma) = \#\text{ 2-cycles in } \pi. \tag{3.5}$$

Proof. Equation (3.2) follows from the fact that odd integers in π are exactly those on the main diagonal and first subdiagonal.

Let π and σ be as above and let $i \in [n]$. Then there are two cases: either $2i - 1 = \pi(2i)$ or $2i - 1 = \pi(2i - 1)$.

Case 1: $\pi(2i) = 2i - 1$. Then $\pi(2i - 1) \geq 2i$, and hence $\pi(2i - 1)$ must be even.

Case 2: $\pi(2i - 1) = 2i - 1$. Then $\pi(2i) \leq 2i - 2$, and hence $\pi(2i)$ must be even.

In either case, for each $i \in [n]$, we have $\{\pi(2i - 1), \pi(2i)\} = \{2i - 1, 2s_i\}$ for some $s_i \in [n]$. Define $\sigma(i) = s_i$. Then $\sigma(i) \geq i$ if $2i - 1 \in \operatorname{fix}_{-1}(\pi)$, and $\sigma(i) \leq i - 1$ if $2i - 1 \in \operatorname{fix}(\pi)$. This proves (3.3) and (3.4).

Finally, let $i \in [n]$ be such that $\sigma(i) = i$. Since $2\sigma(i) \in \{\pi(2i - 1), \pi(2i)\}$ and $\pi(2i) < 2i$, it follows that $2i = 2\sigma(i) = \pi(2i - 1)$, so $2i - 1 = \pi(2i)$, and thus π contains a 2-cycle $(2i - 1, 2i)$.

Conversely, let (ab) be a 2-cycle of π, and assume that $b > a$. Then $\pi(a) > a$, so a must be odd, say $a = 2i - 1$ for some $i \in [n]$. Then $b = \pi^{-1}(a) \in \{2i - 1, 2i\}$, so $b = 2i$, and thus $(ab) = (2i - 1, 2i)$. This proves (3.5). ∎

Theorem 3.2. *Let* $A(q,t,x) = \sum_{n\geq 0}\sum_{\pi\in\mathfrak{D}^2_{2n}(3142)} q^{\mathrm{fix}(\pi)} t^{\#\text{ 2-cycles in }\pi} x^n$ *be the generating function for* 3142*-avoiding Dumont permutations of the second kind with respect to the number of fixed points and the number of 2-cycles. Then*

$$A(q,t,x) = \frac{1 + x(q-t) - \sqrt{1-2x(q+t)+x^2((q+t)^2-4q)}}{2xq(1+x(1-t))}. \tag{3.6}$$

Proof. By the correspondences in Theorem 3.1, it follows that

$$A(q,t,x) = \sum_{n\geq 0}\sum_{\sigma\in E_n} q^{\mathrm{def}(\sigma)} t^{\mathrm{fix}(\sigma)} x^n.$$

For convenience, let us define a related generating function $B(q,t,x) = \sum_{n\geq 0}\sum_{\sigma\in E_n} q^{\mathrm{def}(\sigma)} t^{\mathrm{fix}_{-1}(\sigma)} x^n$. From the block decomposition of permutations $\sigma \in E_n$ as $\sigma = (k, r\circ c(\sigma'), k+\sigma'')$ for some $\sigma' \in E_{k-1}$, $\sigma'' \in E_{n-k}$, it follows that

$$A(q,t,x) = 1 + xtA(q,t,x) + x(B(1/q,t,xq)-1)A(q,t,x). \tag{3.7}$$

The term $xtA(q,t,x)$ corresponds to the case $k = 1$, in which σ' is empty and k is a fixed point. When $k > 1$, σ'' still contributes as $A(q,t,x)$, and the contribution of σ' is $B(1/q,t,xq)-1$, since elements with $\sigma'(i) = i-1$ become fixed points of σ, and all elements of σ' other than its deficiencies become deficiencies of σ.

A similar reasoning gives the following equation for $B(q,t,x)$:

$$B(q,t,x) = 1 + xA(1/q,t,xq)B(q,t,x).$$

Solving for B we have $B(q,t,x) = \frac{1}{1-xA(1/q,t,xq)}$, and plugging $B(1/q,t,xq) = \frac{1}{1-xqA(q,t,x)}$ into (3.7) gives

$$A(q,t,x) = 1 + x\left(\frac{1}{1-xqA(q,t,x)} + t - 1\right)A(q,t,x).$$

Solving this quadratic equation gives the desired formula for $A(q,t,x)$. □

3.1.2. *Cycle decomposition.* Letting $t = 1$ in (3.6), we obtain

Corollary 3.3. We have

$$\sum_{n\geq 0}\sum_{\pi\in\mathfrak{D}^2_{2n}(3142)} q^{\mathrm{fix}(\pi)} x^n = A(q,1,x) = \frac{1+x(q-1)-\sqrt{1-2x(q+1)+x^2(q-1)^2}}{2xq},$$

i.e. the number of permutations in $\pi \in \mathfrak{D}^2_{2n}(3142)$ with k fixed points is the Narayana number $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$, which is also the number of noncrossing partitions of $[n]$ into $n-k$ parts.

Proof. Even though the generating function is an immediate consequence of Theorem 3.2, we will give a combinatorial proof of the corollary, by exhibiting a natural bijection $\psi : \mathfrak{D}^2_{2n}(3142) \to NC(n)$, where $NC(n)$ is the set of noncrossing partitions of $[n]$. We start by considering a permutation $\pi \in \mathfrak{D}^2_{2k}(3142)$. Iterating the block decomposition (3.1), we obtain

$$\begin{aligned}\pi &= (2k_1, 1, c\circ r(\pi_1)+1, 2k_2, 2k_1+1, c\circ r(\pi_2)+2k_1+1, \cdots, 2k_r, 2k_{r-1}+1, c\circ r(\pi_r)+2k_{r-1}+1)\\ &= (2k_1, 1, 2k_1 - r(\pi_1), 2k_2, 2k_1+1, 2k_2 - r(\pi_2), \cdots, 2k_r, 2k_{r-1}+1, 2k_r - r(\pi_r)),\end{aligned}$$

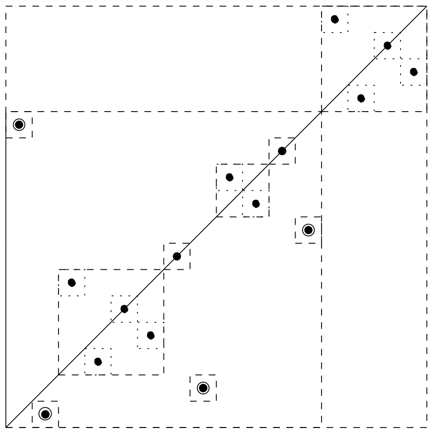

FIGURE 2. The cycle decomposition of a permutation in $D^2_{2n}(3142)$. The circled dots correspond to one of the cycles.

where $1 \le k_1 < k_2 < \cdots < k_r = k$, $\pi_i \in \mathfrak{D}^2_{2(k_i-k_{i-1}-1)}(3142)$ $(1 \le i \le r)$, and we define $k_0 = 0$. Note that each permutation $c \circ r(\pi_i) + 2k_{i-1} + 1 = 2k_i - r(\pi_i)$ of $[2k_{i-1}+2, 2k_i - 1]$ occurs at positions $[2k_{i-1}+3, 2k_i]$ in π.

Now consider

$$\pi' = (2k+2, 1, c \circ r(\pi) + 1) = (2k_r + 2, 1, 2k_r + 2 - r(\pi)).$$

Let $k'_i = k - k_i = k_r - k_i$. By (3.1), we have $\pi \in \mathfrak{D}^2_{2k+2}(3142)$, $\pi_i \in \mathfrak{D}^2_{2(k'_{i-1}-k'_i-1)}(3142)$ $(1 \le i \le r)$, $k'_r = 0$, $k'_0 = k$, and

$$\pi' = (2k+2, 1, \pi_r+2, 2k'_{r-1}+1, 2, \pi_{r-1}+2k'_{r-1}+2, 2k'_{r-2}+1, 2k'_{r-1}+2, \dots, \pi_1+2k'_1+2, 2k+1, 2k'_1+2).$$

Note that, for each $i = 1, 2, \dots, r$, the permutation $\pi_i + 2k'_i + 2$ of $[2k'_i + 3, 2k'_{i-1}]$ occurs at positions $[2k'_i + 3, 2k'_{i-1}]$ in π'. Moreover, the entries $2k'_i + 1$ $(0 \le i \le r-1)$ occur at positions $2k'_i + 1$ in π', and thus are fixed points of π'. Finally, each entry $2k'_i + 2$ $(1 \le i \le r)$ occurs at position $2k'_{i-1} + 2$, 1 occurs at position $2 = 2k'_r + 2$, and $2k + 2 = 2k'_0 + 2$ occurs at position 1.

Thus, $\gamma = (2k'_0 + 2, 2k'_1 + 2, 2k'_2 + 2, \dots, 2k'_{r-1} + 2, 2k'_r + 2, 1) = (2k + 2, 2k'_1 + 2, 2k'_2 + 2, \dots, 2k'_{r-1} + 2, 2, 1)$ is a cycle of π' (such as the one consisting of circled dots in Figure 2), and each remaining nontrivial cycle of π' is completely contained in some $\pi_i + 2k'_i + 2$, which is a 3142-avoiding Dumont permutation of the second kind of $[2k'_i + 3, 2k'_{i-1}]$. Note that

$$2k'_i + 2 < 2k'_i + 3 < 2k'_{i-1} < 2k'_{i-1} + 2,$$

so all entries of each remaining cycle of π' are contained between two consecutive entries of γ.

Now let G be the subset of $[2k + 2]$ consisting of the entries of γ. Then, clearly,

$$G/\{2k'_{r-1} + 1\}/\dots/\{2k'_1 + 1\}/\{2k'_0 + 1\}/[2k'_r + 3, 2k'_{r-1}]/\dots/[2k'_1 + 3, 2k'_0]$$

is a noncrossing partition of $[2k + 2]$. Now it is easy to see by induction on the size of π' that the subsets of π' formed by entries of the cycles in cycle decomposition of π' form a noncrossing partition of π'. Moreover, all the entries of G except the smallest entry are even, so likewise the cycle decomposition of π' determines a unique noncrossing partition of π'_{ev}, hence a unique noncrossing partition of $[n]$.

Finally, any permutation $\hat{\pi} \in \mathfrak{D}^2_{2n}(3142)$ can be written as $\hat{\pi} = (\pi', \pi''+2k+2)$, where π' is as above and $\pi'' \in \mathfrak{D}^2_{2n-2k-2}(3142)$, so the cycles of any permutation in $\mathfrak{D}^2_{2n}(3142)$ determine a unique noncrossing partition of $[n]$.

Notice also that each cycle in the decomposition of $\hat{\pi}$ contains exactly one odd entry, the least entry in each cycle, so the number of odd entries of $\hat{\pi}$ which are not fixed points, $\mathrm{fix}_{-1}(\hat{\pi}) = n - \mathrm{fix}(\hat{\pi})$, is the number of parts in $\psi(\hat{\pi})$. This finishes the proof. □

For example (see Figure 2), if

$$\begin{aligned}\hat{\pi} &= 12, 1, 6, 3, 5, 4, 7, 2, 10, 9, 11, 8, 16, 13, 15, 14\\ &= (12, 8, 2, 1)(6, 4, 3)(10, 9)(16, 14, 13)(15)(11)(7)(5) \in \mathfrak{D}^2_{16}(3142),\end{aligned}$$

then $\psi(\hat{\pi}) = 641/32/5/87 \in NC(8)$. Note also that $\hat{\pi}_{ev} = 63215487 = (641)(32)(5)(87)$.

3.2. **Avoiding 4132.** For Dumont permutations of the second kind avoiding the pattern 4132 we have the following result.

Theorem 3.4. *For any $n \geq 0$, $\mathfrak{D}^2_{2n}(4132) = \mathfrak{D}^2_{2n}(321)$. Moreover, $|\mathfrak{D}^2_{2n}(4132)| = C_n$, where C_n is the nth Catalan number. Thus, 4132 and 3142 are $\mathfrak{D}^2$-Wilf-equivalent.*

Proof. The pattern 321 is contained in 4132. Therefore, if π avoids 321, then π avoids 4132, so $\mathfrak{D}^2_{2n}(321) \subseteq \mathfrak{D}^2_{2n}(4132)$. Now let us prove that $\mathfrak{D}^2_{2n}(4132) \subseteq \mathfrak{D}^2_{2n}(321)$. Let $n \geq 4$ and let $\pi \in \mathfrak{D}^2_{2n}(4132)$ contain an occurrence of 321. Choose the occurrence of 321 in π, say $\pi(i_1) > \pi(i_2) > \pi(i_3)$ with $1 \leq i_1 < i_2 < i_3 \leq 2n$, such that $i_1 + i_2 + i_3$ is minimal. If i_1 is an even number, then $\pi(i_1 - 1) \geq i_1 - 1 \geq \pi(i_1)$, so the occurrence $\pi(i_1 - 1)\pi(i_1)\pi(i_2)$ of pattern 321 contradicts minimality of our choice. Therefore, i_1 is odd. If $i_2 \neq i_1 + 1$, then from the minimality of the occurrence we get that $\pi(i_1 + 1) < \pi(i_3)$. Hence, π contains 4132, a contradiction. So $i_2 = i_1 + 1$. If i_3 is odd, then $\pi(i_3) \geq i_3 > i_1 + 1 \geq \pi(i_1 + 1)$, which contradicts $\pi(i_1) > \pi(i_1 + 1) > \pi(i_3)$. So i_3 is even.

Therefore, the our chosen occurrence of 321 is given by $\pi(2i + 1)\pi(2i + 2)\pi(j)$ where $4 \leq 2i + 2 \leq j \leq 2n$ (since $\pi(2) = 1$, we must have $i \geq 1$). By minimality of the occurrence, we have $\pi(m) \leq 2i$ for all $m \leq 2i$. On the other hand, $\pi(i_3) < \pi(2i + 2) \leq 2i + 1$ which means that $\pi(i_3) \leq 2i$. Hence, π must contain at least $2i + 1$ letters smaller than $2i$, a contradiction.

Thus, if $\pi \in \mathfrak{D}^2_{2n}(4132)$ then $\pi \in \mathfrak{D}^2_{2n}(321)$. The rest follows from [11, Theorem 4.3]. □

3.3. **Avoiding 2143.** Dumont permutations of the second kind that avoid 2143 are enumerated by the following theorem, which we prove in this section.

Theorem 3.5. *For any $n \geq 0$, $|\mathfrak{D}^2_{2n}(2143)| = a_n a_{n+1}$, where*

$$\begin{aligned}a_{2m} &= \frac{1}{2m+1}\binom{3m}{m},\\ a_{2m+1} &= \frac{1}{2m+1}\binom{3m+1}{m+1} = \frac{1}{m+1}\binom{3m+1}{m}.\end{aligned}$$

Remark 3.6. Note that the sequence $\{a_n\}$ also enumerates northeast lattice paths in $\mathbb{Z}^2$ from $(0, 0)$ to $(n, \lfloor \frac{n}{2} \rfloor)$ that stay on or below $y = x/2$, as well as symmetric ternary trees on $3n$ edges and symmetric diagonally convex directed polyominoes with n squares (see [17, A047749] and references therein). Also note that $\{a_{2m+1}\}$ is the convolution of $\{a_{2m}\}$ with itself, while the convolution of $\{a_{2m}\}$ with $\{a_{2m+1}\}$ is $\{a_{2m+2}\}$. Alternatively, if $f(x)$ and $g(x)$ are the ordinary generating functions for $\{a_{2m}\}$ and $\{a_{2m+1}\}$, then $f(x) = 1+xf(x)g(x)$ and $g(x) = f(x)^2$, so $f(x) = 1+xf(x)^3$. Now the Lagrange inversion applied to the last two equations yields the formulas for a_n.

Lemma 3.7. *Let* $\pi \in \mathfrak{D}^2_{2n}(2143)$. *Then the subsequence* $(\pi(1), \pi(3), \ldots, \pi(2n-1))$ *is a permutation of* $\{n+1, n+2, \ldots, 2n\}$ *and the subsequence* $(\pi(2), \pi(4), \ldots, \pi(2n))$ *is a permutation of* $\{1, 2, \ldots, n\}$.

Proof. Assume the lemma is false. Let i be the smallest integer such that $\pi(2i) \geq n+1$. Then $\pi(2i-1) \geq 2i-1 \geq \pi(2i) \geq n+1$. Therefore, if $j \geq i$, then $\pi(2j-1) \geq 2j-1 \geq 2i-1 \geq n+1$. In fact, note that for any $1 \leq j \leq n$, $\pi(2j-1) \geq 2j-1 \geq \pi(2j)$.

By minimality of i, we have $\pi(2j) \leq n$ for $j < i$. Hence, if $\pi(2j-1) \leq n$ for some $j < i$, then $(\pi(2j-1), \pi(2j), \pi(2i-1), \pi(2i))$ is an occurrence of pattern 2143 in π. Therefore, $\pi(2j-1) \geq n+1$ for all $j < i$.

Thus, we have $\pi(2j-1) \geq n+1$ for any $1 \leq j \leq n$, and $\pi(2i) \geq n+1$, so π must have at least $n+1$ entries between $n+1$ and $2n$, which is impossible. The lemma follows. □

For $\pi \in \mathfrak{D}^2_{2n}(2143)$, we denote $\pi_o = (\pi(1), \pi(3), \ldots, \pi(2n-1)) - n$ and $\pi_e = (\pi(2), \pi(4), \ldots, \pi(2n))$. By Lemma 3.7, $\pi_o, \pi_e \in \mathfrak{S}_n(2143)$. For example, given $\pi = 71635482 \in \mathfrak{D}^2_8(2143)$, we have $\pi_o = 3214$ and $\pi_e = 1342$. Note that $\pi(2i-1) = \pi_o(i) + n$ and $\pi(2i) = \pi_e(i)$.

Lemma 3.8. *For any permutation* $\pi \in \mathfrak{D}^2_{2n}(2143)$, *and* π_o *and* π_e *defined as above, the following is true:*

(1) $\pi_o \in \mathfrak{S}_n(132)$ *and the entries of* π_o *are on a board with* n *top-justified columns of sizes* $2, 4, 6, \ldots, 2\left\lfloor \frac{n}{2} \right\rfloor, n, \ldots, n$ *from right to left (see the first and third boards in Figure 3).*

(2) $\pi_e \in \mathfrak{S}_n(213)$ *and the entries of* π_e *are on a board with* n *bottom-justified columns of sizes* $1, 3, 5, \ldots, 2\left\lfloor \frac{n}{2} \right\rfloor - 1, n, \ldots, n$ *from left to right (see the second and fourth boards in Figure 3).*

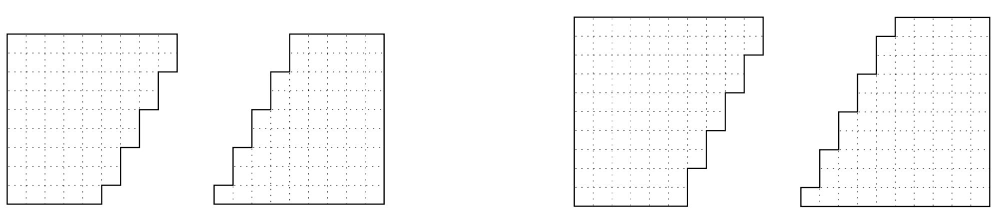

FIGURE 3. The boards of Lemma 3.8 for $n = 9$ (left) and $n = 10$ (right).

Proof. If 132 occurs in π_o at positions $i_1 < i_2 < i_3$, then 2143 occurs in π at positions $2i_1 - 1 < 2i_1 < 2i_2 - 1 < 2i_3 - 1$ since $\pi(2i_1) < \pi(2i_1 - 1)$. Similarly, if 213 occurs in π_e at positions $i_1 < i_2 < i_3$, then 2143 occurs in π at positions $2i_1 < 2i_2 < 2i_3 - 1 < 2i_3$ since $\pi(2i_3 - 1) > \pi(2i_3)$. The rest simply follows from the definition of $\mathfrak{D}^2_{2n}$ and Lemma 3.7. □

Let us call a permutation as in part (1) of Lemma 3.8 an *upper board*, and a permutation as in part (2) of Lemma 3.8 a *lower board*. Note that $\pi_e(1) = 1$ and $213 = r \circ c(132)$. Hence it is easy to see that $\pi_e = (1, r \circ c(\pi') + 1)$ with $\pi' \in \mathfrak{S}_{n-1}(132)$ of upper type. Let b_n be the number of lower boards in $\mathfrak{S}_n(213)$. Then the number of upper boards in $\mathfrak{S}_n(132)$ is b_{n+1}.

Lemma 3.9. *Let $\pi_1 \in \mathfrak{S}_n(132)$ be an upper board and $\pi_2 \in \mathfrak{S}_n(213)$ be a lower board. Let $\pi \in \mathfrak{S}_{2n}$ be defined by $\pi = (\pi_1(1)+n, \pi_2(1), \pi_1(2)+n, \pi_2(2), \ldots, \pi_1(n)+n, \pi_2(n))$ (i.e. such that $\pi_o = \pi_1$ and $\pi_e = \pi_2$). Then $\pi \in \mathfrak{D}^2_{2n}(2143)$.*

Proof. Clearly $\pi \in \mathfrak{D}^2_{2n}$. It is not difficult to see that if π contains 2143, then "2" and "1" are deficiencies (i.e., they are at even positions and come from π_2) and "4" and "3" are excedances or fixed points (i.e. they are at odd positions and come from π_1). Such an occurrence is represented in Figure 4, where an entry $\pi(i)$ is plotted by a dot with abscissa i and ordinate $\pi(i)$, and the two diagonal lines indicate the positions of the fixed points and elements with $\pi(i) = i-1$.

Say the pattern 2143 occurs at positions $2i_1 < 2i_2 < 2i_3 - 1 < 2i_4 - 1$. We have $\pi(2j) \leq 2j-1 < 2i_2 - 1$ for any $j < i_2$. On the other hand, the subdiagonal part of π avoids 213, so $\pi(2j) < \pi(2i_1) \leq 2i_1 - 1 < 2i_2 - 1$ for any $j \geq i_2$. Thus, $\pi(2j) < 2i_2 - 1$ for any $1 \leq j \leq n$. Similarly, $\pi(2j-1) \geq 2j-1 > 2i_3-1$ for any $j > i_3$, and $\pi(2j-1) > \pi(2i_4) \geq 2i_4-1 > 2i_3-1$ for any $j \leq i_3$ since the superdiagonal part of π avoids 132. Thus, $\pi(2j-1) > 2i_3 - 1$ for any $1 \leq j \leq n$.

Therefore, no entry of π lies in the interval $[2i_2 - 1, 2i_3 - 1]$, which is nonempty since $2i_2 < 2i_3 - 1$. This is, of course, impossible, so the lemma follows. □

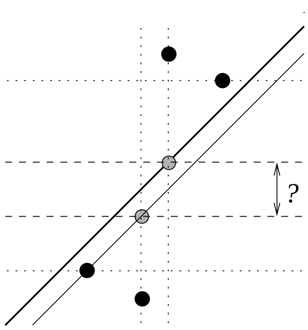

FIGURE 4. This situation is impossible in Lemma 3.9: no value between the grey points (inclusive) can occur in π.

Hence, there is a 1-1 correspondence between permutations $\pi \in \mathfrak{D}^2_{2n}(2143)$ and pairs of permutations (π_1, π_2), where $\pi_1 \in \mathfrak{S}_n(132)$ is an upper board and $\pi_2 \in \mathfrak{S}_n(213)$ is a lower board. Thus, $|\mathfrak{D}^2_{2n}(2143)| = b_n b_{n+1}$, where b_n is the number of lower boards $\pi \in \mathfrak{S}_n(213)$ and b_{n+1} is the number of upper boards $\pi \in \mathfrak{S}_n(132)$ (see the paragraph before Lemma 3.9).

Lemma 3.10. *Let $F(x) = \sum_{m=0}^{\infty} b_{2m} x^m$ and $G(x) = \sum_{m=0}^{\infty} b_{2m+1} x^m$. Then we have $b_0 = 1$ and*

$$b_{2m} = \sum_{i=0}^{m-1} b_{2i} b_{2m-2i-1}, \quad b_{2m+1} = \sum_{i=0}^{m} b_{2i} b_{2m-2i},$$
$$F(x) = 1 + xF(x)G(x), \quad G(x) = F(x)^2.$$

Proof. Let $\pi \in S_n(213)$ be a lower board, and let $i \geq 0$ be maximal such that $\pi(i+1) = 2i+1$. Such an i always exists since $\pi(1) = 1$. Then $\pi(j) \leq 2j - 2$ for $j \geq i+2$. Furthermore, π avoids 213, so if $j_1, j_2 > i+1$, and $\pi(j_1) > \pi(i+1) > \pi(j_2)$, then $j_1 < j_2$. In other words, all entries of π greater than and to the right of $2i+1$ must come before all entries less than and to the right of $2i+1$ (see Figure 5, the areas that cannot contain entries of π are shaded). In addition, $\pi(j) \le 2i+1$ for $j \le i+1$, so $\pi(j) > 2i+1$ only if $j > i+1$. There are $n-2i-1$ values greater than $2i+1$ in π, hence they must occupy the $n-2i-1$ positions immediately to the right of $\pi(i+1)$, i.e. positions $i+2$ through $n-i$. It is not difficult now to see from the above argument that all entries of π greater than $2i+1$ must lie on a board of lower type in $\mathfrak{S}_{n-2i-1}(213)$, while the entries less than $2i+1$ in π must lie on two boards whose concatenation is a lower board in $\mathfrak{S}_{2i}(213)$ (unshaded areas in Figure 5). □

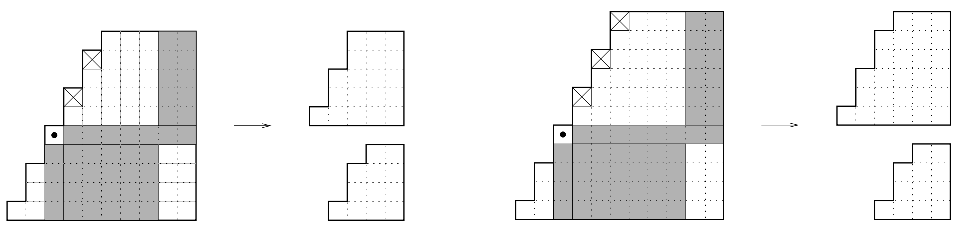

FIGURE 5. A lower board $\pi \in \mathfrak{S}_n(213)$ ($n = 10$ (even), left, and $n = 11$ (odd), right) decomposed into two lower boards according to the largest i such that $\pi(i+1) = 2i+1$ (here $i = 2$).

Thus, we get the same generating function equations as in Remark 3.6, so $F(x) = f(x)$, $G(x) = g(x)$, and hence $b_n = a_n$ for all $n \ge 0$. This proves Theorem 3.5.

We can give a direct bijection showing that $b_n = a_n$. It is well-known that a_{2n} (resp. a_{2n+1}) is the number of northeast lattice paths from $(0,0)$ to $(2n,n)$ (resp. from $(0,0)$ to $(2n+1,n)$) that do not get above the line $y = x/2$. The following bijection uses the same idea as a bijection of Krattenthaler [10] from the set of 132-avoiding permutations in $\mathfrak{S}_n$ to Dyck paths of semilength n, which is described in Section 2.1.

We introduce a bijection between the set of lower boards in $\mathfrak{S}_n(213)$ and northwest paths from $(n,0)$ to $(\lceil n/2\rceil, n)$ that stay on or above the line $y = 2n-2x$ (see Figure 6). Given a lower board in $\mathfrak{S}_n(213)$ represented as an $n \times n$ binary array, consider a lattice path from $(n,0)$ to $(\lceil n/2\rceil, n)$ that leaves all dots to the left and stays as close to the $y = 2n-2x$ as possible. We claim that such a path must stay on or above the line $y = 2n-2x$. Indeed, considering rows of a lower board from top to bottom, we see that at most one extra column appears on the left for every two consecutive rows. Therefore, our path must shift at least r columns to the right for every $2r$ consecutive rows starting from the top. The rest is easy to see.

Conversely, given a northwest path from $(n,0)$ to $(\lceil n/2\rceil, n)$ not below the line $y = 2n-2x$, fill the corresponding board from top to bottom (i.e. from row n to row 1) so that the dots are in the rightmost column to the left of the path that still contains no dots.

Theorem 3.5 implies that $\lim_{n\to\infty} |\mathfrak{D}^2_{2n}(2143)|^{\frac{1}{2n}} = \frac{3^3}{2^2} = \frac{27}{4}$. In comparison, [13] and [21] imply that $|\mathfrak{S}_n(2143)| = |\mathfrak{S}_n(1234)|$ and hence $\lim_{n\to\infty} |\mathfrak{S}_n(2143)|^{\frac{1}{n}} = \lim_{n\to\infty} |\mathfrak{S}_n(1234)|^{\frac{1}{n}} = (4-1)^2 = 9$.

The *median Genocchi number* (or *Genocchi number of the second kind*) H_n [17, A005439] counts the number of derangements in $\mathfrak{D}^2_{2n}$ (also, the number of permutations in $\mathfrak{D}^1_{2n}$ which begin with n or $n+1$). Using the preceding argument, we can also count the number of derangements in $\mathfrak{D}^2_{2n}(2143)$.

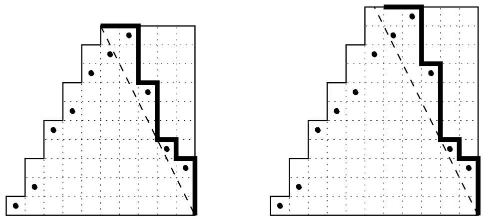

Figure 6. A bijection between lower boards in $\mathfrak{S}_n(213)$, for $n = 10$ (left) and $n = 11$ (right), and northwest paths from $(n, 0)$ to $(\lceil n/2\rceil, n)$ not below $y = 2n - 2x$.

Theorem 3.11. *The number of derangements in $\mathfrak{D}^2_{2n}(2143)$ is a_n^2, where a_n is as in Theorem 3.5.*

Proof. Notice that the fixed points of a permutation $\pi \in \mathfrak{D}^2_{2n}(2143)$ correspond to the dots in the lower right (southeast) corner cells on its upper board (except the lowest right corner when n is odd) (see Figure 3). It is easy to see that deletion of those cells on an upper board produces a rotation of a lower board by 180°. This, together with the preceding lemmas, implies the theorem. □

The following theorem gives the generating function for the distribution of the number of fixed points among permutations in $\mathfrak{D}^2_{2n}(2143)$.

Theorem 3.12. *We have*

$$\begin{aligned}\sum_{\pi\in\mathfrak{D}^2_{2n}(2143)} q^{\mathrm{fix}(\pi)} &= a_n\cdot[x^{n+1}]\left(\frac{1}{1-xf(x^2)}\cdot\frac{1}{1-qx^2f(x^2)^2}\right)\\ &= a_n\cdot[x^{n+1}]\frac{f(x^2)}{(1-xf(x^2))(q+(1-q)f(x^2))}.\end{aligned}\tag{3.8}$$

where $f(x) = \sum_{n\geq 0} a_{2n}x^n$ is a solution of $f(x) = 1 + xf(x)^3$, and $[x^n]h(x)$ is the coefficient at x^n in the power series representation of $h(x)$.

Note that $\sum_{n\geq 0} a_{2n}x^{2n} = f(x^2)$, and that $g(x) = \sum_{n\geq 0} a_{2n+1}x^n = f(x)^2$ implies that $\sum_{n\geq 0} a_{2n+1}x^{2n+1} = xf(x^2)^2$. Hence,

$$\sum_{n\geq 0} a_n x^n = f(x^2) + xf(x^2)^2 = \frac{1}{1-xf(x^2)}.$$

Proof. Let $\pi \in \mathfrak{D}^2_{2n}(2143)$. Note that all fixed points must be on the upper board of π. Therefore, the lower board of π may be any 213-avoiding lower board. This accounts for the factor a_n. Now consider the product of two rational functions on the right. This product corresponds to the fact that the upper board B of π is a concatenation of two objects: the upper board B' of rows below the lowest (smallest) fixed point, and the upper board B'' of rows not below the lowest fixed point. It is easy to see that B' may be any 132-avoiding upper board. Note that B'' must necessarily have an even number of rows and that B'' is a concatenation of a sequence of "slices" between consecutive fixed points, where the ith slice consists of an even number of rows below the $(i + 1)$-th smallest fixed point but not below the ith smallest fixed point.

Thus, we obtain a block decomposition of the upper board B (similar to the one in the Figure 5 for lower boards) into an possibly empty upper board B' and a sequence B'' of nonempty upper boards $B''_1, B''_2, \ldots$, where each B''_i contains an even number of rows and exactly 1 fixed point of π. Taking generating functions yields the product of functions on the right-hand side of (3.8). □

4. Block Decomposition and Dumont permutations avoiding a pair of 4-letter patterns

The core of the block decomposition approach initiated by Mansour and Vainshtein lies in the study of the structure of 132-avoiding permutations, and permutations containing a given number of occurrences of 132 (see [12] and references therein). In this section, using the block decomposition approach, we consider those Dumont permutations in $\mathfrak{S}_n$ that avoid a pair of patterns of length 4 and an arbitrary pattern.

4.1. **$\{1342, 1423\}$-avoiding Dumont permutations of the first kind.** Let $A_\tau(x)$ be the generating function for the number of Dumont permutations of the first kind in $\mathfrak{D}^1_{2n}(1342, 1423, \tau)$, that is,

$$A_\tau(x) = \sum_{n\geq 0} |\mathfrak{D}^1_{2n}(1342, 1423, \tau)| x^n.$$

We say a permutation τ is *decreasing-decomposable* (resp. *increasing-decomposable*) if there exist nonempty subpermutations τ' and τ'' such that $\tau = \tau'\tau''$ and each entry of τ' is bigger (resp. smaller) than each entry of τ''.

Theorem 4.1. *Let $\tau \in \mathfrak{S}_\ell$ be any pattern which is not decreasing-decomposable with $\tau_i \neq \ell$ for $i = 1, \ell-1, \ell$. Then*

$$A_\tau(x) = s(x) = \frac{1 + x - \sqrt{1 - 6x + x^2}}{2x}.$$

Proof. By [3, Theorem 3.4], we have exactly two possibilities for the block decomposition of an arbitrary Dumont permutation of the first kind in $\mathfrak{D}^1_{2n}(1342, 1423)$. Let us write an equation for $A_\tau(x)$. The contribution of the first decomposition above is $xA_\tau(x)(A_\tau(x) - 1)$. The contribution of the second possible decomposition is $x(A_\tau(x))^2$. Therefore, by using the three contributions above we have that $A_\tau(x) = 1 + xA_\tau(x)(A_\tau(x) - 1) + x(A_\tau(x))^2$, where 1 is the contribution of the empty permutation. Solving this equation gives the desired result. □

Similarly, we have the following results.

Theorem 4.2.

(1) *If $\tau' \in \mathfrak{S}_{\ell-1}$ is is not decreasing-decomposable, $\tau'_{\ell-1} \neq \ell - 1$, and $\tau = \tau'\ell$, then*

$$A_\tau(x) = 1 + \frac{x(A_{\tau'}(x))^2}{1 - xA_{\tau'}(x)}.$$

(2) *If $\tau = \tau'(\ell-1)\ell \in \mathfrak{S}_\ell$ with no restrictions on $\tau'\mathfrak{S}_{\ell-2}$, then*

$$A_\tau(x) = 1 + \frac{x(A_{\tau'(\ell-1)}(x))^2}{1 - xA_{\tau'}(x)}.$$

(3) *If* $\tau=\tau'\ell(\ell-1)\in\mathfrak{S}_\ell$, *with no restrictions on* $\tau'\mathfrak{S}_{\ell-2}$, *then*

$$A_\tau(x)=\frac{1+x(1-A_{\tau'}(x))-\sqrt{(1+x(1-A_{\tau'}(x)))^2-4x}}{2x}.$$

For example, if $\tau=13245$, then by Theorem 4.2 we have $A_{13245}(x)=1+\frac{x(A_{1324}(x))^2}{1-xA_{132}(x)}$. Now, using Theorem 4.2 for $\tau=1324$ we get that $A_{1324}(x)=1+\frac{x(A_{132}(x))^2}{1-xA_{132}(x)}$, so

$$A_{13245}(x)=1+\frac{x(1+xA_{132}(x)(A_{132}(x)-1))^2}{(1-xA_{132}(x))^3}.$$

Finally, using Theorem 4.2 together with $A_1(x)=1$, we get that $A_{132}(x)=\frac{1-\sqrt{1-4x}}{2x}=C(x)$. Hence, we can use $C(x)=\frac{1}{1-xC(x)}$ to obtain $A_{13245}(x)=1+(1-x)^2C^3(x)$. Another interesting example obtained by Theorem 4.2 is $A_{2143}(x)=C(x)$ (since $A_{21}(x)=1$). In other words, $|\mathfrak{D}^1_{2n}(1342,1423,2143)|=C_n$. In fact, it is easy to see using block decomposition that $\mathfrak{D}^1_{2n}(1342,1423,2143)=\mathfrak{D}^1_{2n}(132)$.

Theorem 4.3. *Let* $\tau'\in\mathfrak{S}_{\ell-1}$ *be any non-decreasing-decomposable pattern with* $\tau=\ell\tau'$ *and* $\tau_\ell\neq\ell-1$. *Then*

$$A_\tau(x)=\frac{1}{1+x-2xA'_\tau(x)}.$$

Proof. By [3, Theorem 3.4], we have exactly two possibilities for the block decomposition of an arbitrary Dumont permutation of the first kind in $\mathfrak{D}^1_{2n}(1342,1423)$. Let us write an equation for $A_\tau(x)$. The contribution of the first decomposition above is $xA_\tau(x)(A_{\tau'}(x)-1)$. The contribution of the second possible decomposition is $xA_\tau(x)A_{\tau'}(x)$. Therefore, by using the three contributions above we have that $A_\tau(x)=1+xA_\tau(x)(A_{\tau'}(x)-1)+xA_\tau(x)A_{\tau'}(x)$, where 1 stands for the empty permutation. Solving this equation gives the desired result. ☐

Using the above theorems together with $A_1(x)=A_{21}(x)=1$ and $A_{12}(x)=1+x$ we get

τ	$A_\tau(x)$	Reference
1234	$1+\frac{x^5(x+2)^2}{(1-x)^2(1-x-x^2)}$	Theorem 4.2
1243	$\frac{1}{1-x^2}C\left(\frac{x}{(1-x^2)^2}\right)$	Theorem 4.2
1324	$1+xC^3(x)$	Theorem 4.2
1342	$s(x)$	Theorem 4.1
1423	$s(x)$	Theorem 4.1
1432	$s(x)$	Theorem 4.1
2134	$1+\frac{x}{(1-x)^3}$	Theorems 4.1 and 4.2.

4.2. **$\{2341,2413\}$-avoiding Dumont permutations of the first kind.** It was noticed in [3, Theorem 3.5] that $\pi\in\mathfrak{D}^1_{2n}(2341,2413)$ if and only if

- $\pi=(\pi',2n-1,2n,\pi''+2k)$ for $0\leq k\leq n-2$, $\pi'\in\mathfrak{D}^1_{2k}(2341,2413)$, $\pi''\in\mathfrak{D}^1_{2n-2k-2}(2341,2413)$;
- $\pi=(\pi',2n,\pi''+2k,2n-1)$ for $0\leq k\leq n-1$, $\pi'\in\mathfrak{D}^1_{2k}(2341,2413)$, $\pi''\in\mathfrak{D}^1_{2n-2k-2}(2341,2413)$.

This representation is called the *block decomposition* of $\pi \in \mathfrak{D}^1_{2n}(2341, 2413)$. Let $B_\tau(x)$ be the generating function for the number of Dumont permutations of the first kind in $\mathfrak{D}^1_{2n}(2341, 2413, \tau)$, that is, $B_\tau(x) = \sum_{n\geq 0} |\mathfrak{D}^1_{2n}(2341, 2413, \tau)|x^n$.

Theorem 4.4. *Let $\tau = \ell\tau' \in \mathfrak{S}_\ell$ be a pattern with $\tau_\ell \neq \ell - 1$. Then $B_\tau(x) = \frac{1}{1+x-2xB_{\tau'}(x)}$.*

Proof. By Theorem [3, Theorem 3.5], we have exactly two possibilities for the block decomposition of an arbitrary $\pi \in \mathfrak{D}^1_{2n}(2341, 2413)$. Let us write an equation for $B_\tau(x)$. The contribution of the first decomposition above is $xB_\tau(x)(B_{\tau'}(x) - 1)$. The contribution of the second possible decomposition is $xB_\tau(x)B_{\tau'}(x)$. Therefore, adding the two cases with the empty permutation we get

$$B_\tau(x) = xB_\tau(x)(B_{\tau'}(x) - 1) + xB_\tau(x)B_{\tau'}(x).$$

Solving this equation we get the desired result. □

Similarly, we have the following result.

Theorem 4.5. *Let $\tau = \ell\tau'(\ell - 1) \in \mathfrak{S}_\ell$ be a pattern. Then*

$$B_\tau(x) = \frac{1 + x(1 - B_{\tau'}(x)) - \sqrt{(1 + x(1 - B_{\tau'}(x)))^2 - 4x}}{2x}.$$

For example, if $\tau = 4123$ or $\tau = 312$, then by Theorem 4.5 together with $B_1(x) = B_{21}(x) = 1$ we have that $B_\tau(x) = C(x)$.

Chebyshev polynomials of the second kind are defined by $U_r(\cos\theta) = \frac{\sin(r+1)\theta}{\sin\theta}$ for $r \geq 0$. Clearly, $U_r(t)$ satisfies the following recurrence:

$$U_0(t) = 1,\ U_1(t) = 2t,\ \text{and } U_r(t) = 2tU_{r-1}(t) - U_{r-2}(t) \text{ for all } r \geq 2. \tag{4.1}$$

and, thus, is a polynomial of degree r in t with integer coefficients. The same recurrence is used to define $U_r(t)$ for $r < 0$ (for example, $U_{-1}(t) = 0$ and $U_{-2}(t) = -1$). The following lemma can be proved by induction and (4.1).

Lemma 4.6. *Define $a_m = \frac{1}{u - va_{m-1}}$ for all $m \geq 1$, with $a_0 = r$. Then*

$$a_m = \frac{U_{m-1}\left(\frac{u}{2\sqrt{v}}\right) - r\sqrt{v}U_{m-2}\left(\frac{u}{2\sqrt{v}}\right)}{\sqrt{v}\left[U_m\left(\frac{u}{2\sqrt{v}}\right) - r\sqrt{v}U_{m-1}\left(\frac{u}{2\sqrt{v}}\right)\right]},$$

where $U_m(t)$ is the m-th Chebyshev polynomial of the second kind.

Corollary 4.7. For any $k \geq 0$,

$$B_{(k+2)\dots 21}(x) = \frac{U_{k-1}\left(\frac{1+x}{2\sqrt{2x}}\right) - \sqrt{2x}U_{k-2}\left(\frac{1+x}{2\sqrt{2x}}\right)}{\sqrt{2x}\left[U_k\left(\frac{1+x}{2\sqrt{2x}}\right) - \sqrt{2x}U_{k-1}\left(\frac{1+x}{2\sqrt{2x}}\right)\right]}.$$

Proof. It is clear that $B_{21}(x) = 1$. Hence, Theorem 4.4 together with Lemma 4.6 yields the desired result. □

4.3. $\{1342, 2413\}$**-avoiding Dumont permutations of the first kind.** It was noticed in [3, Theorem 3.6] that $\pi \in \mathfrak{D}^1_{2n}(2341, 2413)$ if and only if

- $\pi = (\pi' + 2k, 2n-1, 2n, \pi'')$ for $1 \leq k \leq n-1$, $\pi' \in \mathfrak{D}^1_{2n-2k-2}(1342, 2413)$, $\pi'' \in \mathfrak{D}^1_{2k}(1342, 2413)$;
- $\pi = (\pi', 2n, \pi'' + 2k, 2n-1)$ for $0 \leq k \leq n-1$, $\pi' \in \mathfrak{D}^1_{2k}(1342, 2413)$, $\pi'' \in \mathfrak{D}^1_{2n-2k-2}(1342, 2413)$.

This representation is called the block decomposition of $\pi \in \mathfrak{D}^1_{2n}(1342, 2413)$. Let $C_\tau(x)$ be the generating function for the number of Dumont permutations of the first kind in $\mathfrak{D}^1_{2n}(1342, 2413, \tau)$, that is, $C_\tau(x) = \sum_{n\geq 0} |\mathfrak{D}^1_{2n}(1342, 2413, \tau)|x^n$.

Theorem 4.8. *For all* $k \geq 3$,

$$C_{12\ldots k}(x) = 1 + \frac{x}{1 - xC_{12\ldots(k-2)}(x)} \sum_{j=1}^{k-1} (C_{12\ldots j}(x) - C_{12\ldots(j-1)}(x))C_{12\ldots(k-j)}(x),$$

with $C_1(x) = 1$ *and* $C_{12}(x) = 1 + x$.

Proof. It is easy to check the theorem for $k = 1, 2$, so we can assume $k \geq 3$. As mentioned before, we have exactly two possibilities for the block decomposition of an arbitrary $\pi \in \mathfrak{D}^1_{2n}(1342, 2413)$. Let us write an equation for $C_{12\ldots k}(x)$. The contribution of the first decomposition above is $xC_{12\ldots(k-2)}(x)(C_{12\ldots k}(x) - 1)$. The contribution of the second possible decomposition is $x\sum_{j=1}^{k-1}(C_{12\ldots j}(x) - C_{12\ldots(j-1)}(x))C_{12\ldots(k-j)}(x)$, since if π' contains $12\ldots(j-1)$ and avoids $12\ldots j$, then π'' avoids $12\ldots(k-j)$ (where $j = 1, \ldots, k-1$). Therefore, adding the two cases with the empty permutation we get

$$C_{12\ldots k}(x) = 1 + xC_{12\ldots(k-2)}(x)(C_{12\ldots k}(x) - 1) + x\sum_{j=1}^{k-1}(C_{12\ldots j}(x) - C_{12\ldots(j-1)}(x))C_{12\ldots(k-j)}(x).$$

Solving this linear equation we get the desired result. □

For example, Theorem 4.8 for $k = 3, 4$ gives $C_{123}(x) = \frac{1+2x^2}{1-x}$ and $C_{1234}(x) = \frac{1-x+x^2+4x^3+x^4}{(1-x)(1-x-x^2)}$.

Theorem 4.9. *For all* $k \geq 3$,

$$C_{k\ldots 21}(x) = \frac{1 + x\sum_{j=2}^{k-1}(C_{j\ldots 21}(x) - C_{(j-1)\ldots 21}(x))C_{(k+1-j)\ldots 21}(x)}{1 - xC_{(k-1)\ldots 21}(x)},$$

with $C_1(x) = C_{21}(x) = 1$.

Proof. It is easy to check the theorem for $k = 1, 2$. Let $k \geq 3$. As mentioned before, we have exactly two possibilities for the block decomposition of an arbitrary $\pi \in \mathfrak{D}^1_{2n}(1342, 2413)$. Let us write an equation for $C_{k\ldots 21}(x)$. The contribution of the first decomposition above is

$$x\sum_{j=2}^{k-1} C_{(k+1-j)\ldots 21}(x)(C_{j\ldots 21}(x) - C_{(j-1)\ldots 21}(x)),$$

where π'' contains $(j-1)\ldots 21$ and avoids $j\ldots 21$ for $j = 2, \ldots, k-1$. The contribution of the second possible decomposition is

$$xC_{k\ldots 21}(x)C_{(k-1)\ldots 21}(x).$$

Therefore, adding the two cases with the empty permutation we get

$$C_{k\ldots 21}(x) = 1 + xC_{(k-1)\ldots 21}(x)C_{k\ldots 21}(x) + x\sum_{j=2}^{k-1}(C_{j\ldots 21}(x) - C_{(j-1)\ldots 21}(x))C_{(k+1-j)\ldots 21}(x).$$

Solving this equation we get the desired expression. □

For example, Theorem 4.9 for $k = 3, 4$ gives $C_{321}(x) = \frac{1}{1-x}$ and $C_{4321}(x) = \frac{1-x+x^2}{1-2x}$.

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