Rational Numbers with Non-Terminating, Non-Periodic Modified Engel-Type Expansions

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Abstract.

Recently, Kalpazidou, Knopfmacher, and Knopfmacher asked if there exist rational numbers whose "modified Engel-type" expansion is neither finite nor ultimately periodic. In this note we answer their question by explicitly providing an infinite sequence of such numbers.

In a recent paper [3], Kalpazidou, Knopfmacher, and Knopfmacher discussed expansions for real numbers of the form

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_1 + 1} \cdot \frac{1}{a_2} + \frac{1}{(a_1 + 1)(a_2 + 1)} \cdot \frac{1}{a_3} - \cdots \tag{1}$$

which they called a "modified Engel-type" alternating expansion. Here a_0 is an integer and a_i is a positive integer for $i \geq 1$. If $a_{i+1} \geq a_i$, this expansion is essentially unique. To save space we will abbreviate Eq. (1) by

$$A = \{a_0, a_1, a_2, \ldots\}.$$

They say, "The question of whether or not all rationals have a finite or recurring expansion has not been settled." (By "recurring" we understand "ultimately periodic".)

In this note, we prove that the rational numbers $\frac{2}{2r+1}$ (r an integer ≥ 2) have modified Engel-type expansions that are neither finite nor ultimately periodic.

Theorem.

Let r be an integer ≥ 1. Then

$$\frac{2}{2r+1} = \{a_0, a_1, a_2, \ldots\}$$

where $a_0 = 0$, and $a_{2i-1} = b_i$, $a_{2i} = 2b_i - 1$ for $i \geq 1$, and $b_1 = r$, $b_{n+1} = 2b_n^2 - 1$ for $n \geq 1$.

Proof.

As in [3], we have $a_0 = \lfloor A \rfloor$, $A_1 = A - a_0$, $a_n = \lfloor 1/A_n \rfloor$ for $n \geq 1$ and $A_{n+1} = (1/a_n - A_n)(a_n + 1)$ for $n \geq 1$.

From this we see that $a_0 = \lfloor \frac{2}{2r+1} \rfloor = 0$.

We now prove the following four assertions by induction on n: (i) $A_{2n-1} = \frac{2}{2b_n+1}$; (ii) $a_{2n-1} = b_n$; (iii) $A_{2n} = \frac{b_n+1}{b_n(2b_n+1)}$; and (iv) $a_{2n} = 2b_n - 1$.

It is easy to verify these assertions for $n = 1$, as we find

(i) $A_1 = \frac{2}{2r+1} = \frac{2}{2b_1+1}$;

(ii) $a_1 = \left\lfloor \frac{1}{A_1} \right\rfloor = r = b_1$;

(iii) $A_2 = (\frac{1}{r} - \frac{2}{2r+1})(r+1) = \frac{r+1}{r(2r+1)} = \frac{b_1+1}{b_1(2b_1+1)}$;

(iv) $a_2 = \left\lfloor \frac{1}{A_2} \right\rfloor = \left\lfloor \frac{r(2r+1)}{r+1} \right\rfloor = \left\lfloor 2r - 1 + \frac{1}{r+1} \right\rfloor = 2r - 1 = 2b_1 - 1$.

Now assume the result is true for all $i \leq n$. We prove it for $n + 1$:

(i)

$$\begin{aligned}
A_{2n+1} &= \left(\frac{1}{a_{2n}} - A_{2n} \right) (a_{2n} + 1) \\
&= \left(\frac{1}{2b_n - 1} - \frac{b_n + 1}{b_n(2b_n + 1)} \right) (2b_n) \\
&= \frac{2}{4b_n^2 - 1} \\
&= \frac{2}{2b_{n+1} + 1}.
\end{aligned}$$

(ii)

$$a_{2n+1} = \left\lfloor \frac{1}{A_{2n+1}} \right\rfloor = \left\lfloor \frac{2b_{n+1}+1}{2} \right\rfloor = b_{n+1}.$$

(iii)

$$\begin{aligned} A_{2n+2} &= \left(\frac{1}{a_{2n+1}} - A_{2n+1} \right) (a_{2n+1}+1) \\ &= \left(\frac{1}{b_{n+1}} - \frac{2}{2b_{n+1}+1} \right) (b_{n+1}+1) \\ &= \frac{b_{n+1}+1}{b_{n+1}(2b_{n+1}+1)}. \end{aligned}$$

(iv)

$$\begin{aligned} a_{2n+2} &= \left\lfloor \frac{1}{A_{2n+2}} \right\rfloor \\ &= \left\lfloor \frac{b_{n+1}(2b_{n+1}+1)}{b_{n+1}+1} \right\rfloor \\ &= \left\lfloor 2b_{n+1} - 1 + \frac{1}{b_{n+1}+1} \right\rfloor \\ &= 2b_{n+1} - 1. \end{aligned}$$

This completes the proof. ■

Corollary.

For $r \geq 2$, the rational numbers $\frac{2}{2r+1}$ have non-terminating, non-ultimately-periodic modified Engel-type expansions.

Additional Remarks.

- For $r = 1$, the theorem gives the ultimately periodic expansion

$$2/3 = \{0, 1, 1, 1, 1, \ldots\}.$$

- For $r \geq 2$, the expansion is not ultimately periodic; e.g.

$$2/5 = \{0, 2, 3, 7, 13, 97, 193, 18817, \ldots\}.$$

In this case, we have the following brief table:

n	a_n	b_n	A_n
1	2	2	2/5
2	3	7	3/10
3	7	97	2/15
4	13	18817	8/105
5	97	708158977	2/195
6	193	1002978273411373057	98/18915

• The sequence $b_1, b_2, \ldots = 2, 7, 97, 18817, 708158977, \ldots$, corresponding to $r = 2$, appears to have been discussed first by G. Cantor in 1869 [1], who gave the infinite product

$$\sqrt{3} = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{7}\right)\left(1 + \frac{1}{97}\right)\cdots.$$

For more on this product of Cantor, see Spiess [9], Sierpiński [7], Engel [2], Stratemeyer [10,11], Ostrowski [6], and Mendès France and van der Poorten [5]. The sequence $2, 7, 97, 18817, \ldots$ was also discussed by Lucas [4]. It is sequence #720 in Sloane [8].

• The sequence $b_1, b_2, \ldots = 3, 17, 577, 665857, \ldots$, corresponding to $r = 3$, was also discussed by Cantor [1], who gave the infinite product

$$\sqrt{2} = \left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{17}\right)\left(1 + \frac{1}{577}\right)\cdots.$$

Also see the papers mentioned above. The sequence was also discussed by Wilf [12]. It sequence #1234 in Sloane [8].

• It is easy to prove that $b_{n+1} = B_{2^n}$ where $B_0 = 1$, $B_1 = r$, and $B_n = 2rB_{n-1} - B_{n-2}$ for $n \geq 2$. This gives a closed form for the sequence (b_n):

$$b_{n+1} = \frac{(r + \sqrt{r^2 - 1})^{2^n} + (r - \sqrt{r^2 - 1})^{2^n}}{2}.$$

• 3/7 is the "simplest" rational for which no simple description of the terms in its modified Engel-type expansion is known. The first forty terms are as follows:

$$3/7 = \{0, 2, 4, 5, 7, 8, 10, 25, 53, 62, 134, 574, 2431, 13147, 27167, 229073, 315416,$$

$$435474, 771789, 1522716, 3853889, 7878986, 7922488, 8844776, 9182596, 9388467,$$

$$14781524, 135097360, 1374449987, 1561240840, 4408239956, 11166053604, 12014224315,$$

$$23110106464, 553192836372, 900447772231, 1189661630241, 2058097840143484,$$
$$6730348855426376, 12928512475357529, \cdots\}.$$

More generally, it would be of interest to know whether it is possible to characterize the modified Engel expansion of every rational number.

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