Generalizing Narayana and Schröder Numbers to Higher Dimensions

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Abstract

Let $\mathcal{C}(d,n)$ denote the set of d-dimensional lattice paths using the steps $X_1 := (1,0,\ldots,0)$, $X_2 := (0,1,\ldots,0)$, $\ldots$, $X_d := (0,0,\ldots,1)$, running from $(0,0,\ldots,0)$ to $(n,n,\ldots,n)$, and lying in $\{(x_1,x_2,\ldots,x_d) : 0 \le x_1 \le x_2 \le \ldots \le x_d\}$. On any path $P := p_1p_2\ldots p_{dn} \in \mathcal{C}(d,n)$, define the statistics $ascs(P) :=|\{i : p_ip_{i+1} = X_jX_\ell, j<\ell\}|$ and $des(P) :=|\{i : p_ip_{i+1} = X_jX_\ell, j>\ell\}|$. Define the generalized Narayana number $N(d,n,k)$ to count the paths in $\mathcal{C}(d,n)$ with $ascs(P)=k$. We derive a formula for $N(d,n,k)$, implicit in MacMahon's work. We use Wegschaider's algorithm, extending the Wilf-Zeilberger method to multiple summation, to obtain recurrences for $N(3,n,k)$. We examine other statistics for $N(d,n,k)$ and show $ascs$ and $des-d+1$ to be equidistributed. We then introduce the generalized Schröder numbers $(\sum_k N(d,n,k)2^k)_{n\ge1}$ to count constrained paths using various step sets which include diagonal steps.

Key phases: Lattice paths, Catalan numbers, Narayana numbers, Schröder numbers, Wilf-Zeilberger method.

1 Introduction

In d-dimensional coordinate space consider lattice paths that use the unit steps

$$X_1 := (1,0,\ldots,0), X_2 := (0,1,\ldots,0), \ldots, X_d := (0,0,\ldots,1).$$

Let $\mathcal{C}(d,n)$ denote the set of lattice paths running from $(0,0,\ldots,0)$ to $(n,n,\ldots,n)$ and lying in the region $\{(x_1,x_2,\ldots,x_d) : 0 \le x_1 \le x_2 \le \ldots \le x_d\}$. On any path $P := p_1p_2\ldots p_{dn}$, we call any step pair p_ip_{i+1} an *ascent* (respectively, a *descent*) if $p_ip_{i+1} = X_jX_\ell$ for $j<\ell$

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(respectively, for $j > \ell$). Such paths are also called *ballot paths for d candidates* or *lattice permutations* (as in MacMahon [10]). Synonyms for "ascent" and "descent", respectively, are "minor contact" and "major contact" in [10]; often they are called "valley" and "peak" when $d = 2$.

To denote the statistics for the number of ascents and the number of descents, we put

$$ascs(P) := |\{i : p_i p_{i+1} = X_j X_\ell \text{ for } j < \ell\}|,$$

$$des(P) := |\{i : p_i p_{i+1} = X_j X_\ell \text{ for } j > \ell\}|.$$

For convenience when $d \leq 3$, put $X := X_1$, $Y := X_2$, and $Z := X_3$. See Table 1.

$P \in \mathcal{C}(3,2)$	$ascs(P)$	$des(P)$	$des(P) - ascs(P)$
$ZZYYXX$	0	2	2
$ZZYXYX$	1	3	2
$ZYZYXX$	1	3	2
$ZYZXYX$	2	3	1
$ZYXZYX$	1	4	3

Table 1: For $d = 3$ and $n = 2$.

For $d = 2$, it is well known that, for $0 \leq k \leq n-1$,

$$|\{P \in \mathcal{C}(2,n) : ascs(P) = k\}| = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}, \tag{1}$$

where the right side is called a Narayana number. See Remark 1.1.

For any dimension $d \geq 2$ and for $0 \leq k \leq (d-1)(n-1)$, we define the *d-Narayana distribution*, as

$$N(d,n,k) := \sum_{j=0}^{k} (-1)^{k-j} \binom{dn+1}{k-j} \prod_{i=0}^{d-1} \binom{n+i+j}{n} \binom{n+i}{n}^{-1}. \tag{2}$$

We will derive this formula while proving

Proposition 1 *For any dimension $d \geq 2$ and for $0 \leq k \leq (d-1)(n-1)$,*

$$|\{P \in \mathcal{C}(d,n) : ascs(P) = k\}| = N(d,n,k). \tag{3}$$

For $d \geq 2$ and $n \geq 1$, we define the n-th *d-Narayana polynomial* to be

$$N_{d,n}(t) := \sum_{k=0}^{(d-1)(n-1)} N(d,n,k)t^k,$$

with $N_{d,0}(t) := 1$. The sequence $(N_{d,n}(1))_{n\geq 0}$ has been called the d-dimensional Catalan numbers. For $n \geq 0$, we have the known formula (See [10, art. 93-103][26]; sequence A005789 in [15].):

$$N_{d,n}(1) = (dn)! \prod_{i=0}^{d-1} \frac{i!}{(n+i)!},$$

which we will reconsider for $d = 3$ in Proposition 6. Further,

$$\begin{aligned} N_{3,0}(t) &= 1 \\ N_{3,1}(t) &= 1 \\ N_{3,2}(t) &= 1 + 3t + t^2 \\ N_{3,3}(t) &= 1 + 10t + 20t^2 + 10t^3 + t^4 \\ N_{3,4}(t) &= 1 + 22t + 113t^2 + 190t^3 + 113t^4 + 22t^5 + t^6 \\ N_{3,5}(t) &= 1 + 40t + 400t^2 + 1456t^3 + 2212t^4 + 1456t^5 + 400t^6 + 40t^7 + t^8 \end{aligned}$$

In Section 2 we will prove Proposition 1 as a consequence of a bijection relating d-tuples of nonintersecting paths to constrained paths together with an application of the Gessel-Viennot method. One can obtain Proposition 1 from a more general q-analogue result of MacMahon [9][10, art. 443, 451, 495]. One can also obtain it from a fundamental theorem on order polynomials on posets developed by Stanley [16][18, Theorem 4.5.14].

In Section 3 we will use an algorithm of Wegschaider [24], which extends the Wilf-Zeilberger methodology to multiple summation, to obtain some recurrences for $N(d, n, k)$ and $N_{3,n}(t)$.

In Section 4 we will examine the statistic *des* and other statistics which are also distributed by the d-Narayana distribution. When $d = 2$, since the locations of the descents and the ascents alternate on any $P \in \mathcal{C}(2, n)$, certainly $des(P) = ascs(P) + 1$. However, when $d = 3$, a relationship between these two statistics is not apparent as Table 1 should show. We will prove

Proposition 2 *For $d \geq 2$ and $n \geq 1$, the statistics ascs and $des - d + 1$ are identically distributed on $\mathcal{C}(d, n)$. Hence,*

$$\sum_{P\in\mathcal{C}(d,n)} t^{ascs(P)} = \sum_{P\in\mathcal{C}(d,n)} t^{des(P)-d+1} = N_{d,n}(t).$$

In Section 5 we will introduce a d-dimensional analogue of the large Schröder numbers as the sequence $(2^{d-1}N_{d,n}(2))_{n\geq 1}$. It will follow from Proposition 2 that this sequence counts paths running from $(0, 0, \ldots, 0)$ to $(n, n, \ldots, n)$, lying in $\{(x_1, x_2, \ldots, x_n) : 0 \leq x_1 \leq x_2 \leq \ldots \leq x_n\}$, and using positive steps of the form $(\xi_1, \xi_2, \ldots, \xi_n)$ where $\xi_i \in \{0, 1\}$. It will also follow that $2^{d+n-2}N_{d,n}(2)$ counts the paths running from $(0, 0, \ldots, 0)$ to $(n, n, \ldots, n)$, lying in $\{(x_1, x_2, \ldots, x_n) : 0 \leq x_1 \leq x_2 \leq \ldots \leq x_n\}$, and using positive steps of the form $(\xi_1, \xi_2, \ldots, \xi_n)$ where ξ_i is a nonnegative integer.

Remarks: 1.1: The right side of (1) is named for Narayana who introduced the formula in 1955 [11]. However, this formula is immediately a special case of an earlier formula of MacMahon [10, art. 495, 5th formula]. Proposition 1 shows that the right side of (1) indeed agrees with (2) for $d = 2$. See [21, 22] for studies of $N(2, n, k)$.

1.2: In 1910 MacMahon [9, 10] introduced the *sub-lattice function of order* k, denoted $L_k(n, d; \infty)$, which is a q-analogue of $N(d, n, k)$. This might be the earliest appearance of the "d-Narayana numbers".

1.3: One can express our results in terms of a many candidate ballot problem [10, art. 93] where candidate i never leads candidate j, $1 \le i < j \le n$ throughout the balloting: $N(d, n, k)$ then counts ballot paths having length dn and ending in a tie where there are k instances of a vote for a weaker candidate being followed immediately by a vote for a stronger one. Equivalently, one can express our results in terms of the number linear extensions of the poset $\mathbf{d} \times \mathbf{n}$ having k descents or in terms of the less common terminology used by MacMahon [10]. However, by expressing our results in terms of lattice paths, our proof of Proposition 1, by way of Proposition 3, will intentionally display a relationship between counting restricted d-dimensional paths with respect to ascents and counting nonintersecting d-tuples of paths. The terminology of lattice paths also facilitates considering results admitting diagonal steps and hence the generalization of the Schröder numbers to higher dimensions.

1.4 In [23] the author studies counting $\mathcal{C}(3, n)$ with respect to the statistic *des* and obtains a formula for 3-Narayana numbers which is quite different from the formula of (2).

2 Counting with respect to ascents on paths

Let $\mathcal{NI}(m, n, d)$ denote the set of d-tuples of nonintersecting planar lattice paths, $(P_1, \ldots, P_j, \ldots, P_d)$, where path P_j, $1 \le j \le d$, uses the steps $(1, 0)$ and $(0, -1)$ and runs from $(j, n+j)$ to $(m + j, j)$. E.g., the triple of paths on the right side of Figure 1 belongs to $\mathcal{NI}(4, 5, 3)$. For positive integer n, let $[n] := \{1, 2, \ldots, n\}$. Let $\mathbf{n}$ denote the chain $1 < 2 < \cdots < n$, and let $J(d, m, n)$ denote the set of order ideals of the partially ordered set (poset) $\mathbf{d} \times \mathbf{m} \times \mathbf{n}$. One can find other definitions of this section in Stanley [18].

Proposition 3 *For $d \ge 2$, $m \ge 1$, and $n \ge 1$,*

$$|\mathcal{NI}(m, n, d)| = \sum_{k \ge 0} \binom{dn + m - k}{dn} N(d, n, k). \tag{4}$$

Proof. It is convenient to place the product poset $\mathbf{d} \times \mathbf{n}$ in the coordinate plane so that each element (x, y) of the poset is identified with a unit square having opposing vertices $(x - 1, y - 1)$ and (x, y). The values of a function on $\mathbf{d} \times \mathbf{n}$ will label the unit squares of a rectangle with d columns and n rows. We do the same for $\mathbf{m} \times \mathbf{n}$.

We will show that $|J(d,n,m)|$ is equal to the right side of (4) and then to its left side. One can check that the correspondences defined the proof are bijective.

Observe that each order ideal $I \in J(d,n,m)$ corresponds to a uniquely determined order reversing function $f : \mathbf{d} \times \mathbf{n} \to \mathbf{m{+}1}$: specifically,

$$(x,y,z) \in I \text{ if and only if } 0 < z < f(x,y).$$

For any order reversing function $f : \mathbf{d} \times \mathbf{n} \to \mathbf{m{+}1}$, let

$$(a_1, \ldots, a_i, a_{i+1}, \ldots, a_{dn}) := (f(x_1,y_1), \ldots, f(x_i,y_i), f(x_{i+1},y_{i+1}), \ldots, f(x_{dn},y_{dn}))$$

be that nonincreasing sequence of the entries of the array $(f(x,y))_{(x,y)\in[d]\times[n]}$ so that

$$\text{if } f(x_i,y_i) = f(x_{i+1},y_{i+1}) \text{ then } x_i \leq x_{i+1}. \tag{5}$$

Let $g : \mathbf{d} \times \mathbf{n} \to \{X_1, \ldots, X_d\}$ be that "step assignment" function where $g(x,y) := X_{d+1-x}$. Hence each order reversing function $f : \mathbf{d} \times \mathbf{n} \to \mathbf{m{+}1}$ corresponds uniquely to a matrix

$$\begin{bmatrix} A \\ P \end{bmatrix} := \begin{bmatrix} a_1 & \ldots & a_i & a_{i+1} & \ldots & a_{dn} \\ p_1 & \ldots & p_i & p_{i+1} & \ldots & p_{dn} \end{bmatrix} \tag{6}$$

where $p_i := g(x_i,y_i)$ and $P \in \mathcal{C}(d,n)$ by (5).

Notice that when there is an ascent in P of (6), say, $p_i p_{i+1} = X_j X_\ell$ where $j < \ell$, then we have $a_i > a_{i+1}$. Thus, given a matrix $\begin{bmatrix} A \\ P \end{bmatrix}$ with k ascents in P, there corresponds a unique matrix

$$\begin{bmatrix} B \\ P \end{bmatrix} := \begin{bmatrix} b_1 & \ldots & b_i & b_{i+1} & \ldots & b_{dn} \\ p_1 & \ldots & p_i & p_{i+1} & \ldots & p_{dn} \end{bmatrix}$$

where b_i is equal to a_i minus the number of ascents in the subpath $p_{i+1}p_{i+2}\cdots p_{dn}$. Since B is an unrestricted nonincreasing sequence with values in $[m-k]$, elementary counting shows that the number of B sequences for which $P \in \mathcal{C}(d,n)$ has k ascents is $\binom{dn+m-k}{dn}$. Hence, summing over all possibilities yields

$$|J(d,n,m)| = \sum_k \binom{dn+m-k}{dn} N(k,n,k).$$

Now, for the left side of (4), let the order ideal $I' \in J(m,n,d)$ be the reflection of $I \in J(d,n,m)$, and hence, $|J(d,n,m)| = |J(m,n,d)|$. Each $I' \in J(m,n,d)$ corresponds to a uniquely determined order reversing function $h : \mathbf{m} \times \mathbf{n} \to \mathbf{d{+}1}$ where $(x,y,z) \in I'$ if and only if $0 < z < h(x,y)$ for (x,y). Each function h corresponds uniquely to a rectangle having opposing vertices $(0,0)$ and (m,n) and consisting of unit squares with labels in $[d+1]$ that are nonincreasing along the rows and columns in the positive directions of the coordinate plane. (One could re-express these ideas in terms of plane partitions.) Equivalent to each rectangle with such a labeling, there is a d-tuple of noncrossing lattice paths on the rectangle such that the paths use the steps $(1,0)$ and $(0,-1)$, run from $(0,0)$ to (m,n), and separate the regions of squares of differing labels. We can then translate each path north-eastwardly so that the d-tuple of such noncrossing paths corresponds to a d-tuple of nonintersecting paths in $\mathcal{NI}(m,n,d)$. $\square$

Example: Take $d := 3$, $m := 4$, and $n := 5$. Take $I \in J(d,n,m)$ to be

$$\begin{aligned} I \;:=\; & \{(1,1,1),(1,1,2),(1,1,3),(1,1,4),(1,2,1),(1,2,2),(1,2,3),(1,2,4),\\ & (1,3,1),(1,3,2),(1,4,1),(1,4,2),(1,5,1),(1,5,2),(2,1,1),(2,1,2),(2,1,3),\\ & (2,1,4),(2,2,1),(2,2,2),(2,3,1),(3,1,1),(3,1,2),(3,1,3),(3,2,1)\} \end{aligned} \tag{7}$$

The corresponding function $f : \mathbf{d} \times \mathbf{n} \to \mathbf{m{+}1}$, augmented by the step assignment g, is represented as

$3,Z$	$1,Y$	$1,X$
$3,Z$	$1,Y$	$1,X$
$3,Z$	$2,Y$	$1,X$
$5,Z$	$3,Y$	$2,X$
$5,Z$	$5,Y$	$4,X$

Hence

$$\begin{bmatrix} A \\ P \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 & 4 & 3 & 3 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ Z & Z & Y & X & Z & Z & Z & Y & Y & X & Y & Y & Y & X & X \end{bmatrix},$$

and

$$\begin{bmatrix} B \\ P \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ Z & Z & Y & X & Z & Z & Z & Y & Y & X & Y & Y & Y & X & X \end{bmatrix}.$$

A reflection of the order ideal I of (7) is the ideal $I' \in J(m,n,d)$:

$$\begin{aligned} I' \;:=\; & \{(1,1,1),(1,1,2),(1,1,3),(1,2,1),(1,2,2),(1,2,3),(1,3,1),(1,3,2),\\ & (1,4,1),(1,5,1),(2,1,1),(2,1,2),(2,1,3),(2,2,1),(2,2,2),(2,3,1),(2,4,1),\\ & (2,5,1),(3,1,1),(3,1,2),(3,1,3),(3,2,1),(4,1,1),(4,1,2),(4,2,1)\} \end{aligned}$$

The corresponding function $h : \mathbf{m} \times \mathbf{n} \to \mathbf{d{+}1}$ is given in the left rectangle of Figure 1 together with the noncrossing paths separating regions with different labels. The right side of Figure 1 shows the corresponding nonintersecting triple of paths belonging to $\mathcal{IN}(m, n, d)$. □

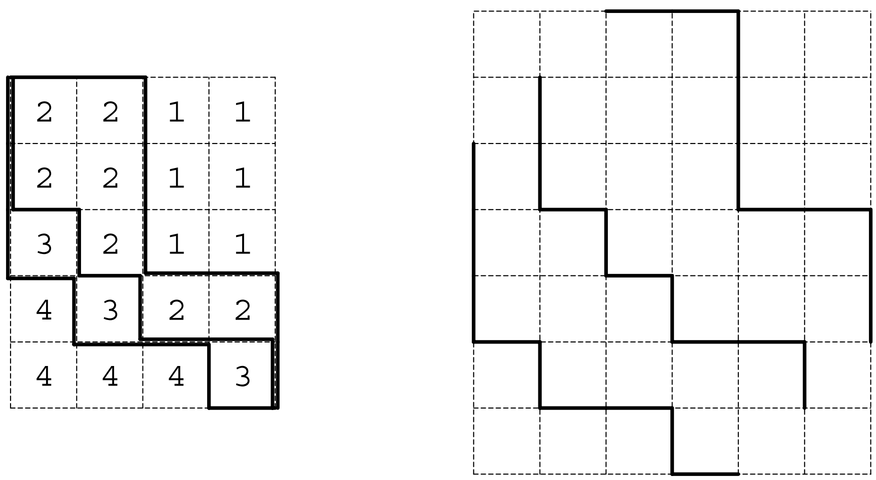

Figure 1: Noncrossing paths and nonintersecting paths for h

Lemma 1 *The cardinality of $\mathcal{NI}(m, n, d)$ is equal to the determinant of a d by d matrix*

$$D(d) := det\left[\binom{m+n}{n+j-i}\right]_{1\leq i,j\leq d}$$

which simplifies to

$$\prod_{j=0}^{d-1}\binom{m+n+j}{n}\binom{n+j}{n}^{-1}.$$

Proof. The well-known Gessel-Viennot method [3, 6, 7] (also known as the Karlin-Lindström-Gessel-Viennot method) yields the first part. Elementary techniques for simplifying determinants, Pascal's rule, and a straight forward induction can establish the second part. □

Proposition 3 and this lemma yield the next proposition which then yields Proposition 1 by a straight forward inversion.

Proposition 4 *For $d \geq 2$, $m \geq 1$, and $n \geq 1$,*

$$\sum_{k\geq 0}\binom{dn+m-k}{dn}N(d,n,k) = \prod_{j=0}^{d-1}\binom{m+n+j}{n}\binom{n+j}{n}^{-1}. \tag{8}$$

Corollary 1 *For $d \geq 2$ and $n \geq 1$, $N_{d,n}(t)$ is a reciprocal polynomial of degree $(d-1)(n-1)$. I.e., the sequence of coefficients of $N_{d,n}(t)$ is symmetric.*

Proof. Recall that for real r, the binomial coefficient is defined so $\binom{r}{k} := (\prod_{j=0}^{k-1} r-j)/k!$ if k is a positive integer and so $\binom{r}{0} := 1$. It is then easily seen that

$$\binom{r}{k} = (-1)^k \binom{k-r-1}{k}. \tag{9}$$

We observe that the degree of $N_{d,n}(t)$ cannot exceed $(dn-1)-(d-2)-n = (d-1)(n-1)$ since there are $dn-1$ step pairs on any path, since each of the final occurrences of the steps $X_2, \ldots, X_{d-1}$ on a path of $\mathcal{C}(d,n)$ cannot immediately precede an ascent, and since every X_d step cannot immediately precede an ascent.

Since the equation (8) is a polynomial equation in m which is valid for all positive integer values of m, it is valid for all real m. Indeed, replacing m by $-d-m-n$ in (8) yields

$$\sum_{k\geq 0} \binom{dn-d-m-n-k}{dn} N(d,n,k) = \prod_{j=0}^{d-1} \binom{-d-m+j}{n} \binom{n+j}{n}^{-1}.$$

Upon applying (9) to each factor of the numerator of the right side and commuting factors, we find

$$\prod_{j=0}^{d-1} \binom{-d-m+j}{n} \binom{n+j}{n}^{-1} = (-1)^{dn} \prod_{j=0}^{d-1} \binom{m+n+j}{n} \binom{n+j}{n}^{-1},$$

Hence,

$$\sum_{k\geq 0} \binom{(d-1)(n-1)-m-1-k}{dn} N(d,n,k) = (-1)^{dn} \sum_{k\geq 0} \binom{dn+m-k}{dn} N(d,n,k). \tag{10}$$

Recalling that the degree of $N_{d,n}(t)$ cannot exceed $(d-1)(n-1)$ and setting $m := 0$, we find that the only nonzero terms in (10) correspond to $k = (d-1)(n-1)$ on the left side and to $k = 0$ on the right side. Hence, $N(d,n,(d-1)(n-1)) = N(d,n,0)$. Next, repeatedly, set $m := 1, 2, \ldots$ and solve. One finds that $N(d,n,(d-1)(n-1)-k) = N(d,n,k)$ for $0 \leq k \leq (d-1)(n-1)$. $\square$

An identity for the 2-Narayana distribution: For $d = 2$, Proposition 1 shows that (1) is indeed equal to $N(2,n,k)$ defined in (2). Moreover, the determinant $D(2)$ of Lemma 1 easily simplifies to $\frac{1}{m+n+1}\binom{m+n+1}{m}\binom{m+n+1}{m+1}$. Thus, Lemma 1 gives a natural analogue of the well-known enumeration of parallelogram polyominoes by Narayana numbers, and Proposition 3 yields the following identity for the common Narayana numbers:

$$N(2,n+m+1,m) = \sum_{j=0}^{m} \binom{2n+j}{j} N(2,n,n-j).$$

Remarks: 2.1: We acknowledge that initially our derivation of (2) was motivated by a study of Kreweras and Niederhausen [8]. Recently Brändén [2] used an approach similar to that in the proof of Proposition 3 in studying the Narayana distribution for $d = 2$. As noted previously, one can deduce the results of this section from the works of MacMahon and of Stanley. However, their results do not consider d-tuples of nonintersecting paths.

2.2: The proof of Corollary 1 is similar to that of [9, art. 29]; the argument in [10, art. 449] seems incomplete. It can also be derived from the results of [16, sect. 18].

3 Recurrences

For $d = 2$ and $n \geq 3$, it is known, with bijective proofs appearing in [20, 22], that

$$(n+1)N_{2,n}(t) = (2n-1)(1+t)N_{2,n-1}(t) - (n-2)(1-t)^2 N_{2,n-2}(t). \tag{11}$$

For $d > 2$, we are interested in finding the recurrences for the d-Narayana polynomial and distribution. Perhaps they are amenable to bijective interpretation.

To find and prove recurrences for the 3-Narayana polynomial, we will apply the algorithm MultiSum of Wegschaider [24] which advances the method of Wilf and Zeilberger [25] for handling multiple summations. We will follow the procedure clearly documented in [24]. Currently the *Mathematica* algorithm MultiSum is being enhanced by, and is available from, Axel Riese [13].

Proposition 5 *For $n \geq 4$, the 3-Narayana polynomial satisfies*

$$\begin{aligned}
&(3n-4)(n+2)(n+1)^2 N_{3,n}(t) = \\
&\qquad (3n-2)(n+1)(4(1+t+t^2) - 5(1+7t+t^2)n + 3(1+7t+t^2)n^2)N_{3,n-1}(t) \\
&\qquad -(n-2)(-12+29n-30n^2+9n^3)(1-t)^4 N_{3,n-2}(t) \\
&\qquad +(3n-1)(n-2)(n-3)(n-4)(1-t)^6 N_{3,n-3}(t)
\end{aligned}$$

Proof. Once MultiSum is installed in a *Mathematica* session, we find and prove this recurrence by executing the following commands, which returns a certificate recurrence, which when summed and then simplified, yields the above recurrence:

```
In[1]:= <<MultiSum.m
In[2]:= summand = 2 (-1)^(k-j) Binomial[3 n + 1, k-j] Binomial[n+j, n]
         Binomial[n+j+1, n] Binomial[n+j+2, n]/(n+1)/(n+2)/(n+1)
In[3]:= certificate1 = FindRecurrence[ summand t^k, n, {j, k} ]
In[4]:= SumCertificate[certificate1]
```

(These commands returned a certificate in approximately 10 minutes on a 667MHz Pentium III cpu. For $d = 2$ and for $d = 4$, similar commands returned certificates in approximately 4 seconds and 24 hours, respectively.) □

Corollary 2 *For $n \geq 1$, $N_{3,n}(t)$ is a reciprocal polynomial.*

Proof. This follows from the proposition by induction. This is also a special case of Corollary 1. □

Proposition 6 *A formula for the 3-dimensional Catalan numbers is*

$$N_{3,n}(1) = \frac{2(3n)!}{n!(n+1)!(n+2)!}$$

Proof. First we find a recurrence for the 3-Narayana polynomial evaluated at $t = 1$.

```
In[5]:= certificate2 = FindRecurrence[ summand, n, {j, k} ];
```

It is then not difficult to guess a simplified formula for $N_{3,n}(1)$ based on the following output:

```
In[6]:= rec = SumCertificate[certificate2][[1]]
Out[6]:= -3 (-2 + 3 n) (-1+3 n) SUM[-1+n] + (1+n) (2+n) SUM[n] == 0
```

We use

```
In[7]:= guess = 2 (3 n)!/n!/(n+1)!/(n+2)!
In[8]:= CheckRecurrence[rec, guess]
```

in order to check that our guess satisfies the recurrence. We complete the proof by checking that the initial value for the guess is correct. (These commands returned a certificate in approximately 6 minutes; for the case $d = 4$, similar commands returned a certificate in approximately 95 minutes.) □

Proposition 7 *For $n \geq 2$, the numbers $N(3, n, k)$ satisfy*

$$\begin{aligned} n(n+1)(n+2)N(3,n,k) \quad = \quad & (n+k+2)(n+k+1)(n+k)N(3,n-1,k) \\ & +3(2n-k-1)(n+k+1)(n+k)N(3,n-1,k-1) \\ & +3(2n-k-1)(2n-k)(n+k)N(3,n-1,k-2) \\ & +(2n-k-1)(2n-k)(2n-k+1)N(3,n-1,k-3). \end{aligned}$$

Proof. To prove this using MultiSum, we execute the following, obtaining a certificate (in just a few seconds) which reduces to the above recurrence:

```
In[9]:= certificate3 = FindRecurrence[ summand, {n, k}, j ]
In[10]:= SumCertificate[certificate3]
```

□

As a generalization, we have the following conjecture, proven for $2 \leq d \leq 5$ using MultiSum:

Conjecture 1 *For any $d \geq 2$ and for $n \geq 2$,*

$$\prod_{i=0}^{d-1}(n+i)N(d,n,k) = \sum_{i=0}^{d}\binom{d}{i}\prod_{j=1}^{i}((d-1)(n-1)-k+j)\prod_{j=0}^{d-i-1}(n+k+j)N(d,n-1,k-i).$$

4 Other statistics having the d-Narayana distribution

4.1 The case for $d=3$

This subsection serves to motivate the following subsection. Its main result is the proof that the *number of descents less two* and the *number of ascents* are equi-distributed on $\mathcal{C}(3,n)$. We will consider 24 statistics for $\mathcal{C}(3,n)$, each of which is expressed in terms of a 3 by 3 0-1 matrix M. Here $(M)_{j\ell}$ denotes the entry in row j and column ℓ of M, while M_{ij} denotes a specific matrix identified by the subscripts. Let Θ_M denote a statistic on $\mathcal{C}(3,n)$ defined so that, for each path $P := p_1p_2\dots p_{3n}$,

$$\Theta_M(P) := \sum_{j=1}^{3}\sum_{\ell=1}^{3}(M)_{j\ell}|\{i : p_ip_{i+1} = X_jX_\ell, i \in [3n-1]\}|.$$

The statistic *ascs* corresponds to the matrix $VM_{22} = M_A := \begin{bmatrix} 0&1&1\\ 0&0&1\\ 0&0&0 \end{bmatrix}$, since

$$ascs(P) = |\{i : p_ip_{i+1} \in \{X_1X_2, X_1X_3, X_2X_3\}\}|.$$

(We explain the "V" momentarily and the "22" in the next section.) Similarly, the statistic *des* corresponds to the matrix $M_D := \begin{bmatrix} 0&0&0\\ 1&0&0\\ 1&1&0 \end{bmatrix}$. A simple search over $\mathcal{C}(3,n)$ for small n shows that any statistic having the form Θ_M and being distributed by $N(3,n,k-c)$ for some $c \in \{0,1,2\}$ must be one of the 24 matrices of Table 2. A series of lemmas will prove

Proposition 8 *For $d=3$ and $n \geq 1$, each matrix M in the first two columns of Table 2 yields a statistic $\Theta_M - (M)_{21} - (M)_{32}$ having the 3-Narayana distribution. In particular (as stated in Proposition 2), des $-$ 2 and ascs are equi-distributed. (The sum $(M)_{21} + (M)_{32}$ adjusts the statistic so $|\{P \in \mathcal{C}(3,n) : \Theta_M(P) - (M)_{21} - (M)_{32} = k\}| = N(3,n,k)$.)*

Conjecture 2 *For $d=3$ and $n \geq 1$, each matrix M in the last two columns of Table 2 yields a statistic $\Theta_M - (M)_{21} - (M)_{32}$ having the 3-Narayana distribution. Consequently, the statistic des$-$ascs-1 has the 3-Narayana distribution in agreement with $M_{12}+VM_{22}+M_E = \begin{bmatrix} 1&1&1\\ 1&1&1\\ 1&1&1 \end{bmatrix}$.*

$$M_D := \begin{bmatrix} 0&0&0 \\ 1&0&0 \\ 1&1&0 \end{bmatrix} \quad M_{21} := \begin{bmatrix} 1&0&1 \\ 0&0&0 \\ 0&0&1 \end{bmatrix} \quad M_E := \begin{bmatrix} 0&0&0 \\ 1&1&0 \\ 0&1&0 \end{bmatrix} \quad M_F := \begin{bmatrix} 0&1&0 \\ 1&1&0 \\ 0&0&0 \end{bmatrix}$$

$$M_{11} := \begin{bmatrix} 0&0&0 \\ 0&1&1 \\ 0&0&1 \end{bmatrix} \quad VM_{21} := \begin{bmatrix} 0&0&0 \\ 1&0&1 \\ 1&0&0 \end{bmatrix} \quad \begin{bmatrix} 0&0&0 \\ 0&0&1 \\ 1&0&1 \end{bmatrix} \quad \begin{bmatrix} 1&0&1 \\ 0&0&1 \\ 0&0&0 \end{bmatrix}$$

$$VM_{11} := \begin{bmatrix} 0&1&1 \\ 0&0&0 \\ 0&1&0 \end{bmatrix} \quad M_{22} := \begin{bmatrix} 0&0&0 \\ 0&1&0 \\ 0&1&1 \end{bmatrix} \quad \begin{bmatrix} 1&0&1 \\ 1&0&0 \\ 0&0&0 \end{bmatrix} \quad \begin{bmatrix} 0&0&0 \\ 1&0&0 \\ 1&0&1 \end{bmatrix}$$

$$M_{12} := \begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 1&0&1 \end{bmatrix} \quad VM_{22} := \begin{bmatrix} 0&1&1 \\ 0&0&1 \\ 0&0&0 \end{bmatrix} \quad \begin{bmatrix} 0&1&0 \\ 0&1&1 \\ 0&0&0 \end{bmatrix} \quad \begin{bmatrix} 0&0&0 \\ 0&1&1 \\ 0&1&0 \end{bmatrix}$$

$$\begin{bmatrix} 0&0&1 \\ 1&0&1 \\ 0&0&0 \end{bmatrix} \quad M_{23} := \begin{bmatrix} 1&0&0 \\ 1&1&0 \\ 0&0&0 \end{bmatrix} \quad \begin{bmatrix} 0&0&1 \\ 0&0&0 \\ 0&1&1 \end{bmatrix} \quad \begin{bmatrix} 0&1&1 \\ 0&0&0 \\ 0&0&1 \end{bmatrix}$$

$$\begin{bmatrix} 1&1&0 \\ 0&1&0 \\ 0&0&0 \end{bmatrix} \quad \begin{bmatrix} 0&1&0 \\ 0&0&0 \\ 1&1&0 \end{bmatrix} \quad \begin{bmatrix} 1&1&0 \\ 0&0&0 \\ 1&0&0 \end{bmatrix} \quad \begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 1&1&0 \end{bmatrix}$$

Table 2: Candidate matrices

P	$\Theta_{M_D}(P)$	$\Theta_{M_A}(P)$	$\Theta_{M_E}(P)$	$\Theta_{M_F}(P)$	$hdes(P)$
$ZZYYXX$	2	0	1	1	2
$ZZYXYX$	3	1	1	2	1
$ZYZYXX$	3	1	1	0	1
$ZYZXYX$	3	2	0	1	1
$ZYXZYX$	4	1	2	1	0

Table 3: Statistical values

If one considers any two statistics Θ_1 and Θ_2 on $\mathcal{C}(3,n)$ to be equivalent when either $\Theta_1+\Theta_2$ or $\Theta_1-\Theta_2$ is a constant statistic for each n, then Table 3 shows the non-equivalency of Θ_{M_D}, Θ_A, Θ_{M_E} and Θ_{M_F}. The statistic *hdes* counting the *high descents* of subsection 4.3 requires $n=3$ to see that it is not equivalent to the others. Lemma 2 shows that each column of Table 2 corresponds to an equivalence class.

For this and the next subsection, for each matrix M being considered, we define the *horizontal complement*, HM, and the *vertical complement*, VM, to be matrices defined so

$$\begin{aligned}(HM)_{j\ell} &:= \begin{cases} 0 & \text{if } j \text{ is a zero row of } M \\ 1-(M)_{j\ell} & \text{if otherwise,}\end{cases} \\ (VM)_{j\ell} &:= \begin{cases} 0 & \text{if } \ell \text{ is a zero column of } M \\ 1-(M)_{j\ell} & \text{if otherwise.}\end{cases}\end{aligned}$$

(E.g., see Table 2, where $M_{11} = HM_D$.)

We express the following lemma in a general form which will be defined in the next subsection. Presently, we read the next lemma for $d=3$.

Lemma 2 *For d by d matrices M having exactly one row and one column of 0's,*

$$\begin{aligned}\Theta_M(P)+\Theta_{HM}(P) &= \begin{cases} (d-1)n & \textit{if row 1 of } M \textit{ is a zero row} \\ (d-1)n-1 & \textit{if otherwise,}\end{cases} \\ \Theta_M(P)+\Theta_{VM}(P) &= \begin{cases} (d-1)n & \textit{if column 1 of } M \textit{ is a zero column} \\ (d-1)n-1 & \textit{if otherwise.}\end{cases}\end{aligned}$$

Proof. We note that each path begins with X_d, ends with X_1, and has a total of $dn-1$ consecutive step pairs. If row 1 of M is a zero row, then the $n-1$ non-final X_1 steps, all of which immediately precede some other step on P, do not contribute to $\Theta_M(P)+\Theta_{HM}(P)$. Hence, $\Theta_M(P)+\Theta_{HM}(P) = (dn-1)-(n-1)$. If row 2 of M is a zero row, then only the n X_2 steps, which must immediately precede some other step on P, do not contribute to $\Theta_M(P)+\Theta_{HM}(P) = (dn-1)-n$. Similarly, the other instances of the lemma are valid. □

We notice that all matrices in Table 2 have exactly one row and one column of 0's. As a consequence of Lemma 2 one can check that

Lemma 3 *For all M in Table 2,*

$$\begin{aligned}\Theta_M(P)-(M)_{23}-(M)_{32}+\Theta_{HM}(P)-(HM)_{23}-(HM)_{32} &= 2n-2. \\ \Theta_M(P)-(M)_{23}-(M)_{32}+\Theta_{VM}(P)-(VM)_{23}-(VM)_{32} &= 2n-2.\end{aligned}$$

Lemma 4 *For any $d\geq 2$, suppose that Θ_1 is distributed by a reciprocal polynomial of degree $(d-1)(n-1)$ on $\mathcal{C}(d,n)$. If $\Theta_1(P)+\Theta_2(P)=(d-1)(n-1)$ for all $P\in\mathcal{C}(d,n)$, then they are equi-distributed.*

$$M_D \xrightarrow{H} M_{11} \xrightarrow{V} VM_{11} \xrightarrow{H} M_{12} \xrightarrow{T} M_{21} \xrightarrow{V} VM_{21} \xrightarrow{H} M_{22} \xrightarrow{V} VM_{22} = M_A$$

Figure 2: The schema for proving that $des - 2$ and $ascs$ are equi-distributed.

Proof.

$$\sum_{P\in\mathcal{C}(d,n)} t^{\Theta_2(P)} = \sum_{P\in\mathcal{C}(d,n)} t^{(d-1)(n-1)-\Theta_1(P)} = \sum_{P\in\mathcal{C}(d,n)} t^{\Theta_1(P)}. \quad \square$$

Lemma 5 *For M_{12} and M_{21} defined in Table 2 and for $1 \le k < 2n-2$, there is an explicit bijection*

$$\beta_2 : \{P \in \mathcal{C}(3,n) : \Theta_{M_{12}}(P) = k\} \to \{P \in \mathcal{C}(3,n) : \Theta_{M_{21}}(P) = k\},$$

and hence $\Theta_{M_{12}}$ and $\Theta_{M_{21}}$ are equi-distributed.

Proof. For any $P \in \mathcal{C}(3,n)$, we split P into maximal blocks (i.e., maximal subpaths) which either contain only Y steps or contain no Y step. In each block of the second type, we exchange the initial maximal subblock (perhaps empty) of X steps with the final maximal subblock (perhaps empty) of X steps. $\beta_2(P)$ is the resulting path. We note that $\beta_2(P) \in \mathcal{C}(3,n)$ since the condition $0 \le x \le y \le z$ for any point (x,y,z) on a path holds during the exchanges. The action of β_2 leaves the number of XX and ZZ pairs fixed and transforms the number of ZX pairs to the number of XZ pairs. Since $M_{21} = TM_{12}$, where T denotes the usual transpose operator, the proof is complete. $\square$

Proof of Proposition 8: This is a consequence of Lemmas 3, 4, 5, and Corollary 1. See Figure 2 where T denotes the transpose operator. In particular, for any $P \in \mathcal{C}(3,n)$,

$$\begin{aligned}
&\Theta_{M_D}(P) + \Theta_{M_{11}}(P) = 2n, && \Theta_{M_{11}}(P) + \Theta_{VM_{11}}(P) = 2n-1,\\
&\Theta_{VM_{11}}(P) + \Theta_{M_{12}}(P) = 2n-1, && \Theta_{M_{21}}(\beta_2(P)) + \Theta_{VM_{21}}(\beta_2(P)) = 2n-1,\\
&\Theta_{VM_{21}}(\beta_2(P)) + \Theta_{M_{22}}(\beta_2(P)) = 2n, && \Theta_{M_{22}}(\beta_2(P)) + \Theta_{VM_{22}}(\beta_2(P)) = 2n-1,
\end{aligned}$$

together with $\Theta_{M_{12}}(P) = \Theta_{M_{21}}(\beta_2(P))$, yield $\Theta_{M_D}(P) - 2 = \Theta_{VM_{22}}(\beta_2(P)) = \Theta_{M_A}(\beta_2(P))$. $\square$

4.2 The case for arbitrary d

Here we prove that $des - d + 1$ and $ascs$ are equi-distributed. We encode the relevant statistics in terms of d by d 0-1 matrices so that, for $P := p_1 p_2 \ldots p_{dn} \in \mathcal{C}(d,n)$,

$$\Theta_M(P) := \sum_{j=1}^{d} \sum_{\ell=1}^{d} (M)_{j\ell} |\{i : p_i p_{i+1} = X_j X_\ell, i \in [dn-1]\}|.$$

$$
\begin{array}{c}
M_{11}\ \ VM_{11}\ \ M_{12} \\
M_{21}\ \ VM_{21}\ \ M_{22}\ \ VM_{22}\ \ M_{23} \\
M_{31}\ \ VM_{31}\ \ M_{32}\ \ VM_{32}\ \ M_{33}\ \ VM_{33}\ \ M_{34} \\
\cdots \\
M_{i1}\ \ VM_{i1}\ \ M_{i2}\ \ VM_{i2}\ \ \ldots\ \ VM_{i,i-1}\ \ M_{ii}\ \ VM_{ii}\ \ M_{i,i+1} \\
\cdots \\
M_{d-1,1}\ \ VM_{d-1,1}\ \ M_{d-1,2}\ \ VM_{d-1,2}\ \ \ldots\ \ VM_{d-1,d-2}\ \ M_{d-1,d-1}\ \ VM_{d-1,d-1}\ \ M_{d-1,d}.
\end{array}
$$

Table 4: The trapezoidal array of matrices.

			(1,1)	(2,1)	(2,2)			
		(2,2)	(1,2)	(1,1)	(3,1)	(3,3)		
	(3,3)	(2,3)	(2,2)	(1,2)	(1,1)	(4,1)	(4,4)	
(4,4)	(3,4)	(3,3)	(2,3)	(2,2)	(1,2)	(1,1)	(5,1)	(5,5)

Table 5: The zero intersections for $d = 5$.

Define the matrices M_A and M_D so

$$
\begin{aligned}
(M_A)_{j\ell} &:= 1 \text{ if } j < \ell, \text{ and } = 0 \text{ if otherwise}, \\
(M_D)_{j\ell} &:= 1 \text{ if } j > \ell, \text{ and } = 0 \text{ if otherwise}.
\end{aligned}
$$

Hence, $ascs(P) = \Theta_{M_A}(P)$ and $des(P) = \Theta_{M_D}(P)$.

Our proof will establish a transition Θ_{M_D} to Θ_{M_A} similar to that in Figure 2, but more extensive, and account for the difference $d-1$. Keeping the definitions of the complement operators H and V from section 4.1, we will define a trapezoidal array of matrices appearing in Table 4. Specifically for $1 \le i \le d-1$, we define M_{i1} so that

$$
(M_{i1})_{j\ell} := \begin{cases} 1 & \text{if } \ell \le j < i \text{ or } j < i < \ell \text{ or } i < j \le \ell \ , \\ 0 & \text{if otherwise.} \end{cases}
$$

(E.g., M_{71} in Figure 3.) Moreover, for $1 \le j \le i \le d-1$, define $M_{i,j+1} := HVM_{i,j}$. (E.g., M_{73} and M_{74} in Figure 3.)

Given each matrix in the trapezoidal array, it is useful to determine the indices of the intersection of its zero row and zero column, called its *zero intersection.* One can check that the array of Table 5 gives the zero intersections corresponding to the the trapezoidal array of matrices for $d = 5$. More generally we state a lemma.

Lemma 6 *Let $d \ge 3$. For $1 \le j < i \le d-1$, the zero intersection of $M_{i,j}$ has indices $(i+1-j, i+1-j)$ and the zero intersection of $VM_{i,j}$ has indices $(i-j, i+1-j)$. The zero* *intersection of $M_{i,i}$ has indices $(1,1)$, the zero intersection of $VM_{i,i}$ has indices $(i+1,1)$, and the zero intersection of $M_{i,i+1}$ has indices $(i+1,i+1)$.*

Proof. Without introducing awkward notation, one can check the validity of this lemma by working through the example of Figure 3 which is sufficiently general to explain the actions of V and H on M_{ij}. Starting with the second matrix, M_{71}, at each stage one should pay particular attention to how the submatrix lying below the zero row and to the right of the zero column is transformed. □.

Lemma 7 *For $1 \le j \le i \le d-1$,*

$$\begin{aligned}\Theta_{M_{ij}}(P) + \Theta_{VM_{ij}}(P) &= (d-1)n-1 \\ \Theta_{VM_{ij}}(P) + \Theta_{M_{i,j+1}}(P) &= \begin{cases} (d-1)n & \text{if } j = i-1 \\ (d-1)n-1 & \text{if otherwise} \end{cases}\end{aligned}$$

Consequently, for $2 \le i \le d-1$,

$$\Theta_{M_{i1}}(P) + \Theta_{M_{i,i+1}}(P) = \begin{cases} 0 & \text{if } i = 1 \\ 1 & \text{if } 2 \le i \le d-1 \end{cases}$$

Proof. Use the above lemma and Lemma 2 whose proof is valid for any $d \ge 2$. The second part relies on telescopic cancellation. □

Lemma 8 *For any $d \ge 2$, since $HM_D = M_{11}$ and $M_A = HM_{d-1,d}$,*

$$\begin{aligned} des(P) + \Theta_{M_{11}}(P) &= (d-1)n \\ ascs(P) + \Theta_{M_{d-1,d}}(P) &= (d-1)n-1 \qquad \square \end{aligned}$$

For any d by d matrix M and for $2 \le i \le d-1$, we define a "restricted transpose", denoted by T_iM, so that

$$(T_iM)_{j\ell} := \begin{cases} (M)_{\ell j} & \text{if } \ell < i < j \text{ or } j < i < \ell \\ (M)_{j\ell} & \text{if otherwise.} \end{cases}$$

Observe that $T_iM_{i-1,i} = M_{i1}$. (E.g., see the top row of Figure 3.)

Also, for $2 \le i \le d-1$, we define a bijection

$$\beta_i : \mathcal{C}(d,n) \to \mathcal{C}(d,n)$$

as follows: For any $P \in \mathcal{C}(d,n)$, break P into maximal blocks which either contain only X_i steps or contain no X_i step. In each block of the second type, we exchange the initial maximal subblock (perhaps empty) of steps belonging to $\{X_1 \ldots X_{i-1}\}$ with the final maximal subblock (perhaps empty) of steps belonging to $\{X_1 \ldots X_{i-1}\}$. $\beta_i(P)$ is the resulting path.

$$M_{67} = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & \mathbf{0} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{0} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & 1 & 1 & 1 & 1 & 1 & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & 1 & 1 & 1 & 1 & 1 & \mathbf{0} & 0 & \mathbf{1} & \mathbf{1} \\ 1 & 1 & 1 & 1 & 1 & 1 & \mathbf{0} & 0 & 0 & \mathbf{1} \end{bmatrix} \quad \overset{T_7}{\longrightarrow} \quad M_{71} = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{0} & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & \mathbf{0} & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & \mathbf{0} & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{0} & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & 1 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & \mathbf{1} \end{bmatrix} \quad \overset{(HV)^2}{\longrightarrow}$$

$$M_{73} = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & \mathbf{0} & 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & \mathbf{0} & 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{0} & 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & 1 & 1 & 1 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{0} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{0} & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{0} & 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{0} & 1 & 1 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{0} & 1 & 1 & 0 & 0 & \mathbf{1} \end{bmatrix} \quad \overset{V}{\longrightarrow} \quad VM_{73} = \begin{bmatrix} 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{0} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & 1 & 1 & 1 & \mathbf{0} & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \mathbf{0} & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \mathbf{0} & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \mathbf{0} & 0 & 0 & \mathbf{1} & 0 & 0 \\ 1 & 1 & 1 & 1 & \mathbf{0} & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \end{bmatrix} \quad \overset{H}{\longrightarrow}$$

$$M_{74} = \begin{bmatrix} \mathbf{1} & 0 & 0 & \mathbf{0} & 1 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & 0 & \mathbf{0} & 1 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & 1 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & 1 & 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{0} & 1 & 1 & 1 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{0} & 1 & 1 & 1 & 0 & 0 & \mathbf{1} \end{bmatrix} \quad \overset{V(HV)^2}{\longrightarrow} \quad VM_{76} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & \mathbf{0} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \mathbf{0} & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \mathbf{0} & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \mathbf{0} & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & \mathbf{0} & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 1 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \end{bmatrix}$$

Figure 3: This illustrates the action of T_7, H, and V.

Example. For $d = 4$, one can check that, if

$$P = X_4X_3X_2X_4X_3X_1X_4X_4X_2X_1X_3X_2X_1X_3X_2X_1$$

then

$$\beta_3(P) = X_4X_3X_4X_2X_3X_2X_1X_4X_4X_1X_3X_2X_1X_3X_2X_1.$$

One can also check that $\Theta_{M_{23}}(P) = \Theta_{T_2M_{23}}(\beta_3(P)) = 4$ where

$$M_{23} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T_3M_{23} = M_{31} := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Lemma 9 *For $2 \le i \le d-1$ and $1 \le k \le (d-1)(n-1)$, the restriction of β_i,*

$$\beta_i : \{P \in \mathcal{C}(d,n) : \Theta_{M_{i-1,i}}(P) = k\} \to \{P \in \mathcal{C}(d,n) : \Theta_{M_{i,1}}(P) = k\},$$

is a bijection with $\Theta_{M_{i-1,i}}(P) = \Theta_{M_{i1}}(\beta_i(P))$. Hence $\Theta_{M_{i-1,i}}$ and $\Theta_{M_{i,1}}$ are equi-distributed.

Proof. We note that $\beta_i(P) \in \mathcal{C}(d,n)$ since the condition $x_1 \leq x_2 \leq \cdots \leq x_d$ holds for any point $(x_1, x_2, \ldots, x_d)$ on P during the exchanging. The action of β_i leaves the number of $X_j X_j$, for $j \neq i$, fixed and exchanges the total number of $X_h X_j$ pairs, for all h, j where $h < i < j$, with an equal total number of $X_j X_h$ pairs. Hence, $\Theta_{M_{i-1,i}}(P) = \Theta_{T_i M_{i-1,i}}(\beta_i(P))$, which equals $\Theta_{M_{i,1}}(\beta_i(P))$. □

Proof of Proposition 2: This uses Lemmas 7, 8, and 9. Specifically, the identities

$$\begin{array}{rcl} des(P) & = & (d-1)n - \Theta_{M_{11}}(P) \\ \Theta_{M_{11}}(P) & = & \Theta_{M_{12}}(P) \\ \Theta_{M_{12}}(P) & = & \Theta_{M_{21}}(\beta_2(P)) \\ \Theta_{M_{21}}(\beta_2(P)) & = & \Theta_{M_{23}}(\beta_2(P)) + 1 \\ \Theta_{M_{23}}(\beta_2(P)) & = & \Theta_{M_{31}}(\beta_3(\beta_2(P))) \\ \Theta_{M_{31}}(\beta_3(\beta_2(P))) & = & \Theta_{M_{34}}(\beta_3(\beta_2(P))) + 1 \\ \vdots & & \vdots \\ \Theta_{M_{d-2,d-1}}(\beta_{d-2}(\cdots(\beta_2(P)))) & = & \Theta_{M_{d-1,1}}(\beta_{d-1}(\beta_{d-2}(\cdots(\beta_2(P))))) \\ \Theta_{M_{d-1,1}}(\beta_{d-1}(\beta_{d-2}(\cdots(\beta_2(P))))) & = & \Theta_{M_{d-1,d}}(\beta_{d-1}(\beta_{d-2}(\cdots(\beta_2(P)))) + 1 \\ \Theta_{M_{d-1,d}}(\beta_{d-1}(\beta_{d-2}(\cdots(\beta_2(P))))) & = & (d-1)n - 1 - ascs(\beta_{d-1}(\beta_{d-2}(\cdots(\beta_2(P)))) \end{array}$$

yield

$$des(P) = ascs(\beta_{d-1}(\beta_{d-2}(\cdots(\beta_2(P)))) + d - 1.$$

□

4.3 High descents

For $d := 2$, on any path a *high peak* is any YX pair whose intermediate vertex (x, y) satisfies $y - x > 1$. If $hpeaks(P)$ denotes the number of high peaks on the path P, Deutsch [4] found that $hpeaks$ has the Narayana distribution on $\mathcal{C}(2,n)$. Now, for arbitrary $d \geq 2$, on any path $P = p_1 p_2 \ldots p_{dn} \in \mathcal{C}(d,n)$, call any step pair $p_i p_{i+1}$ a *high descent* if $p_i p_{i+1} = X_j X_\ell$ for $j > \ell$ and its intermediate vertex $(x_1, x_2, \ldots, x_d)$ satisfies $x_j - x_\ell > 1$. Let $hdes(P)$ denote the number of high descents on the path P.

Counting with respect to high descents seems much closer to counting with respect to ascents than with respect to descents. Specifically, if we simply change the requirement of (5) to

$$\text{if } f(x_i, y_i) = f(x_{i+1}, y_{i+1}) \text{ then either } y_i = y_{i+1} \text{ and } x_{i+1} = x_i + 1 \text{ or } y_i < y_{i+1}$$

in the proof of Proposition 3, then we can modify section 2 to show

Proposition 9 *For any $d \geq 2$ and $0 \leq k \leq (d-1)(n-1)$,*

$$|\{P \in \mathcal{C}(d,n) : P \text{ has } k \text{ high descents}\}| = N(d,n,k).$$

5 d-Schröder numbers and a "2^{n-1} result"

During the past decade the *Schröder numbers* have received considerable attention, for instance in [1, 12, 14, 19, 17]. For arbitrary $d \geq 2$, we generalize the definitions of the *small* and *large* Schröder numbers (as seen in [22]): Let the *small* and *large d-Schröder numbers,* respectively, be the sequences $(N_{d,n}(2))_{n\geq 1}$ and $(2^{d-1}N_{d,n}(2))_{n\geq 1}$, respectively. In each sequence we will set the term for $n = 0$ to be 1. For $d = 3$ we have

$$(N_{3,n}(2))_{n\geq 0} = 1, 1, 11, 197, 4593, 126289, 3888343, 130016393, 4629617873, \ldots$$

$$(4N_{3,n}(2))_{n\geq 1} = 4, 44, 788, 18372, 505156, 15553372, 520065572, 18518471492, \ldots$$

Consider d-dimensional lattice paths that use the nonzero steps of the form $(\xi_1, \xi_2, \ldots, \xi_d)$ where $\xi_i \in \{0,1\}$ for $1 \leq i \leq d$. Let $\mathcal{D}(n)$ denote the set of paths running from $(0,0,\ldots,0)$ to $(n,n,\ldots,n)$, using these steps, and lying in the region $\{(x_1, x_2, \ldots, x_d) : 0 \leq x_1 \leq x_2 \leq \ldots \leq x_d\}$. For $d = 2$, such paths are known as (large) Schröder paths, and it is well known that $|\mathcal{D}(n)| = 2N_{2,n}(2)$ for $n \geq 1$.

Proposition 10 *For any $d \geq 2$ and $n \geq 1$, $|\mathcal{D}(n)| = 2^{d-1}N_{d,n}(2)$.*

Proof. This proof for $d = 3$ can easily be generalized. Let $\mathcal{C}'(n)$ denote the set of replicated paths formed from the paths of $\mathcal{C}(3,n)$ by independently coloring with B or R the intermediate vertices of YX, ZX, and ZY, i.e., intermediate vertices of descents. Color all other vertices with R. Define

$$\mu : \mathcal{D}(n) \longrightarrow \mathcal{C}'(n)$$

to be the bijection that first sequentially applies the following replacement rules to the diagonal steps of each path:

$$\begin{array}{lcl} (1,1,0) & \longrightarrow & YBX \\ (1,0,1) & \longrightarrow & ZBX \\ (0,1,1) & \longrightarrow & ZBY \\ (1,1,1) & \longrightarrow & ZBYBX, \end{array}$$

and then leaves the steps $(1,0,0)$,$(0,1,0)$, and $(0,0,1)$ unaltered, and finally assigns the color R to all non-B vertices on the resulting path. Since $|\mathcal{D}(n)| = |\mathcal{C}'(n)| = \sum_{P\in\mathcal{C}(3,n)} 2^{des(P)} = 2^2 N_{3,n}(2)$ the result follows. □

Next we relate the d-Schröder numbers to constrained paths using steps of arbitrary length. Consider those d-dimensional lattice paths that use the nonzero steps of the form $(\xi_1, \xi_2, \ldots, \xi_d)$ where ξ_i is a nonnegative integer. Let $\mathcal{S}(n)$ denote the set of paths running from $(0,0,\ldots,0)$ to $(n,n,\ldots,n)$, using these steps, and lying in the region $\{(x_1, x_2, \ldots, x_d) : 0 \le x_1 \le x_2 \le \ldots \le x_d\}$.

Lemma 10 *For $d = 3$ and the notation for Θ_M of the previous section, let $M^\star = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.*

$$|\mathcal{S}(n)| = \sum_{P \in \mathcal{C}(3,n)} 2^{\Theta_{M^\star}(P)}.$$

This result generalizes to any $d \ge 2$ where matrix $M^\star$ is the d by d matrix where $(M^\star)_{j\ell} := 1$ if $j \ge \ell$, and $= 0$ if otherwise.

Proof. This proof for $d = 3$ can easily be generalized. Let $\mathcal{C}''(n)$ denote the set of replicated paths formed from the paths of $\mathcal{C}(3,n)$ by independently coloring with B or R the intermediate vertices of XX, YX, YY, ZX, ZY, and ZZ. Color all other vertices with R. We define

$$\nu : \mathcal{S}(n) \longrightarrow \mathcal{C}''(n)$$

to be the bijection that first sequentially applies the following replacement rules to the steps of each path: for $x > 0$, $y > 0$, and $z > 0$,

$$\begin{array}{lcl} (x,0,0) & \longrightarrow & X(BX)^{x-1} \\ (0,y,0) & \longrightarrow & Y(BY)^{y-1} \\ (0,0,z) & \longrightarrow & Z(BZ)^{z-1} \\ (x,y,0) & \longrightarrow & Y(BY)^{y-1}(BX)^{x} \\ (x,0,z) & \longrightarrow & Z(BZ)^{z-1}(BX)^{x} \\ (0,y,z) & \longrightarrow & Z(BZ)^{z-1}(BY)^{y} \\ (x,y,z) & \longrightarrow & Z(BZ)^{z-1}(BY)^{y}(BX)^{x}, \end{array}$$

and then assigns color R to all non-B vertices on the resulting path. Here the exponents indicate multiple factors in a concatenation; the color B marks intermediate vertices. Since $|\mathcal{S}(n)| = |\mathcal{C}''(n)| = \sum_{P \in \mathcal{C}(3,n)} 2^{\Theta_{M^\star}(P)}$ the result follows. □

Proposition 11 *For any $d \ge 2$ and $n \ge 1$, $|\mathcal{S}(n)| = 2^{d+n-2} N_{d,n}(2)$.*

Proof. This proof for $d = 3$ can easily be generalized. Since $M^\star + M_A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $\Theta_{M^\star}(P) + ascs(P) = 3n - 1$. This fact and Corollary 1 show

$$\sum_{P \in \mathcal{C}(3,n)} 2^{\Theta_{M^\star}(P)} = 2^{n+1} \sum_{P \in \mathcal{C}(3,n)} 2^{2n-2-ascs(P)} = 2^{n+1} \sum_{P \in \mathcal{C}(3,n)} 2^{ascs(P)}.$$

Using the Lemma 10 completes the proof. □

Corollary 3 [A 2^{n-1} result.] For any $d \geq 2$ and $n \geq 1$, $|\mathcal{S}(n)| = 2^{n-1}|\mathcal{D}(n)|$.

Proof. This is a consequence of Propositions 10 and 11. □

Remarks:

3.1: We observe that $\mathcal{D}(n)$ is counted using the statistic *des* while $\mathcal{S}(n)$ is counted using the statistic *ascs* together with the reciprocity of the d-Narayana polynomial.

3.2: The classic "2^{n-1} result" is for $d = 1$: one can easily see that $|\mathcal{S}(n)| = 2^{n-1}|\mathcal{D}(n)| = 2^{n-1}$ (See [10, art. 123].) Our interest in such results, which relate paths using "super steps" (perhaps diagonal) to those using "short steps" (perhaps diagonal), originated from Stanley's exercise [19, ex. 6.16]. For $d = 2$ and $n \geq 1$, paper [21] gives a bijection showing that $|\mathcal{S}(n)| = 2^{n-1}|\mathcal{D}(n)| = 2^n N_{2,n}(2)$. Duchi and Sulanke [5] give a bijective proof indicating that for any d, $|\mathcal{S}(n)| = 2^{n-1}|\mathcal{D}(n)|$ is true when the constraint $0 \leq x_1 \leq x_2 \leq \ldots \leq x_d$ is absent. Remarkably, the formula of the "2^{n-1} result" is independent of d.

3.3: Our encoding of the paths of $\mathcal{D}(n)$ in the proof of Proposition 10 and paths of $\mathcal{S}(n)$ in the proof of Lemma 10 in terms of paths of $\mathcal{C}(3, n)$ with colored vertices is consistent with the encoding of such steps by MacMahon [10, sect. IV].

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