THE NUMBER OF PERMUTATIONS CONTAINING EXACTLY ONE INCREASING SUBSEQUENCE OF LENGTH THREE

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Abstract. It is proved that the number of permutations on $\{1, 2, \ldots, n\}$ with exactly one increasing subsequence of length 3 is $\frac{3}{n}\binom{2n}{n+3}$ $[0, 0, 1, 6, 27, 110, 429, \ldots$ (Sloane A3517)].

Given $\sigma \in S_n$, an *abc* subsequence is a set of three elements of a permutation, $\sigma(i)$, $\sigma(j)$, $\sigma(k)$ with $\sigma(i) < \sigma(j) < \sigma(k)$ and $i < j < k$. It is known[2,3,4] that the number of permutations on $\{1, 2, \ldots, n\}$ with no *abc* subsequences is given by the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. A natural question is: is there a nice expression for the generating function $\sum_{r=0}^{\binom{n}{3}} B(n, r)q^r$, where $B(n, r)$ is the number of permutations on $\{1, 2, \ldots, n\}$ with exactly r *abc*'s, for $1 \leq r \leq \binom{n}{3}$? Another question is, for a fixed r, what can one say about the sequence in n, $B(n, r)$. Doron Zeilberger conjectures that for any given r, the coefficients, $B(n, r)$ of the generating function are P-recursive in n, that is, they satisfy a linear recurrence with polynomial coefficients. This is supported by the fact that $B(n, 0)$, being closed form, satisfies a *first order* recurrence and hence, of course, is P-recursive.

It would be too much to hope for a closed form formula for $B(n, r)$ for general r, and a priori, there is no reason to hope that even $B(n, 1)$ is closed form. To our surprise $B(n, 1)$ did turn out to be closed form, and in this paper we present and prove such a formula. We hope to treat $B(n, r)$, for $r > 1$, in a subsequent paper.

Theorem. *The number of permutations on n objects that have exactly one abc subsequence is*

$$\frac{3}{n}\binom{2n}{n+3}. \tag{1}$$

As is the case with many results, in order to prove this, we must first look at a more general result. For $\sigma \in S_n$, let $\phi_k(\sigma) = |\{(i, j)\ ; \sigma(i) < \sigma(j) = k \text{ and } i < j\}|$. Let $\boldsymbol{P}_{(n,I)}$ denote the set of all $\sigma \in S_n$ with no *abc* subsequences and for which $\phi_j(\sigma) = 0$ for all $j \leq I$. Let $P(n, I) = |\boldsymbol{P}_{(n,I)}|$. The result that the number of permutations on n elements with no *abc* subsequences can be stated as $P(n, 1) = \frac{1}{n+1}\binom{2n}{n}$. Notice that $P(n, n) = 1$. Furthermore, from our definition of $\boldsymbol{P}_{(n,I)}$ it follows that $\boldsymbol{P}_{(n,0)} = \boldsymbol{P}_{(n,1)}$.

Lemma 1.

$$P(n, I) = \binom{2n - I - 1}{n - I} - \binom{2n - I - 1}{n - I - 2} \tag{2}$$

These are the famous ballot numbers, and the proof below can be easily bijectified. Erikson and Linusson[1] had a similar result. We will show that both sides of (2) satisfy the same recursion;

$$F(n, I) = F(n - 1, I - 1) + F(n, I + 1), \qquad \text{for } n > 0 \text{ and } I > 0, \tag{2'}$$

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with initial conditions

$$F(n,0) = F(n,1), \qquad \text{for } n > 0 \tag{$2''$}$$

and

$$F(n,n) = 1, \qquad \text{for } n > 0. \tag{$2'''$}$$

That the right hand side of (2) satisfies $(2'), (2'')$ and $(2''')$ is purely routine and is left to the reader. As a result of our definition, $\boldsymbol{P}_{(n,0)} = \boldsymbol{P}_{(n,1)}$. Furthermore, from the definition of $\boldsymbol{P}_{(n,I)}$, $\boldsymbol{P}_{(n,n)}$ is the set of permutations on $\{1, 2, \ldots, n\}$ with no *abc* subsequences and no non-inversions. There is only one such permutation, namely $[n, n-1, \ldots, 2, 1]$, hence $P(n,n) = 1$.

Separate the set $\boldsymbol{P}_{(n,I)}$ into two sets, K_1 and K_2. Let $K_1 := \{\sigma \in \boldsymbol{P}_{(n,I)}; \phi_{I+1}(\sigma) = 0\}$ and $K_2 := \{\sigma \in \boldsymbol{P}_{(n,I)}; \phi_{I+1}(\sigma) > 0\}$. The set K_1 is $\boldsymbol{P}_{(n,I+1)}$.

Sublemma 1.1. *If $\sigma \in K$ then $\sigma(n) = 1$ or $\sigma(n) = I + 1$.*

Let $\sigma \in K$. Assume $1 < \sigma(n) = j < I + 1$. We must have $\sigma(i) = 1$ for some $i < n$, Thus $\phi_j(\sigma) > 0$ contradicting our construction of K_2. Assume $\sigma(n) > I + 1$. By our construction of K_2, we know that $\phi_{I+1}(\sigma) > 0$. Let i and j be chosen so that $\sigma(i) < \sigma(j) = I + 1$ and $i < j$. Then $\sigma(i) < \sigma(j) < \sigma(n)$ and $i < j < n$. Hence σ has an *abc* subsequence contradicting our construction of K_2. □

Let $\sigma \in K_2$ and let $\sigma_1 \in S_{n-1}$ be defined by

$$\sigma_1(i) = \begin{cases} \sigma(i) - 1, & \text{if } \sigma(n) = 1 \\ \sigma(i), & \text{if } \sigma(n) = I + 1 \text{ and } \sigma(i) < I + 1 \\ \sigma(i) - 1, & \text{if } \sigma(n) = I + 1 \text{ and } \sigma(i) > I + 1 \end{cases}.$$

Notice that σ_1 has no *abc* subsequences and $\phi_j(\sigma_1) = 0$ for $j \leq I - 1$. Let $\psi : K_2 \to \boldsymbol{P}_{(n-1,I-1)}$ be defined by $\psi(\sigma) = \sigma_1$. We will show that ψ is a bijection between K_2 and $\boldsymbol{P}_{(n-1,I-1)}$.

First we prove that ψ is one-to-one. Suppose σ, $\pi \in K$ such that $\psi(\sigma) = \psi(\pi)$ and $\sigma \neq \pi$. Then $\sigma(n) \neq \pi(n)$. From sublemma 1.1, we must have $\sigma(n), \pi(n) \in \{1, I+1\}$. Without loss of generality we may assume that $\sigma(n) = 1$ and $\pi(n) = I+1$. Let $\sigma_1 = \psi(\sigma)$ and $\pi_1 = \psi(\pi)$. Since $\sigma \in K$, $\sigma(i) < \sigma(j) = I + 1$ for some $i < j$. It follows that $\sigma_1(i) < \sigma_1(j) = I$. Now $\pi_1 = \sigma_1$ and so $\pi_1(i) < \pi_1(j) = I$. It follows that $\pi(i) < \pi(j) = I < \pi(n) = I + 1$ and $i < j < n$, contradicting our assumption that π has no *abc* subsequences. Therefore $\pi = \sigma$ and ψ is one-to-one.

Now we prove that ψ is onto. Suppose $\sigma_1 \in \boldsymbol{P}_{(n-1,I-1)}$. If $\phi_I(\sigma_1) > 0$ then let σ be defined as

$$\sigma(i) = \begin{cases} \sigma_1(i) + 1, & \text{if } i \neq n \\ 1, & \text{if } i = n \end{cases}.$$

If $\phi_I(\sigma_1) = 0$ then let σ be defined as

$$\sigma(i) = \begin{cases} \sigma_1(i), & \text{if } \sigma_1(i) \leq I \\ \sigma_1(i) + 1, & \text{if } \sigma_1(i) > I \\ I + 1, & \text{if } i = n \end{cases}.$$

In both cases notice that σ has no *abc* subsequences and that $\phi_j(\sigma) = 0$ for $j \leq I$. So $\sigma \in K_2$ and $\psi(\sigma) = \sigma_1$. Therefore ψ is onto and a bijection. We have $|\boldsymbol{P}_{(n,I)}| = |\boldsymbol{P}_{(n,I+1)}| + |\boldsymbol{P}_{(n-1,I-1)}|$, so $P(n,I) = P(n,I+1) + P(n-1,I-1)$. □

Notice that when $I = 1$, using (2) for $P(n, I)$, we rederive the above-mentioned result that the number of permutations with no abc's is

$$P(n,1)=\binom{2n-2}{n-1}-\binom{2n-2}{n-3}=\frac{(2n-2)![n(n+1)-(n-2)(n-1)]}{(n-1)!(n+1)!}=\frac{(2n)!}{n!(n+1)!}=C_n.$$

Let $\boldsymbol{P}^{(1)}_{(n,I)} = \{\sigma \in S_n; \sigma$ has no abc subsequences and $\phi_j(\sigma) = 0$ for $j \leq I\}$. Let $P^{(1)}(n, I) = |\boldsymbol{P}^{(1)}_{(n,I)}|$. Thus $P^{(1)}(n, 1)$ is the number of permutations on $\{1, 2, \ldots, n\}$ with exactly one abc subsequence. Notice that $P^{(1)}(n, n-1) = 0$ for all n, and $P^{(1)}(3, 1) = 1$.

Lemma 2. $P^{(1)}(n, I)=$

$$\binom{2n-I-1}{n}-\binom{2n-I-1}{n+3}+\binom{2n-2I-2}{n-I-4}-\binom{2n-2I-2}{n-I-1}+\binom{2n-2I-3}{n-I-4}-\binom{2n-2I-3}{n-I-2} \qquad (3)$$

To prove this, we prove that both sides of this equation satisfy the recursion

$$F(n, I) = F(n-1, I-1) + F(n, I+1) + P(n-I, 2), \qquad \text{for } n > 0 \text{ and } I > 0, \qquad (3')$$

where $P(n - I, 2)$ is as defined above and with the initial conditions

$$F(n, 0) = F(n, 1), \qquad \text{for } n > 0 \qquad (3'')$$

and

$$F(n, n-2) = n-2, \qquad \text{for } n > 0. \qquad (3''')$$

That the right hand side of (3) satisfies $(3')$, $(3'')$, and $(3''')$ is routine. As a result of our definition, $\boldsymbol{P}^{(1)}_{(n,0)} = \boldsymbol{P}^{(1)}_{(n,1)}$ and so $P^{(1)}(n, 0) = P^{(1)}(n, 1)$. We can easily compute $P^{(1)}(n, n-2)$. If $\sigma \in \boldsymbol{P}^{(1)}_{(n,n-2)}$ then $\phi_j(\sigma) = 0$ for $j \leq n-2$ and σ has exactly 1 abc subsequence. Thus, σ is of the form $[n-2, n-1, n-3, \ldots, n-i, n, n-i-1, \ldots, 2, 1]$. There are exactly $n-2$ such permutations, hence $P^{(1)}(n, n-2) = n-2$. So we see that $P^{(1)}(n, I)$ satisfies $(3'')$ and $(3''')$.

We prove $P^{(1)}(n, I)$ satisfies $(3')$ by separating the set $\boldsymbol{P}^{(1)}_{(n,I)}$ into three sets K_1, K_2, and K_3. Let $K_1 = \{\sigma \in \boldsymbol{P}^{(1)}_{(n,I)}; \phi_{I+1}(\sigma) = 0\}$, $K_2 = \{\sigma \in \boldsymbol{P}^{(1)}_{(n,I)}; \phi_{I+1}(\sigma) > 0$ and $\sigma(n)$ participates in the abc subsequence$\}$, and $K_3 = \{\sigma \in \boldsymbol{P}^{(1)}_{(n,I)}; \phi_{I+1}(\sigma) > 0$ and $\sigma(n)$ does not participate in the abc subsequence$\}$. The first set is $\boldsymbol{P}^{(1)}_{(n,I+1)}$.

We must show that $|K_2| = |\boldsymbol{P}^{(1)}_{(n-1,I-1)}|$ and $|K_3| = |\boldsymbol{P}_{(n-I,2)}|$.

Sublemma 2.1. *If $\sigma \in K_2$ then $\sigma(n) \in \{1, I+1\}$.*

Let $\sigma \in K_2$. If $1 < \sigma(n) = j < I+1$ then $\sigma(i) = 1$ for some $i < n$, but then $\phi_j(\sigma) > 0$ contradicting our construction of K_2. If $\sigma(n) > I+1$ then by our construction of K_2, we know that $\phi_{I+1}(\sigma) > 0$. Let i and j be chosen so that $\sigma(i) < \sigma(j) = I+1$ and $i < j$. Then $\sigma(i) < \sigma(j) < \sigma(n)$ and $i < j < n$. Hence $\sigma(n)$ participates in an abc subsequence which contradicts our construction of K_2. □

Let $\sigma \in K_2$ and let $\sigma_1 \in S_{n-1}$ be defined by

$$\sigma 1(i) = \begin{cases} \sigma(i) - 1, & \text{if } \sigma(n) = 1 \\ \sigma(i), & \text{if } \sigma(n) = I+1 \text{ and } \sigma(i) < I+1 \\ \sigma(i) - 1, & \text{if } \sigma(n) = I+1 \text{ and } \sigma(i) > I+1 \end{cases}.$$

Notice that σ_1 has precisely one abc subsequences and $\phi_j(\sigma_1) = 0$ for $j \leq I - 1$. Let $\psi : K_2 \to \boldsymbol{P}_{(n-1,I-1)}$ be defined by $\psi(\sigma) = \sigma_1$. First we prove that ψ is one-to-one. Suppose there exists $\sigma, \pi \in K_2$ such that $\psi(\sigma) = \psi(\pi)$ and $\sigma \neq \pi$. Let $\sigma_1 = \psi(\sigma)$ and $\pi_1 = \psi(\pi)$. We must have $\sigma(n) \neq \pi(n)$. By sublemma 2.1, $\sigma(n)$ and $\pi(n)$ are in $\{1, I+1\}$. Without loss of generality we may assume that $\sigma(n) = 1$ and $\pi(n) = I + 1$. If $\sigma \in K_2$ then $\sigma(i) < \sigma(j) = I + 1$ for some $i < j < n$. It follows that $\sigma_1(i) < \sigma_1(j) = I$. Thus $\pi_1(i) < \pi_1(j) = I$. But then $\pi(i) < \pi(j) < \pi(n) = I + 1$ which contradicts our construction of K_2. Therefore ψ is one-to-one.

Now we prove that ψ is onto. Suppose $\sigma_1 \in \boldsymbol{P}^{(1)}_{(n-1,I-1)}$. If $\phi_I(\sigma_1) > 0$ then let $\sigma \in S_n$ be defined by

$$\sigma(i) = \begin{cases} \sigma_1(i) + 1, & \text{if } 1 \leq i < n \\ 1, & \text{if } i = n \end{cases}.$$

If $\phi_I(\sigma_1) = 0$ then let $\sigma \in S_n$ defined by

$$\sigma(i) = \begin{cases} \sigma_1(i), & \text{if } \sigma_1(i) < I + 1 \\ \sigma_1(i) + 1, & \text{if } \sigma_1(i) \geq I + 1 \\ I + 1, & \text{if } i = n \end{cases}.$$

In either case, it follows that σ has exactly one abc subsequence, $\phi_{I+1}(\sigma) > 0$, and $\sigma(n)$ does not participate in its abc subsequence. So $\sigma \in K_2$ and $\psi(\sigma) = \sigma_1$. Therefore ψ is onto and a bijection and $|K_2| = |\boldsymbol{P}^{(1)}_{(n-1,I-1)}|$.

Finally, we must construct a bijection between $\boldsymbol{P}_{(n-I,2)}$ and K_3. Let $\sigma \in \boldsymbol{P}_{(n-I,2)}$. Let k be chosen so that $\sigma(k) = 1$. If $\sigma(k-1) \neq 2$ then let $\sigma_1 \in S_n$ be defined by

$$\sigma_1 = \begin{cases} I + \sigma(i), & \text{if } i < k - 1 \text{ or } k < i < n - I \\ I, & \text{if } i = k - 1 \\ I + \sigma(i-1), & \text{if } i = k \\ I + 1, & \text{if } i = n - I \\ n - i, & \text{if } k < i < n \\ I + \sigma(n - I), & \text{if } i = n \end{cases}.$$

If $\sigma(k-1) = 2$ then let $\sigma_1 \in S_n$ be defined by

$$\sigma_1 = \begin{cases} I + \sigma(i), & \text{if } i < k - 1 \\ I, & \text{if } i = k - 1 \\ I + \sigma(i+1), & \text{if } k - 1 < i < n - I \\ I + 1, & \text{if } i = n - I \\ n - i, & \text{if } k < i < n \\ I + 2, & \text{if } i = n \end{cases}.$$

Notice that if $\sigma \in \boldsymbol{P}_{(n-I,2)}$ then $\sigma_1 \in K_3$. Indeed by the way we constructed it, $\phi_j(\sigma_1) = 0$ for $j \leq I$. Furthermore, $\phi_{I+1}(\sigma_1) > 0$, and σ_1 has exactly one abc subsequence, consisting of I, $I + 1$, and the last element of $\varphi(\pi)$.

Let $\varphi : \boldsymbol{P}_{(n-I,2)} \to K_3$ be define as $\varphi(\sigma) = \sigma_1$. We will prove that φ is a bijection.

First we prove φ is one-to-one. Suppose $\pi, \sigma \in \boldsymbol{P}_{(n-I,2)}$ and $\varphi(\pi) = \varphi(\sigma)$. Let

$$\sigma = [\gamma_1, \gamma_2, \ldots, \gamma_{k_1}, \eta_1, 1, \eta_2, \eta_3, \ldots, \eta_{m_1}], \text{ and}$$
$$\pi = [\alpha_1, \alpha_2, \ldots, \alpha_{k_2}, \beta_1, 1, \beta_2, \beta_3, \ldots, \beta_{m_2}].$$

Then by the position of I in $\varphi(\sigma)$ and $\varphi(\pi)$, we may conclude that $m_1 = m_2$ and $k_1 = k_2$. Next we note that the last element of $\varphi(\sigma)$ must be the same as the last element of $\varphi(\pi)$, and so either $\beta_1 = \eta_1 = 2$ or $\beta_m = \eta_m$. Similarly, we may conclude that $\beta_i = \eta_i$ for $1 \le i \le m_1 = m_2$ and $\alpha_i = \gamma_i$ for $1 \le i \le k_1 = k_2$. Thus $\sigma = \pi$ and φ is one-to-one.

Now we prove that φ is onto. Suppose $\sigma = [I+\alpha_1, I+\alpha_2, \ldots, I+\alpha_k, I, I+\beta_1, I+\beta_2, \ldots, I+\beta_m, I+1, I-1, I-2, \ldots, 2, 1, I+j]$, where $j \ne 2$. It is easy to see that $\sigma_1 = [\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, 1, \beta_2, \ldots, \beta_m, j] \in \boldsymbol{P}_{(n-I,2)}$ and $\varphi(\sigma_1) = \sigma$. If $\sigma = [I+\alpha_1, I+\alpha_2, \ldots, I+\alpha_k, I, I+\beta_1, I+\beta_2, \ldots, I+\beta_m, I+1, I-1, I-2, \ldots, 2, 1, I+2]$ then $\sigma_1 = [\alpha_1, \alpha_2, \ldots, \alpha_k, 2, 1, \beta_1, \beta_2, \ldots, \beta_m,] \in \boldsymbol{P}_{(n-I,2)}$ and $\varphi(\sigma_1) = \sigma$. Therefore φ is onto and a bijection and $|K_3| = |P(n-I,2)|$.

We have $|\boldsymbol{P}^{(1)}_{(n,I)} = |\boldsymbol{P}^{(1)}_{(n,I+1)}| + |K_2| + |K_3| = |\boldsymbol{P}^{(1)}_{(n,I+1)}| + |\boldsymbol{P}^{(1)}_{(n-1,I-1)}| + |\boldsymbol{P}_{(n-I,2)}|$. Therefore $P^{(1)}(n,I) = P^{(1)}(n,I+1) + P^{(1)}(n-1,I-1) + P(n-I,2)$. □

From the definition of $P^{(1)}(n,I)$, we see that $P^{(1)}(n,1)$ is the number of permutations on n objects with exactly one *abc* subsequence and no other restrictions. Using (3) with $I=1$, we have

$$P^{(1)}(n,1) = \binom{2n-2}{n} - \binom{2n-2}{n+3} + \binom{2n-4}{n-5} - \binom{2n-4}{n-2} + \binom{2n-5}{n-5} - \binom{2n-5}{n-3} = \frac{3}{n}\binom{2n}{n+3}.$$

□

We observe that $P^{(1)}(n,1) = P(n+2,5)$, so the number of permutations on $\{1,2,\ldots,n\}$ with exactly 1 *abc* equals the number of permutations, σ, on $\{1,2,\ldots,n+3\}$ with no *abc*'s and with $\phi_j(\sigma) = 0$ for $j \le 6$. Doron Zeilberger offers 25 dollars for a *nice* bijective proof.

Note: A small Maple package, 1abc.maple, accompanying this paper, can be obtained using your favorite world wide web browser at `http://www.math.temple.edu/~noonan` or anonymous ftp to `ftp.math.temple.edu`, directory `/pub/noonan`.

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