SUMMA SUMMARUM

Mogens Esrom Larsen

Institut for Matematiske Fag
Matematisk Afdeling
Københavns Universitet

April 11, 2006

Dedicated to the memory of
Erik Sparre Andersen (1919–2003).

PREFACE

You never know when you will encounter a binomial coefficient sum.
Doron Zeilberger [120].

This textbook aims to provide a "summa" i.e. a collection of "all" known algebraic finite sums and a guide to find the sum you need. Of course, the hope is to find this summa on your desk – like Thomas' original was found on the altar!

The kinds of sums we consider are often presented as sums of products of binomial coefficients, rational functions and occasionally harmonic numbers. Closed forms of such sums are usually called "combinatorial identities" though this term ought to include identities between sums, double sums and other equations too.

Our favorite tool is to recognize the identity as some sum already known. To do so we need to write the sum in a standard form and to advice a simple way to change a given sum to this standard form. The standard has to be unique, hence we cannot use more than one binomial coefficient, but must replace the others with factorials. And the standard form must apply to the whole variety of sums considered, so we cannot use the hypergeometric form which does not apply to the formulas in chapters 17 and 18.

The ideal formula has the form – as does the fundamental theorem of algebra – a sum of products equals a product of sums. But we have to weaken this demand; we want to write a sum of products as a sum with as few terms as possible of products of sums. The ideal is one term, but if such expressions simply fail to exist, sums of two or three similar terms may help.

If the wanted sum is unknown, we provide the reader with a variety of tools to attack the problem. We explain the famous algorithms of Gosper and Zeilberger, and to assist the latter we include some advice to solve difference equations with rational coefficients.

In order to classify the formulas we introduce a standard form which deviates from the recently popular forms. We dislike the hypergeometric form for two reasons, it conceals the important properties: symmetric, balanced and well–balanced, and it is insufficient in relation to the sums due to Abel, Cauchy, Hagen–Rothe and Jensen as well as sums containing harmonic numbers. And we dislike the formulations with extensive use of binomial coefficients because they are not uniquely defined so it seems difficult to prefer one form to another.

We prefer to use descending factorials and have found it convenient also to have step lengths different from 1. Hence we introduce the notation, cf. chapter 1, formula 1.6,

$$[x,d]_n = x(x-d)\cdots(x-(n-1)d)$$

which allows powers and ascending factorials to be written as $[x,0]_n$ and $[x,-1]_n$ respectively. In several cases it has been natural to consider $d=2$ too. This is a true generalization to be preferred to the analogy between powers and factorials usually presented.

We have added an appendix on the most elementary generalizations to a q–basic form with their proofs.

I am deeply indepted to my teacher of this field and dear friend, the late professor, dr. phil. Erik Sparre Andersen.

Furthermore, I want to thank my collegue Professor Jørn Børling Olsson for improving my language.

Mogens Esrom Larsen

Mogens Esrom Larsen

CONTENTS

	PREFACE	5
	CONTENTS	7
1	NOTATION	11
2	ELEMENTARY PROPERTIES	13
	Factorial formulas	13
	Binomial coefficient formulas	13
	Antidifferences	13
3	POLYNOMIALS	16
	Geometric progression	16
	Sums of polynomials	16
	Bernoulli polynomials	18
	Stirling numbers	23
	Renate Golombek's problem	25
4	LINEAR DIFFERENCE EQUATIONS	29
	General linear difference equations	29
	The homogeneous equation	29
	First order inhomogeneous equations	31
	First order equations with constant coefficients	31
	Arbitrary order equations with constant coefficients	32
	Systems of equations with constant coefficients	33
	Proof of the Cayley–Hamilton theorem	35
	Generating functions	36
	Kwang–Wu Chen's problem	37
	Equations with polynomial coefficients	38
	Shalosh B. Ekhad's squares	40
	David Doster's problem	40
	Ira Gessel's problem	42
	Emre Alkan's problem	43
5	CLASSIFICATION OF SUMS	46
	Introduction	46
	Classification	46
	Canonical forms of sums of types I–II	47
	Sums of arbitrary limits	49
	Hypergeometric form	50
	The classification recipe	51
	Symmetric and balanced sums	51
	Useful transformation	53
	Polynomial factors	55
6	GOSPER'S ALGORITHM	55
	Gosper's algorithm	55
	An example of Gosper's algorithm	58

7	SUMS OF TYPE II$(1,1,z)$	61
	The binomial theorem	61
	Galperin's and Gauchman's problem	62
8	SUMS OF TYPE II$(2,2,z)$	64
	The Chu–Vandermonde convolution	64
	A simple example	67
	The Laguerre Polynomials	67
	Moriarty's formulas	68
	An example of matrices as arguments	70
	Joseph M. Santmyer's problem	73
	The number of parenthesis	74
	An indefinite sum of type II$(2,2,1)$	75
	Transformations of sums of type II$(2,2,z)$	75
	Kummer's, Gauß' and Bailey's formulas	77
	The factor $\frac{1}{2}+i\frac{\sqrt{3}}{2}$	83
	Sums of types II$(2,2,z)$	84
	The general difference equation	91
9	SUMS OF TYPE II$(3,3,z)$	92
	The Pfaff–Saalschütz and Dixon formulas	92
	Transformations of sums of type II$(3,3,1)$	92
	Generalizations of Dixon's formulas	97
	The balanced and quasi–balanced Dixon identities	99
	Watson's formulas and their contiguous companions	102
	Whipple's formulas and their contiguous companions	103
	Ma Xin–Rong and Wang Tian–Ming's problem	104
	Comment	107
	C. C. Grosjean's problem	108
	Peter Larcombe's problem	110
10	SUMS OF TYPE II$(4,4,\pm 1)$	111
	Introduction	111
	Sum formulas for $z=1$	111
	Sum formulas for $z=-1$	116
11	SUMS OF TYPE II$(5,5,1)$	120
	Indefinite sums	120
	Symmetric and balanced sums	120
12	SUMS OF TYPE II$(6,6,\pm 1)$	126
13	SUMS OF TYPE II$(7,7,1)$	128
14	SUMS OF TYPE II$(8,8,1)$	134
15	SUMS OF TYPES II(p,p,z)	137
16	ZEILBERGER'S ALGORITHM	135
	Zeilberger's algorithm	137
	A simple example of Zeilberger's algorithm	139

	A less simple example of Zeilberger's algorithm	140
	Sporadic formulas of types $II(2,2,z)$	142
	The factor 2	142
	The factor 4	142
	The factor 5	143
	The factor 9	143
	Sporadic formulas of types $II(4,4,z)$	145
17	SUMS OF TYPE III–IV	148
	The Abel, Hagen–Rothe, Cauchy and Jensen formulas	148
	A Polynomial Identity	154
	Joseph Sinyor and Ted Speevak's problem	156
	Changing the problem to a polynomial identity	157
	Rewriting the double sum as two single sums	158
	The integral zeros	159
	The evaluation of the polynomial	159
	The non–integral zeros	160
18	SUMS OF TYPE V, HARMONIC SUMS	167
	Harmonic sums	167
	Harmonic sums of power $m = 1$	167
	Harmonic sums of power $m > 1$	174
	Seung–Jin Bang's problem	175
	The Larcombe identities	177
19	APPENDIX ON INDEFINITE SUMS	181
	Rational functions	181
	Other indefinite sums	182
20	APPENDIX ON BASIC IDENTITIES	183
	Introduction	183
	Factorials and binomial coefficients	184
	Two basic binomial theorems	185
	Two basic Chu–Vandermonde convolutions	186
	Two special cases of the basic Chu–Vandermonde convolutions	187
	The symmetric Kummer identity	188
	The quasi–symmetric Kummer identity	190
	The balanced Kummer identity	191
	The quasi–balanced Kummer identity	194
	A basic transformation of a II$(2,2,z)$ sum	199
	A basic Gauß' theorem	201
	A basic Bailey's theorem	201
	Notes	202
	INDEX	204
	REFERENCES	206

CHAPTER 1. NOTATION

$\mathbb{N}$ The natural numbers, $1, 2, 3, \cdots$

$\mathbb{Z}$ The integers, $\cdots, -2, -1, 0, 1, 2, 3, \cdots$

$\mathbb{R}$ The real numbers.

$\mathbb{C}$ The complex numbers, $z = x + iy$.

For the **nearest integers** to a real number, $x \in \mathbb{R}$, we use the standard notation for the **ceiling** and the **floor**

$$\lceil x \rceil := \min\{n \in \mathbb{Z} | n \geq x\} \tag{1.1}$$

$$\lfloor x \rfloor := \max\{n \in \mathbb{Z} | n \leq x\} \tag{1.2}$$

and the **sign,** $\sigma(x)$, of a real number, $x \in \mathbb{R}$, is defined by

$$\sigma(x) := \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} \tag{1.3}$$

Furthermore, we denote the **maximum** and **minimum** of two numbers, $x, y \in \mathbb{R}$, as

$$x \vee y := \max\{x, y\} \tag{1.4}$$

$$x \wedge y := \min\{x, y\} \tag{1.5}$$

The **factorial** $[x, d]_n$ is defined for any **number,** $x \in \mathbb{C}$, any **stepsize,** $d \in \mathbb{C}$, and any **length,** $n \in \mathbb{Z}$, except for $-x \in \{d, 2d, \cdots, -nd\}$, by

$$[x, d]_n := \begin{cases} \prod_{j=0}^{n-1} (x - jd) & n \in \mathbb{N} \\ 1 & n = 0 \\ \prod_{j=1}^{-n} \frac{1}{x + jd} & -n \in \mathbb{N},\ -x \notin \{d, 2d, \cdots, -nd\} \end{cases} \tag{1.6}$$

As special cases we remark, that

$$[x, 0]_n = x^n \quad n \in \mathbb{Z} \tag{1.7}$$

and furthermore, we want to apply the shorthands

$$[x]_n := [x, 1]_n \qquad \big(x(x-1)\cdots(x-n+1) \text{ for } n > 0\big) \tag{1.8}$$

$$(x)_n := [x, -1]_n \qquad \big(x(x+1)\cdots(x+n-1) \text{ for } n > 0\big) \tag{1.9}$$

The **binomial coefficients,** $\binom{x}{n}$, are defined for $x \in \mathbb{C}$ and $n \in \mathbb{Z}$ by

$$\binom{x}{n} := \begin{cases} \frac{[x]_n}{[n]_n} & \text{for } n \in \mathbb{N}_0 \\ 0 & \text{for } -n \in \mathbb{N} \end{cases} \tag{1.10}$$

The **identity operator** is defined for $f : \mathbb{Z} \to \mathbb{C}$ as

$$\mathbf{I}(f)(k) := f(k) \tag{1.11}$$

and the **shift operator** is defined as

$$\mathbf{E}(f)(k) := f(k+1) \tag{1.12}$$

from which two we get the **difference operator,** Δ, defined by the expressions

$$\Delta := \mathbf{E} - \mathbf{I} \tag{1.13}$$

$$\Delta f(k) := f(k+1) - f(k)\,. \tag{1.14}$$

An **indefinite sum** or **anti–difference** of a function, $g(k)$, is defined as any solution, $f(k)$, to the equation

$$g(k) = \Delta f(k)\,. \tag{1.15}$$

(The anti–difference $f(k)$ is uniquely determined up to a constant (or a periodic function with period 1) by the function $g(k)$.)

We denote this indefinite sum by

$$\sum g(k)\delta k := f(k) \tag{1.16}$$

The **definite sum** of a function, $g(k)$, is defined for any indefinite sum (1.16)

$$\sum\nolimits_a^b g(k)\delta k := f(b) - f(a) \tag{1.17}$$

and we remark that the connection to the usual step–by–step sum is

$$\sum\nolimits_a^b g(k)\delta k = \sum_{k=a}^{b-1} g(k) \tag{1.18}$$

The **harmonic** numbers are defined by

$$H_n := \sum_{k=1}^{n} \frac{1}{k} = \sum\nolimits_0^n [k]_{-1}\delta k \tag{1.19}$$

The **generalized harmonic** numbers are for $n, m \in \mathbb{N}$ and $c \in \mathbb{C}$ defined by

$$H_{c,n}^{(m)} := \sum_{k=1}^{n} \frac{1}{(c+k)^m} \tag{1.20}$$

CHAPTER 2. ELEMENTARY PROPERTIES

Factorial formulas. The factorial satisfies some obvious, but very useful rules of computation. The most important ones are

$$[x,d]_k = [-x+(k-1)d,d]_k(-1)^k \qquad x,d\in\mathbb{C},\quad k\in\mathbb{Z} \tag{2.1}$$

$$[x,d]_k = [x,d]_h[x-hd,d]_{k-h} \qquad x,d\in\mathbb{C},\quad k,h\in\mathbb{Z} \tag{2.2}$$

$$[x,d]_k = 1/[x-kd,d]_{-k} \qquad x,d\in\mathbb{C},\quad k\in\mathbb{Z} \tag{2.3}$$

$$[x,d]_k = [x-d,d]_k + kd[x-d,d]_{k-1} \qquad x,d\in\mathbb{C},\quad k\in\mathbb{Z} \tag{2.4}$$

$$[xd,d]_k = d^k[x]_k \qquad x,d\in\mathbb{C},\quad k\in\mathbb{Z} \tag{2.5}$$

Applying the difference operator, (1.13), we may rewrite (2.4) for $d=1$ as

$$\Delta[k]_n = n[k]_{n-1}\,. \tag{2.6}$$

For sums this formula gives us by (1.17)

$$\sum\nolimits_0^m [k]_n\delta k = \begin{cases} \frac{[m]_{n+1}}{n+1} - \frac{[0]_{n+1}}{n+1} & \text{for } n\neq -1 \\ H_m & \text{for } n=-1 \end{cases} \tag{2.7}$$

using the harmonic numbers (1.19).

Binomial coefficient formulas. Similarly, the binomial coefficients (1.10) satisfy a series of rules.

$$\binom{x}{k} = \binom{x-1}{k-1} + \binom{x-1}{k} \qquad x\in\mathbb{C},\quad k\in\mathbb{N}_0 \tag{2.8}$$

$$\binom{x}{k}[k]_m = [x]_m\binom{x-m}{k-m} \qquad x\in\mathbb{C},\quad m\in\mathbb{Z},\quad k\in\mathbb{N}_0 \tag{2.9}$$

$$\binom{x}{k}\binom{k}{m} = \binom{x}{m}\binom{x-m}{k-m} \qquad x\in\mathbb{C},\quad k,m\in\mathbb{Z} \tag{2.10}$$

$$\binom{x}{k}[y]_k = [x]_k\binom{y}{k} \qquad x,y\in\mathbb{C},\quad k\in\mathbb{Z} \tag{2.11}$$

$$\binom{x}{k} = (-1)^k\binom{k-x-1}{k} \qquad x\in\mathbb{C},\quad k\in\mathbb{Z} \tag{2.12}$$

$$\binom{m}{k} = \binom{m}{m-k} \qquad m\in\mathbb{N}_0,\quad k\in\mathbb{Z} \tag{2.13}$$

Antidifferences. Applying the difference operator, (1.13), we may rewrite (2.8) as

$$\Delta\binom{k}{n} = \binom{k}{n-1} \tag{2.14}$$

equivalent to

$$\sum \binom{k}{n} \delta k = \binom{k}{n+1} \tag{2.15}$$

If we introduce the alternating sign, (2.8) may be written

$$\Delta (-1)^k \binom{x}{k} = (-1)^{k+1} \binom{x+1}{k+1} \tag{2.16}$$

which is equivalent to

$$\sum (-1)^k \binom{x}{k} \delta k = (-1)^{k+1} \binom{x-1}{k-1} \tag{2.17}$$

We omit the proofs, since the formulas (2.8–2.13) are easily proved using (1.10) and the elementary properties (2.1–2.5) of the factorials.

The binomial coefficient itself has a less "nice" difference, but we may note that

$$\Delta \binom{x-1}{k-1} = \binom{x}{k} \left(1 - 2\tfrac{k}{x}\right) \tag{2.18}$$

Proof. We have

$$\begin{aligned} \Delta \binom{x-1}{k-1} &= \binom{x-1}{k} - \binom{x-1}{k-1} = \frac{[x-1]_k - k[x-1]_{k-1}}{[k]_k} = \\ &= \frac{[x-1]_{k-1}}{[k]_k} (x - 1 - k + 1 - k) = \frac{[x]_k}{[k]_k} \frac{1}{x} (x - 2k) = \\ &= \binom{x}{k} \left(1 - 2\tfrac{k}{x}\right) \end{aligned}$$

using (2.4). □

The difference and shift operators in (1.13) and (1.12) commute and satisfy

$$\Delta(fg) = f\Delta g + \mathbf{E} g \Delta f \tag{2.19}$$

and

$$\mathbf{E}k - k\mathbf{E} = \mathbf{E} \tag{2.20}$$

$$\Delta k - k\Delta = \mathbf{E} \tag{2.21}$$

where k is the operator $(k \to f(k)) \to (k \to kf(k))$. This just means that $(k+1)f(k+1) - kf(k+1) = f(k+1)$, etc.

The definite sum is by (1.17)

$$\sum_{k=0}^{n} g(k) = \sum\nolimits_0^{n+1} g(k)\delta k \tag{2.22}$$

Summation by parts is by (2.19) as

$$\sum f(k)\Delta g(k)\delta k = f(k)g(k) - \sum \mathbf{E}g(k)\Delta f(k)\delta k \tag{2.23}$$

$$\sum\nolimits_a^b f(k)\Delta g(k)\delta k = f(b)g(b) - f(a)g(a) - \sum\nolimits_a^b g(k+1)\Delta f(k)\delta k \tag{2.24}$$

and is also called **Abelian summation**.

As we have the identities

$$\Delta = \mathbf{E} - \mathbf{I} \tag{2.25}$$

$$\mathbf{E} = \Delta + \mathbf{I} \tag{2.26}$$

we get for $n \in \mathbb{N}$ the obvious relations between their iterations

$$\Delta^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \mathbf{E}^{n-k} \tag{2.27}$$

$$\mathbf{E}^n = \sum_{k=0}^{n} \binom{n}{k} \Delta^k \tag{2.28}$$

Inversion means that the following formulas are equivalent:

$$g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k) \iff f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} g(k) \tag{2.29}$$

Proof. If f is independent of the limit n we get using (2.10)

(2.30)

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} g(k) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(j) =$$

$$\sum_{j=0}^{n} \sum_{k=j}^{n} (-1)^k \binom{n}{k} (-1)^j \binom{k}{j} f(j) = \sum_{j=0}^{n} f(j) \binom{n}{j} \sum_{k=j}^{n} (-1)^{k-j} \binom{n-j}{k-j} =$$

$$\sum_{j=0}^{n} f(j) \binom{n}{j} 0^{n-j} = f(n)$$

If f depends on k the proof doesn't work. □

CHAPTER 3. POLYNOMIALS

Geometric progression.

We have for $q \neq 1$

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} \tag{3.1}$$

or more generally

$$\sum_{k=m}^{n} q^k = q^m \frac{1-q^{n-m+1}}{1-q} \tag{3.2}$$

Of course, for $q = 1$ we get

$$\sum_{k=0}^{n} q^k = \sum_{k=0}^{n} 1 = n+1 \tag{3.3}$$

Proof. We have $\Delta q^k = q^{k+1} - q^k = (q-1)q^k$. So according to (1.18) and (1.17) we get (3.2)

$$\sum_{k=m}^{n} q^k = \sum\nolimits_m^{n+1} q^k \delta k = \frac{q^{n+1}}{q-1} - \frac{q^m}{q-1} = \frac{q^{n+1}-q^m}{q-1} = q^m \frac{1-q^{n-m+1}}{1-q}$$

$\square$

Sums of polynomials.

According to an anecdote, C. F. Gauß (1777–1855) when he went to school in the age of 7, he annoyed his teacher by counting very fast. To get a rest the teacher asked him to add the numbers from 1 to 100. Gauß immediately answered 5050.

He added the numbers one by one backwards to obtain the sum 101 for each pair of 100 pairs and obtained the double of the sum. It could be written like this

$$\sum_{k=1}^{100} k = \sum_{k=1}^{100} (101-k) = \frac{1}{2}\sum_{k=1}^{100}(k+101-k) = \frac{1}{2}\sum_{k=1}^{100} 101 = \frac{1}{2} 100 \times 101 = 5050$$

We have learned something, the advantage of changing the order of summation and the formula

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{3.4}$$

If we have a polynomial, $p(k) = a_0 + a_1 k + \cdots + a_m k^m$, and want to determinate $\sum_{k=0}^{n} p(k)$, we need to know the sums $\sum_{k=0}^{n} k^m$ for all $m \geq 0$. We found the formula for $m = 1$ above. The formulas for small values of m was determinded in 1631 by J. Faulhaber (1580–1635), [82]:

Polynomials to be evaluated with $x = n(n+1)$.

$$\sum_{k=1}^{n} k^0 = n \tag{3.5}$$

$$\sum_{k=1}^{n} k^1 = \frac{x}{2} \tag{3.6}$$

$$\sum_{k=1}^{n} k^2 = \frac{x(2n+1)}{2\cdot 3} \tag{3.7}$$

$$\sum_{k=1}^{n} k^3 = \frac{x^2}{4} \tag{3.8}$$

$$\sum_{k=1}^{n} k^4 = \frac{xp_4(x)(2n+1)}{2\cdot 3\cdot 5} \qquad p_4(x) = 3x - 1 \tag{3.9}$$

$$\sum_{k=1}^{n} k^5 = \frac{x^2p_5(x)}{2\cdot 6} \qquad p_5(x) = 2x - 1 \tag{3.10}$$

$$\sum_{k=1}^{n} k^6 = \frac{xp_6(x)(2n+1)}{2\cdot 3\cdot 7} \qquad p_6(x) = 3x^2 - 3x + 1 \tag{3.11}$$

$$\sum_{k=1}^{n} k^7 = \frac{x^2p_7(x)}{3\cdot 8} \qquad p_7(x) = 3x^2 - 4x + 2 \tag{3.12}$$

$$\sum_{k=1}^{n} k^8 = \frac{xp_8(x)(2n+1)}{2\cdot 5\cdot 9} \qquad p_8(x) = 5x^3 - 10x^2 + 9x - 3 \tag{3.13}$$

$$\sum_{k=1}^{n} k^9 = \frac{x^2p_9(x)}{2\cdot 10} \qquad p_9(x) = 2x^3 - 5x^2 + 6x - 3 \tag{3.14}$$

$$\sum_{k=1}^{n} k^{10} = \frac{xp_{10}(x)(2n+1)}{2\cdot 3\cdot 11} \qquad p_{10}(x) = 3x^4 - 10x^3 + 17x^2 - 15x + 5 \tag{3.15}$$

$$\sum_{k=1}^{n} k^{11} = \frac{x^2p_{11}(x)}{2\cdot 12} \qquad p_{11}(x) = 2x^4 - 8x^3 + 17x^2 - 20x + 10 \tag{3.16}$$

$$\sum_{k=1}^{n} k^{12} = \frac{xp_{12}(x)(2n+1)}{2\cdot 3\cdot 5\cdot 7\cdot 13} \tag{3.17}$$

$$p_{12}(x) = 105x^5 - 525x^4 + 1435x^3 - 2360x^2 + 2073x - 691$$

$$\sum_{k=1}^{n} k^{13} = \frac{x^2p_{13}(x)}{2\cdot 3\cdot 5\cdot 14} \tag{3.18}$$

$$p_{13}(x) = 30x^5 - 175x^4 + 574x^3 - 1180x^2 + 1382x - 691$$

$$\sum_{k=1}^{n} k^{14} = \frac{xp_{14}(x)(2n+1)}{2\cdot 3\cdot 15} \tag{3.19}$$

$$p_{14}(x) = 3x^6-21x^5+84x^4-220x^3+359x^2-315x+105$$

$$\sum_{k=1}^{n} k^{15} = \frac{x^2p_{15}(x)}{3\cdot 16} \tag{3.20}$$

$$p_{15}(x) = 3x^6-24x^5+112x^4-352x^3+718x^2-840x+420$$

$$\sum_{k=1}^{n} k^{16} = \frac{xp_{16}(x)(2n+1)}{2\cdot 3\cdot 5\cdot 17} \tag{3.21}$$

$$p_{16}(x) = 15x^7-140x^6+770x^5-2930x^4+7595x^3-12370x^2+10851x-3617$$

$$\sum_{k=1}^{n} k^{17} = \frac{x^2p_{17}(x)}{2\cdot 5\cdot 18} \tag{3.22}$$

$$p_{17}(x) = 10x^7-105x^6+660x^5-2930x^4+9114x^3-18555x^2+21702x-10851$$

$$\sum_{k=1}^{n} k^{18} = \frac{xp_{18}(x)(2n+1)}{2\cdot 3\cdot 5\cdot 7\cdot 19} \quad p_{18}(x) = 105x^8-1260x^7+9114x^6-47418x^5 \tag{3.23}$$

$$+178227x^4-460810x^3+750167x^2-658005x+219335$$

$$\sum_{k=1}^{n} k^{19} = \frac{x^2p_{19}(x)}{2\cdot 3\cdot 7\cdot 20} \quad p_{19}(x) = 42x^8-560x^7+4557x^6-27096x^5 \tag{3.24}$$

$$+118818x^4-368648x^3+750167x^2-877340x+438670$$

In principle, the proofs of all these formulas are trivial. Let's take (3.7):

$$\sum_{k=1}^{n} k^2 = \sum\nolimits_0^{n+1} k^2\delta k = \frac{x(2n+1)}{2\cdot 3} = \frac{n(n+1)(2n+1)}{2\cdot 3}$$

Taking the difference of the right side, we obtain

$$\begin{aligned}\Delta\frac{(k-1)k(2k-1)}{2\cdot 3} &= \frac{k(k+1)(2k+1)}{2\cdot 3} - \frac{k(k-1)(2k-1)}{2\cdot 3} \\ &= \frac{k(2k^2+3k+1-2k^2+3k-1)}{6} = k^2\end{aligned}$$

Sums of polynomials by Bernoulli polynomials.

The Faulhaber formulas are special cases of the much more powerful discovery of J. Bernoulli (1654–1705), cf. *Ars Conjectandi* (1713).

The Bernoulli polynomials are the polynomial solutions to the equations,

$$\Delta f_n(k) = nk^{n-1}, \quad n\in\mathbb{N} \tag{3.25}$$

and are uniquely determined up to the constant term. For any solution to (3.25), e.g. for the Bernoulli polynomials to be found below, we then from (1.16) get the formula for the sums:

$$\sum k^n \delta k = \frac{B_{n+1}(k)}{n+1} \tag{3.26}$$

The solution using Bernoulli numbers goes as follows. Differentiation of (3.25) with respect to k yields

$$\Delta f_n'(k) = n(n-1)k^{n-2} \tag{3.26}$$

proving that $\frac{1}{n}f_n'(k)$ solves (3.25) for $n-1$, hence that

$$\frac{1}{n}f_n'(k)-f_{n-1}(k) \tag{3.27}$$

is a constant.

The choice of polynomial solutions to (3.25) for which the constant term in (3.27) is zero, are called the **Bernoulli polynomials** and are denoted as $B_n(k)$.

Suppose we have written the Bernoulli polynomials on the form

$$B_n(k) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j}^n k^j \tag{3.28}$$

for suitable constants B_{n-j}^n.

The index is chosen to have B_0^n as coefficient to the leading term, k^n, and B_n^n as the constant term.

Differentiation of B_n yields

$$B_n'(k) = \sum_{j=1}^{n} \binom{n}{j} B_{n-j}^n j k^{j-1} = \sum_{j=1}^{n} n\binom{n-1}{j-1} B_{n-j}^n k^{j-1} = n\sum_{j=0}^{n-1} \binom{n-1}{j} B_{n-1-j}^n k^j \tag{3.29}$$

From (3.27) this is known to be equal to

$$nB_{n-1}(k) = n\sum_{j=0}^{n-1} \binom{n-1}{j} B_{n-1-j}^{n-1} k^j$$

The conclusion from comparing the coefficients is, that

$$B_{n-1-j}^n = B_{n-1-j}^{n-1}, \qquad \text{for } j = 0, 1, \cdots, n-1 \tag{3.30}$$

The common values in (3.30) are called the **Bernoulli numbers,** and are denoted as B_{n-1-j}, omitting the superfluous superscript.

Hence we may rewrite (3.28) as

$$B_n(k) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j} k^j \tag{3.31}$$

such that the Bernoulli numbers take their appropriate part of the description of the Bernoulli polynomials. The fact that they satisfy (3.25) may lead to a computation of their coefficients. We compute

(3.32)

$$\begin{aligned}
\Delta B_n(k) &= \sum_{j=0}^{n} \binom{n}{j} B_{n-j}((k+1)^j - k^j) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j} \sum_{i=0}^{j-1} \binom{j}{i} k^i \\
&= \sum_{i=0}^{n-1} k^i \sum_{j=i+1}^{n} \binom{n}{j}\binom{j}{i} B_{n-j} = \sum_{i=0}^{n-1} k^i \sum_{j=i+1}^{n} \binom{n}{i}\binom{n-i}{j-i} B_{n-j} \\
&= \sum_{i=0}^{n-1} \binom{n}{i} k^i \sum_{j=i+1}^{n} \binom{n-i}{j-i} B_{n-j} = \sum_{i=0}^{n-1} \binom{n}{n-i} k^i \sum_{j=1}^{n-i} \binom{n-i}{j} B_{n-i-j} \\
&= \sum_{\ell=1}^{n} \binom{n}{\ell} k^{n-\ell} \sum_{j=1}^{\ell} \binom{\ell}{j} B_{\ell-j} = \sum_{\ell=1}^{n} \binom{n}{\ell} k^{n-\ell} \sum_{i=0}^{\ell-1} \binom{\ell}{i} B_i
\end{aligned}$$

where we have applied the binomial formula, (7.1), formulas (2.10), (2.13) and reversed teh direction of summation, e.g., $\ell = n-i$ and $i = \ell-j$.

Now we know from (3.25) that this final polynomial equals nk^{n-1}. This means that the coefficients are n for $\ell = 1$ and 0 else. This gives the formulas for the Bernoulli numbers:

$$\begin{gathered}
B_0 = 1 \\
\sum_{i=0}^{\ell-1} \binom{\ell}{i} B_i = 0 \text{ for } \ell > 1
\end{gathered} \tag{3.33}$$

Sometimes people like to confuse the reader by adding the number B_ℓ to the last sum to get the "implicit" recursion formula

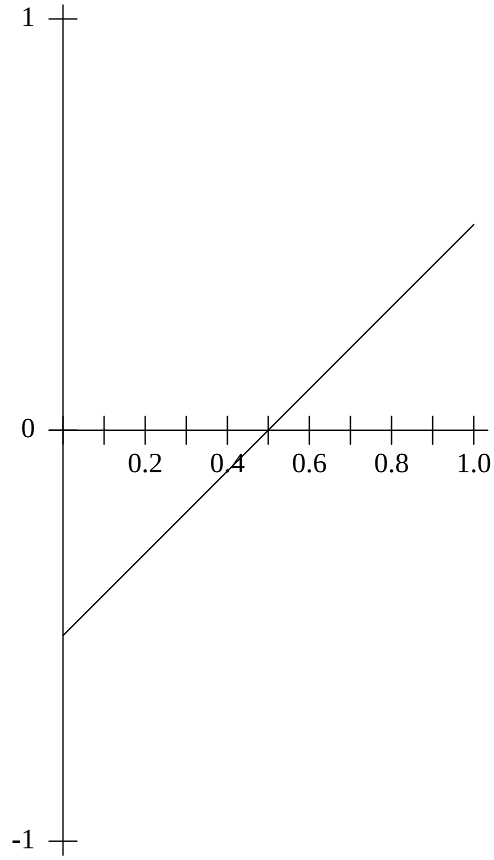
$$\sum_{i=0}^{\ell} \binom{\ell}{i} B_i = B_\ell \tag{3.34}$$

maybe it looks nicer.

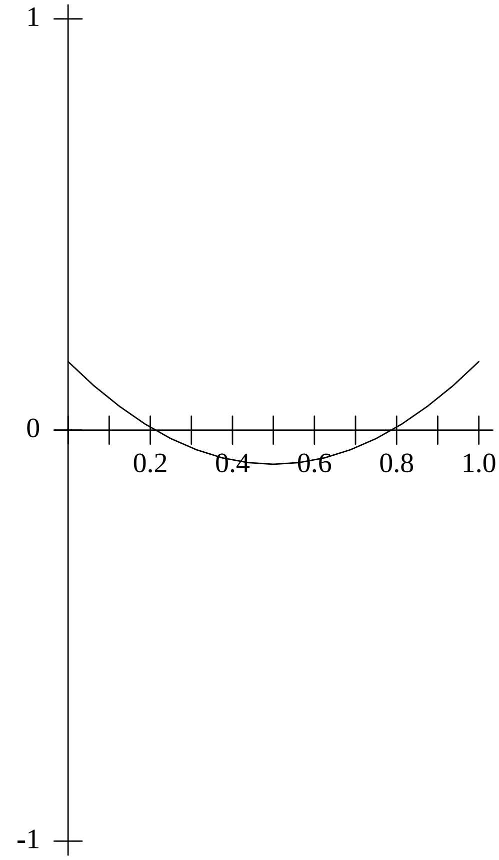
The formulas (3.33) or (3.34) allows the computing of the Bernoulli numbers, we get for the first few (the odd indexed are 0 from 3 on)

$$\begin{aligned}
&B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42} \\
&B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66},\ B_{12} = -\frac{691}{2730}
\end{aligned} \tag{3.35}$$

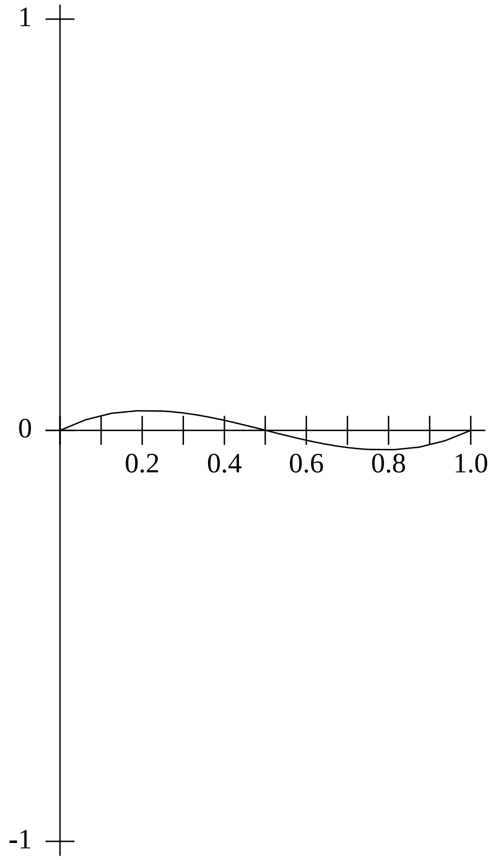
As soon as we have the numbers, we get the polynomials straightaway. The first four of them look like the following graphs:

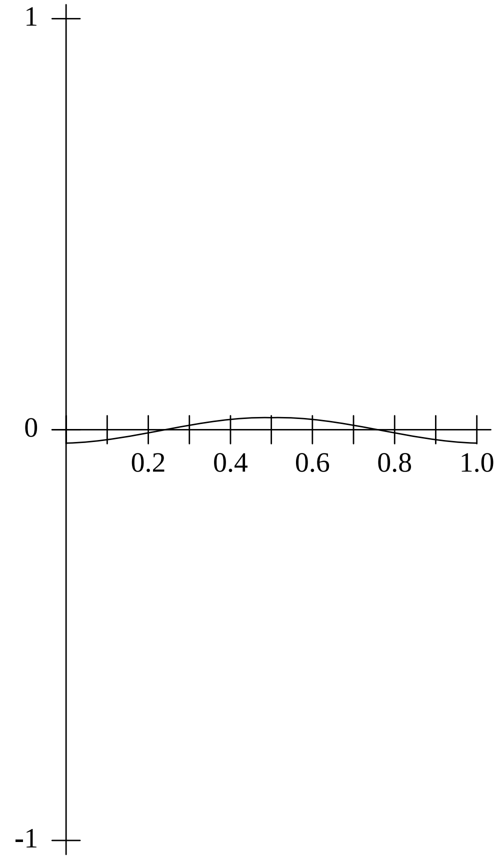


$B_1(k)$



$$
\begin{aligned}
B_0 &= 1\\
B_1 &= k-\tfrac{1}{2}\\
B_2 &= k^2-k+\tfrac{1}{6}\\
B_3 &= k(k-1)\left(k-\tfrac{1}{2}\right)\\
B_4 &= k^4-2k^3+k^2-\tfrac{1}{30}\\
B_5 &= k(k-1)\left(k-\tfrac{1}{2}\right)\left(k^2-k-\tfrac{1}{3}\right)\\
B_6 &= k^6-3k^5+\tfrac{5}{2}k^3-\tfrac{1}{2}k^2+\tfrac{1}{42}\\
B_7 &= k(k-1)\left(k-\tfrac{1}{2}\right)\left(k^4-2k^3+2k+\tfrac{1}{3}\right)\\
B_8 &= k^8-4k^7+\tfrac{14}{3}k^6-\tfrac{7}{3}k^4+\tfrac{2}{3}k^2-\tfrac{1}{30}\\
B_9 &= k(k-1)\left(k-\tfrac{1}{2}\right)\left(k^6-3k^5+k^4-\tfrac{1}{5}k^2-\tfrac{9}{5}k-\tfrac{3}{5}\right)\\
B_{10} &= k^{10}-5k^9+\tfrac{15}{2}k^8-7k^6+5k^4-\tfrac{3}{2}k^2+\tfrac{5}{66}
\end{aligned}
\tag{3.36}
$$





$B_3(k)$

Actually, all odd Bernoulli polynomials look like $\pm B_3$ and the even ones look like $\pm B_4$, e.g., the odd ones for 3 and up have 3 zeros in the interval, the even ones have 2 zeros. But they grow fast in absolute values, e.g.,

$$B_{20} = -\frac{174611}{330},\ B_{40} = -\frac{261082718496449122051}{13530},$$
$$B_{60} = -\frac{1215233140483755572040304994079820246041491}{56786730}$$

Actually, they grow like $\frac{2(2n)!}{(2\pi)^{2n}}$.

The fact that

$$B_n'(k) = nB_{n-1}(k) \tag{3.37}$$

yields the general formula for the derivatives of the Bernoulli polynomials,

$$B_n^{(j)}(k) = [n]_j B_{n-j}(k) \tag{3.38}$$

This formula allows the Taylor development of the polynomials as

$$B_n(k+h) = \sum_{j=0}^{n} [n]_j B_{n-j}(k) \frac{h^j}{j!} = \sum_{j=0}^{n} \binom{n}{j} B_{n-j}(k) h^j \tag{3.39}$$

With the choice of $h = 1$ we get from the defining equation, (3.25), that

$$nk^{n-1} = B_n(k+1) - B_n(k) = \sum_{j=1}^{n} \binom{n}{j} B_{n-j}(k) \tag{3.40}$$

Consider the function

$$f(k) = (-1)^{n+1} B_{n+1}(1-k) \tag{3.41}$$

We get immediately

$$\begin{aligned} f(k+1) - f(k) &= (-1)^{n+1}\left(B_{n+1}(-k) - B_{n+1}(1-k)\right) = \\ &= -(-1)^{n+1}(n+1)(-k)^n = (n+1)k^n \end{aligned}$$

This means that $f(k)$ deviates from $B_{n+1}(k)$ with a constant. Rather than finding this we just differentiate the two functions and obtain the formula

$$B_n(1-k) = (-1)^n B_n(k) \tag{3.42}$$

The equation

$$f(k + \tfrac{1}{2}) - f(k) = (n+1)k^n \tag{3.43}$$

has the solutions $B_{n+1}(k) + B_{n+1}(k + \frac{1}{2})$ and $2^{-n} B_{n+1}(2k)$, both polynomials in k. Their difference is again a polynomial which is periodic and hence constant. By differentiation of the two functions we get the same function and hence the formula

$$B_n(2k) = 2^{n-1}\left(B_n(k) + B_n(k + \tfrac{1}{2})\right) \tag{3.44}$$

For $k = 0$ we get the value

$$B_n(\tfrac{1}{2}) = -\left(1 - 2^{1-n}\right) B_n \tag{3.45}$$

For n even the absolute value in $\frac{1}{2}$ is just a little smaller than the absolute value in 0 and 1.

Sums of polynomials by Stirling numbers.

The sums of a polynomial may be found by writing the polynomial in k in the basis of $[k]_m$, $m = 0, 1, \cdots$. This may be done by the use of the **Stirling numbers of the second kind,** after J. Stirling (1692–1770):

$$\begin{pmatrix} k \\ k^2 \\ k^3 \\ k^4 \\ k^5 \\ k^6 \\ k^7 \\ k^8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 & 0 & 0 & 0 \\ 1 & 15 & 25 & 10 & 1 & 0 & 0 & 0 \\ 1 & 31 & 90 & 65 & 15 & 1 & 0 & 0 \\ 1 & 63 & 301 & 350 & 140 & 21 & 1 & 0 \\ 1 & 127 & 966 & 1701 & 1050 & 266 & 28 & 1 \end{pmatrix} \cdot \begin{pmatrix} [k]_1 \\ [k]_2 \\ [k]_3 \\ [k]_4 \\ [k]_5 \\ [k]_6 \\ [k]_7 \\ [k]_8 \end{pmatrix} \tag{3.46}$$

This matrix can be extended infinitely, see. [1], p. 835. It simply says that e.g.,

$$k^5 = 1 \times [k]_1 + 15 \times [k]_2 + 25 \times [k]_3 + 10 \times [k]_4 + 1 \times [k]_5$$

The Stirling numbers of the second kind are denoted as $\mathfrak{S}_n^{(j)}$ and appear as coefficients in the formula:

$$k^n = \sum_{j=1}^{n} \mathfrak{S}_n^{(j)} [k]_j \tag{3.47}$$

From (3.47) we get by multiplying with k,

$$\begin{aligned} k^{n+1} &= \sum_{j=1}^{n} \mathfrak{S}_n^{(j)} k[k]_j = \sum_{j=1}^{n} \mathfrak{S}_n^{(j)} (k-j+j)[k]_j = \sum_{j=1}^{n} \mathfrak{S}_n^{(j)} [k]_{j+1} + \sum_{j=1}^{n} \mathfrak{S}_n^{(j)} j[k]_j \\ &= \sum_{j=1}^{n+1} \mathfrak{S}_n^{(j-1)} [k]_j + \sum_{j=1}^{n} \mathfrak{S}_n^{(j)} j[k]_j = \sum_{j=1}^{n+1} \left(\mathfrak{S}_n^{(j-1)} + j\mathfrak{S}_n^{(j)} \right) [k]_j \end{aligned}$$

From this we derive the recurrence formula:

$$\mathfrak{S}_{n+1}^{(j)} = \mathfrak{S}_n^{(j-1)} + j\mathfrak{S}_n^{(j)} \tag{3.48}$$

This formula proves that the Stirling numbers of the second kind may be interpreted as:

$\mathfrak{S}_n^{(j)}$ *is the number of ways a set of* n *objects can be divided in* j *nonempty subsets.*

It may also be used to prove the explicit formula,

$$\mathfrak{S}_n^{(j)} = \frac{1}{j!} \sum_{k=0}^{n} (-1)^{j-k} \binom{j}{k} k^n \tag{3.49}$$

The Stirling numbers of the first kind are the solutions to the inverse problem, i.e., the coefficients to the expressions of the factorials, $[k]_n$, in terms of the monomials, k^j, i.e.

$$[k]_n = \sum_{j=1}^{n} S_n^{(j)} k^j \tag{3.50}$$

They may conveniently be arranged in a matrix too:

(3.51)

$$\begin{pmatrix} [k]_1 \\ [k]_2 \\ [k]_3 \\ [k]_4 \\ [k]_5 \\ [k]_6 \\ [k]_7 \\ [k]_8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 & 0 & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 & 0 & 0 & 0 \\ 24 & -50 & 35 & -10 & 1 & 0 & 0 & 0 \\ -120 & 274 & -225 & 85 & -15 & 1 & 0 & 0 \\ 720 & -1764 & 1624 & -735 & 175 & -21 & 1 & 0 \\ -5040 & 13068 & -13132 & 6769 & -1960 & 322 & -28 & 1 \end{pmatrix} \cdot \begin{pmatrix} k \\ k^2 \\ k^3 \\ k^4 \\ k^5 \\ k^6 \\ k^7 \\ k^8 \end{pmatrix}$$

This matrix just states, that e.g.,

$$[k]_5 = k^5 - 10k^4 + 35k^3 - 50k^4 + 24k \tag{3.52}$$

The matrix is simply the inverse of the matrix in (3.46). These numbers may be found in [1] p. 833.

If we multiply (3.50) with $k-n$, we obtain

$$[k]_{n+1} = \sum_{j=1}^{n} S_n^{(j)}(k-n)k^j = \sum_{j=1}^{n} S_n^{(j)}k^{j+1} - \sum_{j=1}^{n} S_n^{(j)}nk^j = \sum_{j=1}^{n+1}\left(S_n^{(j-1)} - nS_n^{(j)}\right)k^j \tag{3.53}$$

From this we derive the recurrence formula:

$$S_{n+1}^{(j)} = S_n^{(j-1)} - nS_n^{(j)} \tag{3.54}$$

This formula proves that the absolute value of the Stirling numbers of the first kind may be interpreted as:

$(-1)^{n-j}S_n^{(j)}$ *is the number of permutations of a set of* n *objects having* j *cycles.*

It may also be used to prove the explicit formula, which this time depend on the Stirling numbers of the second kind,

$$S_n^{(j)} = \sum_{k=0}^{n-j}(-1)^k\binom{n-1+k}{n-j+k}\binom{2n-j}{n-j-k}\mathfrak{S}_{n-j+k}^{(k)} \tag{3.55}$$

The solution of (3.25) by Stirling numbers goes

$$\begin{aligned} f_n(k) &= n\sum k^{n-1}\delta k = n\sum\sum_{j=1}^{n-1}\mathfrak{S}_{n-1}^{(j)}[k]_j\delta k = \\ &= n\sum_{j=1}^{n-1}\mathfrak{S}_{n-1}^{(j)}\frac{[k]_{j+1}}{j+1} = n\sum_{j=1}^{n-1}\mathfrak{S}_{n-1}^{(j)}\frac{1}{j+1}\sum_{i=1}^{j+1}S_{j+1}^{(i)}k^i = \\ &= \sum_{i=1}^{n}k^i\cdot n\sum_{j=i-1}^{n-1}\frac{1}{j+1}\mathfrak{S}_{n-1}^{(j)}S_{j+1}^{(i)} = \sum_{i=1}^{n}k^i\cdot n\sum_{j=i}^{n}\frac{1}{j}\mathfrak{S}_{n-1}^{(j-1)}S_j^{(i)} \end{aligned} \tag{3.56}$$

Renate Golombek's problem. In 1994 Renate Golombek, Marburg, [50], posed the following problem, to determine the sums for $r = 1, 2, 3, \dots$

$$P(n,r) = \sum_{j=0}^{n}\binom{n}{j}j^r \tag{3.57}$$

$$Q(n,r) = \sum_{j=0}^{n}\binom{n}{j}^2 j^r \tag{3.58}$$

The solution applies the Stirling numbers of the second kind (3.46). We apply (3.47) to write

$$P(n,r) = \sum_{k=1}^{r} \mathfrak{S}_r^{(k)} \sum_{j=0}^{n} \binom{n}{j} [j]_k \tag{3.59}$$

The inner sum is a binomial sum, cf. (7.1), i.e.,

$$\sum_{j=0}^{n} \binom{n}{j} [j]_k = [n]_k \sum_{j=k}^{n} \binom{n-k}{j-k} = [n]_k 2^{n-k} \tag{3.60}$$

hence we obtain the formula

$$P(n,r) = \sum_{k=1}^{r} \mathfrak{S}_r^{(k)} [n]_k 2^{n-k} \tag{3.61}$$

Proceeding the same way with the squares we get

$$Q(n,r) = \sum_{k=1}^{r} \mathfrak{S}_r^{(k)} \sum_{j=0}^{n} \binom{n}{j}^2 [j]_k \tag{3.62}$$

This time we shall evaluate the sums

$$\sum_{j=0}^{n} \binom{n}{j}^2 [j]_k \tag{3.63}$$

The terms less than $j = k$ vanish, so by reversing the direction of summation we get

$$\sum_{j=0}^{n-k} \binom{n}{j}^2 [n-j]_k \tag{3.64}$$

Now we apply the formula (2.9) to write the sum as

$$[n]_k \sum_{j=0}^{n-k} \binom{n}{j} \binom{n-k}{n-k-j} \tag{3.65}$$

which is a Chu–Vandermonde convolution, (8.1), so we get

$$[n]_k \binom{2n-k}{n-k} \tag{3.66}$$

Substitution of (3.66) in (3.62) yields

$$Q(n,r) = \sum_{k=1}^{r} \mathfrak{S}_r^{(k)} [n]_k \binom{2n-k}{n-k} \tag{3.67}$$

This formula is the general solution to the problem. In order to make it easy to see, how much it coincides with the solutions suggested, we shall appreciate a reformulation of the factor using (2.2):

$$[n]_k \binom{2n-k}{n-k} = \binom{2n-r}{n-1} \frac{[2n-k]_{r-k}[n-1]_{k-1}}{[n-k]_{r-k-1}} \tag{3.68}$$

With this substituted in (3.67) we get

$$Q(n,r) = \binom{2n-r}{n-1} \sum_{k=1}^{r} \mathfrak{S}_r^{(k)} \frac{[2n-k]_{r-k}[n-1]_{k-1}}{[n-k]_{r-k-1}} \tag{3.69}$$

Application of (3.69) yields for $r=1$

$$Q(n,1) = \binom{2n-1}{n-1} \frac{[2n-1]_0[n-1]_0}{[n-1]_{-1}} = \binom{2n-1}{n-1}(n-1+1) = n\binom{2n-1}{n} \tag{3.70}$$

For $r=2$ we get

(3.71)

$$\begin{aligned} Q(n,2) &= \binom{2n-2}{n-1} \left(\frac{[2n-1]_1[n-1]_0}{[n-1]_0} + \frac{[2n-1]_0[n-1]_1}{[n-2]_{-1}} \right) \\ &= \binom{2n-2}{n-1} \left(2n-1+(n-1)^2\right) = \binom{2n-2}{n-1} n^2 \end{aligned}$$

And for $r=3$ we get

(3.72)

$$\begin{aligned} &Q(n,3) = \\ &\binom{2n-3}{n-1} \left(\frac{[2n-1]_2[n-1]_0}{[n-1]_1} + 3\frac{[2n-2]_1[n-1]_1}{[n-2]_0} + \frac{[2n-3]_0[n-1]_2}{[n-3]_{-1}} \right) \\ &= \binom{2n-3}{n-1} \left((2n-1)2 + 3(2n-2)(n-1) + (n-1)(n-2)^2\right) \\ &= \binom{2n-3}{n-1} n^2(n+1) \end{aligned}$$

But the pattern does not proceed any further. The binomial coefficient in front is too much to want in general. Already for $r=4$ we get a denominator.

(3.73)

$$\begin{aligned} &Q(n,4) = \\ &\binom{2n-4}{n-1} \left(\frac{[2n-1]_3}{[n-1]_2} + 7\frac{[2n-2]_2[n-1]_1}{[n-2]_1} + 6[2n-3]_1[n-1]_2 + \frac{[n-1]_3}{[n-4]_{-1}} \right) \\ &= \binom{2n-4}{n-1} n^2 \frac{n^3+n^2-3n-1}{n-2} \end{aligned}$$

If we should rewrite (3.67), we should prefer to replace the binomial coefficient with factorials and write it

$$Q(n,r) = \sum_{k=1}^{r\wedge n} \mathfrak{S}_r^{(k)} \frac{[2n-k]_n}{[n-k]_{n-k}} \tag{3.74}$$

CHAPTER 4. LINEAR DIFFERENCE EQUATIONS

General linear difference equations. The general linear difference equations of order m has the form

$$f(n)=\sum_{k=1}^{m} a_k(n)f(n-k)+g(n) \tag{4.1}$$

where a_k and g are given functions of n, where a_m is not identically zero, and f is wanted. Iteration of (4.1) gives the consecutive values of $f(n)$, $n=m+1,m+2,\cdots$ for given initial values, $f(1),f(2),\cdots,f(m)$. The question is whether it is possible to find a closed form for $f(n)$.

The complete solution is usually divided in a sum of two, the solution to the homogeneous equation, i.e., the equation with $g(n)=0$, and a particular solution to the equation (4.1) as given.

The homogeneous equation. A set of m solutions to the homogeneous equation, (4.1), $f_1,\cdots,f_m$, is called *linearly independent,* if any equation of the form

$$\sum_{j=1}^{m} c_j f_j(n)=0, \qquad n\in\mathbb{Z} \tag{4.2}$$

must have coefficients equal to zero, $c_1=\cdots=c_m=0$.

In analogy to the Wronskian we may consider the determinant

$$W(n)=\begin{vmatrix} f_1(n-m) & f_2(n-m) & \cdots & f_m(n-m) \\ f_1(n-m+1) & f_2(n-m+1) & \cdots & f_m(n-m+1) \\ \vdots & & \cdots & \vdots \\ f_1(n-1) & f_2(n-1) & \cdots & f_m(n-1) \end{vmatrix} \tag{4.3}$$

We may easily compute $W(n)$ by the use of (4.1) with $g=0$. Row operations yield

(4.4)

$$\begin{aligned} W(n+1)&=\begin{vmatrix} f_1(n-m+1) & f_2(n-m+1) & \cdots & f_m(n-m+1) \\ \vdots & & \cdots & \vdots \\ a_m(n)f_1(n-m) & a_m(n)f_2(n-m) & \cdots & a_m(n)f_m(n-m) \end{vmatrix} \\ &=(-1)^{m-1}a_m(n)W(n) \end{aligned}$$

Hence, if $a_m(n)\neq 0$ for all values of n, then the determinant is either identically zero or never zero. In the latter case it may be computed as

$$W(n)=(-1)^{(m-1)n}W(0)\prod_{j=0}^{n-1} a_m(j) \tag{4.5}$$

This means that if we start with m independent m–dimensional vectors, $(f_1(1), \cdots, f_1(m))$, ..., $(f_m(1), \cdots, f_m(m))$, the solutions defined by (4.2) will remain independent provided $a_m(n) \neq 0$ for all values of n. Furthermore, any other solution is uniquely defined by its starting values $f(1), \cdots, f(m)$, which are a linear combination of the m vectors above. This proves that the space of solutions to (4.2) is an m–dimensional vector space.

The special case of $m = 1$,

$$f(n) = a(n)f(n-1) \tag{4.6}$$

We assume that $a(n) \neq 0$, and write the homogeneous equation (4.6) as

$$\frac{f(n)}{f(n-1)} = a(n) \tag{4.7}$$

giving the solutions for different choices of initial value $f(0)$,

$$f(n) = f(0) \prod_{k=1}^{n} a(k) \tag{4.8}$$

The special case of $m = 2$,

$$f(n) = a_1(n)f(n-1) + a_2(n)f(n-2) \tag{4.9}$$

with two solutions, $f_1(n)$ and $f_2(n)$, giving the determinant

$$W(n) = \begin{vmatrix} f_1(n-2) & f_2(n-2) \\ f_1(n-1) & f_2(n-1) \end{vmatrix} = f_1(n-2)f_2(n-1) - f_2(n-2)f_1(n-1) \tag{4.10}$$

computable by (4.5) as

$$W(n) = (-1)^n W(0) \prod_{j=0}^{n-1} a_2(j) \tag{4.11}$$

If we know – by guessing perhaps – a solution $f_1(n) \neq 0$, then we may use the determinant to find the other solution, $f_2(n)$. If we try the solution $f_2(n) = \phi(n)f_1(n)$ in (4.10), we get

$$f_1(n)\phi(n+1)f_1(n+1) - \phi(n)f_1(n)f_1(n+1) = W(n+2) \tag{4.12}$$

from which it is easy to find $\phi(n)$ from the form

$$\Delta\phi(n) = \phi(n+1) - \phi(n) = \frac{W(n+2)}{f_1(n)f_1(n+1)} \tag{4.13}$$

having the solution by the definition (1.16)

$$\phi(n) = \sum \frac{W(n+2)}{f_1(n)f_1(n+1)}\delta n \tag{4.14}$$

and hence providing us with the second solution to the equation (4.9)

$$f_2(n) = \phi(n)f_1(n) = f_1(n) \sum \frac{W(n+2)}{f_1(n)f_1(n+1)}\delta n \tag{4.15}$$

First order inhomogeneous equations. If $m = 1$, we have the form

$$f(n) = a(n)f(n-1) + g(n) \tag{4.16}$$

We assume that $a(n) \neq 0$, and let $f \neq 0$ be any solution of form (4.8) to the homogeneous equation. Then we consider a solution to (4.16) of the form $f\phi$ and get

$$f(n)\phi(n) - a(n)f(n-1)\phi(n-1) = g(n) \tag{4.17}$$

together with (4.6) we obtain

$$\phi(n) - \phi(n-1) = \frac{g(n)}{f(n)} \tag{4.18}$$

which by telescoping yields

$$\phi(n) = \sum_{k=1}^{n} \frac{g(k)}{f(k)} + \phi(0) \tag{4.19}$$

and hence the complete solution to (4.16)

$$f(n)\phi(n) = f(n)\sum_{k=1}^{n} \frac{g(k)}{f(k)} + \phi(0)f(n) \tag{4.20}$$

where the term $\phi(0)f(n)$ is "any" solution to the homogeneous equation.

First order equations with constant coefficients. For $m = 1$ we choose $a(n) = a$ independent of n in (4.16), then the solution, f, (4.8), to the homogeneous equation becomes

$$f(n) = f(0)a^n \tag{4.21}$$

giving the complete solution (4.20) of (4.16)

$$\begin{aligned} f(n)\phi(n) &= a^n \sum_{k=1}^{n} \frac{g(k)}{a^k} + f(0)\phi(0)a^n = \\ &= \sum_{k=1}^{n} g(k)a^{n-k} + f(0)\phi(0)a^n \end{aligned} \tag{4.22}$$

Arbitrary order equations with constant coefficients. If $m > 1$, we may consider the operator polynomial in $\mathbf{E}$ from (4.1),

$$\mathbf{E}^m - \sum_{k=1}^{m} a_k \mathbf{E}^{m-k} \tag{4.23}$$

with which the equation (4.1) takes the form

$$\left(\mathbf{E}^m - \sum_{k=1}^{m} a_k \mathbf{E}^{m-k} \right) f(n) = g(n) \tag{4.24}$$

Now let the characteristic polynomial of (4.23),

$$p(x) = x^m - \sum_{k=1}^{m} a_k x^{m-k} \tag{4.25}$$

according to the fundamental theorem of algebra have the different roots in $\mathbb{C}$, $\alpha_1, \cdots \alpha_q$ of order respectively $\nu_1, \cdots, \nu_q$. Then the operator may be split into

$$\mathbf{E}^m - \sum_{k=1}^{m} a_k \mathbf{E}^{m-k} = \prod_{j=1}^{q} (\mathbf{E} - \alpha_j \mathbf{I})^{\nu_j} \tag{4.26}$$

Then the equation (4.24) may be solved successively by use of the solution (4.22).

Theorem 4.1. *If the homogeneous equation*

$$\left(\mathbf{E}^m - \sum_{k=1}^{m} a_k \mathbf{E}^{m-k} \right) f(n) = 0 \tag{4.27}$$

may be written in the operator form with different α's

$$\prod_{j=1}^{q} (\mathbf{E} - \alpha_j \mathbf{I})^{\nu_j} f(n) = 0 \tag{4.28}$$

then the complete solution is

$$f(n) = \sum_{j=1}^{q} p_j(n) \alpha_j^n \tag{4.29}$$

where p_j is any polynomial of degree at most $\nu_j - 1$.

Proof. By induction after m. For $m = 1$ (4.29) reduces to (4.21).

Presume the theorem for m, and consider the equation of form

$$\left(\mathbf{E}^m - \sum_{k=1}^{m} a_k \mathbf{E}^{m-k}\right)(\mathbf{E} - \alpha\mathbf{I})f(n) = 0 \tag{4.30}$$

Then we get the presumed solution

$$(\mathbf{E} - \alpha\mathbf{I})f(n) = \sum_{j=1}^{q} p_j(n)\alpha_j^n \tag{4.31}$$

By the linearity of the operator it is enough to solve each of the equations

$$(\mathbf{E} - \alpha\mathbf{I})f(n) = [n]_k \alpha_j^n,\ k = 0,\cdots,\nu_j - 1; j = 0,\cdots,q \tag{4.32}$$

Case 1. $\alpha = \alpha_j$. We put $f(n) = [n]_{k+1}\alpha^{n-1}$. This is allowed because the degree of $\alpha = \alpha_j$ is now $\nu_j + 1$. Then we get by (3.4)

$$(\mathbf{E} - \alpha\mathbf{I})f(n) = [n+1]_{k+1}\alpha^n - [n]_{k+1}\alpha^n = \Delta[n]_{k+1}\alpha^n = (k+1)[n]_k\alpha^n \tag{4.33}$$

proving that $f(n) = \frac{[n]_{k+1}\alpha^{n-1}}{k+1}$ solves (4.32).

Case 2. $\alpha \neq \alpha_j$. Analogously we try $f(n) = [n]_k\alpha_j^n$. Then we get by (3.4)

(4.34)
$$\begin{aligned}(\mathbf{E} - \alpha\mathbf{I})f(n) = [n+1]_k\alpha_j^{n+1} - \alpha[n]_k\alpha_j^n &= (([n]_k + k[n]_{k-1})\alpha_j - \alpha[n]_k)\alpha_j^n \\ &= (\alpha_j - \alpha)[n]_k\alpha_j^n + k[n]_{k-1}\alpha_j^{n+1}\end{aligned}$$

showing that $f(n) = \frac{[n]_k\alpha_j^n}{\alpha_j - \alpha}$ solves the problem for $k = 0$ and otherwise reduces the problem from k to $k - 1$. □

Systems of equations with constant coefficients. If we define a vector of functions by $f_j(n) = f(n - j + 1),\quad j = 1,\cdots,m$, then the equation (4.24) may be written as a first order equation in vectors, $\mathbf{f} = (f_1,\cdots,f_m)$, $\mathbf{g}(n) = (g(n),0,\cdots,0)$,

$$\mathbf{f}(n) = \mathbf{A}\mathbf{f}(n-1) + \mathbf{g}(n) \tag{4.35}$$

with matrix of coefficients

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_m \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \cdots & \vdots \end{pmatrix} \tag{4.36}$$

In the form (4.35) the solution in dimension 1 may be copied. The homogeneous system has the solution

$$\mathbf{f}(n) = \mathbf{A}^n \mathbf{f}(0) \tag{4.37}$$

and if a solution to the original system takes the form

$$\mathbf{f}(n) = \mathbf{A}^n \phi(n) \tag{4.38}$$

with any vector function $\phi(n)$, then we may get

$$\mathbf{A}^n \phi(n) - \mathbf{A}\mathbf{A}^{n-1}\phi(n-1) = \mathbf{g}(n) \tag{4.39}$$

which may be written for a regular matrix, $\mathbf{A}$, as

$$\phi(n) - \phi(n-1) = \mathbf{A}^{-n}\mathbf{g}(n) \tag{4.40}$$

From this follows by summing that

$$\phi(n) = \sum_{k=1}^{n} \mathbf{A}^{-k}\mathbf{g}(k) + \phi(0) \tag{4.41}$$

and hence that the solution (4.38) becomes

$$\mathbf{f}(n) = \sum_{k=1}^{n} \mathbf{A}^{n-k}\mathbf{g}(k) + \phi(0)\mathbf{A}^n \tag{4.42}$$

The only problem with this solution is that powers of matrices are cumbersome to compute. But, we may apply the Cayley–Hamilton theorem, cf. (4.51), to reduce the computation to the first $m-1$ powers of the matrix, $\mathbf{A}$. That is, if the characteristic polynomial of $\mathbf{A}$ is

$$p_{\mathbf{A}}(x) = \det(x\mathbf{U} - \mathbf{A}) = x^m + a_{m-1}x^{m-1} + \cdots + a_0 \tag{4.43}$$

where $\mathbf{U}$ is the unit matrix, then we have by the Cayley–Hamilton theorem, that $p_{\mathbf{A}}(\mathbf{A}) = \mathbf{O}$, and hence that

$$\mathbf{A}^m = -a_{m-1}\mathbf{A}^{m-1} - \cdots - a_0\mathbf{U} \tag{4.44}$$

Furthermore, if we apply this equation to the solution (4.37) then we get

$$\mathbf{A}^m\mathbf{f}(n) = -a_{m-1}\mathbf{A}^{m-1}\mathbf{f}(n) - \cdots - a_0\mathbf{f}(n) \tag{4.45}$$

or better,

$$\mathbf{f}(n+m) = -a_{m-1}\mathbf{f}(n+m-1) - \cdots - a_0\mathbf{f}(n) \tag{4.46}$$

which equation simply states that each coordinate function of the vector solution to the homogeneous system of equations satisfies the higher order equation with the same characteristic polynomial as the system.

Proof of the Cayley–Hamilton theorem.

Let $\mathbf{A}$ be any $n \times n$–matrix, $\mathbf{A} = (a_{ij})$. Then the determinant may be computed by the development after a column or row, e.g.,

$$\det \mathbf{A} = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} \det \mathbf{A}^{(i,j)} \tag{4.47}$$

where $\mathbf{A}^{(i,j)}$ is the (i,j)–th complement, i.e., the matrix obtained by omitting the i–th row and the j–th column from the matrix, $\mathbf{A}$. This means, that if we define a matrix, $\mathbf{B} = (b_{jk})$, as

$$b_{jk} = (-1)^{k+j} \det \mathbf{A}^{(k,j)} \tag{4.48}$$

then we have traced the inverse of the matrix, $\mathbf{A}$, provided it exists. At least we may write

$$\sum_{j=1}^{n} a_{ij} b_{jk} = \delta_{ik} \det \mathbf{A} \tag{4.49}$$

or, in matrix form,

$$\mathbf{AB} = (\det \mathbf{A})\mathbf{U} \tag{4.50}$$

The Cayley–Hamilton theorem. *If*

$$p(\xi) = \det (\xi \mathbf{U} - \mathbf{A}) = \xi^n + a_{n-1}\xi^{n-1} + \cdots + a_0 \tag{4.51}$$

is the characteristic polynomial for the matrix $\mathbf{A}$*, then the matrix* $p(\mathbf{A})$ *satisfies*

$$p(\mathbf{A}) = \mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \cdots + a_0 \mathbf{U} = \mathbf{O} \tag{4.52}$$

Proof. We shall apply (4.50) to the matrix $\lambda\mathbf{U} - \mathbf{A}$ to get

$$p(\lambda)\mathbf{U} = (\lambda \mathbf{U} - \mathbf{A})\, \mathbf{B}(\lambda) \tag{4.53}$$

where $\mathbf{B}(\lambda) = (b_{ij}(\lambda))$ is a matrix of polynomials in λ, defined as the (j,i)–th complement of the matrix $\lambda\mathbf{U} - \mathbf{A}$. Hence we may write

$$\mathbf{B}(\lambda) = \lambda^{n-1}\mathbf{B}_{n-1} + \cdots + \lambda \mathbf{B}_1 + \mathbf{B}_0 \tag{4.54}$$

as a polynomial in λ with coefficients which are matrices independent of λ.

For any $k \geq 1$ we may write

$$\mathbf{A}^k - \lambda^k \mathbf{U} = (\mathbf{A} - \lambda \mathbf{U}) \left(\mathbf{A}^{k-1} + \lambda \mathbf{A}^{k-2} + \cdots + \lambda^{k-1}\mathbf{U} \right) \tag{4.55}$$

Hence we can write

(4.56)
$$
\begin{aligned}
p(\mathbf{A}) - p(\lambda)\mathbf{U} &= \mathbf{A}^n - \lambda^n\mathbf{U} + a_{n-1}\left(\mathbf{A}^{n-1} - \lambda^{n-1}\mathbf{U}\right) + \cdots + a_1\left(\mathbf{A}^1 - \lambda^1\mathbf{U}\right) \\
&= (\mathbf{A} - \lambda\mathbf{U})\,\mathbf{C}(\lambda) = (\mathbf{A} - \lambda\mathbf{U})\left(\lambda^{n-1}\mathbf{U} + \lambda^{n-2}\mathbf{C}_{n-2} + \cdots + \mathbf{C}_0\right)
\end{aligned}
$$

where $\mathbf{C}(\lambda)$ is a polynomial in λ with coefficients which are matrices independent of λ.

Adding (4.53) and (4.56) we get

(4.57)
$$
\begin{aligned}
p(\mathbf{A}) &= (\mathbf{A} - \lambda\mathbf{U})\,(\mathbf{C}(\lambda) - \mathbf{B}(\lambda)) \\
&= (\mathbf{A} - \lambda\mathbf{U})\left(\lambda^{n-1}(\mathbf{U} - \mathbf{B}_{n-1}) + \cdots + \lambda(\mathbf{C}_1 - \mathbf{B}_1) + \mathbf{C}_0 - \mathbf{B}_0\right) \\
&= \lambda^{k+1}(\mathbf{B}_k - \mathbf{C}_k) + \lambda^k \cdots
\end{aligned}
$$

where k is the degree of the second factor to the right.

But this polynomial in λ can only be constant, i.e., independent of λ, if the second factor is zero. But in that case it is all zero, or $p(\mathbf{A}) = \mathbf{O}$. □

Generating functions. The equation (4.1) with $g = 0$,

$$f(n+m) = \sum_{k=0}^{m-1} a_k f(n+k) \tag{4.58}$$

may be solved by defining the series

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f(n) \tag{4.59}$$

called the **exponential generating function** for (4.58). It satisfies that

$$F^{(k)}(x) = \sum_{n=k}^{\infty} \frac{x^{n-k}}{(n-k)!} f(n) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f(n+k) \tag{4.60}$$

Hence we get

$$\sum_{k=0}^{m-1} a_k F^{(k)}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{m-1} a_k f(n+k) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f(n+m) = F^{(m)}(x) \tag{4.61}$$

This is a differential equation we may solve.

Take e.g. the difference equation

$$f(2+n) = f(1+n) + f(n) \tag{4.62}$$

The exponential generating function satisfies the equation

$$F" = F' + F \tag{4.63}$$

With the roots $\xi_{\pm} = \frac{1 \pm \sqrt{5}}{2}$ so that F becomes a combination of

$$e^{\xi_{\pm} x} = \sum_{n=0}^{\infty} \xi_{\pm}^{n} \frac{x^n}{n!} \tag{4.64}$$

Even the Bernoulli polynomials have an exponential generating function:

$$\frac{xe^{kx}}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n(k)x^n}{n!} \tag{4.65}$$

Consider the product

$$e^x \sum_{n=0}^{\infty} \frac{B_n(k)x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{j=0}^{n} \binom{n}{j} B_j(k)$$

now from (3.40) we have this equal to

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \left(nk^{n-1} + B_n(k)\right) = xe^{kx} + \sum_{n=0}^{\infty} \frac{B_n(k)x^n}{n!}$$

Kwang–Wu Chen's problem. In 1994 Kwang–Wu Chen, Chia–Yi, Taiwan, Republic of China, [28], posed the following problem, to prove the identity for Bernoulli polynomials for $m \geq 1$,

$$\sum_{k=0}^{m} \binom{m}{k} B_k(\alpha) B_{m-k}(\beta) = -(m-1)B_m(\alpha+\beta) + m(\alpha+\beta-1)B_{m-1}(\alpha+\beta) \tag{4.66}$$

Proof.

Multiplication of two copies of (4.65), taken for $t = x$, $k = \alpha$ and $k = \beta$ respectively, yields

$$\frac{t^2 e^{(\alpha+\beta)t}}{\left(e^t - 1\right)^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} B_k(\alpha) B_{n-k}(\beta) \tag{4.67}$$

We have also by differentiation of (4.65) with respect to t and then multiplication with t the formula

$$\frac{te^{xt}}{e^t - 1} + xt\frac{te^{xt}}{e^t - 1} - \frac{t^2 e^{(x+1)t}}{\left(e^t - 1\right)^2} = \sum_{n=0}^{\infty} \frac{nB_n(x)t^n}{n!} \tag{4.68}$$

Substitution of the series from (4.65) in this formula for the choice of $x = \alpha+\beta-1$ yields

(4.69)
$$\frac{t^2 e^{(\alpha+\beta)t}}{(e^t-1)^2} =$$
$$= 1+\sum_{n=1}^{\infty}\frac{t^n}{n!}\left(-(n-1)B_n(\alpha+\beta-1)+n(\alpha+\beta-1)B_{n-1}(\alpha+\beta-1)\right)$$

From the very difference equation, the Bernoulli polynomials were invented to solve, cf. (3.25), we get

$$B_m(\alpha+\beta-1) = B_m(\alpha+\beta) - m(\alpha+\beta-1)^{m-1} \tag{4.70}$$

for $m = n$ and $m = n-1$ respectively in (4.69), and the formula (4.66) follows from comparing the terms in (4.69) and (4.67).

Equations with polynomial coefficients. If in (4.1) the functions $a_k(n)$ are polynomials in n, certain higher order equations may be solved. It is always possible to transform the equation to a differential equation, not necessarily of the same order as the difference equation.

The simplest case is the difference equation of order 2 with coefficients of order 1:

$$(c_0+b_0n)f(n)+(c_1+b_1(n-1))f(n-1)+(c_2+b_2(n-2))f(n-2)=0 \tag{4.100}$$

One attack consists in the introduction of the *generating function* for the solution, $f(n)$,

$$F(x) = \sum_{n=0}^{\infty} f(n)x^n \tag{4.71}$$

This function obeys two simple rules of our concern, written conveniently at once as

$$\begin{aligned} x^k F^{(j)}(x) &= \sum_{n=j}^{\infty} x^k f(n)[n]_j x^{n-j} \\ &= \sum_{n=j}^{\infty}[n]_j f(n)x^{n+k-j} \\ &= \sum_{n=k}^{\infty}[n-k+j]_j f(n-k+j)x^n \end{aligned} \tag{4.72}$$

To handle (4.100) it is enough to consider $j = 0, 1$, hence the differential equation gets order 1, and to consider $k = 0, 1, 2, 3$ to obtain the shift of order 2. The differential equation in F becomes

$$(c_0 + c_1 x + c_2 x^2)\, F(x) + (b_0 x + b_1 x^2 + b_2 x^3)\, F'(x) = 0 \tag{4.73}$$

If we find the zeros of the polynomial $b_0 + b_1 x + b_2 x^2$, say they are a_1, a_2, then we may use partial fractions to write the differential equation (4.73) in general as

$$\frac{F'(x)}{F(x)} = -\frac{c_0 + c_1 x + c_2 x^2}{b_0 x + b_1 x^2 + b_2 x^3} = \frac{\beta_0}{x} + \frac{\beta_1}{x - a_1} + \frac{\beta_2}{x - a_2} \tag{4.74}$$

In the cases of double roots and common zeros the expressions are similar. We remark that the constant $\beta_0 = -\frac{b_2 c_0}{b_0}$. This may be solved as

$$F(x) = x^{\beta_0} (x - a_1)^{\beta_1} (x - a_2)^{\beta_2} \tag{4.75}$$

If β_0 is an integer, we may develop the other factors as series in x, cf. (7.2),

$$(x - a)^{\beta} = \sum_{k=0}^{\infty} \binom{\beta}{k} (-a)^{\beta - k} x^k \tag{4.76}$$

and get their product to find

(4.77)

$$\begin{aligned} F(x) &= x^{\beta_0} \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{\beta_1}{k} \binom{\beta_2}{n-k} (-a_1)^{\beta_1 - k} (-a_2)^{\beta_2 + k - n} \\ &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n-\beta_0} \binom{\beta_1}{k} \binom{\beta_2}{n - \beta_0 - k} (-a_1)^{\beta_1 - k} (-a_2)^{\beta_2 + k - n + \beta_0} \end{aligned}$$

yielding the solution to the equation (4.100) as

$$f(n) = \sum_{k=0}^{n-\beta_0} \binom{\beta_1}{k} \binom{\beta_2}{n - \beta_0 - k} (-a_1)^{\beta_1 - k} (-a_2)^{\beta_2 + k - n + \beta_0} \tag{4.78}$$

In order to write this solution in the forms treated in this book we may prefer to rewrite (4.78) as

$$f(n) = (-a_1)^{\beta_1} (-a_2)^{\beta_2 - n + \beta_0} \sum_{k=0}^{n-\beta_0} \binom{n - \beta_0}{k} [\beta_1]_k [\beta_2]_{n - \beta_0 - k} \left(\frac{a_2}{a_1}\right)^k \tag{4.79}$$

Shalosh B. Ekhad's squares. In 1994 Shalosh B. Ekhad, Princeton, NJ, [37] posed the following problem:

Let X_n be defined by $X_0 = 0, X_1 = 1, X_2 = 0, X_3 = 1$, and for $n \geq 1$

$$X_{n+3} = \frac{\left(n^2+n+1\right)(n+1)}{n} X_{n+2} + \left(n^2+n+1\right) X_{n+1} - \frac{n+1}{n} X_n \tag{4.80}$$

Prove that X_n is a square of an integer for every $n \geq 0$.

We consider the general second order difference equation,

$$x_{n+2} = f(n)x_{n+1} + g(n)x_n \qquad f, g : \mathbb{N} \to \mathbb{Z} \setminus \{0\} \tag{4.81}$$

With any integral initial values, x_1, x_2, this equation will generate a sequence of integers. We shall see, that with the suitable choice of functions and initial values the solution to (4.81) shall be the sequence of square roots asked for in the problem.

We consider the equation (4.81) in the two forms:

$$x_{n+3} = f(n+1)x_{n+2} + g(n+1)x_{n+1} \tag{4.82}$$

$$g(n)x_n = x_{n+2} - f(n)x_{n+1} \tag{4.83}$$

By squaring these two equations and then eliminating the mixed products, $x_{n+2}x_{n+1}$, we get a third order difference equations in the squares:

(4.84)

$$\frac{1}{f(n+1)g(n+1)} x_{n+3}^2 = \left(\frac{f(n+1)}{g(n+1)} + \frac{1}{f(n)}\right) x_{n+2}^2 + \left(\frac{g(n+1)}{f(n+1)} + f(n)\right) x_{n+1}^2 - \frac{g(n)^2}{f(n)} x_n^2$$

With the choice of the functions $f(n) = n,\ g(n) = 1$ we obtain the equation for $X_n = x_n^2$, equivalent to the one, posed in the problem:

$$\frac{1}{n+1} X_{n+3} = \left((n+1) + \frac{1}{n}\right) X_{n+2} + \left(\frac{1}{n+1} + n\right) X_{n+1} - \frac{1}{n} X_n \tag{4.85}$$

With the initial values $x_1 = 1,\ x_2 = 0$ we obtain $x_3 = 1$ and the initial values of the problem – (the value of X_0 is irrelevant, provided it is a square). Hence the solution to the posed problem consists of integral squares.

David Doster's problem. In 1994, David Doster, Choate Rosemary Hall, Wallingford, CT, [33], posed the following problem:

Define a sequence $< y_n >$ recursively by $y_0 = 1,\ y_1 = 3$ and

$$y_{n+1} = (2n+3)y_n - 2ny_{n-1} + 8n \tag{4.86}$$

for $n \geq 1$. Find an asymptotic formula for y_n.

A solution is

$$2^{n+1}n!\sqrt{e} \tag{4.87}$$

while an exact expression for y_n may be

$$y_n = 1 + 2n + \sum_{k=2}^{n} [n]_k 2^{k+1} \tag{4.88}$$

How to solve it?

We define a new sequence, z_n, by the formula

$$y_n = 2^n n! z_n \tag{4.89}$$

Then the new sequence is defined by $z_0 = 1$, $z_1 = \frac{3}{2}$ and for $n \geq 1$ the recursion

$$z_{n+1} = z_n + \frac{z_n - z_{n-1}}{2(n+1)} + \frac{4n}{2^n(n+1)!} \tag{4.90}$$

This is really a difference equation in the difference,

$$x_n = \Delta z_n = z_{n+1} - z_n \tag{4.91}$$

which sequence is defined by $x_0 = \frac{1}{2}$ and for $n \geq 1$ the recursion

$$x_n = \frac{x_{n-1}}{2(n+1)} + \frac{4n}{2^n(n+1)!} \tag{4.92}$$

with the straightforward solution for $n \geq 1$

$$x_n = \frac{1}{2^{n-1}(n-1)!} + \frac{1}{2^{n+1}(n+1)!} \tag{4.93}$$

Hence we find

(4.94)

$$\begin{aligned} z_n &= z_0 + \sum_{k=0}^{n-1} x_k = 1 + \tfrac{1}{2} + \sum_{k=1}^{n-1} \frac{1}{2^{k-1}(k-1)!} + \sum_{k=1}^{n-1} \frac{1}{2^{k+1}(k+1)!} \\ &= \sum_{k=0}^{n-2} \frac{\left(\frac{1}{2}\right)^k}{k!} + 1 + \tfrac{1}{2} + \sum_{k=2}^{n} \frac{\left(\frac{1}{2}\right)^k}{k!} = \sum_{k=0}^{n-2} \frac{\left(\frac{1}{2}\right)^k}{k!} + \sum_{k=0}^{n} \frac{\left(\frac{1}{2}\right)^k}{k!} \end{aligned}$$

It follows that the limit must be

$$\lim_{n\to\infty} z_n = 2e^{\frac{1}{2}} \tag{4.95}$$

and that we may compute y_n as

$$\begin{aligned} y_n &= 2^n n! \left(\sum_{k=0}^{n-2} \frac{\left(\frac{1}{2}\right)^k}{k!} + \sum_{k=0}^{n} \frac{\left(\frac{1}{2}\right)^k}{k!} \right) = \\ &= \sum_{k=0}^{n-2} 2^{n-k}[n]_{n-k} + \sum_{k=0}^{n} 2^{n-k}[n]_{n-k} = \\ &= 1 + 2n + \sum_{k=2}^{n} 2^{k+1}[n]_k \end{aligned} \tag{4.96}$$

Ira Gessel's problem. In 1995 Ira Gessel, Brandeis University, Waltham, Mass, [47], [17], posed the following problem, to evaluate the sum for all $n \in \mathbb{N}$,

$$S(n) = \sum_{3k \le n} 2^k \frac{n}{n-k} \binom{n-k}{2k} \tag{4.97}$$

Solution. *For all $n \in \mathbb{N}$, we have*

$$S(n) = \sum_{3k \le n} 2^k \frac{n}{n-k} \binom{n-k}{2k} = 2^{n-1} + \cos\left(n \cdot \tfrac{\pi}{2}\right) \tag{4.98}$$

Proof. The function $S(n)$ defined by (4.97) satisfies the difference equation,

$$S(n) - 2S(n-1) + S(n-2) - 2S(n-3) = 0 \tag{4.99}$$

Hence the solution must take the form

$$S(n) = \alpha 2^n + \beta i^n + \gamma(-i)^n \tag{4.100}$$

Finding $S(1) = 1$, $S(2) = 1$ and $S(3) = 4$ yields the solution, $\alpha = \beta = \gamma = \frac{1}{2}$.

In order to establish (4.99) it is convenient to split the sum of $S(n)$ in two parts,

$$S(n) = U(n) + V(n) \tag{4.101}$$

$$U(n) = \sum_{3k \le n} 2^k \binom{n-k}{2k} \tag{4.102}$$

$$V(n) = \sum_{3k \le n-3} 2^k \binom{n-2-k}{2k+1} \tag{4.103}$$

These two functions satisfy the simultaneous difference equations,

$$U(n+1) - U(n) = 2V(n+1) \tag{4.104}$$

$$V(n+1) - V(n) = U(n-2) \tag{4.105}$$

obviously from splitting the binomial coefficients in the sums.

From these follows that both $U(n)$ and $V(n)$ satisfy the difference equation (4.99), and hence does their sum, $S(n)$.

Remark 1: The solution for $U(n)$ is as ugly as

$$U(n) = \tfrac{1}{5}\left(2^{n+1} + 3\cos\left(n \cdot \tfrac{\pi}{2}\right) + \sin\left(n \cdot \tfrac{\pi}{2}\right)\right) \tag{4.106}$$

while it for $V(n)$ is similar

$$V(n) = \tfrac{1}{5}\left(2^{n-1} + 2\cos\left(n \cdot \tfrac{\pi}{2}\right) - \sin\left(n \cdot \tfrac{\pi}{2}\right)\right) \tag{4.107}$$

It is seen that adding the two gives the simpler form (4.98).

Emre Alkan's problem. In 1995 Emre Alkan, Bosphorus University, Istanbul, Turkey, [3], posed the problem:

Prove that there are infinitely many positive integers, m, such that

$$\frac{1}{5\cdot 2^m}\sum_{k=0}^{m}\binom{2m+1}{2k}3^k$$

is an odd integer.

This expression is not an integer for all non–negative arguments, $m\in\mathbb{N}_0$, so we shall consider the function of m,

$$f(m)=\frac{1}{2^m}\sum_{k=0}^{m}\binom{2m+1}{2k}3^k \tag{4.108}$$

which function shall be proved to be a sequence of odd integers, sometimes divisible by 5.

The sum in (4.108) is the half of a binomial sum, so let us add and subtract the remaining terms and write

$$f(m)=\frac{1}{2^{m+1}}\sum_{k=0}^{2m+1}\binom{2m+1}{k}\left(\left(\sqrt{3}\right)^k+\left(-\sqrt{3}\right)^k\right) \tag{4.109}$$

then we may apply the binomial theorem twice to (2) and get

$$f(m)=\frac{1}{2^{m+1}}\left(\left(1+\sqrt{3}\right)^{2m+1}+\left(1-\sqrt{3}\right)^{2m+1}\right) \tag{4.110}$$

Now we distribute the 2's to get a sum of two plain powers:

$$f(m)=\frac{1+\sqrt{3}}{2}\left(\frac{\left(1+\sqrt{3}\right)^2}{2}\right)^m+\frac{1-\sqrt{3}}{2}\left(\frac{\left(1-\sqrt{3}\right)^2}{2}\right)^m \tag{4.111}$$

hence the function f must satisfy a difference equation with the two roots, $\frac{\left(1\pm\sqrt{3}\right)^2}{2}$, i.e. the characteristic polynomial

$$x^2-4x+1 \tag{4.112}$$

giving the difference equation

$$f(m+2)=4f(m+1)-f(m) \tag{4.113}$$

As we have easily $f(0)=1, f(1)=5$ which are odd integers, all of the values of f must be odd integers. Hence, if there are some divisible by 5, the quotient must be odd for each of them.

Rather than finding other values divisible by 5, we shall prove the following property, from which it follows that there are infinitely many such values:

Theorem. *For any integer, $z \in \mathbb{Z}$, the sums of integers for $m \in \mathbb{N}_0$,*

$$S_m(z) = \sum_{k=0}^{m} \binom{2m+1}{2k} z^k \tag{4.114}$$

satisfies for all $m \in \mathbb{N}_0$, that $S_m(z)$ divides $S_{3m+1}(z)$.

Proof. For any real or complex square root of z we have

$$\begin{aligned} S_m(z) &= \frac{1}{2} \sum_{k=0}^{2m+1} \binom{2m+1}{k} \left(\left(\sqrt{z}\right)^k + \left(-\sqrt{z}\right)^k \right) = \\ &= \frac{1}{2} \left(\left(1+\sqrt{z}\right)^{2m+1} + \left(1-\sqrt{z}\right)^{2m+1} \right) \end{aligned} \tag{4.115}$$

Using the identity

$$x^3 + y^3 = (x+y)\left(x^2 - xy + y^2\right) \tag{4.116}$$

we get from (4.115) that

(4.117)

$$\begin{aligned} &S_{3m+1}(z) = \frac{1}{2} \left(\left(1+\sqrt{z}\right)^{6m+3} + \left(1-\sqrt{z}\right)^{6m+3} \right) = \\ &= \frac{1}{2} \left(\left(1+\sqrt{z}\right)^{2m+1} + \left(1-\sqrt{z}\right)^{2m+1} \right) \cdot \\ &\cdot \left(\left(1+\sqrt{z}\right)^{4m+2} - \left(1+\sqrt{z}\right)^{2m+1} \left(1-\sqrt{z}\right)^{2m+1} + \left(1-\sqrt{z}\right)^{4m+2} \right) = \\ &= S_m(z) \cdot \left(\left(1+\sqrt{z}\right)^{4m+2} - (1-z)^{2m+1} + \left(1-\sqrt{z}\right)^{4m+2} \right) \end{aligned}$$

where the second factor in (4.116) is an integer since the odd powers of the square roots must cancel.

From this theorem we get that $5 = f(1)$ is a divisor in $f(4)$, which divides $f(13)$, etc.

Comments. For any $j \in \mathbb{N}$ the sequence (4.114) satisfies the difference equation

$$S_{m+2j}(z) = 2 \sum_i \binom{j}{2i} (z+1)^{j-2i} (4z)^i \, S_{m+j}(z) - (z-1)^{2j} \, S_m(z) \tag{4.118}$$

From this follows with $j = 2m+1$ that each term of the sequence $S_m(z)$, $S_{3m+1}(z)$, $S_{5m+2}(z)$ etc. is divisible by the first term, $S_m(z)$.

From this result with $z = 3$ follows that 5 is divisor in $f(7)$, $f(10)$, etc.

The divisor 5 may be replaced by any divisor in any term of the sequence, but what about 3 in 3^k? Of course, it may be replaced by any integer, $z \in \mathbb{Z}$, and for odd integers the sum (4.114) becomes divisible by a power of 2. We actually get three cases, z even, $z \equiv 3\ (4)$ and $z \equiv 1\ (4)$.

The first result is that for z even we have that all values are odd integers and the sequence satisfies the difference equation

$$S_{m+2}(z) = 2(z+1)S_{m+1}(z) - (z-1)^2 S_m(z) \tag{4.119}$$

easily derived from (4.115), and the start values $S_0(z) = 1$ and $S_2(z) = 1 + 3z$, easily derived from (4.114).

The second that we for $3 + 4z$ get that the sum

$$U_m(z) = \frac{1}{2^m} \sum_{k=0}^{m} \binom{2m+1}{2k} (3+4z)^k \tag{4.120}$$

satisfies the difference equation

$$U_{m+2}(z) = 4(z+1)U_{m+1}(z) - (2z+1)^2 U_m(z) \tag{4.121}$$

and the start values are $U_0(z) = 1$ and $U_1(z) = 5 + 6z$. So, in this case we get only odd integers.

And the third result is that we for $1 + 4z$ get that the sum

$$T_m(z) = \frac{1}{4^m} \sum_{k=0}^{m} \binom{2m+1}{2k} (1+4z)^k \tag{4.122}$$

satisfies the difference equation

$$T_{m+2}(z) = (1+2z)T_{m+1}(z) - z^2 T_m(z) \tag{4.123}$$

and the start values are $T_0(z) = 1$ and $T_1(z) = 1 + 3z$. If z is even then the terms are all odd, but if z is odd, they must alternate with a period of length 3.

In the binomial coefficients $\binom{2m+1}{2k}$ we may replace $2m+1$ with $2m$ or $2k$ by $2k+1$, e.g., $\binom{2m}{2k+1}$, and obtain similar results.

CHAPTER 5. CLASSIFICATION OF SUMS

Introduction. This classification is due to the late Erik Sparre Andersen (1919–2003), (1989) [7]. We consider sums of the form

$$T(c,n)=\sum_{k=0}^{n} t(c,n,k) \tag{5.1}$$

with $n \in \mathbb{N}_0$ the limit of summation, $c \in \mathbb{C}^{\ell}$, an argument–vector, and $k \in \mathbb{N}_0$ the summation variable.

We may assume the lower limit to be equal to zero, because it is otherwise trivial to translate the formula to obtain this limit.

The question is, do we know a formula, which by a trivial transformation gives us the value of the sum, (5.1)? By "trivial" we mean obtained by the following three operations:

1) Multiplying with a non–zero constant which may depend on c and n.
2) Special choice of the arguments and limits.
3) Reversing the direction of summation.

The way to treat the first triviality is to consider the *formal* quotient

$$q_t(c,n,k):=\frac{t(c,n,k+1)}{t(c,n,k)} \tag{5.2}$$

which characterizes the expressions up to proportionality, because of the formula:

$$T(c,n)=t(c,n,0)\sum_{k=0}^{n}\prod_{i=0}^{k-1} q_t(c,n,i) \tag{5.3}$$

Hence it is tempting to classify mainly according to the character of the quotient, (5.2).

Classification. We shall classify the sums (5.1) according to the nature of the terms and the quotient (5.3) in five types numbered I–V.

If the quotients are independent of the limit, n, we shall call the sum *indefinite,* otherwise it is called *definite.*

We shall apply the main classification as follows:

I) The terms take the form

$$t(c,n,k)=r(c,n,k)\cdot z^k \tag{5.4}$$

where $r(c,n,k)$ is a rational function of k. Typical examples are the quotient series, (3.1),

$$\sum_{k=0}^{n} z^k=\begin{cases}\frac{z^{n+1}-1}{z-1} & (z\neq 1)\\ n+1 & (z=1)\end{cases} \tag{5.5}$$

and the sum of polynomials, e.g., (2.7),

$$\sum\nolimits_0^n [k]_m \delta k = \begin{cases} \frac{[n]_{m+1}}{m+1} & \text{for } m \neq -1 \\ H_n & \text{for } m = -1 \end{cases} \tag{5.6}$$

II) Not of type I, but the quotients (5.2) are rational functions of k. Typical example is the binomial formula, cf. (7.1),

$$\sum_{k=0}^{n} \binom{n}{k} \cdot z^k = (1+z)^n \tag{5.7}$$

with quotient equal to

$$q(x,n,k) = \frac{n-k}{-1-k} \cdot (-z) \tag{5.8}$$

III) Not of types I or II, but the terms are products or quotients of terms, which might be of types I or II, and factors or divisors of the form $[x+ky]_k$, where $0 \neq y \neq \pm 1$. The typical example is the Hagen–Rothe formula, cf. (16.17),

$$\sum_{k=0}^{n} \binom{n}{k} [x+kz-1]_{k-1} [y+(n-k)z-1]_{n-k-1} = \frac{x+y}{xy} [x+y+nz-1]_{n-1} \tag{5.9}$$

IV) Not of types I or II, but the terms are products or quotients of terms, which might be of types I or II, and factors or divisors of the form $(x+ky)^k$, where $y \neq 0$. The typical example is the Abel formula, cf. (16.18),

$$\sum_{k=0}^{n} \binom{n}{k} (x+kz)^{k-1} (y-kz)^{n-k-1} = \frac{x+y-nz}{x(y-nz)} (x+y)^{n-1} \tag{5.10}$$

V) Not of types I or II, but the terms are products of factors, one as if it was a term of types I or II, and a harmonic number, H_k or $H_{c,k}^{(m)}$, cf. (1.19–20). A typical example is the formula

$$\sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} H_k = \frac{1}{n} \tag{5.11}$$

Canonical forms of sums of types I–II. In order to recognize a sum it is convenient to write it in a standard form. By definition the quotients (5.2) are rational functions of k, so they must take the form

$$q_t(c,n,k) = \frac{(\alpha_1 - k)\cdots(\alpha_p - k)}{(\beta_1 - k)\cdots(\beta_q - k)} \cdot z \tag{5.12}$$

with $z \neq 0$ and $\alpha_i \neq \beta_j$ independent of k, but dependent of c and n.

The type of the sum is said to have the *parameters* (p, q, z). So, we shall say, a sum is of type II(p, q, z), if the quotient is as in (5.12), but provided it is not of type I.

The products of quotients in (5.3) may by use of (5.12) be written as

$$r(k) := \frac{\prod_{j=1}^{p}[\alpha_j]_k}{\prod_{j=1}^{q}[\beta_j]_k} z^k \tag{5.13}$$

The function $r(k)$ in (5.13) may vanish for certain values of k, if $\alpha_j \in \mathbb{N}_0$ or $-\beta_j \in \mathbb{N}$. Similarly, it may be undefined or infinity, if $-\alpha_j \in \mathbb{N}$ or $\beta_j \in \mathbb{N}_0$.

Definition. *We shall call a sum (5.3) with quotient of the form (5.12) a* **sum of natural limits,** *if the function, $t(c, n, k)$, is defined for all $0 \le k \le n$ and $\beta_j = -1$, $\alpha_i = n$ for some (i, j).*

In this case we shall write

$$\frac{[\alpha_i]_k}{[\beta_j]_k} = \frac{[n]_k}{[-1]_k} = \binom{n}{k}(-1)^k \tag{5.14}$$

Without loss of generality we may assume that $i = j = 1$. If we further have the situation, that $\beta_j < 0$ or $n \le \beta_j$ for $j = 1, 2, \cdots q$, then we may apply

$$\frac{1}{[\beta_j]_k} = \frac{[\beta_j - k]_{n-k}}{[\beta_j]_n} = \frac{[n-1-\beta_j]_{n-k}(-1)^{n-k}}{[\beta_j]_n} = \frac{[b_j]_{n-k}(-1)^{n-k}}{[\beta_j]_n} \tag{5.15}$$

to replace the denominator $[\beta_j]_k$ by the numerator $[b_j]_{n-k}(-1)^{n-k}$, where we have set $b_j = n - 1 - \beta_j$.

With the replacements $a_j = \alpha_j$ and $x = (-1)^q z$ we may define:

Definition. *The* **canonical** *form of a sum of natural limits (5.3) with quotient of the form (5.12) is*

$$S(c, n) = \sum_{k=0}^{n} s(c, n, k) = \sum_{k=0}^{n} \binom{n}{k} \prod_{j=2}^{p}[a_j]_k \prod_{j=2}^{q}[b_j]_{n-k}\, x^k \tag{5.16}$$

From the knowledge of (5.16) it is easy to find the wanted sum from (5.3) as

$$T(c, n) = \frac{t(c, n, 0)}{s(c, n, 0)} S(c, n) \tag{5.17}$$

If the difference between a pair of roots equals one, say $\beta_2 - \alpha_2 = 1$, i.e., $a_2 + b_2 = n - 2$, we may replace the corresponding product by

(5.18)

$$\begin{aligned}
[a_2]_k\,[b_2]_{n-k} &= [a_2]_{k-1}\,(a_2 - k + 1)\,[n - a_2 - 2]_{n-k} = \\
&= (a_2 + 1 - k)\,[a_2]_{k-1}\,[a_2 - k + 1]_{n-k}\,(-1)^{n-k} = \\
&= [a_2]_{n-1}\,(-1)^n\,(a_2 + 1 - k)\,(-1)^k
\end{aligned}$$

The constant terms may be ignored, so we shall define

Definition. *The* **special canonical** *form of a sum with a pair of roots with difference one, i.e,* $a_2 + b_2 = n - 2$ *or* $\beta_2 - \alpha_2 = 1$ *is*

$$S(c,n) = \sum_{k=0}^{n} s(c,n,k) = \sum_{k=0}^{n} \binom{n}{k} \prod_{j=3}^{p} [a_j]_k \prod_{j=3}^{q} [b_j]_{n-k} \, (a_2 + 1 - k) \; (-x)^k \tag{5.19}$$

Remark. *In the form (5.17) it is particularly easy to change the direction of summation, as the sum must be equal to*

$$S(c,n) = \sum_{k=0}^{n} \binom{n}{k} \prod_{j=2}^{q} [b_j]_k \prod_{j=2}^{p} [a_j]_{n-k} \left(\frac{1}{x}\right)^k \tag{5.20}$$

while the roots are interchanged between numerator and denominator, with the b_j*'s in the numerator and the* $n - 1 - a_j$*'s in the denominator.*

Sums of arbitrary limits. We may try to reconstruct the sums of form (5.12) from the quotients (5.12) for all possible limits, $n \in \mathbb{N}$. The problem is that we have difficulties summing past the integral roots of the rational function (5.12).

The obvious idea is to sum the products for the different situations of no roots in the interval, $[0, n[$. But does it really matter? Suppose we have two integral roots, say α and β, satisfying

$$0 \le \alpha < \beta < n \tag{5.21}$$

and that we have either of the two cases

$$t((\alpha,\beta),n,k) = \begin{cases} [\beta - k]_{\beta-\alpha} \\ [k - \alpha - 1]_{\beta-\alpha} \end{cases} \tag{5.22}$$

Then the term factor becomes in both cases

$$\frac{t((\alpha,\beta),n,k)}{t((\alpha,\beta),n,0)} = g(k) = \frac{[\beta - k]_{\beta-\alpha}}{[\beta]_{\beta-\alpha}} \tag{5.23}$$

gives the quotient

$$q_g(k) = \frac{[\beta - k - 1]_{\beta-\alpha}}{[\beta - k]_{\beta-\alpha}} = \frac{\alpha - k}{\beta - k} \tag{5.24}$$

But, we must admit that

$$g(k) = 0 \quad \text{for} \quad \alpha < k \le \beta \tag{5.25}$$

The natural question is, are there other non–vanishing term factors giving the same quotient inside the interval $]\alpha, \beta]$?

The answer is yes, we may consider

$$h(k)=\frac{(-1)^k}{\binom{\beta-\alpha-1}{k-\alpha-1}} \tag{5.26}$$

which is defined for $\alpha<k\le\beta$ and gives the quotient wanted,

$$q_h(k)=-\frac{\binom{\beta-\alpha-1}{k-\alpha-1}}{\binom{\beta-\alpha-1}{k-\alpha}}=-\frac{[\beta-\alpha-1]_{k-\alpha-1}[k-\alpha]_{k-\alpha}}{[\beta-\alpha-1]_{k-\alpha}[k-\alpha-1]_{k-\alpha-1}}=-\frac{k-\alpha}{\beta-k}=\frac{\alpha-k}{\beta-k} \tag{5.27}$$

Similarly, if $\beta<\alpha$, either of the two term factors

$$g(k)=\begin{cases}\frac{1}{[\alpha-k]_{\alpha-\beta}}=[\beta-k]_{\beta-\alpha}\\ [k-\alpha-1]_{\beta-\alpha}\end{cases} \tag{5.28}$$

gives the quotient as above, and is defined for $k\le\beta$ and for $\alpha<k$.

For the interval $]\beta,\alpha]$ we may choose the term factor

$$h(k)=(-1)^k\binom{\alpha-\beta-1}{k-\beta-1} \tag{5.29}$$

which has the quotient

$$q_h(k)=-\frac{\binom{\alpha-\beta-1}{k-\beta}}{\binom{\alpha-\beta-1}{k-\beta-1}}=\frac{\alpha-k}{\beta-k} \tag{5.30}$$

and is defined for $\beta<k\le\alpha$.

Hypergeometric form. Sums of types I–II may be written as hypergeometric sums as well. To do that it is only necessary to use (5.15) and the ascending factorial from (1.8) and (2.5), possibly (2.1). As an example consider the Gaußian hypergeometric function,

$${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!} \tag{5.31}$$

With this we may write a canonical sum of type $\mathrm{II}(2,2,z)$ as

$$\sum_{k=0}^{n}\binom{n}{k}[\alpha]_k[\beta]_{n-k}z^k=[\beta]_n{}_2F_1(-n,-\alpha;\beta-n+1;z) \tag{5.32}$$

In general, the transformations are

$${}_pF_q(-n,a_1,\cdots;b_1,\cdots;z)=\frac{(-1)^{nq}}{[-b_1]_n\cdots}\sum_{k=0}^{n}\binom{n}{k}[-a_1]_k\cdots[b_1+n-1]_{n-k}\cdots((-1)^pz)^k \tag{5.33}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}[\alpha_1]_k\cdots[\beta_1]_{n-k}\cdots z^k=[\beta_1]_n\cdots{}_pF_q(-n,-\alpha_1,\cdots;\beta_1-n+1,\cdots;(-1)^pz) \tag{5.34}$$

The classification recipe. Given the formula (5.1), we first check whether the terms, $t(c,n,k)$ are rational functions in k, in which case the sum is of type I. Next, if this is not the case, we extract factors of the form of harmonic numbers, $H_{c,k}^{(m)}$, or the form (where we assume $0 \neq y \neq \pm d$)

$$[x + yk, d]_k \tag{5.35}$$

to see, if the types III–V may apply. Then we take the quotients of the rest of the terms, and if they are rational functions in k, we find the roots of the numerator and denominator. Now we know the possible type, II–V.

Eventually we compare the roots with the limits to figure out, if the limits are natural, or if it is convenient to divide the sum as sums over several different intervals. This might be the case, if there are several integral roots of the quotient in the original interval of summation.

Then we look in the table under the heading $X(p,q,z)$ to see, if the sum is known. If not, we may, in the case of type II(p,q,z) try the Zeilberger algorithm, and in the case of type I, the Gosper algorithm.

Symmetric and balanced sums. The majority of know sums with several factors are either symmetric or balanced. To recognize these cases the canonical form (5.16) is most convenient. But remark, that the binomial coefficient corresponds to a pair of factors by the transformation

$$\binom{n}{k} = \frac{[n]_k}{[k]_k} = \frac{[n]_k [n]_{n-k}}{[n]_n} \tag{5.36}$$

So, in the canonical form we may have the arguments $a_1 = n, a_2, \ldots$ for k and $b_1 = n, b_2, \ldots$ for $n-k$.

Definition. *A sum of type II$(p,p,\pm x)$ is called* **symmetric** *if we may write it in a canonical form (5.16) with $a_j = b_j$, $j = 2, \ldots, p$.*

This means that the canonical form of a symmetric sum becomes

$$S(c,n) = \sum_{k=0}^{n} \binom{n}{k} \prod_{j=2}^{p} [a_j]_k [a_j]_{n-k} x^k \tag{5.37}$$

Remark. *Symmetric sums of form II$(p,p,(-1)^{p-1})$ share the property, that the sums for n odd are zero, because of the change of sign by reversing the direction of summation.*

Definition. *A sum of type II$(p,p,\pm x)$ is called* **balanced** *if we may write it in a canonical form (5.16) with $a_j - b_j = 2a$, $j = 1, \ldots, p$ for some constant a.*

Remark. *In the case of balanced sums we have to let* $a_1 = n$ *and* $b_2 = n$*, such that we get* $b_1 = n - 2a$ *and* $a_2 = n + 2a$*, while the rest of the arguments may be written as* $a_j = c_j + a$ *and* $b_j = c_j - a$*.*

This means that the canonical form of a balanced sum becomes

$$S(c,n) = \sum_{k=0}^{n} \binom{n}{k} [n+2a]_k [n-2a]_{n-k} \prod_{j=3}^{p} [c_j + a]_k [c_j - a]_{n-k}\, x^k \tag{5.38}$$

Definition. *A sum is called* **well–balanced** *if it is balanced and may be written in a canonical form (5.38) for some constant* a*, and such that for some* $j > 2$ *we further have* $a_j + b_j = n - 2$*.*

If we have the canonical form and want to find out whether the sum is symmetric or balanced, the symmetric case is easy enough, and in the balanced case we shall have, that the roots of the quotient, (5.12), must satisfy, that the pairs have the constant sum, $n - 1 + 2a$, so a may be found from the sum of all the roots divided by $2p$.

The well–balanced case means that for one pair of corresponding roots the difference is one, $\beta_j - \alpha_j = 1$, which gives the strange expressions, $a_j = \alpha_j = \frac{n}{2} - 1 + a$, and therefore $\beta_j = \frac{n}{2} + a$ and $b_j = \frac{n}{2} - 1 - a$.

Remark. *In the case of well–balanced sums we may let* $a_2 = \frac{n}{2} - 1 + a$ *and* $b_2 = \frac{n}{2} - 1 - a$*. Hence we may write the product*

(5.39)
$$\begin{aligned}
[a_2]_k\,[b_2]_{n-k} &= \left[\tfrac{n}{2} - 1 + a\right]_k \left[\tfrac{n}{2} - 1 - a\right]_{n-k} = \\
&= \left[\tfrac{n}{2} - 1 + a\right]_k \left[\tfrac{n}{2} + a - k\right]_{n-k} (-1)^{n-k} = \\
&= (-1)^n \tfrac{1}{2} \left[\tfrac{n}{2} - 1 + a\right]_{n-1} \cdot (n + 2a - 2k)\,(-1)^k
\end{aligned}$$

Dividing the constants out we may get the special canonical form for well–balanced sums,

(5.40)
$$S(c,n) = \sum_{k=0}^{n} \binom{n}{k} [n+2a]_k [n-2a]_{n-k} \left(\prod_{j=3}^{p} [c_j + a]_k [c_j - a]_{n-k} \right) (n + 2a - 2k)\, x^k$$

Sometimes it is interesting and possible to find small deviations from these two basic demands. If we have the condition fulfilled except for $a_1 = b_1 + p$ or $a_1 - b_1 = 2a + p$ for some integers $p \in \mathbb{Z}$, we shall denote these sums as **quasi-**, i.e., we shall talk about **quasisymmetric**, **quasibalanced** and **quasiwell–balanced** sums.

Useful transformation. In order to write a given sum in the canonical form, it is often convenient to make one of the following transformations.

Lemma. *The products of constant length may be transformed to the canonical form for a sum of limits 0 and n by*

(5.41)
$$[a-k]_p = [a-p]_k[n-1-a]_{n-k}(-1)^k \cdot \frac{(-1)^n}{[a-p]_{n-p}} \qquad a \notin \{p, p+1, \ldots, n-1\}$$
(5.42)
$$[a+k]_p = [-a-1]_k[n-p+a]_{n-k}(-1)^k \cdot \frac{(-1)^{n+p}}{[-a-1]_{n-p}} \quad -a \notin \{1, 2, \ldots, n-p\}$$

Proof. Trivial. □

Some formulas seems occasionally very useful

(5.43) $$[2n]_n = 2[2n-1]_n$$
(5.44) $$[2n]_n = 4^n\left[n-\tfrac{1}{2}\right]_n = (-4)^n\left[-\tfrac{1}{2}\right]_n$$
(5.45) $$\binom{2n}{n} = (-4)^n\binom{-\frac{1}{2}}{n}$$

Other times it is convenient to extract consecutive binomial coefficients from a sequence of every second such,

(5.46) $$\binom{2n}{2k} = \frac{1}{\left[n-\frac{1}{2}\right]_n}\binom{n}{k}\left[n-\tfrac{1}{2}\right]_k\left[n-\tfrac{1}{2}\right]_{n-k}$$
(5.47) $$\binom{2n+1}{2k} = \frac{1}{\left[n-\frac{1}{2}\right]_n}\binom{n}{k}\left[n+\tfrac{1}{2}\right]_k\left[n-\tfrac{1}{2}\right]_{n-k}$$
(5.48) $$\binom{2n+1}{2k+1} = \frac{1}{\left[n-\frac{1}{2}\right]_n}\binom{n}{k}\left[n-\tfrac{1}{2}\right]_k\left[n+\tfrac{1}{2}\right]_{n-k}$$
(5.49) $$\binom{2n+2}{2k+1} = \frac{n+1}{\left[n+\frac{1}{2}\right]_{n+1}}\binom{n}{k}\left[n+\tfrac{1}{2}\right]_k\left[n+\tfrac{1}{2}\right]_{n-k}$$

The formulas (5.46) and (5.47) may be united and the formulas (5.48) and (5.49) united in two ways as

(5.50)
$$\binom{n}{2k} = \frac{1}{\left[\lfloor\frac{n}{2}\rfloor-\frac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor}}\binom{\lfloor\frac{n}{2}\rfloor}{k}\left[\lceil\tfrac{n}{2}\rceil-\tfrac{1}{2}\right]_k\left[\lfloor\tfrac{n}{2}\rfloor-\tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor-k}$$
(5.51)
$$\binom{n}{2k+1} = \frac{n}{2\left[\lfloor\frac{n-1}{2}\rfloor+\frac{1}{2}\right]_{\lfloor\frac{n-1}{2}\rfloor+1}}\binom{\lfloor\frac{n-1}{2}\rfloor}{k}\left[\lceil\tfrac{n-1}{2}\rceil-\tfrac{1}{2}\right]_k\left[\lfloor\tfrac{n-1}{2}\rfloor+\tfrac{1}{2}\right]_{\lfloor\frac{n-1}{2}\rfloor-k}$$

(5.52)
$$\binom{n+1}{2k+1} = \frac{n+1}{2\left[\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor+1}}\binom{\lfloor\frac{n}{2}\rfloor}{k}\left[\lceil\tfrac{n}{2}\rceil-\tfrac{1}{2}\right]_k\left[\lfloor\tfrac{n}{2}\rfloor+\tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor-k}$$

Another useful transformation is the following

$$\binom{n-k}{k} = \frac{(-1)^{\lfloor\frac{n}{2}\rfloor}}{[n]_{\lfloor\frac{n}{2}\rfloor}}\binom{\lfloor\frac{n}{2}\rfloor}{k}\left[\lceil\tfrac{n}{2}\rceil-\tfrac{1}{2}\right]_k\left[-\lceil\tfrac{n}{2}\rceil-1\right]_{\lfloor\frac{n}{2}\rfloor-k}(-4)^k \tag{5.53}$$

Or, if we divide in the even and the odd case, we get

$$\binom{2n-k}{k} = \frac{(-1)^n}{[2n]_n}\binom{n}{k}\left[n-\tfrac{1}{2}\right]_k[-n-1]_{n-k}(-4)^k \tag{5.54}$$

(5.55)
$$\binom{2n+1-k}{k} = \frac{(-1)^n}{[2n+1]_n}\binom{n}{k}\left[n+\tfrac{1}{2}\right]_k[-n-2]_{n-k}(-4)^k$$

It is furthermore convenient to have computed for $m \in \mathbb{Z}$ and $m\vee 0 \leq k \leq \lfloor\frac{n}{2}\rfloor$ the binomial of (5.53) with a factorial,

(5.56)
$$[k]_m\binom{n-k}{k} =$$
$$\frac{(-1)^{\lfloor\frac{n}{2}\rfloor-m}}{[n-2m]_{\lfloor\frac{n}{2}\rfloor-2m}}\binom{\lfloor\frac{n}{2}\rfloor-m}{k-m}\left[\lceil\tfrac{n}{2}\rceil-m-\tfrac{1}{2}\right]_{k-m}\left[-\lceil\tfrac{n}{2}\rceil-1\right]_{\lfloor\frac{n}{2}\rfloor-k}(-4)^{k-m}$$

Or, if we decide to split in even and odd cases,

(5.57)
$$[k+m]_m\binom{2n+m-k}{k+m} = \frac{(-1)^n}{[2n]_{n-m}}\binom{n}{k}\left[n-\tfrac{1}{2}\right]_k[-n-m-1]_{n-k}(-4)^k$$

(5.58)
$$[k+m]_m\binom{2n+m+1-k}{k+m} = \frac{(-1)^n}{[2n+1]_{n-m}}\binom{n}{k}\left[n+\tfrac{1}{2}\right]_k[-n-m-2]_{n-k}(-4)^k$$

The similar formulas for k added are much simpler, valid for $k, n \in \mathbb{N}$, $m \in \mathbb{Z}$, $-n \leq m$

$$\binom{n+k}{k} = \binom{-n-1}{k}(-1)^k \tag{5.59}$$

$$[k]_m\binom{n+k}{k} = [-n-1]_m\binom{-n-1-m}{k-m}(-1)^k \tag{5.60}$$

Polynomial factors. We may get rid of a polynomial factor in k by writing the polynomial as a sum of factorials, cf. (3.47), and then change the expression using (2.9) and (2.2) as follows:

$$\binom{n}{k}[k]_m[a]_k[b]_{n-k} = [n]_m[a]_m\binom{n-m}{k-m}[a-m]_{k-m}[b]_{n-m-(k-m)} \tag{5.61}$$

CHAPTER 6. GOSPER'S ALGORITHM

Gosper's algorithm. In 1978 R. William Gosper in [49] gave a beautiful algorithm to establish possible indefinite sums of the types I and II.

Assume we have a sum of the form

$$T(a,k) = \sum t(a,k)\delta k \tag{6.1}$$

with quotient function of the form

$$q_t(a,k) = \frac{(\alpha_1 - k)\cdots(\alpha_p - k)}{(\beta_1 - k)\cdots(\beta_q - k)} \cdot \chi \tag{6.2}$$

If it is possible, we shall choose the indices such that $\beta_i - \alpha_i \in \mathbb{N}, \quad i = 1, \cdots, j$, and $\beta_i - \alpha_\ell \notin \mathbb{N}, \quad i, \ell > j$. For each of the pairs with integral difference we shall write with $\alpha = \beta - m$ the quotient as

$$\frac{\alpha - k}{\beta - k} = \frac{\beta - m - k}{\beta - k} = \frac{(\beta - k - m)[\beta - k - 1]_{m-1}}{(\beta - k)[\beta - k - 1]_{m-1}} = \frac{[\beta - k - 1]_m}{[\beta - k]_m} \tag{6.3}$$

Using (6.3) we may define the polynomial $f(k)$ of degree $m_1 + \cdots + m_j$ as

$$f(k) = [\beta_1 - k]_{m_1} \cdots [\beta_j - k]_{m_j} \tag{6.4}$$

If we further define the polynomials of the rest of the factors,

$$g(k) = (\alpha_{j+1} - k)\cdots(\alpha_p - k) \cdot \chi \tag{6.5}$$

$$h(k) = (\beta_{j+1} + 1 - k)\cdots(\beta_q + 1 - k) \tag{6.6}$$

then they satisfy, that for no pair of roots do we have

$$\beta_i - \alpha_\ell \in \mathbb{N} \tag{6.7}$$

and we may write the quotient (6.2) as

$$q_t(a,k) = \frac{f(k+1)}{f(k)} \frac{g(k)}{h(k+1)} \tag{6.8}$$

With the quotient given in this form, we shall write the candidate for a solution to the sum, (6.1), as

$$T(a,k) = \frac{h(k)s(k)t(a,k)}{f(k)} \tag{6.9}$$

where $s(k)$ is an unknown function, we shall find, if it is possible.

If the function defined by (6.9) solves the equation (6.1), then we must have using the equation (6.8)

$$\begin{aligned}
t(a,k) &= \Delta T(a,k) = \\
&= \frac{h(k+1)s(k+1)t(a,k+1)}{f(k+1)} - \frac{h(k)s(k)t(a,k)}{f(k)} = \\
&= \frac{s(k+1)t(a,k)g(k)}{f(k)} - \frac{h(k)s(k)t(a,k)}{f(k)} = \\
&= t(a,k)\frac{s(k+1)g(k) - h(k)s(k)}{f(k)}
\end{aligned} \tag{6.10}$$

This means that if $s(k)$ shall solve the problem, it must satisfy the difference equation

$$f(k) = s(k+1)g(k) - s(k)h(k) \tag{6.11}$$

Now, it is obvious that the function $s(k)$ must be rational, i.e., we have polynomials, P and Q, without common roots, such that

$$s(k) = \frac{P(k)}{Q(k)} \tag{6.12}$$

We want to prove that $s(k)$ is a polynomial, so we assume, that Q is not constant. Then we may consider a non–root, β, such that $\beta+1$ is a root of Q, and there exists a greatest integer, $N \in \mathbb{N}$, such that $\beta+N$ is a root of Q.

If we substitute (6.12) in (6.11) and multiply with Q, we get

$$f(k)Q(k+1)Q(k) = g(k)P(k+1)Q(k) - h(k)P(k)Q(k+1) \tag{6.13}$$

Now we apply this equation for $k=\beta$ and $k=\beta+N$ to obtain

$$0 = g(\beta)P(\beta+1)Q(\beta) \tag{6.14}$$

$$0 = h(\beta+N)P(\beta+N)Q(\beta+N+1) \tag{6.15}$$

Neither P nor Q are zeros in the chosen points, hence we have

$$0 = g(\beta) \tag{6.16}$$

$$0 = h(\beta+N) \tag{6.17}$$

contradicting the property of g and h, that no pair of roots may have an integral difference of this sign. Hence s is a polynomial. Assume it looks like

$$s(k) = \alpha_d k^d + \alpha_{d-1}k^{d-1} + \cdots + \alpha_0, \quad \alpha_d \neq 0 \tag{6.18}$$

In order to solve (6.11) in s, we shall estimate the size of d.

We shall introduce the polynomials

$$G(k) = g(k) - h(k) \tag{6.19}$$

$$H(k) = g(k) + h(k) \tag{6.20}$$

Then we rewrite (6.11) as

$$2f(k) = G(k)(s(k+1) + s(k)) + H(k)(s(k+1) - s(k)) \tag{6.21}$$

and appreciate that

$$s(k+1) + s(k) = 2\alpha_d k^d + \cdots \qquad \text{of degree } d \tag{6.22}$$

$$s(k+1) - s(k) = d\alpha_d k^{d-1} + \cdots \quad \text{of degree } d-1 \tag{6.23}$$

With *deg* meaning "degree" we then have three possibilities.

1) If $deg(G) \geq deg(H)$, then we simply conclude that

$$d = deg(f) - deg(G) \tag{6.24}$$

2) If $d' = deg(H) > deg(G)$, then we consider

$$G(k) = \lambda' k^{d'-1} + \cdots \tag{6.25}$$

$$H(k) = \lambda k^{d'} + \cdots \quad \lambda \neq 0 \tag{6.26}$$

Hence the right hand side of (6.21) begins

$$(2\lambda'\alpha_d + \lambda d\alpha_d)k^{d+d'-1} \tag{6.27}$$

This means that we in general, namely if further $2\lambda' + \lambda d \neq 0$, have the formula

$$d = deg(f) - deg(H) + 1 \tag{6.28}$$

3) The last case is the exceptional, that $deg(H) > deg(G)$ and $2\lambda' + \lambda d = 0$. But then the latter equation yields

$$d = -2\frac{\lambda'}{\lambda} \tag{6.29}$$

This tells us that in the case of $deg(H) > deg(G)$ we may try one or two values for d, according to the integrability of the solution (6.29).

As soon as d is established, one may try to solve (6.11) in the $d+1$ coefficients.

An example of Gosper's algorithm. Let us consider the following indefinite sum of the type II(2,2,1) for a, b, c and d any complex numbers,

$$T(a,b,c,d,k) = \sum \frac{[a]_k[b]_k}{[c-1]_k[d-1]_k}\delta k \tag{6.30}$$

with quotient

$$q(k) = \frac{(a-k)(b-k)}{(c-1-k)(d-1-k)} \tag{6.31}$$

We shall apply the Gosper's algorithm to prove the following formula,

Theorem 6.1. *For any complex numbers, $a,b,c,d \in \mathbb{C}$ satisfying the condition that $p = a+b-c-d \in \mathbb{N}_0$, we have the indefinite summation formula*

$$\sum \frac{[a]_k[b]_k}{[c-1]_k[d-1]_k}\delta k = \frac{[a]_k[b]_k}{[c-1]_{k-1}[d-1]_{k-1}} \sum_{j=0}^{p} \frac{[p]_j[k-c-1]_j}{[a-c]_{j+1}[b-c]_{j+1}} \tag{6.32}$$

Proof. As we in general shall have the differences (6.7) not as positive integers, we have immediately the polynomial, $f(k) = 1$. The others are

$$g(k) = k^2 - (a+b)k + ab \tag{6.33}$$

$$h(k) = k^2 - (c+d)k + cd \tag{6.34}$$

Hence sum and difference becomes

$$G(k) = (c+d-a-b)k + ab - cd \tag{6.35}$$

$$H(k) = 2k^2 - (a+b+c+d)k + ab + cd \tag{6.36}$$

So we are in cases 2)–3) with $deg(G) < deg(H)$, and after (6.28) we compute $deg(f) - deg(H) + 1 = 0 - 2 + 1 = -1$, leaving us with the case 3) as the only possibility. Hence we may have

$$deg(s) = -2\frac{c+d-a-b}{2} = a+b-c-d \tag{6.37}$$

So we get the restriction on the parameters, that

$$p = a+b-c-d \in \mathbb{N}_0 \tag{6.38}$$

If $p = 0$, we have $s(k) = \alpha$ to be found by (6.11), i.e. from

$$1 = \alpha\,(g(k) - h(k)) = \alpha G(k) = \alpha(ab - cd) \tag{6.39}$$

So we have using $p = a + b - c - d = 0$

$$\alpha = \frac{1}{ab - cd} = \frac{1}{(a-c)(b-c)} \tag{6.40}$$

According to (6.9) we get the indefinite sum

$$\begin{aligned} T(a,b,c,d,k) =& \frac{(c-k)(d-k)}{(a-c)(b-c)} \cdot \frac{[a]_k[b]_k}{[c-1]_k[d-1]_k} = \\ =& \frac{1}{(a-c)(b-c)} \cdot \frac{[a]_k[b]_k}{[c-1]_{k-1}[d-1]_{k-1}} \end{aligned} \tag{6.41}$$

If $p = 1$, we have $s(k) = \alpha_1 k + \alpha_0$ to be found by (6.11), i.e. from

$$\begin{aligned} 1 &= (\alpha_1 k + \alpha_1 + \alpha_0)g(k) - (\alpha_1 k + \alpha_0)h(k) \\ &= (\alpha_1(ab - cd - a - b) - \alpha_0)k + \alpha_1 ab + \alpha_0(ab - cd) \end{aligned} \tag{6.42}$$

yielding us two equations in (α_1, α_0),

$$0 = \alpha_1(ab - cd - a - b) - \alpha_0 \tag{6.43}$$

$$1 = \alpha_1 ab + \alpha_0(ab - cd) \tag{6.44}$$

with the solutions

$$\alpha_0 = \frac{ab - cd - a - b}{ab + (ab - cd)(ab - cd - a - b)} \tag{6.45}$$

$$\alpha_1 = \frac{1}{ab + (ab - cd)(ab - cd - a - b)} \tag{6.46}$$

To make it look nicer, we shall write the denominator as

$$\begin{aligned} & ab + (ab - cd)(ab - cd - a - b) = \\ =& ab + (ab - cd)^2 - (ab - cd)(a + b) = \\ =& ab + (ab - cd)^2 - (ab - cd)(1 + c + d) = \\ =& (ab - cd)^2 - (ab - cd)(c + d) + cd = \\ =& (ab - cd - c)(ab - cd - d) = \\ =& (ab - c(a + b - c - 1) - c)(ab - (c + 1)(a + b - c - 1) = \\ =& (a - c)(b - c)(a - b - 1)(b - c - 1) = \\ =& [a - c]_2[b - c]_2 \end{aligned} \tag{6.47}$$

and the numerator similarly as

$$ab - cd - a - b = ab - c(a + b - c - 1) - a - b = (a - c - 1)(b - c - 1) - c - 1 \tag{6.48}$$

According to (6.9) we get the indefinite sum

(6.49)
$$\begin{aligned}
&T(a,b,c,d,k) = \\
&= \frac{k+(a-c-1)(b-c-1)-c-1}{[a-c]_2[b-c]_2} \cdot \frac{[a]_k[b]_k}{[c-1]_{k-1}[d-1]_{k-1}} = \\
&= \left(\frac{1}{(a-c)(b-c)} + \frac{k-c-1}{[a-c]_2[b-c]_2} \right) \cdot \frac{[a]_k[b]_k}{[c-1]_{k-1}[d-1]_{k-1}}
\end{aligned}$$

We may in principle continue for each positive integral value of p to end up with the formula for $p = a+b-c-d$

$$\sum \frac{[a]_k[b]_k}{[c-1]_k[d-1]_k}\delta k = \frac{[a]_k[b]_k}{[c-1]_{k-1}[d-1]_{k-1}} \sum_{j=0}^{p} \frac{[p]_j[k-c-1]_j}{[a-c]_{j+1}[b-c]_{j+1}} \tag{6.50}$$

but it is easier to proceed with guessing (6.50) and prove it by induction. □

Corollary 6.1. *For any complex numbers, $a,b,c,d \in \mathbb{C}$ satisfying the condition that $a+b-c-d=0$, we have the indefinite summation formula*

$$\sum \frac{[a]_k[b]_k}{[c-1]_k[d-1]_k}\delta k = \frac{1}{(a-c)(b-c)} \cdot \frac{[a]_k[b]_k}{[c-1]_{k-1}[d-1]_{k-1}} \tag{6.51}$$

Proof. Obvious. □

CHAPTER 7. SUMS OF TYPE II$(1,1,z)$

The binomial theorem. The most important, famous and oldest formula is the binomial theorem,

$$\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}=(x+y)^n \tag{7.1}$$

discovered around year 1000 by Al–Karaǧi in Baghdad, [36]. This formula is of type $II(1,1,z)$ with $z=\frac{x}{y}$. (For proof see theorem 8.1.)

Remark. *The binomial formula generalizes to $n\in\mathbb{C}$ in the form of an infinite series, the Taylor series, applying the general binomial coefficients, defined by (1.10)*

$$\sum_{k=0}^{\infty}\binom{n}{k}x^k y^{n-k}=(x+y)^n \ \ \textit{for}\ |x|<|y| \tag{7.2}$$

The formula (7.1) is of type $II(1,1,z)$ with natural limits and quotient

$$q_{binom}((x,y),n,k)=\frac{n-k}{-1-k}\left(-\frac{x}{y}\right) \tag{7.3}$$

An indefinite formula of type $II(1,1,1)$ is

$$\sum\frac{[\alpha]_k}{[\beta]_k}\delta k=\frac{1}{\alpha-\beta-1}\frac{[\alpha]_k}{[\beta]_{k-1}} \tag{7.4}$$

with quotient

$$q(\alpha,\beta,k)=\frac{\alpha-k}{\beta-k} \tag{7.5}$$

Proof.

$$\Delta\frac{[\alpha]_k}{[\beta]_{k-1}}=\frac{[\alpha]_{k+1}}{[\beta]_k}-\frac{[\alpha]_k}{[\beta]_{k-1}}=\frac{[\alpha]_k(\alpha-k-\beta+k-1)}{[\beta]_k}$$

We have a formula of type II(1,1,–1) due to Tor B. Staver, cf. [110],

$$\sum_{k=0}^{n}\frac{1}{\binom{n}{k}}=(n+1)\cdot 2^{-n-1}\sum_{k=1}^{n+1}\frac{2^k}{k} \tag{7.6}$$

with quotient

$$q_t(k)=\frac{-1-k}{n-k}(-1) \tag{7.7}$$

Proof of (7.6), Staver's formula. Define

$$S(n,p)=\sum_{k=0}^{n}\frac{k^p}{\binom{n}{k}} \tag{7.8}$$

Then we get

$$S(n,0)=\frac{1}{n}\left(\sum_{k=0}^{n}\frac{n-k}{\binom{n}{k}}+\sum_{k=0}^{n}\frac{k}{\binom{n}{k}}\right)=\frac{2}{n}S(n,1) \tag{7.9}$$

On the other hand we also get

(7.10)

$$\begin{aligned}S(n,0)=1+\sum_{k=1}^{n}\frac{k}{n\binom{n-1}{k-1}}&=1+\frac{1}{n}\sum_{k=0}^{n-1}\frac{k}{\binom{n-1}{k}}+\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}}=\\&=1+\frac{1}{n}S(n-1,1)+\frac{1}{n}S(n-1,0)\end{aligned}$$

When we apply (7.9) for $n-1$, we get

$$S(n,0)=1+\frac{1}{n}\frac{n-1}{2}S(n-1,0)+\frac{1}{n}S(n-1,0)=\frac{n+1}{2n}S(n-1,0)+1 \tag{7.11}$$

The homogeneous equation has the solution $\frac{n+1}{2^n}$, so we are looking for a solution of the form $\phi(n)\frac{n+1}{2^n}$, i.e., we look at the equation

$$\phi(n)\frac{n+1}{2^n}=\phi(n-1)\frac{n+1}{2n}\frac{n}{2^{n-1}}+1 \tag{7.12}$$

or

$$\phi(n)-\phi(n-1)=\frac{2^n}{n+1} \tag{7.13}$$

which solves the problem on the form

$$S(n,0)=\frac{n+1}{2^n}\sum_{k=0}^{n}\frac{2^k}{k+1}=(n+1)\cdot 2^{-n-1}\sum_{k=1}^{n+1}\frac{2^k}{k} \tag{7.14}$$

□

Gregory Galperin's and Hillel Gauchman's problem. In 2004 Gregory Galperin and Hillel Gauchman, Eastern Illinois University, Charleston, posed in Am. Math. Monthly, 111 as no. 11103, [43], the problem to prove the identity:

$$\sum_{k=1}^{n}\frac{1}{k\binom{n}{k}}=\frac{1}{2^{n-1}}\sum_{k=1,k\text{ odd}}^{n}\frac{\binom{n}{k}}{k} \tag{7.15}$$

The left side is

$$\sum_{k=1}^{n}\frac{1}{k\binom{n}{k}}=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\binom{n-1}{k-1}}=2^{-n}\sum_{k=1}^{n}\frac{2^k}{k}=\frac{1}{2^{n-1}}\sum_{k=1}^{n}\frac{2^{k-1}}{k} \tag{7.16}$$

by (7.6).

Now consider the sum on the right side of (7.15). The difference in n is

$$\sum_{k\text{ odd}}\frac{\binom{n}{k}}{k}-\sum_{k\text{ odd}}\frac{\binom{n-1}{k}}{k}=\sum_{k\text{ odd}}\frac{\binom{n-1}{k-1}}{k}=\frac{1}{n}\sum_{k\text{ odd}}\binom{n}{k} \tag{7.17}$$

As we have

$$\sum_{k\text{ odd}}\binom{n}{k}+\sum_{k\text{ even}}\binom{n}{k}=\sum_{k}\binom{n}{k}=2^n \tag{7.18}$$

$$\sum_{k\text{ even}}\binom{n}{k}-\sum_{k\text{ odd}}\binom{n}{k}=\sum_{k}\binom{n}{k}(-1)^k=0 \tag{7.19}$$

We get the wanted result from (7.16)

$$\sum_{k\text{ odd}}\frac{\binom{n}{k}}{k}=\sum_{k=1}^{n}\frac{2^{k-1}}{k} \tag{7.20}$$

CHAPTER 8. SUMS OF TYPE II$(2,2,z)$

The Chu–Vandermonde convolution. The most important sum at all is the generalization of one of the oldest formulas at all, the *Chu–Vandermonde convolution,*

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n} \tag{8.1}$$

with quotient

$$q_{CV} = \frac{(n-k)(x-k)}{(-1-k)(n-1-y-k)} \tag{8.2}$$

which appeared in the famous Chinese algebra text by Shih–Chieh Chu, *Precious Mirror of the Four Elements,* in 1303, [29,77], while its discovery in Europe was much later. (This is the usual title, the translation should rather be: *Textbook on four variables.*) As variables or unknowns he uses the four directions, north, east, south, west. It was discovered in 1772 of A. T. Vandermonde, [112]. The combinatorial interpretation is to consider an urn with x red and y blue balls. Now we may take n balls in $n+1$ ways, k red and $n-k$ blue. Together these numbers add up to the number of ways to take n out of $x+y$ balls ignoring the color. The generalization we appreciate here is to allow $x, y \in \mathbb{C}$ and the general step size, d, which for $d = 0$ includes the binomial theorem of type II$(1,1,z)$, (7.1)

Rather than considering the expression (8.1) with generalized binomial coefficients, it becomes nice if we multiply the equation with the integral denominator, $[n]_n$. Then (8.1) may be written as:

$$\sum_{k=0}^{n} \binom{n}{k} [x]_k [y]_{n-k} = [x+y]_n \tag{8.3}$$

But this expression allows a generalization with arbitrary stepsize, i.e.,

Theorem 8.1. *For $n \in \mathbb{N}_0$, $x, y \in \mathbb{C}$ and $d \in \mathbb{C}$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} [x,d]_k [y,d]_{n-k} = [x+y,d]_n \tag{8.4}$$

Proof. Let

$$S(n) := \sum_{k} \binom{n}{k} [x,d]_k [y,d]_{n-k}$$

We replace using (2.8) $\binom{n}{k}$ by $\binom{n-1}{k-1} + \binom{n-1}{k}$ and split the sum in two sums. In the sum with $\binom{n-1}{k-1}$ we substitute $k+1$ for k. Corresponding terms in the two sums now have $\binom{n-1}{k}[x,d]_k[y,d]_{n-k-1}$ as common factor. Using this we obtain

$$\sum_{k} \binom{n-1}{k} [x,d]_k [y,d]_{n-k-1} (x - k\cdot d + y - (n-k-1)\cdot d) = S(n-1)\cdot(x+y-(n-1)\cdot d)$$

Recursion now yields

$$S(n) = S(n-k) \cdot [x + y - (n-k) \cdot d, d]_k$$

Since $S(0) = 1$ we obtain (8.4). □

The quotient becomes in the case of $d \neq 0$

(8.5)
$$q_t(a,b,d,n,k) = \frac{(n-k)(\frac{a}{d} - k)}{(-1-k)(n-1-\frac{b}{d}-k)}$$

This means that we may always compare with a formula with $d = 1$, provided we have $d \neq 0$. Hence, if we come across a formula,

(8.6)
$$T(a,b,n) = \sum_{k=0}^{n} t(a,b,n,k)$$

with quotient

(8.7)
$$q_t(a,b,n,k) = \frac{t(a,b,n,k+1)}{t(a,b,n,k)} = \frac{(n-k)(a-k)}{(-1-k)(b-k)}$$

Then the formula has the type II(2,2,1) and is equivalent to the Chu–Vandermonde above, (8.4), and hence the sum must be

(8.8)
$$T(a,b,n) = \frac{t(a,b,n,0)}{[n-1-b]_n}[a+n-1-b]_n$$

As an example consider the formulas

(8.9)
$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} [c+kd,d]_m = [c,d]_{m-n} d^n [m]_n$$

(8.10)
$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k} [c-kd,d]_m = [c-nd,d]_{m-n} d^n [m]_n$$

The quotient in (8.9) is

(8.11)
$$-\frac{\binom{n}{k+1}}{\binom{n}{k}} \cdot \frac{[c+(k+1)d,d]_m}{[c+kd,d]_m} = \frac{n-k}{-1-k} \cdot \frac{-\frac{c}{d}-1-k}{m-\frac{c}{d}-1-k}$$

Hence we get from (8.8) that the sum must be
(8.12)
$$\frac{(-1)^n [c,d]_m}{\left[n-1-m+1+\frac{c}{d}\right]_n} \left[-\frac{c}{d} - 1 + n - 1 - m + 1 + \frac{c}{d}\right]_n = [c,d]_{m-n} d^n [m]_n$$

where we have used formulas (2.1), (2.2), (2.3) and (2.5). The formula (8.10) follows from (8.9) be reversing the direction of summation.

With $d = 1$ the formulas (8.9–8.10) are very frequently used in the forms:

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} [c+k]_m = [c]_{m-n} [m]_n \tag{8.13}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k} [c-k]_m = [c-n]_{m-n} [m]_n \tag{8.14}$$

First remark. *For $0 \leq m < n$ the right side becomes 0.*

Second remark. *For $c = 0$ the right side becomes 0, except for $m = n$. Hence*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} [k]_m = \delta_{mn} \cdot [n]_n = \delta_{mn} \cdot [m]_m \tag{8.15}$$

Third remark. *As we may write*

$$k^m = \sum_{j=1}^{m} \mathfrak{S}_m^{(j)} [k]_j \tag{8.16}$$

where $\mathfrak{S}_m^{(j)}$ are the Stirling numbers of the second kind as defined in chapter 3, cf. formula (3.47), we get the general formula

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^m = \begin{cases} 0 & \text{for } 0 \leq m < n \\ \mathfrak{S}_m^{(n)} \cdot [n]_n & \text{for } n \leq m \end{cases} \tag{8.17}$$

Fourth remark. *For $m = -1$ we may apply (2.3) to get (with c replaced by $c-d$)*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{c+kd} = \frac{d^n [-1]_n}{[c+nd, d]_{n+1}} = \frac{d^n (-1)^n [n]_n}{[c+nd, d]_{n+1}} \tag{8.18}$$

or

$$\frac{1}{c(c+d)\cdots(c+nd)} = \frac{1}{d^n [n]_n} \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{c+kd} \tag{8.19}$$

i.e., the partial fraction for a polynomial with equidistant roots; for $d = 1$ we even get

$$\frac{1}{(c)_{n+1}} = \frac{1}{[c+n]_{n+1}} = \frac{1}{[n]_n} \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{c+k} \tag{8.20}$$

i.e., the partial fraction for a polynomial with consecutive integral roots.

A simple example. Consider the formula

$$\sum_{k=0}^{n}(-1)^k\binom{n-k}{k}\binom{n-2k}{m-k}=1 \quad m\le n \tag{8.21}$$

In order to analyze such a formula we may find the quotients, we find

$$\begin{aligned} q(k) &= -\frac{[n-k-1]_{k+1}}{[n-k]_k}\cdot\frac{[k]_k}{[k+1]_{k+1}}\cdot\frac{[n-2k-2]_{m-k-1}}{[n-2k]_{m-k}}\cdot\frac{[m-k]_{m-k}}{[m-k-1]_{m-k-1}} \\ &= -\frac{[n-2k]_2}{n-k}\cdot\frac{1}{k+1}\cdot\frac{n-m-k}{[n-2k]_2}\cdot\frac{m-k}{1}=\frac{(m-k)(n-m-k)}{(-1-k)(n-k)} \end{aligned} \tag{8.22}$$

This is a quotient of type II$(2,2,1)$, i.e. of the Chu–Vandermonde convolution, cf. (8.3), if the limits are right. Now, it is obvious that terms for $k>m$ do vanish, so we may compute the value of (8.21) as

$$\begin{aligned} &\sum_{k=0}^{m}(-1)^k\binom{n-k}{k}\binom{n-2k}{m-k}= \\ &=\frac{\binom{n}{m}}{[m-n-1]_m}\sum_{k=0}^{m}\binom{m}{k}[n-m]_k[m-n-1]_{m-k}= \\ &=\frac{[n]_m[-1]_m}{[m]_m[m-n-1]_m}=1 \end{aligned} \tag{8.23}$$

After realizing that the sum is just a plain Chu–Vandermonde, one may look for a shortcut to this formula. First, we may assume $m\le\frac{n}{2}$, as otherwise the terms vanish for $k>n-m$. Then we may multiply and divide by $m!$ to write

$$\begin{aligned} &\sum_{k=0}^{m}(-1)^k\binom{n-k}{k}\binom{n-2k}{m-k}= \\ &=\frac{1}{m!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}[n-k]_k[n-2k]_{m-k}= \\ &=\frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}[n-k]_m(-1)^k= \\ &=\frac{1}{m!}[n-m]_{m-m}[m]_m=1 \end{aligned} \tag{8.24}$$

according to (8.14).

The Laguerre Polynomials. When we learn some smart formula like the expressions for the Laguerre polynomials,

$$L_n(x)=e^x\frac{d^n}{dx^n}(x^ne^{-x})=n!\sum_{j=0}^{n}\binom{n}{j}\frac{(-x)^j}{j!} \tag{8.25}$$

none of the equalities are immediately verified. The proof of the second equality goes like the following

(8.26)
$$\begin{aligned}
e^x \cdot \frac{d^n}{dx^n}(x^n e^{-x}) &= \left(\sum_{j=0}^{\infty} \frac{x^j}{j!}\right) \cdot \frac{d^n}{dx^n}\left(x^n \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}\right) \\
&= \left(\sum_{j=0}^{\infty} \frac{x^j}{j!}\right) \cdot \left(\sum_{k=0}^{\infty} \frac{(-1)^k [n+k]_n x^k}{k!}\right) = \sum_{k=0}^{\infty} x^k \sum_{j=0}^{k} \frac{(-1)^j [n+j]_n}{j!(k-j)!} \\
&= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j [n+j]_n = \sum_{k=0}^{\infty} \frac{x^k}{k!} (-1)^k [n]_k [n]_{n-k} \\
&= n! \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{k!}
\end{aligned}$$

The crucial point was of course the step

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^j [n+j]_n = (-1)^k [n]_k [n]_{n-k} \tag{8.27}$$

which is just (8.14).

The systematic way of recognizing the relevant form is to consider the quotient of two consecutive terms as a rational function of j,

$$q(j) = \frac{(k-j)\cdot(-k-1-j)}{(-1-j)\cdot(-1-j)} \tag{8.28}$$

and then recognize this as the special case with $m = 0$, $c = p = k$ of

$$q(j) = \frac{(k-j)\cdot(m-c-1-j)}{(m-1-j)\cdot(m-1+p-c-j)} \tag{8.29}$$

the quotient of the Chu–Vandermonde, (8.3), translated to

$$\sum_{j=m}^{k} \binom{k-m}{j-m} (-1)^{k-j} [c-m+j]_p = [p]_{k-m} [c]_{p-k+m} \tag{8.30}$$

Moriarty's formulas. The so–called Moriarty formulas are as given in H. W. Gould's table, [64], formulas 3.177 and 3.178:

$$\sum_{k=0}^{n-p} \binom{2n+1}{2p+2k+1} \binom{p+k}{k} = \binom{2n-p}{p} 2^{2n-2p} \tag{8.31}$$

$$\sum_{k=0}^{n-p} \binom{2n+1}{2p+2k} \binom{p+k}{k} = \frac{n}{2n-p} \binom{2n-p}{p} 2^{2n-2p} \tag{8.32}$$

These formulas were coined after professor Moriarty of H. T. Davis in 1962, cf. [31] because of their ugliness with reference to the short story by A. Conan Doyle, *The adventure of the final problem* from 1883, where he writes about the professor: "At the age of twenty–one he wrote a treatise upon the Binomial Theorem which had a European vogue. On the strength of it, he won the Mathematical Chair at one of our smaller Universities."

As pointed out by H. W. Gould, cf. [67, 65, 66], Moriarty was a master of disguise, and these formulas are nothing but disguises of the formulas for the coefficients to the Chebysheff polynomials. Furthermore, eventually they prove to be the Chu–Vandermonde formula, (8.3), in disguise.

The relation to the Chebysheff polynomials comes straightforward from the binomial theorem (7.1). Let $c = \cos x$ and $s = \sin x$, then we compute the real and imaginary parts of

(8.33)

$$\begin{aligned}
(c+is)^n &= \sum_{k=0}^{n} \binom{n}{k} c^{n-k} (is)^k = \\
&= \sum_k \binom{n}{2k} c^{n-2k} \left(-s^2\right)^k + is \sum_k \binom{n}{2k+1} c^{n-1-2k} \left(-s^2\right)^k = \\
&= \sum_k \binom{n}{2k} c^{n-2k} \left(c^2-1\right)^k + is \sum_k \binom{n}{2k+1} c^{n-1-2k} \left(c^2-1\right)^k = \\
&= \sum_k \binom{n}{2k} c^{n-2k} \sum_j \binom{k}{j} c^{2(k-j)} (-1)^j + \\
&+ is \sum_k \binom{n}{2k+1} c^{n-1-2k} \sum_j \binom{k}{j} c^{2(k-j)} (-1)^j = \\
&= \sum_j (-1)^j c^{n-2j} \sum_k \binom{n}{2k}\binom{k}{j} + is \sum_j (-1)^j c^{n-1-2j} \sum_k \binom{n}{2k+1}\binom{k}{j}
\end{aligned}$$

Rectifying the limits of summation yields the left sides of the formulas (8.31) and (8.32).

Rather than working on the cumbersome expressions, we just compute the quotients. Of (8.32) it is

(8.34)

$$q(k) = \frac{(2n-2p-2k)(2n-2p-2k+1)(p+k+1)}{(2p+2k+2)(2p+2k+1)(k+1)} = \frac{(n-p-k)\left(n-p+\frac{1}{2}-k\right)}{(-1-k)\left(-p-\frac{1}{2}-k\right)}$$

Hence it is a Chu–Vandermonde with limit $n-p$ and arguments $n-p+\frac{1}{2}$ and $n-\frac{1}{2}$. So, the sum must be equal to

$$\binom{2n+1}{2p} \cdot \frac{[2n-p]_{n-p}}{\left[n-\frac{1}{2}\right]_{n-p}} = \frac{2n+1}{2n+1-2p}\binom{2n-p}{p} 4^{n-p} \tag{8.35}$$

The quotient of (8.31) is

(8.36)
$$q(k) = \frac{(2n-2p-2k)(2n-2p-2k-1)(p+k+1)}{(2p+2k+2)(2p+2k+3)(k+1)} = \frac{(n-p-k)\left(n-p-\frac{1}{2}-k\right)}{(-1-k)\left(-p-\frac{3}{2}-k\right)}$$

Hence it is a Chu–Vandermonde with limit $n-p$ and arguments $n-p-\frac{1}{2}$ and $n+\frac{1}{2}$. So, the sum must be equal to

$$\binom{2n+1}{2p+1} \cdot \frac{[2n-p]_{n-p}}{\left[n+\frac{1}{2}\right]_{n-p}} = \binom{2n-p}{p} 4^{n-p} \tag{8.37}$$

An example of matrices as arguments. The solution to a **difference equation** is essentially the computation of the powers of some matrix, $\mathbf{A}^n$, for $n \in \mathbb{N}$.

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{8.38}$$

We shall need the half trace and the determinant of the matrix, and the discriminant for the characteristic equation too,

$$\Theta = \frac{a+d}{2} \tag{8.39}$$

$$D = ad - bc \tag{8.40}$$

$$\Delta = \Theta^2 - D \tag{8.41}$$

With these notations we may write the Cayley–Hamilton formula, cf. (4.52), for $\mathbf{A}$ as

$$\mathbf{A}^2 - 2\Theta\mathbf{A} + D\mathbf{E} = \mathbf{O} \tag{8.42}$$

or more convenient for us,

$$(\mathbf{A} - \Theta\mathbf{E})^2 = \Delta\mathbf{E} \tag{8.43}$$

For this reason we shall appreciate to write

$$\mathbf{A} = \Theta\mathbf{E} + (\mathbf{A} - \Theta\mathbf{E}) \tag{8.44}$$

Because $\mathbf{E}$ commutes with any matrix, we may apply the binomial formula, cf. (7.1), on the sum (8.44).

(8.45)
$$\begin{aligned}\mathbf{A}^n &= (\Theta\mathbf{E} + (\mathbf{A} - \Theta\mathbf{E}))^n = \\ &= \sum_k \binom{n}{2k}\Theta^{n-2k}(\mathbf{A}-\Theta\mathbf{E})^{2k} + \sum_k \binom{n}{2k+1}\Theta^{n-2k-1}(\mathbf{A}-\Theta\mathbf{E})^{2k+1} = \\ &= \sum_k \binom{n}{2k}\Theta^{n-2k}\Delta^k\mathbf{E} + \sum_k \binom{n}{2k+1}\Theta^{n-1-2k}\Delta^k(\mathbf{A}-\Theta\mathbf{E})\end{aligned}$$

In the case of $\Delta = 0$ we shall only get the terms for $k = 0$, i.e.,

$$\mathbf{A}^n = \Theta^n\mathbf{E} + n\Theta^{n-1}(\mathbf{A}-\Theta\mathbf{E}) = \Theta^n\left(\mathbf{E} + \frac{n}{\Theta}(\mathbf{A}-\Theta\mathbf{E})\right) \tag{8.46}$$

If $D = 0$, we have $\Delta = \Theta^2$ and hence the whole sum reduces to

$$\begin{aligned}\mathbf{A}^n &= \sum_k \binom{n}{2k}\Theta^n\mathbf{E} + \sum_k \binom{n}{2k+1}\Theta^{n-1}(\mathbf{A}-\Theta\mathbf{E}) = \\ &= 2^{n-1}\Theta^n\mathbf{E} + 2^{n-1}\Theta^{n-1}(\mathbf{A}-\Theta\mathbf{E}) = \\ &= (2\Theta)^{n-1}\mathbf{A}\end{aligned} \tag{8.47}$$

In the case of $\Delta > 0$ we shall remark, that

$$\left(\Theta \pm \sqrt{\Delta}\right)^n = \sum_k \binom{n}{2k}\Theta^{n-2k}\Delta^k \pm \sqrt{\Delta}\sum_k \binom{n}{2k+1}\Theta^{n-1-2k}\Delta^k \tag{8.48}$$

Solving these two equations we get

$$\sum_k \binom{n}{2k}\Theta^{n-2k}\Delta^k = \frac{\left(\Theta+\sqrt{\Delta}\right)^n + \left(\Theta-\sqrt{\Delta}\right)^n}{2} \tag{8.49}$$

$$\sum_k \binom{n}{2k+1}\Theta^{n-1-2k}\Delta^k = \frac{\left(\Theta+\sqrt{\Delta}\right)^n - \left(\Theta-\sqrt{\Delta}\right)^n}{2\sqrt{\Delta}} \tag{8.50}$$

So we get the formula:

(8.51)
$$\mathbf{A}^n = \frac{\left(\Theta+\sqrt{\Delta}\right)^n + \left(\Theta-\sqrt{\Delta}\right)^n}{2}\mathbf{E} + \frac{\left(\Theta+\sqrt{\Delta}\right)^n - \left(\Theta-\sqrt{\Delta}\right)^n}{2\sqrt{\Delta}}(\mathbf{A}-\Theta\mathbf{E})$$

In the case of $\Delta < 0$ we shall remark, that from (8.41) we have $\Delta = \Theta^2 - D$, and hence

$$\Delta^k = \left(\Theta^2 - D\right)^k = \sum_j \binom{k}{j}\Theta^{2k-2j}(-D)^j \tag{8.52}$$

Substituting (8.52) in (8.45) allows us to proceed as follows:

(8.53)

$$\begin{aligned}\mathbf{A}^n =&\sum_k \binom{n}{2k}\Theta^{n-2k}\sum_j\binom{k}{j}\Theta^{2k-2j}(-D)^j\mathbf{E}+\\&+\sum_k\binom{n}{2k+1}\Theta^{n-1-2k}\sum_j\binom{k}{j}\Theta^{2k-2j}(-D)^j(\mathbf{A}-\Theta\mathbf{E}) =\\=&\sum_j\Theta^{n-2j}(-D)^j\sum_k\binom{n}{2k}\binom{k}{j}\mathbf{E}+\\&+\sum_j\Theta^{n-1-2j}(-D)^j\sum_k\binom{n}{2k+1}\binom{k}{j}(\mathbf{A}-\Theta\mathbf{E}) =\\=&\sum_j\Theta^{n-2j}(-D)^j\frac{n}{2}\binom{n-j}{j}\frac{2^{n-2j}}{n-j}\mathbf{E}+\\&+\sum_j\Theta^{n-1-2j}(-D)^j\binom{n-1-j}{j}2^{n-1-2j}(\mathbf{A}-\Theta\mathbf{E})\end{aligned}$$

In the case of $\Delta < 0$ we always have $D > 0$, hence we may proceed

(8.54)

$$\begin{aligned}\mathbf{A}^n =&\left(\sqrt{D}\right)^n\frac{n}{2}\sum_j\binom{n-j}{j}\frac{(-1)^j}{n-j}\left(\frac{2\Theta}{\sqrt{D}}\right)^{n-2j}\mathbf{E}+\\&+\left(\sqrt{D}\right)^{n-1}\sum_j\binom{n-1-j}{j}(-1)^j\left(\frac{2\Theta}{\sqrt{D}}\right)^{n-1-2j}(\mathbf{A}-\Theta\mathbf{E})\end{aligned}$$

The sums may be recognized as the Chebysheff polynomials, cf. (8.33), so using the rewriting

$$\sin\left(\arccos\left(\frac{\Theta}{\sqrt{D}}\right)\right) = \sqrt{1-\left(\frac{\Theta}{\sqrt{D}}\right)^2} = \sqrt{\frac{D-\Theta^2}{D}} = \sqrt{\frac{-\Delta}{D}} \tag{8.55}$$

we go on to

(8.56)

$$\begin{aligned}\mathbf{A}^n =&\left(\sqrt{D}\right)^n\left(T_n\left(\frac{\Theta}{\sqrt{D}}\right)\mathbf{E}+\frac{1}{\sqrt{D}}U_{n-1}\left(\frac{\Theta}{\sqrt{D}}\right)(\mathbf{A}-\Theta\mathbf{E})\right) =\\=&\left(\sqrt{D}\right)^n\left(\cos\left(n\arccos\left(\frac{\Theta}{\sqrt{D}}\right)\right)\mathbf{E}+\frac{\sin\left(n\arccos\left(\frac{\Theta}{\sqrt{D}}\right)\right)}{\sqrt{-\Delta}}(\mathbf{A}-\Theta\mathbf{E})\right)\end{aligned}$$

Joseph M. Santmyer's problem. In 1994 Joseph M. Santmyer, [103], California University of Pennsylvania, California PA, posed the following problem:

If M, N are integers satisfying $1 \le m \le n-1$, prove that

$$\binom{2n-m-1}{2n-2m-1} - \binom{n-1}{m} = \sum_k \sum_j \binom{k+j}{k} \binom{2n-m-2k-j-3}{2(n-m-k-1)}. \tag{8.57}$$

We have nonzero terms only for $0 \le k \le n-m-1$ and $0 \le j \le m-1$. Hence the sum is finite.

We write the sum as

(8.58)

$$\begin{aligned} S &= \sum_{k=0}^{n-m-1} \sum_{j=0}^{m-1} \frac{[k+j]_k}{[k]_k} \cdot \frac{[2n-m-2k-j-3]_{m-1-j}}{[m-1-j]_{m-1-j}} \\ &= \sum_{k=0}^{n-m-1} \frac{1}{[k]_k} \sum_{j=0}^{m-1} [k+j]_k \cdot \frac{[2n-m-2k-3]_{m-1}[m-1]_j}{[2n-m-2k-3]_j[m-1]_{m-1}} \\ &= \frac{1}{[m-1]_{m-1}} \sum_{k=0}^{n-m-1} \frac{[2n-m-2k-3]_{m-1}}{[k]_k} \sum_{j=0}^{m-1} \frac{[k+j]_k[m-1]_j}{[2n-m-2k-3]_j} \end{aligned}$$

In this form the inner sum is recognizable. We need to compute the quotient of two consecutive terms. It is

$$q(j) = \frac{(m-1-j)(-k-1-j)}{(-1-j)(2n-m-2k-3-j)} \tag{8.59}$$

This quotient is the quotient of the Chu–Vandermonde formula with parameters $-k-1$ and $2m-2n+2k+1$. Hence the sum is easily computed as

(8.60)

$$\frac{[k]_k}{[2m-2n+2k+1]_{m-1}} \cdot [2m-2n+k]_{m-1} = \frac{[k]_k[2n-2m-k+m-2]_{m-1}}{[2n-2m-2k-1+m-2]_{m-1}}$$

Substitution of this result in the sum gives

(8.61)

$$\begin{aligned} S &= \frac{1}{[m-1]_{m-1}} \sum_{k=0}^{n-m-1} \frac{[2n-m-2k-3]_{m-1}[k]_k[2n-m-2-k]_{m-1}}{[k]_k[2n-m-2k-3]_{m-1}} = \\ &= \frac{1}{[m-1]_{m-1}} \sum_{k=0}^{n-m-1} [2n-m-2-k]_{m-1} \end{aligned}$$

This sum is easily calculated, but in order to get the wanted result we shall apply the reversing of the order of summation,

$$S = \frac{1}{[m-1]_{m-1}} \sum_{k=0}^{n-m-1} [n-1+k]_{m-1} \tag{8.62}$$

The quotient of this sum becomes

$$q(k) = \frac{n+k}{n-m+1+k} = \frac{-n-k}{-n+m-1-k} \tag{8.63}$$

so that the sum is equivalent to the sum (7.4) with $a = -n$ and $b = -n+m-1$, with arbitrary limits,

(8.64)
$$\begin{aligned}\sum_{k=0}^{n-m-1} \frac{[-n]_k}{[m-n-1]_k} &= \frac{1}{-m}\left(\frac{[-n]_{n-m}}{[m-n-1]_{n-m-1}} - \frac{1}{[m-n-1]_{-1}}\right)\\ &= \frac{1}{-m}\left(\frac{(-1)^{n-m}[n+n-m-1]_{n-m}}{(-1)^{n-m-1}[n-m+1+(n-m-2)]_{n-m-1}} - m + n\right)\\ &= \frac{1}{m}\left(\frac{[2n-m-1]_{n-m}}{[2n-2m-1]_{n-m-1}} - (n-m)\right)\\ &= \frac{1}{m}\left(\frac{[2n-m-1]_m}{[n-1]_{m-1}} - (n-m)\right)\end{aligned}$$

Hence we get for the very sum, S,

(8.65)
$$\begin{aligned}S &= \frac{1}{[m-1]_{m-1}} \cdot \frac{[n-1]_{m-1}}{1} \cdot \frac{1}{m}\left(\frac{[2n-m-1]_m}{[n-1]_{m-1}} - (n-m)\right) =\\ &= \frac{1}{[m]_m}\left([2n-m-1]_m - [n-1]_m\right) =\\ &= \frac{[2n-m-1]_m}{[m]_m} - \frac{[n-1]_m}{[m]_m} =\\ &= \binom{2n-m-1}{2n-2m-1} - \binom{n-1}{m}\end{aligned}$$

The number of parenthesis. Let P_n denote the number of ways we can place parentheses legally between n objects. We want to prove the formula

$$P_n = \frac{1}{n}\binom{2n-2}{n-1} \tag{8.66}$$

from the obvious recursion

$$P_{n+1} = \sum_{k=1}^{n} P_k P_{n+1-k} \tag{8.67}$$

Substitution means that we shall prove the formula

$$\frac{1}{n+1}\binom{2n}{n} = \sum_{k=1}^{n} \frac{1}{k}\binom{2k-2}{k-1}\frac{1}{n+1-k}\binom{2n-2k}{n-k} \tag{8.68}$$

Using the quotient we find that the sum is proportional to the sum

$$\sum_{k=1}^{n} \binom{n+1}{k} \left[\tfrac{1}{2}\right]_k \left[\tfrac{1}{2}\right]_{n+1-k} \tag{8.69}$$

Now, Chu–Vandermonde says the sum from 0 to $n+1$ is 0, so we must have that

$$\sum_{k=1}^{n} \binom{n+1}{k} \left[\tfrac{1}{2}\right]_k \left[\tfrac{1}{2}\right]_{n+1-k} = -2 \left[\tfrac{1}{2}\right]_{n+1} \tag{8.70}$$

Hence we get
(8.71)

$$\sum_{k=1}^{n} \frac{1}{k} \binom{2k-2}{k-1} \frac{1}{n+1-k} \binom{2n-2k}{n-k} = -\frac{[2n-2]_{n-1}}{(n+1)!\frac{1}{2}\left[\frac{1}{2}\right]_n} \cdot 2 \left[\tfrac{1}{2}\right]_{n+1} = \frac{1}{n+1} \binom{2n}{n}$$

An indefinite sum of type II$(2, 2, 1)$**.**

For any complex numbers, $a, b, c, d \in \mathbb{C}$ satisfying the condition that $a + b - c - d = 0$, we have the indefinite summation formula

$$\sum \frac{[a]_k [b]_k}{[c-1]_k [d-1]_k} \delta k = \frac{1}{(a-c)(b-c)} \cdot \frac{[a]_k [b]_k}{[c-1]_{k-1} [d-1]_{k-1}} \tag{8.72}$$

with the excess generalization: For any complex numbers, $a, b, c, d \in \mathbb{C}$ satisfying the condition that $p = a+b-c-d \in \mathbb{N}_0$, we have the indefinite summation formula

$$\sum \frac{[a]_k [b]_k}{[c-1]_k [d-1]_k} \delta k = \frac{[a]_k [b]_k}{[c-1]_{k-1} [d-1]_{k-1}} \sum_{j=0}^{p} \frac{[p]_j [k-c-1]_j}{[a-c]_{j+1} [b-c]_{j+1}} \tag{8.73}$$

We proved these formulas as corollary 6.1 and theorem 6.1, formulas (6.51) and (6.32).

Transformations of sums of type II$(2, 2, z)$**.** The standard form of a sum of type II$(2, 2, z)$ with natural limits is

$$T(a, b, n, z) = \sum_{k=0}^{n} \binom{n}{k} [a]_k [b]_{n-k} z^k \tag{8.74}$$

with quotient

$$q_t(a, b, n, k) = \frac{(n-k)(a-k)}{(-1-k)(n-b-1-k)} z \tag{8.75}$$

Explicit forms of the sum in (8.74) are in general not known. But for certain values of z, formulas exist with some restraints on the parameters, (a, b). Furthermore, the possible values of z appear in groups in general of order 6. This is due to the following transformations:

Transformation theorem. *The sums of the form (8.74) allows the following identities:*

$$T(a,b,n,z) = z^n T(b,a,n,1/z) \tag{8.76}$$
$$= (-1)^n T(a, n-a-b-1, n, 1-z) \tag{8.77}$$
$$= (z-1)^n T(n-a-b-1, a, n, 1/(1-z)) \tag{8.78}$$
$$= (1-z)^n T(n-a-b-1, b, n, z/(z-1)) \tag{8.79}$$
$$= (-z)^n T(b, n-a-b-1, n, 1-1/z) \tag{8.80}$$

Proof. The formulas follow from repetition of the two of them, (8.76) and (8.77). The former is obtained by reversing the direction of summation in (8.74). We only need to consider the formula (8.77), the right side becomes:

$$
\begin{aligned}
&(-1)^n T(a, n-a-b-1, n, 1-z) = \\
=&(-1)^n \sum_{k=0}^{n} \binom{n}{k} [a]_k [n-a-b-1]_{n-k} (1-z)^k = \\
=&(-1)^n \sum_{k=0}^{n} \binom{n}{k} [a]_k [n-a-b-1]_{n-k} \sum_{j=0}^{k} \binom{k}{j} (-z)^j = \\
=&(-1)^n \sum_{j=0}^{n} \binom{n}{j} (-z)^j \sum_{k=j}^{n} \binom{n-j}{k-j} [a]_k [n-a-b-1]_{n-k} = \\
=&(-1)^n \sum_{j=0}^{n} \binom{n}{j} (-z)^j [a]_j \sum_{k=0}^{n-j} \binom{n-j}{k} [a-j]_k [n-a-b-1]_{n-j-k} = \\
=&(-1)^n \sum_{j=0}^{n} \binom{n}{j} (-z)^j [a]_j [n-j-b-1]_{n-j} = \\
=&(-1)^n \sum_{j=0}^{n} \binom{n}{j} (-z)^j [a]_j [-n+j+b+1+(n-j)-1]_{n-j} (-1)^{n-j} = \\
=&\sum_{j=0}^{n} \binom{n}{j} z^j [a]_j [b]_{n-j} = \\
=&T(a,b,n,z)
\end{aligned}
$$

where we have interchanged the order of summation, applied (2.10), (2.2), (2.1) and eventually the Chu–Vandermonde formula, (8.3). □

This theorem tells us, that with each possible value of z, there are up to five other formulas easily obtainable giving the six values

$$z, 1/z, 1-z, 1/(1-z), z/(z-1), 1-1/z \tag{8.81}$$

The exceptions giving less than six different possibilities are $z = 1$, giving at most $1 - z = 0$, and $z = 1/z = -1$ related to $1 - z = 1 - 1/z = 2$ and $1/(1-z) = z/(z-1) = \frac{1}{2}$, giving a group of 3, and the sixth root of unit, $z = 1/(1-z) = e^{i\frac{\pi}{3}} = 1 - 1/z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ related to $1/z = 1 - z = z/(z-1) = e^{-i\frac{\pi}{3}} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$, giving a group of 2 complex conjugates.

The factor -1 or 2; Kummer's, Gauß' and Bailey's formulas. The formulas of types II(2, 2, -1) and II(2, 2, 2) are due to E. E. Kummer, [85], 1836, C. F. Gauß, [45], 1813, and W. N. Bailey, [20], 1935. As may be seen from the transformations (8.76–79), these formulas are closely related.

Consider a sum of the form

$$S(a,b,n,d) = \sum_{k=0}^{n} \binom{n}{k} [a,d]_k [b,d]_{n-k} (-1)^k \tag{8.82}$$

with the quotient

$$q_k = \frac{(n-k)(\frac{a}{d} - k)}{(-1-k)(n-1-\frac{b}{d}-k)} \cdot (-1) \tag{8.83}$$

It reduces to the binomial theorem for $d = 0$, and otherwise we may reduce the sum to the form of $d = 1$ due to (4.13). The factor (-1) makes the sum alternating, therefore, for $a = b$ and n odd we must have $S(a, a, 2n+1, d) = -S(a, a, 2n+1, d) = 0$.

Theorem 8.3. *The general Kummer formula for $a, b \in \mathbb{C}$ is*

$$S(a,b,n) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{b-a}{n-2j} [n]_{n-j} [a]_j (-1)^j \tag{8.84}$$

Proof. Let us define

$$S = \sum_{k=0}^{n} \binom{n}{k} [a]_k [b]_{n-k} (-1)^k$$

Then we apply (8.4) to the second factorial as

$$[b]_{n-k} = \sum_j \binom{n-k}{j-k} [a-k]_{j-k} [b-a+k]_{n-j}$$

Hence we obtain

$$S = \sum_k \sum_j \binom{n}{k} \binom{n-k}{j-k} (-1)^k [a]_k [a-k]_{j-k} [b-a+k]_{n-j}$$

We apply (2.10) and (2.2) and we interchange the order of summation to get

$$S = \sum_j \sum_k \binom{n}{j}\binom{j}{k}(-1)^k [a]_j [b-a+k]_{n-j} =$$
$$= \sum_j \binom{n}{j}[a]_j \sum_k \binom{j}{k}[b-a+k]_{n-j}(-1)^k$$

Now (8.12) implies that this is zero for $0 \le n-j < j$, but for $j \le \frac{n}{2}$ we get

$$S = \sum_j \binom{n}{j}[a]_j (-1)^j [b-a]_{n-2j}[n-j]_j$$

We apply (2.11) and (2.13) to write

$$\binom{n}{j}[n-j]_j = \binom{n-j}{j}[n]_j = \binom{n-j}{n-2j}[n]_j$$

and then (2.11) again to write

$$\binom{n-j}{n-2j}[b-a]_{n-2j} = \binom{b-a}{n-2j}[n-j]_{n-2j}$$

and finally (2.2) to write

$$[n]_j[n-j]_{n-2j} = [n]_{n-j}$$

The result is then

$$S = \sum_j \binom{b-a}{n-2j}[n]_{n-j}[a]_j(-1)^j$$

□

In general this theorem is the only formula known, but for two special cases there are improvements. If either $b-a$ or $b+a$ are integers, we can shorten the sum.

In the first case we have the quasi–symmetric Kummer formula with $p = b-a \in \mathbb{Z}$:

Theorem 8.4. *The quasi–symmetric Kummer formula for $a \in \mathbb{C}$ and $p \in \mathbb{Z}$ is*

$$S(a, a+p, n) = \sigma(p)^n \sum_{j=\lceil \frac{n-|p|}{2} \rceil}^{\lfloor \frac{n}{2} \rfloor} \binom{|p|}{n-2j}[n]_{n-j}[a+(p \wedge 0)]_j(-1)^j \tag{8.85}$$

Proof. Let $p \ge 0$. Then the change to the natural limits gives the formula (8.85). The general formula is obtained by reversing the order of summation. □

The special case of difference zero is the symmetric Kummer identity

Corollary 8.1. *The symmetric Kummer identity for $a \in \mathbb{C}$:*

$$\sum_{k=0}^{n} \binom{n}{k} [a]_k [a]_{n-k} (-1)^k = \begin{cases} [n]_m [a]_m (-1)^m & \text{for } n = 2m \\ 0 & \text{for } n \text{ odd} \end{cases} \tag{8.86}$$

By division with $[n]_n = [n]_m [m]_m$ we may rewrite the formula (8.85) to look analogous to (8.1):

Corollary 8.2. *The quasi–symmetric Kummer formula*

$$\sum_{k=0}^{n} \binom{x}{k} \binom{x+p}{n-k} (-1)^k = \sigma(p)^n \sum_{j=\lceil \frac{n-|p|}{2} \rceil}^{\lfloor \frac{n}{2} \rfloor} \binom{|p|}{n-2j} \binom{x + (p \wedge 0)}{j} (-1)^j \tag{8.87}$$

In the second case, $a + b \in \mathbb{Z}$, we have the different quasi–balanced and the balanced Kummer formulas. To prove these we need the following

Theorem 8.5. *The first quasi–balanced Kummer formula for $a \in \mathbb{C}$ and $0 \le n \ge p \in \mathbb{Z}$*

(8.88)

$$\sum_{k=0}^{n} \binom{n}{k} [n-p+a]_k [n-a]_{n-k} (-1)^k =$$

$$\frac{2^{n-(p\vee 0)}}{[n-p]_{-(p\wedge 0)}} \sum_{j=0}^{|p|} \binom{|p|}{j} (-\sigma(p))^{p+j} \cdot [n-a]_j [n-p+a]_{|p|-j} [n-j-1-a, 2]_{n-(p\vee 0)}$$

Proof. We apply (2.1) to the first factorial to obtain

$$S = \sum_{k=0}^{n} \binom{n}{k} [n-p+a]_k [n-a]_{n-k} (-1)^k = \sum_{k} \binom{n}{k} [n-a]_{n-k} [p-n-a+k-1]_k$$

The product runs from $n-a$ to $p-n-a$ with the exception of the factors in the factorial $[k-a]_{n-p+1}$. Therefore we may write using (2.2) twice

$$\begin{aligned} S &= \sum_{k} \binom{n}{k} \frac{[n-a]_{n-k} [k-a]_{n-p+1} [p-n-a+k-1]_k}{[k-a]_{n-p+1}} \\ &= \sum_{k} \binom{n}{k} \frac{[n-a]_{2n-p+1}}{[k-a+p-n+(n-p)]_{n-p+1}} \end{aligned}$$

We shall now apply (8.20) to the denominator with $c = k-a+p-n$ and $n = n-p$ to get

$$S = [n-a]_{2n-p+1} \sum_{k} \binom{n}{k} \frac{1}{[n-p]_{n-p}} \sum_{j} \binom{n-p}{j} \frac{(-1)^j}{k-a+p-n+j}$$

We substitute $i = k + j$ as summation variable in the second sum and get

$$S = \frac{[n-a]_{2n-p+1}}{[n-p]_{n-p}} \sum_k \binom{n}{k} \sum_i \binom{n-p}{i-k} \frac{(-1)^{i-k}}{p-n-a+i}$$

Then we interchange the order of summation and receive

$$S = \frac{[n-a]_{2n-p+1}}{[n-p]_{n-p}} \sum_i \frac{(-1)^i}{p-n-a+i} \sum_k \binom{n}{k} \binom{n-p}{i-k} (-1)^k$$

Now the integral quasi–symmetric Kummer formula (8.87) applies to the second sum (with the sign of p changed). Hence we get

$$S = \frac{[n-a]_{2n-p+1}}{[n-p]_{n-p}} \sum_i \frac{(-1)^i}{p-n-a+i} (-\sigma(p))^i \sum_j \binom{|p|}{i-2j} \binom{n-(p \vee 0)}{j} (-1)^j$$

Next we interchange the order of summation and get

$$S = \frac{[n-a]_{2n-p+1}}{[n-p]_{n-p}} \sum_j \binom{n-(p \vee 0)}{j} (-1)^j \sum_i \binom{|p|}{i-2j} \frac{(\sigma(p))^i}{p-n-a+i}$$

Now we substitute $k = i - 2j$ as summation variable in the second sum and get

$$S = \frac{[n-a]_{2n-p+1}}{[n-p]_{n-p}} \sum_j \binom{n-(p \vee 0)}{j} (-1)^j \sum_k \binom{|p|}{k} \frac{(\sigma(p))^k}{p-n-a+k+2j}$$

Again we interchange the order of summation while we divide the denominator by 2

$$S = \frac{[n-a]_{2n-p+1}}{[n-p]_{n-p}} \sum_k \binom{|p|}{k} \frac{(\sigma(p))^k}{2} \sum_j \binom{n-(p \vee 0)}{j} \frac{(-1)^j}{\frac{p-n-a+k}{2} + j}$$

Now we may again apply (8.20) with $c = \frac{p-n-a+k}{2}$, $n = n - (p \vee 0)$ on the inner sum. It becomes

$$S = \frac{[n-a]_{2n-p+1}}{[n-p]_{n-p}} \sum_k \binom{|p|}{k} \frac{(\sigma(p))^k}{2} \cdot \frac{[n-(p \vee 0)]_{n-(p \vee 0)}}{\left[\frac{p-n-a+k}{2} + n - (p \vee 0)\right]_{n-(p \vee 0)+1}}$$

Next step is to double each factor in the factorial of the denominator, which also doubles the stepsize, and we cancel common factors of the factorials before the sum, and finally we get

$$S = \frac{2^{n-(p \vee 0)}}{[n-p]_{-(p \wedge 0)}} [n-a]_{2n-p+1} \sum_k \binom{|p|}{k} \frac{(\sigma(p))^k}{[n-a+k-|p|, 2]_{n-(p \vee 0)+1}}$$

We reverse the direction of summation and get

$$S = \frac{2^{n-(p\vee 0)}}{[n-p]_{-(p\wedge 0)}} \sum_k \binom{|p|}{k} (\sigma(p))^{p-k} \frac{[n-a]_{2n-p+1}}{[n-a-k,2]_{n-(p\vee 0)+1}}$$

We use (2.2) to split the numerator as

$$[n-a]_{2n-p+1} = \\ [n-a]_k [n-a-k]_{2n-2(p\vee 0)+1} [-a-n-k+2(p\vee 0)-1]_{2(p\vee 0)-p-k}$$

and (2.8) to split

$$[n-a-k]_{2n-2(p\vee 0)+1} = [n-a-k,2]_{n-(p\vee 0)+1} [n-a-k-1,2]_{n-(p\vee 0)}$$

and finally (2.1) to write the last factorial as

$$[-a-n-k+2(p\vee 0)-1]_{2(p\vee 0)-p-k} = (-1)^{p-k} [a+n-p]_{|p|-k}$$

Then the result follows. □

Theorem 8.6. *The balanced Kummer identity for $a \in \mathbb{C}$:*

(8.89) $$\sum_{k=0}^{n} \binom{n}{k} [n+a]_k [n-a]_{n-k} (-1)^k = 2^n [n-1-a,2]_n$$

Proof. We just have to apply (8.88) for $p = 0$. □

Theorem 8.7. *The second quasi–balanced Kummer formula for $a \in \mathbb{C}$ and $p \in \mathbb{Z}$*

(8.90)
$$\sum_{k=0}^{n} \binom{n}{k} [n-p+a]_k [n-a]_{n-k} (-1)^k =$$
$$= \frac{2^{n-(p\vee 0)}}{[n-p]_{-(p\wedge 0)}} \sum_{j=0}^{|p|} \binom{|p|}{j} \sigma(p)^{p+j} [-a+n+j-1,2]_{n-(p\wedge 0)}$$

Proof. We shall define

$$S(n,a,p) := \sum_{k=0}^{n} \binom{n}{k} [n-p+a]_k [n-a]_{n-k} (-1)^k$$

We shall apply (2.4) to the second factorial to split the sum in two, and then the formula (2.8) to the second sum to get

$$\begin{aligned}S(n,a,p) = \sum_{k=0}^{n}\binom{n}{k}[(n-(p+1))+(a+1)]_k[n-1-a]_{n-k}(-1)^k \\ + n\sum_{k=0}^{n-1}\binom{n-1}{k}[((n-1)-(p-1))+a]_k[n-1-a]_{n-1-k}(-1)^k = \\ = S(n,a+1,p+1)+nS(n-1,a,p-1)\end{aligned}$$

Next we shall apply (2.4) to the first factorial to split the sum in two, and then the formula (2.8) to the second sum to get

$$\begin{aligned}S(n,a,p) = \sum_{k=0}^{n}\binom{n}{k}[(n-(p+1))+a]_k[n-a]_{n-k}(-1)^k \\ + n\sum_{k=1}^{n}\binom{n-1}{k-1}[((n-1)-(p-1))+(a-1)]_{k-1}\cdot \\ \cdot [n-1-(a-1)]_{n-1-(k-1)}(-1)^k = \\ = S(n,a,p+1)-nS(n-1,a-1,p-1)\end{aligned}$$

By eliminating respectively the second and the first terms of the right sides of the two formulas, we obtain two useful recursions

$$\begin{aligned}2S(n,a,p) &= S(n,a-1,p-1)+S(n,a,p-1) \\ 2(n+1)S(n,a,p) &= S(n+1,a,p+1)-S(n+1,a+1,p+1)\end{aligned}$$

Repeating the first formula p times or the second $-p$ times and then using the formula (8.89) valid for $p=0$ and cancelling common powers of 2 yields the form (8.90) after reversing the direction of summation. □

Corollary 8.3. *For arbitrary c and arbitrary $n, p \in \mathbb{N}_0$ and $\delta = \pm 1$ we have*

(8.91)
$$\begin{aligned}\sum_{j=0}^{p}\binom{p}{j}(\delta)^{p+j}[c+j,2]_{n+p} = \\ \sum_{j=0}^{p}\binom{p}{j}(-\delta)^{p+j}[c-1]_j[2n+p-1-c]_{p-j}[c-j,2]_n\end{aligned}$$

Proof. We compare the right sides of (8.88) and (8.90) and cancel the common factors. □

Related to the factor $z=-1$ is the factor 2, which appears in known formulas. E.g., the formulas of Gauß from 1813, [9], and Bailey from 1935, [6], which are the following formulas for the choice $p=0$.

Theorem 8.8. *The generalized Gauß identity for $a \in \mathbb{C}$, $p \in \mathbb{Z}$ and $n \in \mathbb{N}_0$*

$$\sum_{k=0}^{n} \binom{n}{k} [a]_k [n-p-1-2a]_{n-k} 2^k = \tag{8.92}$$
$$= (-\sigma(p))^n \sum_{j=\lceil \frac{n-|p|}{2} \rceil}^{\lfloor \frac{n}{2} \rfloor} \binom{|p|}{n-2j} [n]_{n-j} [a+(p \wedge 0)]_j (-1)^j$$

The quotient of Gauß is

$$q_g = \frac{(n-k)(a-k)}{(-1-k)(2a-p-k)} \cdot 2 \tag{8.93}$$

making it of type II$(2,2,2)$.

Theorem 8.9. *The generalized Bailey identity for $a \in \mathbb{C}$, $p \in \mathbb{Z}$ and $n \in \mathbb{N}_0$*

(8.94)
$$2^{(p-n)\vee 0} [n-p]_{-(p\wedge 0)} \sum_{k=0}^{n} \binom{n}{k} [a]_k [(p-n-1)]_{n-k} 2^k =$$
$$(-1)^n 2^{(n-(p\vee 0))\vee 0} \sum_{j=0}^{|p|} \binom{|p|}{j} \sigma(p)^{p+j} [-a+2n-p+j-1, 2]_{n-(p\wedge 0)}$$

The quotient of Bailey is

$$q_b = \frac{(n-k)(a-k)}{(-1-k)(2n-p-k)} \cdot 2 \tag{8.95}$$

making it of type II$(2,2,2)$.

Proof. We apply (8.77) to (8.85) to obtain the generalized Gauß formula, (8.92).

We replace a by $a+p-n$ in (8.90) to obtain as the left side

$$\sum_{k=0}^{n} \binom{n}{k} [a]_k [2n+p-a]_{n-k} (-1)^k \tag{8.96}$$

When we apply (8.77) to (8.96), we obtain the generalized Bailey formula, (8.94).
□

The factor $\frac{1}{2}+i\frac{\sqrt{3}}{2}$. If we choose the three parameters in the transformation formulas (8.76–79) equal, i.e., $a=b=n-a-b-1$, or precisely, $a=b=\frac{n-1}{3}$, then the transformation (8.78) yields with $\rho=\frac{1}{2}+i\frac{\sqrt{3}}{2}$

$$T(\tfrac{n-1}{3},\tfrac{n-1}{3},n,\rho)=(\rho-1)^nT(\tfrac{n-1}{3},\tfrac{n-1}{3},n,\rho)=\rho^{2n}T(\tfrac{n-1}{3},\tfrac{n-1}{3},n,\rho) \tag{8.97}$$

This means that we get 0, except if $n=3m$, because $\rho^6=1$. Furthermore, for m even the sum must be real, while it is purely imaginary for m odd.

Actually we have the formula

$$\sum_{k=0}^{n}\binom{n}{k}\left[\tfrac{n-1}{3}\right]_k\left[\tfrac{n-1}{3}\right]_{n-k}\rho^k=\begin{cases}0 & 3\not| n\\ \frac{[3m]_{2m}\left[-\frac{2}{3}\right]_m i^m}{(\sqrt{27})^m} & n=3m\end{cases} \tag{8.98}$$

This formula is a special case of no. 15.1.31 in [2], see p. 557, which has as other special cases with the factor mentioned the following 3 forms.

$$\sum_{k=0}^{n}\binom{n}{k}[3n+1]_k[-n-1]_{n-k}\rho^k=\left[-\tfrac{2}{3}\right]_n\left(-i\sqrt{27}\right)^n \tag{8.99}$$

$$\sum_{k=0}^{n}\binom{n}{k}[-n-1]_k[3n+1]_{n-k}\rho^k=\left[-\tfrac{2}{3}\right]_n\left(-i\sqrt{27}\rho\right)^n \tag{8.100}$$

(8.101)
$$\sum_{k=0}^{n}\binom{n}{k}[-n-1]_k[-n-1]_{n-k}\rho^k=\left[-\tfrac{2}{3}\right]_n\left(i\sqrt{27}\bar{\rho}\right)^n$$

Sums of types II$(2,2,z)$. From chapter 15, formulas (15.1–5), we know the sum for the four values of the constants in the factorial, $n\pm\frac{1}{2}$, and any choice of z.

(8.102)
$$\sum_{k=0}^{n}\binom{n}{k}\left[n-\tfrac{1}{2}\right]_k\left[n-\tfrac{1}{2}\right]_{n-k}z^k=[2n-1]_n\left(\left(\tfrac{\sqrt{z}+1}{2}\right)^{2n}+\left(\tfrac{\sqrt{z}-1}{2}\right)^{2n}\right)$$

(8.103)
$$\sum_{k=0}^{n}\binom{n}{k}\left[n+\tfrac{1}{2}\right]_k\left[n-\tfrac{1}{2}\right]_{n-k}z^k=[2n]_n\left(\left(\tfrac{\sqrt{z}+1}{2}\right)^{2n+1}-\left(\tfrac{\sqrt{z}-1}{2}\right)^{2n+1}\right)$$

(8.104)
$$\sum_{k=0}^{n}\binom{n}{k}\left[n+\tfrac{1}{2}\right]_k\left[n+\tfrac{1}{2}\right]_{n-k}z^k=\frac{[2n+1]_n}{\sqrt{z}}\left(\left(\tfrac{\sqrt{z}+1}{2}\right)^{2n+2}-\left(\tfrac{\sqrt{z}-1}{2}\right)^{2n+2}\right)$$

The formulas (8.102–8.104) are in Gould's table, [64], no. 1.38–39, 1.70–71. Some of the formulas above for the choice of $z=5,9,25$ are repeated in Gould's table as no's 1.74, 1.77 and 1.69, respectively. To recognize them it may be necessary to apply some of the transformations (8.23–8.24). For completeness we shall give the formulas obtained by these transformations in the nine other forms possible. The first one, (8.105), appears in Gould's table, [64], as no. 1.64.

(8.105)
$$\sum_{k=0}^{n}\binom{n}{k}\left[n-\tfrac{1}{2}\right]_k[-n]_{n-k}\,z^k=[-n]_n\left(\left(\tfrac{\sqrt{1-z}+1}{2}\right)^{2n}+\left(\tfrac{\sqrt{1-z}-1}{2}\right)^{2n}\right)$$

(8.106)
$$\sum_{k=0}^{n}\binom{n}{k}[-n]_k\left[n-\tfrac{1}{2}\right]_{n-k}z^k=[-n]_n\left(\left(\tfrac{\sqrt{z}+\sqrt{z-1}}{2}\right)^{2n}+\left(\tfrac{\sqrt{z}-\sqrt{z-1}}{2}\right)^{2n}\right)$$

(8.107)
$$\sum_{k=0}^{n}\binom{n}{k}\left[n+\tfrac{1}{2}\right]_k[-n-1]_{n-k}\,z^k=[-n-1]_n\left(\left(\tfrac{\sqrt{1-z}+1}{2}\right)^{2n+1}-\left(\tfrac{\sqrt{1-z}-1}{2}\right)^{2n+1}\right)$$

(8.108)
$$\sum_{k=0}^{n}\binom{n}{k}[-n-1]_k\left[n+\tfrac{1}{2}\right]_{n-k}z^k=\tfrac{[-n-1]_n}{\sqrt{z}}\left(\left(\tfrac{\sqrt{z-1}+\sqrt{z}}{2}\right)^{2n+1}-\left(\tfrac{\sqrt{z-1}-\sqrt{z}}{2}\right)^{2n+1}\right)$$

(8.109)
$$\sum_{k=0}^{n}\binom{n}{k}\left[n-\tfrac{1}{2}\right]_k[-n-1]_{n-k}\,z^k=\tfrac{[-n-1]_n}{\sqrt{1-z}}\left(\left(\tfrac{1+\sqrt{1-z}}{2}\right)^{2n+1}-\left(\tfrac{1-\sqrt{1-z}}{2}\right)^{2n+1}\right)$$

(8.110)
$$\sum_{k=0}^{n}\binom{n}{k}[-n-1]_k\left[n-\tfrac{1}{2}\right]_{n-k}z^k=\tfrac{[-n-1]_n}{\sqrt{z-1}}\left(\left(\tfrac{\sqrt{z}+\sqrt{z-1}}{2}\right)^{2n+1}-\left(\tfrac{\sqrt{z}-\sqrt{z-1}}{2}\right)^{2n+1}\right)$$

(8.111)
$$\sum_{k=0}^{n}\binom{n}{k}\left[n-\tfrac{1}{2}\right]_k\left[n+\tfrac{1}{2}\right]_{n-k}z^k=\tfrac{[2n]_n}{\sqrt{z}}\left(\left(\tfrac{1+\sqrt{z}}{2}\right)^{2n+1}-\left(\tfrac{1-\sqrt{z}}{2}\right)^{2n+1}\right)$$

(8.112)
$$\sum_{k=0}^{n}\binom{n}{k}\left[n+\tfrac{1}{2}\right]_k[-n-2]_{n-k}\,z^k=\tfrac{[-n-2]_n}{\sqrt{1-z}}\left(\left(\tfrac{\sqrt{1-z}+1}{2}\right)^{2n+2}-\left(\tfrac{\sqrt{1-z}-1}{2}\right)^{2n+2}\right)$$

(8.113)
$$\sum_{k=0}^{n}\binom{n}{k}[-n-2]_k\left[n+\tfrac{1}{2}\right]_{n-k}z^k=\tfrac{[-n-2]_n}{\sqrt{z}\sqrt{z-1}}\left(\left(\tfrac{\sqrt{z-1}+\sqrt{z}}{2}\right)^{2n+2}-\left(\tfrac{\sqrt{z-1}-\sqrt{z}}{2}\right)^{2n+2}\right)$$

Remark. *In the formulas the choice of the square roots shall be consistent throughout the right side.*

It is possible to generalize these formulas introducing a parameter, $m\in\mathbb{Z}$, to add to one of or both of the arguments. For $m<0$ we only know some recurrence formulas, but for $m \in \mathbb{N}_0$ we may give exact expressions. We have with the arbitrary z replace by different convenient expressions in the resulting variable, r, e.g.

(8.114)
$$\sum_{k=0}^{n} \binom{n}{k} \left[n-\tfrac{1}{2}\right]_k \left[n+m+\tfrac{1}{2}\right]_{n-k} \left(\frac{1+r}{1-r}\right)^{2k} =$$
$$= \frac{[2n]_{n-m}}{(r-1)^{2n}} \sum_{k=0}^{m} \frac{[2n+2m]_k[-m]_{m-k}}{(1+r)^{2m+1-k}} \cdot$$
$$\left(\left(\binom{m}{k} r - \binom{m-1}{k}\right) r^{2n+2m-k} + \binom{m}{k} - r\binom{m-1}{k}\right)$$

(8.115)
$$\sum_{k=0}^{n} \binom{n}{k} \left[n+\tfrac{1}{2}\right]_k \left[n+m+\tfrac{1}{2}\right]_{n-k} \left(\frac{1+r}{1-r}\right)^{2k} =$$
$$= \frac{[2n+1]_{n-m}}{(r-1)^{2n+1}} \sum_{k=0}^{m} \frac{[2n+2m+1]_k[-m]_{m-k}}{(1+r)^{2m+1-k}} \cdot$$
$$\left(\left(\binom{m}{k} r - \binom{m-1}{k}\right) r^{2n+2m+1-k} + \binom{m}{k} - r\binom{m-1}{k}\right)$$

(8.116)
$$\sum_{k=0}^{n} \binom{n}{k} \left[n-\tfrac{1}{2}\right]_k [-n-m-1]_{n-k} \left(\frac{-4r}{(1-r)^2}\right)^{k} =$$
$$= \frac{[2n]_{n-m}(-1)^n}{(r-1)^{2n}} \sum_{k=0}^{m} \frac{[2n+2m]_k[-m]_{m-k}}{(1+r)^{2m+1-k}} \cdot$$
$$\left(\left(\binom{m}{k} r - \binom{m-1}{k}\right) r^{2n+2m-k} + \binom{m}{k} - r\binom{m-1}{k}\right)$$

(8.117)
$$\sum_{k=0}^{n} \binom{n}{k} \left[n+\tfrac{1}{2}\right]_k [-n-m-2]_{n-k} \left(\frac{-4r}{(1-r)^2}\right)^{k} =$$
$$= \frac{[2n+1]_{n-m}(-1)^n}{(r-1)^{2n+1}} \sum_{k=0}^{m} \frac{[2n+2m+1]_k[-m]_{m-k}}{(1+r)^{2m+1-k}} \cdot$$
$$\left(\left(\binom{m}{k} r - \binom{m-1}{k}\right) r^{2n+2m+1-k} + \binom{m}{k} - r\binom{m-1}{k}\right)$$

(8.118)
$$\sum_{k=0}^{n}\binom{n}{k}\left[n+m+\tfrac{1}{2}\right]_k[-n-m-1]_{n-k}\left(\frac{-4r}{(1+r)^2}\right)^k=$$
$$=\frac{[2n]_{n-m}(-1)^n}{(r+1)^{2n}}\sum_{k=0}^{m}\frac{[2n+2m]_k[-m]_{m-k}}{(1+r)^{2m+1-k}}\cdot$$
$$\left(\left(\binom{m}{k}r-\binom{m-1}{k}\right)r^{2n+2m-k}+\binom{m}{k}-r\binom{m-1}{k}\right)$$

(8.119)
$$\sum_{k=0}^{n}\binom{n}{k}\left[n+m+\tfrac{1}{2}\right]_k[-n-m-2]_{n-k}\left(\frac{-4r}{(1+r)^2}\right)^k=$$
$$=\frac{[2n+1]_{n-m}(-1)^n}{(r+1)^{2n}(r-1)}\sum_{k=0}^{m}\frac{[2n+2m+1]_k[-m]_{m-k}}{(1+r)^{2m+1-k}}\cdot$$
$$\left(\left(\binom{m}{k}r-\binom{m-1}{k}\right)r^{2n+2m+1-k}+\binom{m}{k}-r\binom{m-1}{k}\right)$$

We prove these formulas by using the transformations (8.23–8.24) and in the forms of (8.116–8.117) the transformations (5.57) and (5.58). These formulas are then joint by the formula (5.56), so all there remains to be proven is the following:

Theorem 8.10. *For $r=1$ and any $m\in\mathbb{N}_0$ we have the formula*

$$\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}[j]_m\binom{n-j}{j}(-1)^j2^{n-2j}=(-1)^m m!\binom{n+1}{2m+1} \tag{8.120}$$

For any $r\in\mathbb{C}\setminus\{1\}$ and any $m\in\mathbb{N}_0$ we have the formula

(8.121)
$$\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}[j]_m\binom{n-j}{j}(-r)^j(1+r)^{n-2j}=(-r)^m\sum_{k=0}^{m}\frac{[n]_k[-m]_{m-k}}{(1-r)^{2m+1-k}}\cdot$$
$$\left(\left(\binom{m}{k}+r\binom{m-1}{k}\right)-\left(\binom{m}{k}r+\binom{m-1}{k}\right)(-1)^k r^{n-k}\right)$$

To prove this formula we shall appreciate a more general lemma, valid for $m\in\mathbb{Z}$, establishing a recursion for the kind of sums we consider.

Lemma. *The function defined for $n\in\mathbb{N}_0$, $m\in\mathbb{Z}$ and $r\in\mathbb{C}$ as*

$$f(n,m,r)=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}[j]_m\binom{n-j}{j}(-r)^j(1+r)^{n-2j} \tag{8.122}$$

satisfies the recursion in n and m,

$$f(n+2,m,r)-(1+r)f(n+1,m,r)+rf(n,m,r)=-mrf(n,m-1,r) \tag{8.123}$$

Proof. We split the binomial coefficient with (2.8) in two to write

$$\begin{aligned} f(n+2,m,r) = &\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor+1}[j]_m\binom{n+1-j}{j}(-r)^j(1+r)^{n+2-2j}+\\ &+\sum_{j=1}^{\lfloor\frac{n}{2}\rfloor+1}[j]_m\binom{n+1-j}{j-1}(-r)^j(1+r)^{n+2-2j}=\\ &=(1+r)f(n+1,m,r)+\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}[j+1]_m\binom{n-j}{j}(-r)^{j+1}(1+r)^{n-2j}\end{aligned}$$

Now we apply (2.4) to split the factorial to establish the formula (8.123). □

Proof of theorem 8.10. The difference equation (8.123) has the characteristic roots r and 1. Hence the homogeneous equation is solved by theorem 4.1 of the functions

$$\alpha r^n+\beta \text{ for } r\neq 1$$
$$\alpha n+\beta \text{ for } r=1$$

In particular this means that the solutions for $m=0$ follows from the equations $f(0,0,r)=1$ and $f(1,0,r)=1+r$ to be

$$f(n,0,r)=\frac{r^{n+1}-1}{r-1} \text{ for } r\neq 1$$
$$f(n,0,1)=n+1$$

The solutions may now be found by splitting the operator and applying (4.20) successively. This procedure must give the solutions in the form

$$f(n,m,r)=p_m(n,r)r^n+q_m(n,r) \text{ for } r\neq 1 \tag{8.124}$$
$$f(n,m,1)=s_m(n) \tag{8.125}$$

where p_m, q_m and s_m are polynomials in n of degree respectively m and $2m+1$.

Let us consider the special case $r=1$. Then we may write

$$s_m(n)=\sum_{k=0}^{2m+1}\gamma_k^m[n]_k$$

for which the lemma tells us that

$$\sum_{k=2}^{2m+1}\gamma_k^m[k]_2[n]_{k-2}=-m\sum_{k=0}^{2m-1}\gamma_k^{m-1}[n]_k$$

This gives the recursion in the coefficients

$$\gamma_k^m = -m\frac{\gamma_{k-2}^{m-1}}{[k]_2} \text{ for } k > 1$$

Hence we get e.g., $\gamma_2^1 = -\frac{1}{2}$ and $\gamma_3^1 = -\frac{1}{6}$ so that we find

$$f(n,1,1) = -\tfrac{1}{6}[n]_3 - \tfrac{1}{2}[n]_2 + \gamma_1^1 n + \gamma_0^1$$

The last two terms must vanish, so the final formula is

$$f(n,1,1) = -\binom{n+1}{3}$$

Induction now proves (8.120), but this case also follows from Chu–Vandermonde, (8.3).

Substitution of (8.124) in (8.123) yields after division with the appropriate power of $r \neq 0$,

(8.126)
$$rp_m(n+2,r) - (1+r)p_m(n+1,r) + p_m(n,r) = -mp_{m-1}(n,r)$$
(8.127)
$$\tfrac{1}{r}q_m(n+2,r) - (1+\tfrac{1}{r})q_m(n+1,r) + q_m(n,r) = -mq_{m-1}(n,r)$$

We may write the two polynomials as

(8.128)
$$p_m(n,r) = \sum_{k=0}^{m} \alpha_k^m(r)[n]_k$$
(8.129)
$$q_m(n,r) = \sum_{k=0}^{m} \beta_k^m(r)[n]_k$$

and denote we already know that

$$p_0(n,r) = \frac{r}{r-1} \quad q_0(n,r) = \frac{1}{r-1} = p_0\left(n, \tfrac{1}{r}\right)$$

Now substitution of (8.128) in (8.126) gives

(8.130)
$$\alpha_k^m(r) = \frac{1}{1-r}\left(\frac{m}{k}\alpha_{k-1}^{m-1}(r) + r(k+1)\alpha_{k+1}^m(r)\right)$$

For $m > 0$ we have the equations

(8.131)
$$\alpha_0^m(r) + \beta_0^m(r) = 0$$
(8.132)
$$\left(\alpha_1^m(r) + \alpha_0^m(r)\right)r + \beta_1^m(r) + \beta_0^m(r) = 0$$

If we assume that

$$\beta_1^m(r) = -\alpha_1^m\left(\tfrac{1}{r}\right)$$

then the solution of the simultaneous equations (8.131) and (8.132) proves that

$$\beta_0^m(r) = -\alpha_0^m\left(\tfrac{1}{r}\right)$$

The similarity of the equations (8.126) and (8.127) insures that we have the general identities

$$\beta_k^m(r) = -\alpha_k^m\left(\tfrac{1}{r}\right)$$

$$q_m(n,r) = -p_m\left(n, \tfrac{1}{r}\right)$$

The recursion (8.130) proves that

$$\alpha_m^m(r) = \frac{1}{1-r}\alpha_{m-1}^{m-1}(r) = \frac{r}{(1-r)^{m+1}}$$

Now we define

$$\sigma_k^m(r) = \frac{(1-r)^{2m+1-k}}{[m]_{m-k}}\alpha_k^m(r)$$

Substitution in (8.130) gives the recursion for this new function,

$$\sigma_k^m(r) = \sigma_{k-1}^{m-1}(r) + r\sigma_{k+1}^m(r) \tag{8.133}$$

In particular, we have $\sigma_m^m(r) = r$. Induction proves that $\sigma_k^m(r)$ is a polynomial in r with only two term, namely

$$\sigma_k^m(r) = \phi_k^m r^{m-k+1} + \psi_k^m r^{m-k} \tag{8.134}$$

Substation of (8.134) in (8.133) yields for the coefficients the very same recurrence, namely

$$\phi_k^m = \phi_{k-1}^{m-1} + \phi_{k+1}^m$$

which has as solution some kind of binomial coefficients. We find the solutions to be

$$\phi_k^m = \binom{2m-k-1}{m}$$

$$\psi_k^m = \binom{2m-k-1}{m-1}$$

Substitution backwards eventually yields

(8.135)

$$p_m(n,r) = -\sum_{k=0}^{m}\left(\binom{m}{k}r + \binom{m-1}{k}\right)\frac{[-m]_{m-k}(-r)^{m-k}}{(1-r)^{2m+1-k}}[n]_k$$

(8.136)

$$q_m(n,r) = \sum_{k=0}^{m}\left(\binom{m}{k} + r\binom{m-1}{k}\right)\frac{[-m]_{m-k}(-r)^{m}}{(1-r)^{2m+1-k}}[n]_k$$

Substitution of the formulas (8.135) and (8.136) in (8.124) gives the formula in the theorem, (8.121). □

Remark. *That the theorem in particular gives the formula for $m = 0$, gives an easy way to find the formula for $m = -1$,*

$$f(n,-1,r) = \frac{1}{(n+2)r}\left((1+r)^n - r^n - 1\right) \tag{8.137}$$

The recursion in the lemma, (8.122), allows the computation of formulas for $m < -1$ as soon as the formula for $m = -1$ is known.

Proof of (8.137). Consider the formula for $m = 0$,

$$f(n,0,r) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} (-r)^j (1+r)^{n-2j} = \frac{r^{n+1}-1}{r-1}$$

Take the first term of the sum, and then remove the two highest factors of the binomial coefficient to get

$$f(n,0,r) = (1+r)^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n-j}{j} \binom{n-j-1}{j-1} (-r)^j (1+r)^{n-2j}$$

In the sum, change the variable to $j-1$,

$$f(n,0,r) = (1+r)^n - r \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} \left(\frac{n}{j+1} - 1\right) \binom{n-2-j}{j} (-r)^j (1+r)^{n-2-2j}$$

which may be written as

$$f(n,0,r) = (1+r)^n - rf(n-2,-1,r) + rf(n-2,0,r)$$

This proves the formula (8.137), which remains valid for $r = 1$. □

The general difference equation. The only general result about sums of the form (8.109) is, that the function $T(a, b, n, z)$ must satisfy the second order difference equation in the variable n,

$$\left(\mathbf{E}^2 + ((1+z)(n+1) - za - b)\,\mathbf{E} + z(n-a-b)(n+1)\mathbf{I}\right) T(a,b,n,z) = 0 \tag{8.138}$$

For fixed a, b, z the equation has coefficients which are polynomials in n of degree two. So, no general solution is obtainable, but for special choices of a, b and z it is of course possible to solve the equation, but for all such choices, we have been able to figure out, we get some of the formulas mentioned above.

This formula is derived by the Zeilberger algorithm, cf. (16,24).

CHAPTER 9. SUMS OF TYPE II$(3,3,z)$

The Pfaff–Saalschütz and Dixon formulas. The most famous of these identities are the Pfaff–Sallschütz identity, first discovered of J. F. Pfaff (1797) and later reformulated by L. Saalschütz (1890) [97,102] and the two Dixon identities (1903) due to A. C. Dixon, [32], all of type II$(3,3,1)$.

Theorem 9.1. *If the numbers satisfy $a_1+a_2+b_1+b_2=n-1$ we have the Pfaff-Saalschütz formula*

(9.1)
$$\sum_{k=0}^{n}\binom{n}{k}[a_1]_k\,[a_2]_k\,[b_1]_{n-k}\,[b_2]_{n-k}\,(-1)^k=[a_1+b_1]_n\,[a_1+b_2]_n$$
$$=[b_1+a_1]_n\,[b_1+a_2]_n\,(-1)^n$$

The quotient is

$$q_{ps}=\frac{(n-k)\,(a_1-k)\,(a_2-k)}{(-1-k)\,(n-1-b_1-k)\,(n-1-b_2-k)} \tag{9.2}$$

Remark. *The formula is reflexive. If we reverse the order of summation, the condition repeats itself. But the sign will change for odd n.*

Theorem 9.2. *The symmetric Dixon formula is*
(9.3)
$$\sum_{k=0}^{n}\binom{n}{k}[a]_k[b]_k[a]_{n-k}[b]_{n-k}(-1)^k=\begin{cases}0 & \textit{for } n \textit{ odd}\\ [a]_m[b]_m[n-a-b-1]_m[n]_m & \textit{for } n=2m\end{cases}$$

Theorem 9.3. *The balanced Dixon formula is*

$$\sum_{k=0}^{n}\binom{n}{k}[n+2a]_k[b+a]_k[b-a]_{n-k}[n-2a]_{n-k}(-1)^k=[n-2a-1,2]_n[n+2b,2]_n \tag{9.4}$$

Besides these formulas there are formulas due to G. N. Watson (1925), [115], and F. J. W. Whipple (1925), [116], but both are transformations of the formulas due to Dixon above.

Transformations of sums of type II$(3,3,1)$. The sums of form

$$S_n\,(a_1,a_2,b_1,b_2)=\sum_{k=0}^{n}\binom{n}{k}[a_1]_k\,[a_2]_k\,[b_1]_{n-k}\,[b_2]_{n-k}\,(-1)^k \tag{9.5}$$

are independent of interchanging of a_1 and a_2, of b_1 and b_2, and by reversing the direction of summation, only by the factor $(-1)^n$ depending on the interchanging of the pair of sets, $\{a_1, a_2\}$ and $\{b_1, b_2\}$.

Let us define the function of two variables,

$$f_n(x, y) = n - 1 - x - y \tag{9.6}$$

and the transformation of the set of four variables,

$$\tau_n (a_1, a_2, b_1, b_2) = (f_n (a_1, b_1), b_2, a_1, f_n (a_2, b_2)) \tag{9.7}$$

With these notation we have the

Transformation theorem. *For S defined by (9.5) and the transformation of indices τ_n defined by (9.7), we have*

$$S_n (\tau_n (a_1, a_2, b_1, b_2)) = S_n (a_1, a_2, b_1, b_2) \tag{9.8}$$

or, written out

$$S_n (n - 1 - a_1 - b_1, b_2, a_1, n - 1 - a_2 - b_2) = S_n (a_1, a_2, b_1, b_2) \tag{9.9}$$

Proof. By Chu–Vandermonde, (8.3) we have using (2.1)

$$[b_1]_{n-k} = (-1)^{n-k} [-b_1 + n - k - 1]_{n-k} =$$
$$(-1)^{n-k} \sum_{j=k}^{n} \binom{n-k}{j-k} [a_1 - k]_{j-k} [n - a_1 - b_1 - 1]_{n-j}$$

Applied to the definition (9.5) it gives the form

$$S_n (a_1, a_2, b_1, b_2) =$$
$$\sum_{k=0}^{n} \sum_{j=k}^{n} \binom{n}{k} \binom{n-k}{j-k} [a_1]_k [a_1 - k]_{j-k} [a_2]_k [n - a_1 - b_1 - 1]_{n-j} [b_2]_{n-k} (-1)^n$$

Now we apply (2.10) and (2.2), and interchange the order of summation

$$S_n (a_1, a_2, b_1, b_2) =$$
$$(-1)^n \sum_{j=0}^{n} \binom{n}{j} [a_1]_j [n - a_1 - b_1 - 1]_{n-j} \sum_{k=0}^{j} \binom{j}{k} [a_2]_k [b_2]_{n-k}$$

Then we apply (2.1) to write $[b_2]_{n-k} = [b_2]_{n-j} [b_2 - n + j]_{j-k}$, so we get

$$S_n (a_1, a_2, b_1, b_2) =$$
$$(-1)^n \sum_{j=0}^{n} \binom{n}{j} [a_1]_j [n - a_1 - b_1 - 1]_{n-j} [b_2]_{n-j} \sum_{k=0}^{j} \binom{j}{k} [a_2]_k [b_2 - n + j]_{j-k}$$

Now we can apply Chu–Vandermonde, (8.3), to the inner sum and get

$$S_n\left(a_1,a_2,b_1,b_2\right)=$$
$$(-1)^n\sum_{j=0}^{n}\binom{n}{j}\left[a_1\right]_j\left[n-a_1-b_1-1\right]_{n-j}\left[b_2\right]_{n-j}\left[a_2+b_2-n+j\right]_j$$

When we apply (2.1) to the last term, we get

$$\left[a_2+b_2-n+j\right]_j=(-1)^j\left[n-a_2-b_2-1\right]_j$$

and the proof is finished after changing the direction of summation. □

A simple consequence of the transformation theorem is the following formula.

Theorem. *For any $a,b,c\in\mathbb{C}$ and $n,p\in\mathbb{N}_0$ we have*

(9.10)
$$\sum_{k=0}^{n}\binom{n}{k}[a]_k[b]_{n-k}[c-k]_p=$$
$$[a+b-p]_{n-p}\sum_{j=0}^{p}\binom{p}{j}[n]_j[b]_j[c-n]_{p-j}[p-1-a-b]_{p-j}(-1)^{p-j}$$

Proof. We apply the lemma in chapter 4 to, formula (4.35), to write

$$[c-k]_p=[c-p]_k[n-1-c]_{n-k}(-1)^k\cdot\frac{(-1)^n}{[c-p]_{n-p}}$$

Then the transformation (9.8) yields

$$[a+b-p]_{n-p}\sum_{j=0}^{n}\binom{n}{j}[p]_j[b]_j[c-n]_{p-j}[p-1-a-b]_{p-j}(-1)^{p-j}$$

And we only have to remark, that $\binom{n}{j}[p]_j=\binom{p}{j}[n]_j$. □

By interchanging the a's or b's or both we may get this transformation in four different forms, all looking like

$$\text{(9.11)}\qquad\left(a_1,a_2,b_1,b_2\right)\rightarrow\left(b,f_n\left(a,\hat{b}\right),a,f_n\left(\hat{a},b\right)\right),\quad\hat{a}\neq a,\hat{b}\neq b$$

Furthermore, by iteration of the transformation, τ_n, one gets another four formulas, where we remark that

$$\text{(9.12)}\qquad f_n\left(f_n\left(a,b\right),f_n\left(\hat{a},\hat{b}\right)\right)=a_1+a_2+b_1+b_2-n+1$$

is independent of interchanges, hence we get the forms for $a\in\{a_1,a_2\}$ and $b\in\{b_1,b_2\}$

$$\text{(9.13)}\qquad\left(a_1,a_2,b_1,b_2\right)\rightarrow\left(a,a_1+a_2+b_1+b_2-n+1,f_n\left(a,b_1\right),f_n\left(a,b_2\right)\right)$$

or

$$\text{(9.14)}\qquad\left(a_1,a_2,b_1,b_2\right)\rightarrow\left(f_n\left(a_1,b\right),f_n\left(a_2,b\right),a_1+a_2+b_1+b_2-n+1,b\right)$$

Not only is the Pfaff–Saalschütz theorem a simple consequence of the transformations (9.13) or (9.14), but they give us a generalization,

The generalized Pfaff-Saalschütz formula. *If the number*

(9.15)
$$p = a_1 + a_2 + b_1 + b_2 - n + 1$$

satisfies $p \in \mathbb{N}_0$, *then we have*

(9.16)
$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} [a_1]_k\, [a_2]_k\, [b_1]_{n-k}\, [b_2]_{n-k}\, (-1)^k = \\
&\sum_{k=0}^{p} \binom{p}{k} [n]_k\, [a_2]_k\, [a_1 + b_1 - p]_{n-k}\, [a_1 + b_2 - p]_{n-k}\, (-1)^k = \\
&[a_1 + b_1 - p]_{n-p}\, [a_1 + b_2 - p]_{n-p} \cdot \\
&\sum_{k=0}^{p} \binom{p}{k} [n]_k\, [a_2]_k\, [a_1 + b_1 - n]_{p-k}\, [a_1 + b_2 - n]_{p-k}\, (-1)^k
\end{aligned}$$

Proof. From (9.13) we get immediately

(9.17)
$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} [a_1]_k\, [a_2]_k\, [b_1]_{n-k}\, [b_2]_{n-k}\, (-1)^k = \\
&\qquad \sum_{k=0}^{n} \binom{n}{k} [p]_k\, [a_2]_k\, [n - 1 - a_2 - b_1]_{n-k}\, [n - 1 - a_2 - b_2]_{n-k}\, (-1)^k
\end{aligned}$$

Now we apply the condition to write $n - 1 - a_2 - b_2 = a_1 + b_1 - p$, and the formula (2.11) to write $\binom{n}{k}[p]_k = \binom{p}{k}[n]_k$. Then we use (2.2) to write $[a + b - p]_{n-k} = [a + b - p]_{n-p}\, [a + b - n]_{p-k}$. The limit of summation may be changed to p because of the binomial coefficient. This establishes (9.16). □

A special case of this formula for $p = 1$ appears in Slater, [105], formula (III.16).

We do not know of any general formulas for p of (9.15) a negative integer, but the special case of $p = -2$ has a two argument family of formulas found by I. M. Gessel and D. Stanton as a generalization of a one argument family due to R. W. Gosper, see [46], formula 1.9.

(9.18)
$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} [a - b - ab]_k [-na - ab - 1]_k [ab - a - 1]_{n-k} [n + na + b + ab - 1]_{n-k} (-1)^k \\
&\; = (n + 1)[n + b - 1]_n [-na - a - 1]_n
\end{aligned}$$

The formula (9.18) has excess (9.15), $p = -2$. This is a transformation by τ_n, (9.7), of the formula mentioned, which looks in our writing

(9.19)

$$\sum_{k=0}^{n}\binom{n}{k}[ab-a-1]_k[1-b]_k[n+b]_{n-k}[-na-ab-1]_{n-k}(-1)^k$$
$$=(n+1)[n+b-1]_n[-na-a-1]_n$$

Proof. From (2.2) and (2.1) we have

$$[n+b]_{n+2} = [n+b]_{n-k}[b+k]_2[1-b]_k(-1)^k$$

dividing both sides of (9.19) with this factor, we get the equivalent identity

$$\sum_{k=0}^{n}\binom{n}{k}\frac{[ab-a-1]_k[-na-ab-1]_{n-k}}{[b+k]_2} = \frac{(n+1)[-na-a-1]_n}{(n+b)(b-1)}$$

Splitting the denominator as $\frac{1}{[b+k]_2} = \frac{1}{b+k-1} - \frac{1}{b+k}$ we may split the sum in two, change the summation variable in one of them and join them again:

$$\sum_{k=0}^{n}\binom{n}{k}\frac{[ab-a-1]_k[-na-ab-1]_{n-k}}{[b+k]_2}$$
$$=\sum_{k=0}^{n}\binom{n}{k}\frac{[ab-a-1]_k[-na-ab-1]_{n-k}}{b+k-1}-$$
$$-\sum_{k=0}^{n}\binom{n}{k}\frac{[ab-a-1]_k[-na-ab-1]_{n-k}}{b+k}$$
$$=\sum_{k=0}^{n}\binom{n}{k}\frac{[ab-a-1]_k[-na-ab-1]_{n-k}}{b+k-1}-$$
$$-\sum_{k=1}^{n+1}\binom{n}{k-1}\frac{[ab-a-1]_{k-1}[-na-ab-1]_{n+1-k}}{b+k-1}$$
$$=\sum_{k=1}^{n+1}\binom{n}{k-1}\frac{[ab-a-1]_{k-1}[-na-ab-1]_{n-k}}{b+k-1}\cdot\frac{(n+1)a(b+k-1)}{k}+$$
$$+\frac{[-na-ab-1]_n}{b-1}$$
$$=a\sum_{k=1}^{n+1}\binom{n+1}{k}[ab-a-1]_{k-1}[-na-ab-1]_{n-k}+a[ab-a-1]_{-1}[-na-ab-1]_n$$
$$=\frac{-1}{a(b-1)(n+b)}\sum_{k=0}^{n+1}\binom{n+1}{k}[ab-a]_k[-na-ab]_{n+1-k}$$

Now we may apply the Chu–Vandermonde formula, (8.3), to obtain

$$\frac{-1}{a(b-1)(n+b)} \cdot [-na-a]_{n+1} = \frac{(n+1)[-na-a-1]_n}{(b-1)(n+b)}$$

as we wanted. □

Generalizations of Dixon's formulas. We want to consider a family of sums, similar to the symmetric Dixon, so let us define for $n \in \mathbb{N}_0$ and $p, q \in \mathbb{Z}$

$$S_n(a+p, b+q, a, b) = \sum_{k=0}^{n} \binom{n}{k} [a+p]_k [b+q]_k [a]_{n-k} [b]_{n-k} (-1)^k \tag{9.20}$$

For $p = q = 0$ we get the sum in the symmetric Dixon formula, (9.3).

We can find expressions for such sums for $p = q$. We shall claim

Theorem 9.4. *The quasi–symmetric Dixon formulas for $p \in \mathbb{N}_0$ are*

$$S_n(a+p, b+p, a, b) = \sum_{j=\lceil \frac{n}{2} \rceil}^{\lfloor \frac{n+p}{2} \rfloor} \binom{p}{2j-n} [n]_j \, [n-p-a-b-1]_j \, [a]_{n-j} \, [b]_{n-j} \tag{9.21}$$

and

$$\begin{aligned} &S_n(a-p, b-p, a, b) = \\ &(-1)^n \sum_{j=\lceil \frac{n}{2} \rceil}^{\lfloor \frac{n+p}{2} \rfloor} \binom{p}{2j-n} [n]_j \, [n+p-a-b-1]_j \, [a-p]_{n-j} \, [b-p]_{n-j} \end{aligned} \tag{9.22}$$

Proof. We prove (9.21). The Pfaff–Saalschütz formula (9.1) yields

$$[a+p]_k [b+p]_k = \sum_{j=0}^{k} \binom{k}{j} [\alpha]_j [\beta]_j [a+p-\alpha]_{k-j} [b+p-\alpha]_{k-j} (-1)^j \tag{9.23}$$

provided $\beta = \alpha + k - 2p - a - b - 1$. When we substitute (9.23) in the sum (9.20) with $q = p$, we get

$$\begin{aligned} &S_n(a+p, b+p, a, b) = \\ &\sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} [\alpha]_j [\beta]_j [a+p-\alpha]_{k-j} [b+p-\alpha]_{k-j} [a]_{n-k} [b]_{n-k} (-1)^{j+k} \end{aligned} \tag{9.24}$$

Now we apply (2.10) to write $\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$. Then we choose $\alpha = n + p - k$, so that we may apply (2.2) to write $[a]_{n-k}[a+p-\alpha]_{k-j} = [a]_{n-j}$ and

$[b]_{n-k}[b+p-\alpha]_{k-j}=[b]_{n-j}$. Eventually we interchange the order of summations to obtain if we remember that $\beta=n-p-a-b-1$,

(9.25)
$$S_n(a+p,b+p,a,b)=$$
$$=\sum_{j=0}^{n}\binom{n}{j}[a]_{n-j}[b]_{n-j}[n-p-a-b-1]_j\sum_{k=j}^{n}\binom{n-j}{k-j}[n+p-k]_j(-1)^{j+k}$$

When we apply (8.14) to the inner sum, it may be evaluated as

$$[n+p-j-n+j]_{j-n+j}j!=[p]_{2j-n}j! \tag{9.26}$$

We only need to write $\binom{n}{j}[p]_{2j-n}[j]_{n-j}=\binom{p}{2j-n}[n]_j$ and restrict summation to nonzero terms to obtain (9.21).

To prove (9.22) we reverse the direction of summation and apply (9.21) to $(a-p,b-p)$. □

We do not know formulas for the general sums (9.20), but we may give you a couple of recursion formulas.

The general recursion formula. *The sums in (9.20) satisfy*
(9.27)
$$S_n(a+p,b+q,a,b)=S_n(a+p-1,b+q,a,b)-n(b+q)S_{n-1}(a+p-1,b+q-1,a,b)$$

Proof. We apply (2.4) to write $[a+p]_k=[a+p-1]_k+k[a+p-1]_{k-1}$. Then the sum splits in the two mentioned in (9.27). □

The particular interesting case of $q=0$ may be treated by a recursion of such sums,

The special recursion formula. *The sums in (9.20) satisfy*

(9.28)
$$\begin{aligned}S_n(a+p,b,a,b)=&S_n(a+p-1,b,a,b)-n(b-a-p+1)S_{n-1}(a+p-1,b,a,b)-\\&n(a+p-1)S_{n-1}(a+p-2,b,a,b)\end{aligned}$$

Proof. We apply (2.4) to write $[a+p]_k=[a+p-1]_k+k[a+p-1]_{k-1}$. Then the sum splits in two, the first one is just $S_n(a+p-1,b,a,b)$, but the second is

(9.29)
$$S_n(a+p,b,a,b)=$$
$$=S_n(a+p-1,b,a,b)+\sum_{k=0}^{n}\binom{n}{k}k[a+p-1]_{k-1}[b]_k[a]_{n-k}[b]_{n-k}(-1)^k=$$
$$=S_n(a+p-1,b,a,b)+n\sum_{k=1}^{n}\binom{n-1}{k-1}[a+p-1]_{k-1}[b]_k[a]_{n-k}[b]_{n-k}(-1)^k=$$
$$=S_n(a+p-1,b,a,b)-nT_{n-1}(a,b,p-1)$$

where we have put

$$T_n(a,b,p) = \sum_{k=0}^{n} \binom{n}{k} [a+p]_k [b]_{k+1} [a]_{n-k} [b]_{n-k} (-1)^k \tag{9.30}$$

If we write $[b]_{k+1} = [b]_k(b-k)$, then we get the new sum, T_n, written as a sum of two,

(9.31)

$$\begin{aligned} T_n(a,b,p) &= \\ &= bS_n(a+p,b,a,b) + n\sum_{k=0}^{n-1} \binom{n-1}{k} [a+p]_{k+1}[b]_{k+1}[a]_{n-1-k}[b]_{n-1-k}(-1)^k = \\ &= bS_n(a+p,b,a,b) + n(a+p)T_{n-1}(a,b,p-1) \end{aligned}$$

If we multiply (9.29) with $a+p$ and add to (9.31), then we get

$$T_n(a,b,p) = (b-a-p)S_n(a+p,b,a,b) + (a+p)S_n(a+p-1,b,a,b) \tag{9.32}$$

When we substitute (9.32) in (9.29), we eventually arrive at the recursion (9.28). □

To get started we need two neighboring values, so we need e.g. to compute $S_n(a+1,b,a,b)$.

From (9.27) we get

$$S_n(a+1,b,a,b) = S_n(a+1,b+1,a,b) + n(a+1)S_{n-1}(a,b,a,b) \tag{9.33}$$

where the two terms follow from (9.21). The result becomes

$$S_n(a+1,b,a,b) = [n]_{\lfloor \frac{n}{2} \rfloor}[n-a-b-2]_{\lceil \frac{n}{2} \rceil}[a]_{\lceil \frac{n}{2} \rceil}[b]_{\lfloor \frac{n}{2} \rfloor}(-1)^n \tag{9.34}$$

The balanced and quasi–balanced Dixon identities. The balanced Dixon identity, (9.4), is not so easy to generalize. And its proof does not follow from the Pfaff–Saalschütz formula or the symmetric Dixon theorem. We shall present a proof following an idea due to G. N. Watson, [114].

To prove also some quasi–balanced Dixon identities, we shall consider for small integers $p \in \mathbb{Z}$, with the balanced case as $p = 0$

(9.35)

$$S_n(n+2a, b+a, b-a, n-2a-p) = \sum_{k=0}^{n} \binom{n}{k} [n+2a]_k [b+a]_k [b-a]_{n-k} [n-2a-p]_{n-k} (-1)^k$$

Now we apply the Chu–Vandermonde formula (8.4) to write

$$[a+b]_k = \sum_{j=0}^{k} \binom{k}{j} [n+2a-k]_j [b-a-n+k]_{k-j} \tag{9.36}$$

Then the sum becomes

(9.37)
$$S_n(n+2a,b+a,b-a,n-2a-p) =$$
$$=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}[n+2a]_k[n+2a-k]_j\cdot$$
$$[b-a]_{n-k}[b-a-n+k]_{k-j}[n-2a-p]_{n-k}(-1)^k$$

Now we shall apply (2.10) to write $\binom{n}{k}\binom{k}{j}=\binom{n}{j}\binom{n-j}{k-j}$. Next we apply (2.2) twice and change the order of summations to get

(9.38)
$$S_n(n+2a,b+a,b-a,n-2a-p) =$$
$$=\sum_{j=0}^{n}\binom{n}{j}[b-a]_{n-j}\sum_{k=j}^{n}\binom{n-j}{k-j}[n+2a]_{k+j}[n-2a-p]_{n-k}(-1)^k$$

Now we apply (2.2) to write $[n+2a]_{k+j}=[n+2a]_{2j}[n+2a-2j]_{k-j}$. Then the sum becomes

(9.39)
$$S_n(n+2a,b+a,b-a,n-2a-p) =$$
$$=\sum_{j=0}^{n}\binom{n}{j}[n+2a]_{2j}[b-a]_{n-j}\sum_{k=j}^{n}\binom{n-j}{k-j}[n+2a-2j]_{k-j}[n-2a-p]_{n-k}(-1)^k$$

Changing variables in the inner sum, one gets

$$(-1)^j\sum_{k=0}^{n-j}\binom{n-j}{k}[n-j-p+(2a-j+p)]_k[n-j-(2a-j+p)]_{n-j-k}(-1)^k \tag{9.40}$$

This sum is a quasi–balanced Kummer sum, to be evaluated by (8.34) as

(9.41)
$$\frac{2^{n-j-(p\vee 0)}}{[n-j-p]_{-(p\wedge 0)}}\sum_{i=0}^{|p|}\binom{|p|}{i}\sigma(p)^{p+i}[-(2a-j+p)+n-j+i-1,2]_{n-j-(p\wedge 0)} =$$
$$\frac{2^{n-j-(p\vee 0)}}{[n-j-p]_{-(p\wedge 0)}}\sum_{i=0}^{|p|}\binom{|p|}{i}\sigma(p)^{p+i}[-2a+p+n+i-1,2]_{n-j-(p\wedge 0)}$$

If $p=0$, we apply (2.1) to (9.41) and substitute the result in (9.39), then we get for the sum

(9.42)
$$S_n(n+2a,b+a,b-a,n-2a) =$$
$$=\sum_{j=0}^{n}\binom{n}{j}[n+2a]_{2j}[b-a]_{n-j}(-1)^j(-2)^{n-j}[n+2a-2j-1,2]_{n-j}$$

If we split $[n+2a]_{2j} = [n+2a,2]_j[n+2a-1,2]_j$, and use (2.4) to $[n+2a-1,2]_j[n+2a-2j-1,2]_{n-j} = [n+2a-1,2]_n$ and (2.5) to $[b-a]_{n-j}2^{n-j} = [2b-2a,2]_{n-j}$, we may write the sum as

(9.43)
$$S_n(n+2a,b+a,b-a,n-2a) = [n+2a-1,2]_n(-1)^n \sum_{j=0}^{n} \binom{n}{j} [n+2a,2]_j[2b-2a,2]_{n-j}$$

Now the Chu–Vandermonde formula (8.4) applies, so we get

$$S_n(n+2a,b+a,b-a,n-2a) = [n+2a-1,2]_n[n+2b,2]_n \tag{9.44}$$

This proves the balanced Dixon identity (9.4).

If $p > 0$, we get

(9.45)
$$S_n(n+2a,b+a,b-a,n-2a-p) = (-1)^n 2^{-p} \sum_{i=0}^{p} \binom{p}{i} \sum_{j=0}^{n} \binom{n}{j}\cdot$$
$$[2b-2a,2]_{n-j}[n+2a,2]_j[n+2a-1,2]_j[n+2a-2j+p-1-i,2]_{n-j}$$

and for $p < 0$ we get

(9.46)
$$S_n(n+2a,b+a,b-a,n-2a-p) = \frac{(-1)^n}{[n-p]_{-p}[2b-2a-2p,2]_{-p}}\cdot$$
$$\sum_{i=0}^{-p} \binom{-p}{i} (-1)^i \sum_{j=0}^{n} \binom{n-p}{j} [2b-2a-2p,2]_{n-p-j}[n+2a,2]_j\cdot$$
$$[n+2a-1,2]_j[n+2a-2j-p-1-i,2]_{n-p-j}$$

Theorem 9.5. *Some quasi–balanced Dixon formulas are*

(9.47)
$$S_n(n+2a,b+a,b-a,n-2a-1) =$$
$$= \tfrac{1}{2}\left([n-2b-1,2]_n[n+2a,2]_n + [n-2a-1,2]_n[n+2b,2]_n\right)$$

(9.48)
$$S_n(n+2a,b+a,b-a,n-2a-2) =$$
$$= \tfrac{1}{4}[n-2a-1,2]_n[n+2b,2]_n + \tfrac{1}{2}[n-2b-1,2]_n[n+2a,2]_n +$$
$$\tfrac{1}{4}[n-2a-3,2]_n[n+2b,2]_n + \tfrac{1}{2}n(n+2a)[n-2b-2,2]_n[n+2a-1,2]_n$$

(9.49)
$$S_n(n+2a,b+a,b-a,n-2a+1) = \frac{(-1)^n}{(n+1)(2b-2a+2)}\cdot$$
$$\left([n+2a,2]_{n+1}[n+2b+1,2]_{n+1} - [n+2a-1,2]_{n+1}[n+2b+2,2]_{n+1}\right)$$

Proof. We substitute $p = 1$ and $p = 2$ in (9.45) and $p = -1$ in (9.46) and eventually apply Chu–Vandermonde (8.4) to each term. □

Remarks. *Surprisingly,* $S_n(n+2a, b+a, b-a, n-2a-1)$ *is symmetric in* (a,b). *Another symmetry for the balanced Dixon formula is*

(9.50)
$$S_n(n+2a, b+a, b-a, n-2a) = S_n\left(n-2b-1, -b-a-1, b-a, n+2b+1\right)$$

Furthermore, it does not matter, which term is changed with the deviation, p. *If we apply (9.8) to* $S_n(n+2a, b+a, b-a, n-2a-p)$, *we may get the symmetric expression:*

(9.51)
$$\sum_{k=0}^{n}\binom{n}{k}[-a-b-1]_k[a+b]_k[b-a]_{n-k}[-1+a-b+p]_{n-k}(-1)^k$$

Proof of (9.50). Apply the transformation (9.8). □

Watson's formulas and their contiguous companions. The two finite versions of Watson's formulas are

(9.52)
$$\sum_{k=0}^{n}\binom{n}{k}[-a]_k[-b]_k\left[\tfrac{1}{2}(n+a-1)\right]_{n-k}[n+2b-1]_{n-k}(-1)^k$$
$$=\begin{cases} 0 & \text{for } n \text{ odd} \\ \left[-\tfrac{1}{2}(a+1)\right]_m[-b]_m\left[\tfrac{1}{2}(a-1)-b\right]_m[n]_m & \text{for } n=2m \end{cases}$$

(9.53)
$$\sum_{k=0}^{n}\binom{n}{k}[-a]_k[-b]_k\left[n+\tfrac{1}{2}(a+b-1)\right]_{n-k}[-n-1]_{n-k}(-1)^k$$
$$=[-a-1,2]_n[-b-1,2]_n(-1)^n$$

Besides we may introduce an excess, p, in one or two of the factorials. We may write down:

(9.54)
$$\sum_{k=0}^{n}\binom{n}{k}[-a]_k[-b]_k\left[\tfrac{1}{2}(n+a-1)\right]_{n-k}[n+2b]_{n-k}(-1)^k$$
$$=[n]_{\lfloor\frac{n}{2}\rfloor}\left[\tfrac{1}{2}(n-a-1)+b\right]_{\lceil\frac{n}{2}\rceil}[-b-1]_{\lceil\frac{n}{2}\rceil}\left[\tfrac{1}{2}(n+a-1)\right]_{\lfloor\frac{n}{2}\rfloor}(-1)^n$$

(9.55)
$$\sum_{k=0}^{n}\binom{n}{k}[-a]_k[-b]_k\left[\tfrac{1}{2}(n+a-1)\right]_{n-k}[n+2b-2]_{n-k}(-1)^k$$
$$=[n]_{\lfloor\frac{n}{2}\rfloor}\left[\tfrac{1}{2}(n-a-3)+b\right]_{\lceil\frac{n}{2}\rceil}[-b]_{\lceil\frac{n}{2}\rceil}\left[\tfrac{1}{2}(n+a-1)\right]_{\lfloor\frac{n}{2}\rfloor}$$

(9.56)

$$\sum_{k=0}^{n}\binom{n}{k}[-a-p]_k[-b]_k\left[\tfrac{1}{2}(n+a-1)\right]_{n-k}[n+2b-1+p]_{n-k}(-1)^k$$
$$=\sigma(p)^n\sum_{j=\lceil\frac{n}{2}\rceil}^{\lfloor\frac{n+|p|}{2}\rfloor}\binom{|p|}{2j-n}[n]_j\left[\tfrac{1}{2}(n-a-1)+b\right]_j$$
$$\left[\tfrac{1}{2}(n+a-1)+(p\wedge 0)\right]_{n-j}[-b-(p\vee 0)]_{n-j}$$

(9.57)

$$\sum_{k=0}^{n}\binom{n}{k}[-a]_k[-b]_k\left[n+\tfrac{1}{2}(a+b-1)\right]_{n-k}[-n]_{n-k}(-1)^k$$
$$=\tfrac{1}{2}\left([-a,2]_n[-b,2]_n+[-a-1,2]_n[-b-1,2]_n\right)(-1)^n$$

(9.58)

$$\sum_{k=0}^{n}\binom{n}{k}[-a]_k[-b]_k\left[n+\tfrac{1}{2}(a+b-1)\right]_{n-k}[-n-2]_{n-k}(-1)^k$$
$$=\frac{1}{(n+1)(2n+a+b+1)}\left([-a,2]_{n+1}[-b,2]_{n+1}+\right.$$
$$\left.[-a-1,2]_{n+1}[-b-1,2]_{n+1}\right)(-1)^n$$

Proof. The formulas (9.52) and (9.54–9.56) are transformed by (9.14) to the different quasi–symmetric Dixon formulas, (9.3), (9.34), and (9.21–9.22). The formulas (9.53) and (9.57–9.58) are similarly transformed by (9.14) to the balanced Dixon formula, (9.4), and the quasi–balanced Dixon formulas, (9.47) and (9.49). □

The formulas with p of size one, (9.54), (9.55), (9.57) and (9.58) were considered by J. L. Lavoie [89] in 1987.

Whipple's formulas and their contiguous companions. The two finite versions of Whipple's formulas are

(9.59)

$$\sum_{k=0}^{n}\binom{n}{k}[-n-1]_k[-a]_k[b+n-1]_{n-k}[2a-b+n]_{n-k}(-1)^k$$
$$=[b+n-2,2]_n[2a-b+n-1,2]_n$$

(9.60)

$$\sum_{k=0}^{n}\binom{n}{k}[a-1]_k[-a]_k[b+n-1]_{n-k}[-b-n]_{n-k}(-1)^k$$
$$=[a-b-1,2]_n[-a-b,2]_n$$

These formulas do also allow an excess of the size 1, giving the following 4 formulas

(9.61)
$$\sum_{k=0}^{n}\binom{n}{k}[-n-1]_k[-a]_k[b+n-1]_{n-k}[2a-b+n+1]_{n-k}(-1)^k$$
$$=\tfrac{1}{2}\left([b+n-2,2]_n[2a-b+n-1,2]_n+[b+n-1,2]_n[2a-b+n-2,2]_n\right)$$

(9.62)
$$\sum_{k=0}^{n}\binom{n}{k}[-n-1]_k[-a]_k[b+n-1]_{n-k}[2a-b+n-1]_{n-k}(-1)^k$$
$$=\frac{(-1)^n}{(n+1)2a}\left([b+n-1,2]_{n+1}[-2a+b+n,2]_{n+1}-\right.$$
$$\left.[b+n-2,2]_{n+1}[2a-b+n-1,2]_{n+1}\right)$$

(9.63)
$$\sum_{k=0}^{n}\binom{n}{k}[a-1]_k[-a]_k[b+n-1]_{n-k}[-b-n+1]_{n-k}(-1)^k$$
$$=\tfrac{1}{2}\left([a-b-1,2]_n[-a-b,2]_n+[a-b,2]_n[-a-b+1,2]_n\right)$$

(9.64)
$$\sum_{k=0}^{n}\binom{n}{k}[a-1]_k[-a]_k[b+n-1]_{n-k}[-b-n-1]_{n-k}(-1)^k$$
$$=\frac{1}{(n+1)2(n+b)}\left([a-b-1,2]_{n+1}[-a-b+2,2]_{n+1}-\right.$$
$$\left.[a-b,2]_{n+1}[-a-b-1,2]_{n+1}\right)$$

Proof. The formulas (9.59) and (9.61–9.62) are transformed by (9.14) to the different quasi–balanced Dixon formulas, (9.4), (9.47) and (9.49). The formulas (9.60) and (9.63–9.64) are respectively the balanced Dixon formula, (9.5) transformed by (9.8) and once more transformed by (9.8) to the quasi–balanced Dixon formulas, (9.47) and (9.49). □

Ma Xin–Rong and Wang Tian–Ming's problem. In 1995 Ma Xin–Rong and Wang Tian–Ming, Dalian University of Technology, Dalian, China, posed the problem:

Show that

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n+k}{k}\binom{n+1}{j-k}=\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{k}\binom{m+1}{j-k} \tag{9.65}$$

and

$$\sum_{k=0}^{m}\binom{2n-m-1-k}{n-k}\binom{m+k}{k}=\binom{2n-1}{n}\quad (m\leq n-1) \tag{9.66}$$

Proof of (9.65).

We shall remark that the upper limits of summation may well be changed to j, because, possible terms with $k > j$ or $k > m$ $(k > n)$ are zeros anyway.

Hence the identity may be written as

$$\sum_{k=0}^{j} \binom{m}{k}\binom{n+k}{k}\binom{n+1}{j-k} = \sum_{k=0}^{j} \binom{n}{k}\binom{m+k}{k}\binom{m+1}{j-k} \tag{9.67}$$

In this form, (9.67), the sums are defined for any complex values of m and n.

We shall prove the generalization of (9.65), that the function,

$$S_j(m,n) = \sum_{k=0}^{j} \binom{m}{k}\binom{n+k}{k}\binom{n+1}{j-k} \tag{9.68}$$

defined for integers j and complex variables m and n, is symmetric in m and n.

The function (9.68) may be changed without disturbing a possible symmetry by multiplication with $j!^2$. Then the function $j!^2 S_j(m,n)$ becomes, using the rewriting (2.1), $[n+k]_k = [-n-1]_k(-1)^k$,

$$j!^2 S_j(m,n) = \sum_{k=0}^{j} \binom{j}{k} [m]_k[-n-1]_k[n+1]_{j-k}[j]_{j-k}(-1)^k \tag{9.69}$$

Now we apply the transformation (9.9) to write (9.69) in a form symmetric in m and n

$$j!^2 S_j(m,n) = \sum_{k=0}^{j} \binom{j}{k} [m]_k[n]_k[j-2-n-m]_{j-k}[j]_{j-k}(-1)^{j-k} \tag{9.70}$$

If we divide again with $j!^2$, where we use once more,

$$[j-2-n-m]_{j-k}(-1)^{j-k} = [m+n+1-k]_{j-k} \tag{9.71}$$

then we get one symmetric form of the original function, (9.68),

$$S_j(m,n) = \sum_{k=0}^{j} \binom{m}{k}\binom{n}{k}\binom{m+n+1-k}{j-k} \tag{9.72}$$

If we apply (9.9) to the function (9.69) interchanging the first two factorials

$$j!^2 S_j(m,n) = \sum_{k=0}^{j} \binom{j}{k} [-n-1]_k[m]_k[n+1]_{j-k}[j]_{j-k}(-1)^k \tag{9.73}$$

then we get another symmetric form, namely

$$j!^2 S_j(m,n) = (-1)^j \sum_{k=0}^{j} \binom{j}{k} [-n-1]_k [-m-1]_k [j-1]_{j-k} [j]_{j-k} (-1)^k \tag{9.74}$$

From this form we may regain as another symmetric form of the function, (9.68),

$$S_j(m,n) = \sum_{k=0}^{j} \binom{-n-1}{k} \binom{-m-1}{k} \binom{j-1}{j-k} (-1)^{j-k} \tag{9.75}$$

or, if you prefer

$$S_j(m,n) = \sum_{k=0}^{j} \binom{n+k}{k} \binom{m+k}{k} \binom{-k}{j-k} \tag{9.76}$$

□

Proof of (9.66).

We remark, that (9.66) is true for $m = 0$.

Then we prove that as long as $m < n$, the sum is the same for m and $m-1$.

To get this equality it is convenient to apply Abelian summation or summation by parts. We remark that we have by (2.8)

(9.77)
$$\binom{2n-m-1-k}{n-k} = \binom{2n-(m-1)-1-k}{n-k} - \binom{2n-(m-1)-1-k-1}{n-k-1}$$

With this we may split the sum in (9.66) as

(9.78)
$$\begin{aligned}
&\sum_{k=0}^{m} \binom{2n-m-1-k}{n-k} \binom{m+k}{k} = \\
&= \sum_{k=0}^{m} \binom{2n-(m-1)-1-k}{n-k} \binom{m+k}{k} - \\
&\qquad - \sum_{k=0}^{m} \binom{2n-(m-1)-1-k-1}{n-k-1} \binom{m+k}{k} = \\
&= \sum_{k=0}^{m} \binom{2n-(m-1)-1-k}{n-k} \binom{m+k}{k} - \\
&\qquad - \sum_{k=0}^{m+1} \binom{2n-(m-1)-1-k}{n-k} \binom{m+k-1}{k-1} = \\
&= \sum_{k=0}^{m-1} \binom{2n-(m-1)-1-k}{n-k} \binom{(m-1)+k}{k} + \\
&\qquad + \binom{2n-2m}{n-m} \binom{2m-1}{m} - \binom{2n-2m-1}{n-m-1} \binom{2m}{m}
\end{aligned}$$

where the last two terms cancel, as long as $m < n$, because one is obtained from the other by moving a factor 2 from one binomial coefficient to the other, cf. (5.43).

If we have $m = n$, then the equation (9.78) looks as

$$\binom{2n}{n} = \binom{2n-1}{n} + \binom{2n-1}{n} - 0 \tag{9.79}$$

where the last two terms do not cancel. □

Comment. If we replace m with $n - m - 1$ in the formula (9.66), then we get

$$\sum_{k=0}^{n-m-1} \binom{n+m-k}{n-k}\binom{n-m-1+k}{k} = \binom{2n-1}{n} \quad (m \le n-1) \tag{9.80}$$

Now, if we change the summation variable replacing k with $n - k$, i.e., changing the direction of summation, then we get

$$\sum_{k=m+1}^{n} \binom{m+k}{k}\binom{2n-m-1-k}{n-k} = \binom{2n-1}{n} \quad (m \le n-1) \tag{9.81}$$

But this is the natural continuation of (9.66), so if we add (9.66) and (9.81), we get

$$\sum_{k=0}^{n} \binom{2n-m-1-k}{n-k}\binom{m+k}{k} = \binom{2n}{n} \tag{9.82}$$

which happens to be valid for all complex values of m.

This is just a simple consequence of the Chu–Vandermonde equation, (8.4). To see this we only need to change the binomial coefficients by (2.12),

(9.83)

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{2n-m-1-k}{n-k}\binom{m+k}{k} = \\
=&(-1)^n \sum_{k=0}^{n} \binom{n-k-2n+m+1+k-1}{n-k}\binom{k-m-k-1}{k} = \\
=&(-1)^n \sum_{k=0}^{n} \binom{m-n}{n-k}\binom{-m-1}{k} = \\
=&(-1)^n \binom{-n-1}{n} = \\
=&\binom{2n}{n}
\end{aligned}$$

C. C. Grosjean's problem. In 1992, C. C. Grosjean, University of Ghent, Belgium, [72], [12], posed the problem:

Determine the sum

$$\sum_{m=0}^{n} \binom{-\frac{1}{4}}{m}^2 \binom{-\frac{1}{4}}{n-m}^2. \tag{9.84}$$

We shall consider the sum

$$\begin{aligned} S = \sum_{m=0}^{n} \binom{n}{m} & [n+x+y]_m \left[x-\tfrac{1}{4}\right]_m \left[y-\tfrac{1}{4}\right]_m \\ & [n-x-y]_{n-m} \left[-x-\tfrac{1}{4}\right]_{n-m} \left[-y-\tfrac{1}{4}\right]_{n-m}, \end{aligned} \tag{9.85}$$

We use the Pfaff–Saalschütz–identity, (9.1),

$$[a+d]_n\,[b+d]_n = \sum_{k=0}^{n} \binom{n}{k} [n-a-b-d-1]_k\,[d]_k\,[a]_{n-k}\,[b]_{n-k}\,(-1)^k. \tag{9.86}$$

We want to write the sum in (9.85) as a double sum using (9.86) to expand $\left[x-\frac{1}{4}\right]_m \left[y-\frac{1}{4}\right]_m$. We therefore replace n by m and let $a = x-\frac{1}{4}-d$, $b = y-\frac{1}{4}-d$ in (9.86). This gives us

$$\begin{aligned} S = & \sum_{m=0}^{n} \binom{n}{m} [n+x+y]_m\,[n-x-y]_{n-m} \left[-x-\tfrac{1}{4}\right]_{n-m} \left[-y-\tfrac{1}{4}\right]_{n-m} \\ & \sum_{k=0}^{m} \binom{m}{k} \left[m+d-x-y-\tfrac{1}{2}\right]_k [d]_k \left[x-\tfrac{1}{4}-d\right]_{m-k} \left[y-\tfrac{1}{4}-d\right]_{m-k} (-1)^k. \end{aligned} \tag{9.87}$$

If we let $d = n-m+x+y$, then

$$\left[x-\tfrac{1}{4}-d\right]_{m-k} \left[-y-\tfrac{1}{4}\right]_{n-m} = \left[-y-\tfrac{1}{4}\right]_{n-k}$$

and

$$\left[y-\tfrac{1}{4}-d\right]_{m-k} \left[-x-\tfrac{1}{4}\right]_{n-m} = \left[-x-\tfrac{1}{4}\right]_{n-k}.$$

Using also $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we obtain after changing the order of summation

$$\begin{aligned} S = & \sum_{k=0}^{n} \binom{n}{k} \left[n-\tfrac{1}{2}\right]_k \left[-x-\tfrac{1}{4}\right]_{n-k} \left[-y-\tfrac{1}{4}\right]_{n-k} (-1)^k \\ & \sum_{m=k}^{n} \binom{n-k}{m-k} [n+x+y]_m\,[n-x-y]_{n-m}\,[n-m+x+y]_k\,. \end{aligned} \tag{9.88}$$

Let T be the inner sum, i.e.

$$T = \sum_{m=k}^{n} \binom{n-k}{m-k} [n+x+y]_m \, [n-x-y]_{n-m} \, [n-m+x+y]_k \, . \tag{9.89}$$

Since $[n+x+y]_m \, [n-m+x+y]_k = [n+x+y]_{2k} \, [n+x+y-2k]_{m-k}$ it follows using the Chu–Vandermonde convolution (8.4) and (5.44) that

(9.90)
$$T = [n+x+y]_{2k} \, [2n-2k]_{n-k} = \left[\tfrac{n+x+y}{2}\right]_k \left[\tfrac{n+x+y-1}{2}\right]_k \left[n-k-\tfrac{1}{2}\right]_{n-k} 4^n.$$

Introducing this in (9.88) yields

(9.91)
$$S = \sum_{k=0}^{n} \binom{n}{k} \left[n-\tfrac{1}{2}\right]_k \left[-x-\tfrac{1}{4}\right]_{n-k} \left[-y-\tfrac{1}{4}\right]_{n-k} (-1)^k$$
$$\left[\tfrac{n+x+y}{2}\right]_k \left[\tfrac{n+x+y-1}{2}\right]_k \left[n-k-\tfrac{1}{2}\right]_{n-k} 4^n =$$
$$4^n \left[n-\tfrac{1}{2}\right]_n \sum_{k=0}^{n} \binom{n}{k} \left[\tfrac{n+x+y}{2}\right]_k \left[\tfrac{n+x+y-1}{2}\right]_k \left[-x-\tfrac{1}{4}\right]_{n-k} \left[-y-\tfrac{1}{4}\right]_{n-k} (-1)^k.$$

This sum is a Pfaff–Saalschütz sum, (9.1), and we get using (9.86) an identity for the sum in (9.85):

(9.92)
$$S = [2n]_n \left[\tfrac{n-x+y}{2}-\tfrac{3}{4}\right]_n \left[\tfrac{n+x-y}{2}-\tfrac{3}{4}\right]_n = [2n]_n \left[n+x-y-\tfrac{1}{2}\right]_{2n} \left(-\tfrac{1}{4}\right)^n,$$

where we have given two different forms of the result. If we divide by $n!^3$ we obtain

(9.93)
$$\sum_{m=0}^{n} \binom{n+x+y}{m} \binom{x-\frac{1}{4}}{m} \binom{y-\frac{1}{4}}{m}$$
$$\binom{n-x-y}{n-m} \binom{-x-\frac{1}{4}}{n-m} \binom{-y-\frac{1}{4}}{n-m} \Bigg/ \binom{n}{m}^2$$
$$= \binom{2n}{n} \binom{\frac{n-x+y}{2}-\frac{3}{4}}{n} \binom{\frac{n+x-y}{2}-\frac{3}{4}}{n} = \binom{2n}{n}^2 \binom{n+x-y-\frac{1}{2}}{2n} (-\tfrac{1}{4})^n.$$

For $x = y = 0$ we obtain

$$\sum_{m=0}^{n} \binom{-\frac{1}{4}}{m}^2 \binom{-\frac{1}{4}}{n-m}^2 = \binom{2n}{n} \binom{\frac{n}{2}-\frac{3}{4}}{n}^2 = \binom{2n}{n}^2 \binom{n-\frac{1}{2}}{2n} (-\tfrac{1}{4})^n. \tag{9.94}$$

this solves the problem.

Peter Larcombe's problem. On June 18, 2005, Peter Larcombe asked the questions, is the following formula true, and how to prove it?

$$16^n \sum_{k=0}^{2n} 4^k \binom{-\frac{1}{2}}{k}\binom{\frac{1}{2}}{k}\binom{-2k}{2n-k} = (4n+1)\binom{2n}{n}^2 \tag{9.95}$$

Well, using the technique of getting a standard form, one may change the left side to

$$\frac{2\cdot 4^{2n}}{(2n)!^2}\sum_{k=0}^{2n}\binom{2n}{k}[-2n]_k[\tfrac{1}{2}]_k[2n]_{2n-k}[2n-1]_{2n-k}(-1)^k \tag{9.96}$$

Using the formula (9.13) to the sum (without the front factor) we get

$$\sum_{k=0}^{2n}\binom{2n}{k}[\tfrac{1}{2}]_k[\tfrac{1}{2}]_k[-\tfrac{1}{2}]_{2n-k}[-1\tfrac{1}{2}]_{2n-k}(-1)^k \tag{9.97}$$

This expression is a generalized Dixon formula, so I have the tools to compute it. Let's define

$$S_n(a,b;p,q) = \sum_{k=0}^{n}\binom{n}{k}[a+p]_k[b+q]_k[a]_{n-k}[b]_{n-k}(-1)^k \tag{9.98}$$

Then we are after

$$S_{2n}(-\tfrac{1}{2},-1\tfrac{1}{2};1,2) \tag{9.99}$$

Now the general recursion formula (9.27) allows us to write it as

$$S_{2n}(-\tfrac{1}{2},-1\tfrac{1}{2};2,2)+nS_{2n-1}(-\tfrac{1}{2},-1\tfrac{1}{2};1,1) \tag{9.100}$$

Both sums are quasi-symmetric Dixon sums to be evaluated be theorem 9.4 with the formula (9.21) as

(9.101)
$$S_{2n}(-\tfrac{1}{2},-1\tfrac{1}{2};2,2) = [2n]_n[2n-1]_n[-\tfrac{1}{2}]_n[-1\tfrac{1}{2}]_n+[2n]_{n+1}[2n-1]_{n+1}[-\tfrac{1}{2}]_{n-1}[-1\tfrac{1}{2}]_{n-1}$$

and

$$S_{2n-1}(-\tfrac{1}{2},-1\tfrac{1}{2};1,1) = [2n-1]_n[2n-1]_n[-\tfrac{1}{2}]_{n-1}[-1\tfrac{1}{2}]_{n-1} \tag{9.102}$$

Now a tedious computation then leads to the result

$$[2n]_n^4\tfrac{1}{2}(\tfrac{1}{4})^{2n}(4n+1) \tag{9.103}$$

proving the formula (9.95).

CHAPTER 10. SUMS OF TYPE II$(4,4,\pm 1)$

Introduction. We do not know any indefinite formulas of this type, so we may as well rectify the limits and consider a sum of terms

$$S(c,n)=\sum_{k=0}^{n} s(c,n,k) \tag{10.1}$$

with quotient

$$q_t(k)=\frac{(n-k)(\alpha_1-k)(\alpha_2-k)(\alpha_3-k)}{(-1-k)(\beta_1-k)(\beta_2-k)(\beta_3-k)}\cdot z \tag{10.2}$$

We shall prefer to define $a_j=\alpha_j$ and $b_j=n-1-\beta_j$, $(j=1,\cdot,3)$, and hence redefine the sum as

$$S_n\left(a_1,\cdot,a_3,b_1,\cdot,b_3\right)=\sum_{k=0}^{n}\binom{n}{k}\left[a_1\right]_k\left[a_2\right]_k\left[a_3\right]_k\left[b_1\right]_{n-k}\left[b_2\right]_{n-k}\left[b_3\right]_{n-k} z^k \tag{10.3}$$

Sum formulas for $z=1$. In the case of $z=1$ we know two important formulas with three arguments and one excess.

The symmetric formula, cf. H. W. Gould, [64], formula (23.1):

Theorem 10.1. *For any $a,b,c\in\mathbb{C}$ and $n\in\mathbb{N}$ we have, that if*

$$a+b+c=n-\tfrac{1}{2} \tag{10.4}$$

then

$$\sum_{k=0}^{n}\binom{n}{k}[a]_k[b]_k[c]_k[a]_{n-k}[b]_{n-k}[c]_{n-k}=\left(\tfrac{1}{4}\right)^n[2a]_n[2b]_n[2c]_n \tag{10.5}$$

This formula allows the generalization with excess:

Theorem 10.2. *For any $a,b,c\in\mathbb{C}$ and $n,p\in\mathbb{N}_0$ we have, that if*

$$p=a+b+c-n+\tfrac{1}{2} \tag{10.6}$$

then

(10.7)

$$\sum_{k=0}^{n}\binom{n}{k}[a]_k[b]_k[c]_k[a]_{n-k}[b]_{n-k}[c]_{n-k}=$$

$$(-1)^p 2^n[a]_{\lceil\frac{n}{2}\rceil}[b]_{\lceil\frac{n}{2}\rceil}[c]_{\lceil\frac{n}{2}\rceil}\left[a-p-\tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor-p}\left[b-p-\tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor-p}\left[c-p-\tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor-p}\cdot$$

$$\sum_{i=0}^{p}\binom{p}{i}\left[\lfloor\tfrac{n}{2}\rfloor\right]_i[c]_i\left[c-\lceil\tfrac{n}{2}\rceil\right]_i\left[p-c-\tfrac{1}{2}\right]_{p-i}\left[a-\lfloor\tfrac{n}{2}\rfloor-\tfrac{1}{2}\right]_{p-i}\left[b-\lfloor\tfrac{n}{2}\rfloor-\tfrac{1}{2}\right]_{p-i}$$

To prove these formulas, we need a lemma,

Lemma 10.1. *For any $a, b, x, y \in \mathbb{C}$ and $n \in \mathbb{N}$ we have,*

(10.8)
$$\sum_{k=0}^{n} \binom{n}{k} [x]_k [y]_{n-k} [a]_k [a]_{n-k} [b]_k [b]_{n-k} =$$
$$(-1)^n \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{[n]_{2j}}{j!} [x]_j [y]_j [n-1-a-b]_j [a]_{n-j} [b]_{n-j} [n-1-x-y]_{n-2j} (-1)^j$$

Proof. We use the Pfaff–Saalschütz identity, (9.1), to write

$$[a]_k [b]_k [a]_{n-k} [b]_{n-k} = \sum_{j=0}^{k} \binom{k}{j} [n-k]_j [n-1-a-b]_j [a]_{n-j} [b]_{n-j} (-1)^j \tag{10.9}$$

Substitution of this in the sum of (10.8) yields after changing the order of summation

$$\sum_{j=0}^{n} \binom{n}{j} [n-1-a-b]_j [a]_{n-j} [b]_{n-j} [x]_j [y]_j (-1)^j \cdot \sum_{k=j}^{n} \binom{n-j}{k-j} [n-k]_j [x-j]_{k-j} [y-j]_{n-j-k} \tag{10.10}$$

After writing $\binom{n-j}{k-j}[n-k]_j = \binom{n-2j}{k-j}[n-j]_j$ we can apply the Chu–Vandermonde formula (8.4) to the inner sum and get

$$[n-j]_j [x+y-2j]_{n-2j}$$

Then we shall write

$$[x+y-2j]_{n-2j} = (-1)^n [n-1-x-y]_{n-2j}$$

and eventually we shall write

$$\binom{n}{j} [n-j]_j = \frac{[n]_{2j}}{j!}$$

Substituting all the changes, the very sum becomes the right side of (10.8). □

Proof. We use lemma 1 with $x = y = c$ and $n-1-a-b = c - \frac{1}{2} - p$ to write the sum as

$$(-1)^n \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{[n]_{2j}}{j!} [c]_j^2 [c - \tfrac{1}{2} - p]_j [a]_{n-j} [b]_{n-j} [n-1-2c]_{n-2j} (-1)^j$$

Then we shall write

$$[n-1-2c]_{n-2j}(-1)^n = [2c-2j]_{n-2j} = 2^{n-2j}[c-j]_{\lceil\frac{n}{2}\rceil-j}\left[c-j-\tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor-j}$$

$$[c]_j[c-j]_{\lceil\frac{n}{2}\rceil-j} = [c]_{\lceil\frac{n}{2}\rceil}$$

$$\left[c-\tfrac{1}{2}-p\right]_j\left[c-j-\tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor-j} = \left[c-\tfrac{1}{2}-p\right]_{\lfloor\frac{n}{2}\rfloor-p}\left[c-j-\tfrac{1}{2}\right]_p$$

$$\left[c-j-\tfrac{1}{2}\right]_p = (-1)^p\left[p-c-\tfrac{1}{2}+j\right]_p$$

Eventually we shall write

$$\frac{[n]_{2j}}{j!} = 2^{2j}\binom{\lfloor\frac{n}{2}\rfloor}{j}\left[\lceil\tfrac{n}{2}\rceil-\tfrac{1}{2}\right]_j$$

Substituting all the changes, the very sum becomes

$$2^n[c]_{\lceil\frac{n}{2}\rceil}\left[c-\tfrac{1}{2}-p\right]_{\lfloor\frac{n}{2}\rfloor-p}(-1)^p\cdot$$
$$\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{n}{2}\rfloor}{j}\left[\lceil\tfrac{n}{2}\rceil-\tfrac{1}{2}\right]_j[c]_j[a]_{n-j}[b]_{n-j}(-1)^j\left[p-c-\tfrac{1}{2}+j\right]_p$$

Now we apply the Chu–Vandermonde convolution, (8.4), to write the last sum as

$$\left[p-c-\tfrac{1}{2}+j\right]_p = \sum_{h=0}^{p}\binom{p}{h}\left[p-c-\tfrac{1}{2}\right]_{p-h}[j]_h$$

and use the formula (2.9) to write

$$\binom{\lfloor\frac{n}{2}\rfloor}{j}[j]_h = \left[\lfloor\tfrac{n}{2}\rfloor\right]_h\binom{\lfloor\frac{n}{2}\rfloor-h}{j-h}$$

Eventually we change the order of summation to get

$$2^n[c]_{\lceil\frac{n}{2}\rceil}\left[c-\tfrac{1}{2}-p\right]_{\lfloor\frac{n}{2}\rfloor-p}(-1)^p\sum_{h=0}^{p}\binom{p}{h}\left[p-c-\tfrac{1}{2}\right]_{p-h}\left[\lfloor\tfrac{n}{2}\rfloor\right]_h\cdot$$
$$\sum_{j=h}^{\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{n}{2}\rfloor-h}{j-h}\left[\lceil\tfrac{n}{2}\rceil-\tfrac{1}{2}\right]_j[c]_j[a]_{n-j}[b]_{n-j}(-1)^j$$

The inner sum may be written as

$$\left[\lceil\tfrac{n}{2}\rceil-\tfrac{1}{2}\right]_h[c]_h[a]_{\lceil\frac{n}{2}\rceil}[b]_{\lceil\frac{n}{2}\rceil}(-1)^h\cdot$$
$$\sum_{j=h}^{\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{n}{2}\rfloor-h}{j-h}\left[\lceil\tfrac{n}{2}\rceil-\tfrac{1}{2}-h\right]_{j-h}[c-h]_{j-h}[a-\lceil\tfrac{n}{2}\rceil]_{\lfloor\frac{n}{2}\rfloor-j}[b-\lceil\tfrac{n}{2}\rceil]_{\lfloor\frac{n}{2}\rfloor-j}(-1)^{j-h}$$

This sum satisfies the condition for applying the generalized Pfaff–Saalschütz identity, (9.4), with excess

$$\lceil\tfrac{n}{2}\rceil - \tfrac{1}{2} - h + c - h + a - \lceil\tfrac{n}{2}\rceil + b - \lceil\tfrac{n}{2}\rceil - \lfloor\tfrac{n}{2}\rfloor + h + 1 = p - h$$

so we may write it

$$\sum_{i=0}^{p-h}\binom{p-h}{i}\left[\lfloor\tfrac{n}{2}\rfloor - h\right]_i [c-h]_i \left[\lceil\tfrac{n}{2}\rceil - \tfrac{1}{2} - h + a - \lceil\tfrac{n}{2}\rceil - p + h\right]_{\lfloor\frac{n}{2}\rfloor - h - i}\cdot$$
$$\left[\lceil\tfrac{n}{2}\rceil - \tfrac{1}{2} - h + b - \lceil\tfrac{n}{2}\rceil - p + h\right]_{\lfloor\frac{n}{2}\rfloor - h - i}(-1)^i$$
$$= \sum_{i=0}^{p-h}\binom{p-h}{i}\left[\lfloor\tfrac{n}{2}\rfloor - h\right]_i [c-h]_i \left[a - p - \tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor - h - i}\left[b - p - \tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor - h - i}(-1)^i$$

Now we want to rewrite

$$\binom{p}{h}\binom{p-h}{i} = \binom{p}{h+i}\binom{h+i}{h}$$
$$\left[\lfloor\tfrac{n}{2}\rfloor\right]_h\left[\lfloor\tfrac{n}{2}\rfloor - h\right]_i = \left[\lfloor\tfrac{n}{2}\rfloor\right]_{h+i}$$
$$[c]_h[c-h]_i = [c]_{h+i}$$

So the very sum becomes

$$2^n[a]_{\lceil\frac{n}{2}\rceil}[b]_{\lceil\frac{n}{2}\rceil}[c]_{\lceil\frac{n}{2}\rceil}\left[c - \tfrac{1}{2} - p\right]_{\lfloor\frac{n}{2}\rfloor - p}(-1)^p\left[a - p - \tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor - p}\left[b - p - \tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor - p}\cdot$$
$$\sum_{h=0}^{p}\left[p - c - \tfrac{1}{2}\right]_{p-h}\left[\lceil\tfrac{n}{2}\rceil - \tfrac{1}{2}\right]_h\cdot$$
$$\sum_{i=0}^{p-h}\binom{p}{h+i}\binom{h+i}{h}\left[\lfloor\tfrac{n}{2}\rfloor\right]_{h+i}[c]_{h+i}\left[a - \lfloor\tfrac{n}{2}\rfloor - \tfrac{1}{2}\right]_{p-h-i}\left[b - \lfloor\tfrac{n}{2}\rfloor - \tfrac{1}{2}\right]_{p-h-i}(-1)^{h+i}$$

Now we only need to change the summation variable. We write i for $i+h$ and obtain

$$2^n[a]_{\lceil\frac{n}{2}\rceil}[b]_{\lceil\frac{n}{2}\rceil}[c]_{\lceil\frac{n}{2}\rceil}\left[c - \tfrac{1}{2} - p\right]_{\lfloor\frac{n}{2}\rfloor - p}(-1)^p\left[a - p - \tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor - p}\left[b - p - \tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor - p}\cdot$$
$$\sum_{h=0}^{p}\left[p - c - \tfrac{1}{2}\right]_{p-h}\left[\lceil\tfrac{n}{2}\rceil - \tfrac{1}{2}\right]_h\cdot$$
$$\sum_{i=h}^{p}\binom{p}{i}\binom{i}{h}\left[\lfloor\tfrac{n}{2}\rfloor\right]_i[c]_i\left[a - \lfloor\tfrac{n}{2}\rfloor - \tfrac{1}{2}\right]_{p-i}\left[b - \lfloor\tfrac{n}{2}\rfloor - \tfrac{1}{2}\right]_{p-i}(-1)^i$$

Now it is time to change the order of summation again to get

$$2^n[a]_{\lceil\frac{n}{2}\rceil}[b]_{\lceil\frac{n}{2}\rceil}[c]_{\lceil\frac{n}{2}\rceil}\left[c - \tfrac{1}{2} - p\right]_{\lfloor\frac{n}{2}\rfloor - p}(-1)^p\left[a - p - \tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor - p}\left[b - p - \tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor - p}\cdot$$
$$\sum_{i=0}^{p}\binom{p}{i}\left[\lfloor\tfrac{n}{2}\rfloor\right]_i[c]_i\left[a - \lfloor\tfrac{n}{2}\rfloor - \tfrac{1}{2}\right]_{p-i}\left[b - \lfloor\tfrac{n}{2}\rfloor - \tfrac{1}{2}\right]_{p-i}(-1)^i\cdot$$
$$\sum_{h=0}^{i}\binom{i}{h}\left[p - c - \tfrac{1}{2}\right]_{p-h}\left[\lceil\tfrac{n}{2}\rceil - \tfrac{1}{2}\right]_h$$

The inner sum may be subject to summation using the Chu–Vandermonde formula, (8.4), it becomes

$$\sum_{h=0}^{i}\binom{i}{h}\left[p-c-\tfrac{1}{2}\right]_{p-h}\left[\lceil\tfrac{n}{2}\rceil-\tfrac{1}{2}\right]_h=\left[p-c-\tfrac{1}{2}\right]_{p-i}\sum_{h=0}^{i}\binom{i}{h}\left[i-c-\tfrac{1}{2}\right]_{i-h}\left[\lceil\tfrac{n}{2}\rceil-\tfrac{1}{2}\right]_h$$
$$=\left[p-c-\tfrac{1}{2}\right]_{p-i}\left[\lfloor\tfrac{n}{2}\rfloor+i-c-1\right]_i=\left[p-c-\tfrac{1}{2}\right]_{p-i}\left[c-\lfloor\tfrac{n}{2}\rfloor\right]_i(-1)^i$$

This proves the formula in the theorem. □

For $p=0$ we get theorem 1.

Theorem 10.3. *For any $a,b,x,y\in\mathbb{C}$ and $n,p\in\mathbb{N}_0$ we have, where $x+y=n-1-p$,*

(10.11)
$$\sum_{k=0}^{n}\binom{n}{k}[x]_k[y]_{n-k}[a]_k[a]_{n-k}[b]_k[b]_{n-k}=$$
$$=\sum_{k=0}^{n}\binom{n}{k}[x]_k[x+p-k]_{n-k}[a]_k[a]_{n-k}[b]_k[b]_{n-k}(-1)^{n-k}=$$
$$=(-1)^n\sum_{j=\lceil\frac{n-p}{2}\rceil}^{\lfloor\frac{n}{2}\rfloor}\frac{[n]_{2j}}{j!}[x]_j[y]_j[n-1-a-b]_j[a]_{n-j}[b]_{n-j}[p]_{n-2j}(-1)^j$$

Proof. Follows immediately from lemma (10.1). □

The special case of $x=y=\frac{n}{2}-1$ which gives $p=1$, is found by L. J. Slater, [105], formula III.22.

There is a balanced identity of type II(4, 4, 1).

Theorem 10.4. *For any $a,b,c\in\mathbb{C}$ and $n\in\mathbb{N}$, satisfying the condition, $b+c=-\frac{1}{2}$, we have*

(10.12)
$$\sum_{k=0}^{n}\binom{n}{k}[n+2a]_k[n-2a]_{n-k}[b+a]_k[b-a]_{n-k}[c+a]_k[c-a]_{n-k}=\left(-\tfrac{1}{4}\right)^n[2n]_n[2b+n]_n$$

Proof. This formula follows as a corollary to theorem 11.4 with $p=0$ and $d=\frac{n-1}{2}$.

We also have a well–balanced identity of type II(4, 4, 1), mentioned by L. J. Slater, [105], formula III.26.

Theorem 10.4. *For any $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$ we have*

(10.13)
$$\sum_{k=0}^{n} \binom{n}{k} [n+2a]_k [n-2a]_{n-k} [b+a]_k [b-a]_{n-k} (n+2a-2k)(-1)^k =$$
$$= 2^{2n+1} \left[\tfrac{n}{2}+a\right]_{n+1} \left[\tfrac{n-1}{2}+b\right]_n (-1)^n$$

Proof. Follows as a corollary to theorem 11.3, inserting $c = \frac{n-1}{2}$ in (11.5). □

Sum formulas for $z = -1$. We also have a well–balanced identity of type II$(4, 4, -1)$, mentioned by L. J. Slater, [105], formula III.11.

Theorem 10.5. *For any $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$ we have*

$$\sum_{k=0}^{n} \binom{n}{k} [n+2a]_k [n-2a]_{n-k} [b+a]_k [b-a]_{n-k} (n+2a-2k) = [n+2a]_{2n+1} \tag{10.14}$$

Proof. This theorem follows as a corollary to a well–balanced formula of type II$(5, 5, 1)$. Consider the equation (11.6) as a polynomial identity in the variable, c, and compare the coefficients to the term, c^n. □

There are a couple of general formulas, neither symmetric nor balanced. A family of these are the following.

Theorem 10.6. *For any $a, b \in \mathbb{C}$ and any $n, p \in \mathbb{N}_0$ we have*

(10.15)
$$\sum_{k=0}^{n} \binom{n}{k} \left[\tfrac{a}{2}\right]_k \left[\tfrac{a-1}{2}\right]_k [n+p-b]_k \left[\tfrac{b}{2}\right]_{n-k} \left[\tfrac{b-1}{2}\right]_{n-k} [n-1-a]_{n-k} =$$
$$= \left(\tfrac{1}{4}\right)^n [a]_n [b]_{n+p+1} [2n-a-b-1]_{n-p} \cdot$$
$$\sum_{j=0}^{p} \binom{p}{j} [n-a-b-1-p]_{p-j} [n]_j [b-p-1-j]_{-p-1} (-1)^j$$

Proof. We shall write the product
(10.16)
$$\left[\tfrac{a}{2}\right]_k \left[\tfrac{a-1}{2}\right]_k [n-1-a]_{n-k} = 4^{-k} [a]_{2k} (-1)^{n-k} [a-k]_{n-k} = (-1)^n (-4)^{-k} [a]_n [a-k]_k$$

and the product

(10.17)
$$\left[\tfrac{b}{2}\right]_{n-k} \left[\tfrac{b-1}{2}\right]_{n-k} [n+p-b]_k = 4^{k-n} [b]_{2n-2k} (-1)^k [b-p-n-1+k]_k =$$
$$= 4^{-n} (-4)^k [b]_{n+p+1} [b-p-n-1+k]_{n-p-1-k}$$

The sum then becomes

$$(-\tfrac{1}{4})^n [a]_n[b]_{n+p+1} \sum_{k=0}^{n} \binom{n}{k} [a-k]_k[b-p-n-1+k]_{n-p-1-k} \tag{10.18}$$

Now we shall apply Chu–Vandermonde (8.3) to write
(10.19)

$$[a-k]_k = (-1)^k[-a-1+2k]_k = (-1)^k \sum_{j=0}^{k} \binom{k}{j} [2n-a-b-1]_j[b-2n+2k]_{k-j}$$

Substitution in the sum and interchanging the order of summation yields using (2.10)

(10.20)

$$\begin{aligned}(-\tfrac{1}{4})^n [a]_n[b]_{n+p+1} &\sum_{j=0}^{n} \binom{n}{j} [2n-a-b-1]_j(-1)^j\cdot \\ &\sum_{k=j}^{n} \binom{n-j}{k-j} (-1)^{k-j}[b-p-n-1+k]_{n-p-1-j}\end{aligned}$$

By the Chu–Vandermonde formula, (8.13), the inner sum is equal to

$$[b-p-n-1+j]_{-p-1}[n-p-j-1]_{n-j}(-1)^{n-j} = [b-p-n-1+j]_{-p-1}[p]_{n-j} \tag{10.21}$$

so the very sum becomes

$$(-\tfrac{1}{4})^n [a]_n[b]_{n+p+1} \sum_{j=0}^{n} \binom{n}{j} [2n-a-b-1]_j[b-p-n-1+j]_{-p-1}[p]_{n-j}(-1)^j \tag{10.22}$$

Reversing the direction of summation, splitting the first factor and using $\binom{n}{j}[p]_j = \binom{p}{j}[n]_j$ proves the theorem. □

The theorem for $p = 0$ reduces to the formula known from J. L. Slater, [105], formula III.20,

Theorem 10.7. *For any $a, b \in \mathbb{C}$ and any $n \in \mathbb{N}$ we have*

(10.23)

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} \left[\tfrac{a}{2}\right]_k \left[\tfrac{a-1}{2}\right]_k [n-b]_k \left[\tfrac{b}{2}\right]_{n-k} \left[\tfrac{b-1}{2}\right]_{n-k} [n-1-a]_{n-k} &= \\ = (\tfrac{1}{4})^n [a]_n[b-1]_n[2n-a-b-1]_n&\end{aligned}$$

Furthermore, a beautiful formula due to H. L. Krall. cf. [84], or equivalent forms by H. W. Gould, in [64], formulas (11.5) and (11.6), in Krall's form:

Theorem 10.8. *For all $x \neq y \in \mathbb{C}$ we have*

(10.24)
$$\sum_k \binom{n-k}{k}[x]_k[y]_k[n-x-y]_{n-2k}(-1)^k = (-1)^n \frac{[x]_{n+1}-[y]_{n+1}}{x-y} = $$
$$= \sum_{k=0}^{n}[x]_k[n-y]_{n-k}(-1)^{n-k}$$

In our standard form, it looks like:

Theorem 10.9. *For all $x \neq y \in \mathbb{C}$ we have*

(10.25)
$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor}{k} \left[\lceil \tfrac{n}{2} \rceil - \tfrac{1}{2}\right]_k [x]_k[y]_k \left[-\lceil \tfrac{n}{2} \rceil - 1\right]_{\lfloor \frac{n}{2} \rfloor - k} \left[\tfrac{n-x-y}{2}\right]_{\lceil \frac{n}{2} \rceil - k} \left[\tfrac{n-x-y-1}{2}\right]_{\lfloor \frac{n}{2} \rfloor - k}$$
$$= (-1)^{\lceil \frac{n}{2} \rceil} \left(\tfrac{1}{2}\right)^n \frac{[x]_{n+1}-[y]_{n+1}}{x-y}$$

Proof. We prove the form (10.24). Let us define

$$S_n(x,y) = \sum_k \binom{n-k}{k}[x]_k[y]_k[n-x-y]_{n-2k}(-1)^k \tag{10.26}$$

Then the formula (10.24) is obvious for $n = 0, 1$, so we consider $S_n(x,y)$. Splitting the binomial coefficient we obtain

$$S_n(x,y) = \sum_k \binom{n-1-k}{k}[x]_k[y]_k[n-x-y]_{n-2k}(-1)^k +$$
$$+ \sum_k \binom{n-1-k}{k-1}[x]_k[y]_k[n-x-y]_{n-2k}(-1)^k =$$
$$= (n-x-y)S_{n-1}(x,y)$$
$$- \sum_k \binom{n-2-k}{k}[x]_{k+1}[y]_{k+1}[n-2-(x-1)-(y-1)]_{n-2-2k}(-1)^k =$$
$$= (n-x-y)S_{n-1}(x,y) - xyS_{n-2}(x-1,y-1)$$

Substitution of the right side of (10.24) gives

$$\frac{(n-x-y)\left([x]_n-[y]_n\right) + xy\left([x-1]_{n-1}-[y-1]_{n-1}\right)}{(-1)^{n-1}(x-y)} =$$
$$= \frac{(n-x)x[x-1]_{n-1}-(n-y)y[y-1]_{n-1}}{(-1)^{n-1}(x-y)} = \frac{[x]_{n+1}-[y]_{n+1}}{(-1)^n(x-y)}$$

which is the correct right side of (10.24) for n. □

We may split the formula in theorem 10.9 in the even and odd case to get

Theorem 10.10. *For all $x \neq y \in \mathbb{C}$ we have*

(10.27)
$$\sum_{k=0}^{n} \binom{n}{k} \left[n-\tfrac{1}{2}\right]_k [x]_k [y]_k \, [-n-1]_{n-k} \left[n-\tfrac{x+y}{2}\right]_{n-k} \left[n-\tfrac{x+y+1}{2}\right]_{n-k}$$
$$= \left(-\tfrac{1}{4}\right)^n \frac{[x]_{2n+1}-[y]_{2n+1}}{x-y}$$

(10.28)
$$\sum_{k=0}^{n} \binom{n}{k} \left[n+\tfrac{1}{2}\right]_k [x]_k [y]_k \, [-n-2]_{n-k} \left[n-\tfrac{x+y}{2}\right]_{n-k} \left[n-\tfrac{x+y+1}{2}\right]_{n-k}$$
$$= \left(-\tfrac{1}{4}\right)^n \frac{[x]_{2n+2}-[y]_{2n+2}}{(x+y-2n-1)(x-y)}$$

CHAPTER 11. SUMS OF TYPE II$(5,5,1)$

Indefinite sums. There is a general summation formula of type II$(5,5,1)$, namely:

Theorem 11.5. *For any $a,b,c\in\mathbb{C}$ we have*

$$\Delta\frac{[a+b+c]_k[a-1]_{k-1}[b-1]_{k-1}[c-1]_{k-1}(-1)^k}{k![b+c-1]_{k-1}[a+c-1]_{k-1}[a+b-1]_{k-1}}= \tag{11.1}$$
$$=\frac{[a+b+c]_k[a]_k[b]_k[c]_k(a+b+c-2k)(-1)^k}{k![b+c-1]_k[a+c-1]_k[a+b-1]_k}$$

Proof. Straightforward computation. □

Symmetric and balanced sums. There are quite a few symmetric and balanced – in particular well–balanced – formulas of type II$(5,5,1)$. We start with a symmetric formula:

Theorem 11.2. *For any $n,p\in\mathbb{N}_0$ and $a,b,c,d\in\mathbb{C}$ satisfying the condition,*

$$a+b+c+d-3n+1=p$$

we have

(11.2)
$$\sum_{k=0}^{2n}\binom{2n}{k}[a]_k[b]_k[c]_k[d]_k[a]_{2n-k}[b]_{2n-k}[c]_{2n-k}[d]_{2n-k}(-1)^k=$$
$$=[2n]_n[a]_n[b]_n[c]_n[d]_n[a+b-n-p]_{n-p}[a+c-n-p]_{n-p}[b+c-n-p]_{n-p}\cdot$$
$$\sum_{j=0}^{p}\binom{p}{j}[n]_j[d]_j[2n-a-b-c+p-1]_j\cdot$$
$$[a+b-2n]_{p-j}[a+c-2n]_{p-j}[b+c-2n]_{p-j}=$$
$$=[2n]_n[a]_n[b]_n[c]_n[d]_n\sum_{j=0}^{p}\binom{p}{j}[n]_j[d]_j[2n-a-b-c+p-1]_j\cdot$$
$$[a+b-n-p]_{n-j}[a+c-n-p]_{n-j}[b+c-n-p]_{n-j}$$

Proof. We apply the Pfaff–Saalschütz identity, (9.1), to write

$$[a]_k[b]_k[a]_{2n-k}[b]_{2n-k}=\sum_{j=0}^{k}\binom{k}{j}[2n-k]_j[2n-a-b-1]_j[a]_{2n-j}[b]_{2n-j}(-1)^j$$

Substitution of this in the sum of (11.2) yields after changing the order of summation

$$\sum_{j=0}^{2n}\binom{2n}{j}[2n-a-b-1]_j[a]_{2n-j}[b]_{2n-j}\left([c]_j[d]_j\right)^2\cdot$$

$$\sum_{k=j}^{2n}\binom{2n-j}{k-j}[2n-k]_j[c-j]_{k-j}[d-j]_{k-j}[c-j]_{2n-j-k}[d-j]_{2n-j-k}(-1)^{k-j}$$

Now, we have $[2n-k]_j=0$ for $j>2n-k$, so we may end summation for $k=2n-j$ the inner sum then starts

$$\sum_{k=j}^{2n-j}\binom{2n-2j}{2n-k-j}[2n-j]_j\cdots$$

The inner sum may be written as

$$[2n-j]_j\sum_{k=j}^{2n-j}\binom{2n-2j}{k-j}[c-j]_{k-j}[d-j]_{k-j}[c-j]_{2n-j-k}[d-j]_{2n-j-k}(-1)^{k-j}$$

On this sum we may apply the Dixon identity, (9.3), and write it as

$$[2n-j]_j[c-j]_{n-j}[d-j]_{n-j}[2n-2j-c+j-d+j-1]_{n-j}[2n-2j]_{n-j}$$

Therefore we get the sum using $\binom{2n}{j}[2n-j]_n=[2n]_n\binom{n}{j}$

$$\sum_{j=0}^{2n}\binom{2n}{j}[2n-a-b-1]_j[a]_{2n-j}[b]_{2n-j}[c]_j[d]_j[2n-j]_n[c]_n[d]_n[2n-c-d-1]_{n-j}$$
$$=[2n]_n[a]_n[b]_n[c]_n[d]_n\cdot$$
$$\sum_{j=0}^{2n}\binom{n}{j}[2n-a-b-1]_j[a-n]_{n-j}[b-n]_{n-j}[c]_j[d]_j[a+b-n-p]_{n-j}$$

Now we may write

$$\begin{aligned}&[a+b-n-p]_{n-j}[2n-a-b-1]_j=\\=&[a+b-n-p]_{n-p}[a+b-2n]_{p-j}[a+b-2n+j]_j(-1)^j=\\=&[a+b-n-p]_{n-p}[a+b-2n+j]_p(-1)^j\end{aligned}$$

And we may apply the Chu–Vandermonde theorem (9.4) to write

$$[a+b-2n+j]_p=\sum_{h=0}^{p}\binom{p}{h}[a+b-2n]_{p-h}[j]_h$$

Interchanging the order of summation yields

$$[2n]_n[a]_n[b]_n[c]_n[d]_n[a+b-n-p]_{n-p}\sum_{h=0}^{p}\binom{p}{h}[a+b-2n]_{p-h}(-1)^h\cdot$$
$$[c]_h[d]_h\sum_{j=0}^{2n}\binom{n}{j}[j]_h[a-n]_{n-j}[b-n]_{n-j}[c-h]_{j-h}[d-h]_{j-h}(-1)^{j-h}$$

As we remark that $\binom{n}{j}[j]_h=[n]_h\binom{n-h}{j-h}$, and we can see that the condition for the generalized Pfaff–Saalschütz identity, (9.15), yields $c-h+d-h+a-n+b-n-n+h+1=p-h$, we may write the sum as

$$[2n]_n[a]_n[b]_n[c]_n[d]_n[a+b-n-p]_{n-p}\sum_{h=0}^{p}\binom{p}{h}[a+b-2n]_{p-h}(-1)^h\cdot$$
$$[c]_h[d]_h[n]_h\sum_{i=0}^{p-h}\binom{p-h}{i}[n-h]_i[d-h]_i\cdot$$
$$[c-h+a-n-(p-h)]_{n-h-i}[c-h+b-n-(p-h)]_{n-h-i}(-1)^i$$

As we may write $\binom{p}{h}\binom{p-h}{i}=\binom{p}{h+i}\binom{h+i}{h}$ and join $[n]_h[n-h]_i=[n]_{h+i}$ and $[d]_h[d-h]_i=[d]_{h+i}$ we can change the variable i to $i-h$ and write the sum as

$$[2n]_n[a]_n[b]_n[c]_n[d]_n[a+b-n-p]_{n-p}\sum_{h=0}^{p}[a+b-2n]_{p-h}[c]_h\cdot$$
$$\sum_{i=h}^{p}\binom{p}{i}\binom{i}{h}[n]_i[d]_i[c+a-n-p]_{n-i}[c+b-n-p]_{n-i}(-1)^i$$

Interchanging the order of summation yields

$$[2n]_n[a]_n[b]_n[c]_n[d]_n[a+b-n-p]_{n-p}$$
$$\sum_{i=0}^{p}\binom{p}{i}[a+b-2n]_{p-i}[n]_i[d]_i[c+a-n-p]_{n-i}[c+b-n-p]_{n-i}(-1)^i\cdot$$
$$\sum_{h=0}^{i}\binom{i}{h}[a+b-2n-p+i]_{i-h}[c]_h$$

Here the inner sum may be object to the Chu–Vandermonde convolution, (8.4), so we have

$$[2n]_n[a]_n[b]_n[c]_n[d]_n[a+b-n-p]_{n-p}\sum_{i=0}^{p}\binom{p}{i}[a+b-2n]_{p-i}[n]_i[d]_i\cdot$$
$$[c+a-n-p]_{n-i}[c+b-n-p]_{n-i}(-1)^i[a+b-2n-p+i+c]_i$$

Rewriting $(-1)^i[a+b-2n-p+i+c]_i = [2n+p-1-a-b-c]_i$ yields the formula (11.2). □

The special case of $p = 0$ is due to W. A. Al–Salam from 1957, [6], cf. H. W. Gould, [64], formula (22.1).

Then we proceed with the following well–balanced formula of type II(5, 5, 1), cf. L. J. Slater, [105], formula III.13, although it also follows as a corollary to the formula of J. Dougall, cf. chapter 12, formula (12.9).

We give the formula in the canonical form, (4.34).

Theorem 11.3. *For any $a, b, c \in \mathbb{C}$ and $n \in \mathbb{N}$ we have*

(11.3)
$$\sum_{k=0}^{n} \binom{n}{k} [n+2a]_k [n-2a]_{n-k} [b+a]_k [b-a]_{n-k} [c+a]_k [c-a]_{n-k} (n+2a-2k) = \\ = [b+c]_n [n+2a]_{2n+1}$$

Proof. We use the Pfaff–Saalschütz formula, (9.1), to write
(11.4)
$$[b+a]_k[c+a]_k = \sum_{j=0}^{k} \binom{k}{j} (-1)^j [b-a-n+k]_{k-j} [c-a-n+k]_{k-j} [2a+n-k]_j [n-b-c-1]_j$$

Then we shall join the factorials,

$$[b-a]_{n-k}[b-a-n+k]_{k-j} = [b-a]_{n-j} \tag{11.5}$$
$$[c-a]_{n-k}[c-a-n+k]_{k-j} = [c-a]_{n-j} \tag{11.6}$$
$$[n+2a]_k[n+2a-k]_j = [n+2a]_{k+j} \tag{11.7}$$

Substitution of (11.4) in (11.3) yields after interchanging the order of summation using (2.10)

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j [n-b-c-1]_j [b-a]_{n-j} [c-a]_{n-j} \cdot \\ \sum_{k=j}^{n} \binom{n-j}{k-j} [n+2a]_{k+j} [n-2a]_{n-k} (n+2a-2k) \tag{11.8}$$

Now we may write

$$[n+2a]_{k+j} = [n+2a]_{2j}[n+2a-2j]_{k-j}$$

and split the factor as

$$n+2a-2k = (n+2a-2j) - 2(k-j)$$

Then we get the inner sum split in two,

(11.9)

$$\sum_{k=j}^{n}\binom{n-j}{k-j}[n+2a]_{k+j}[n-2a]_{n-k}(n+2a-2k)=$$
$$=[n+2a]_{2j}(n+2a-2j)\sum_{k=j}^{n}\binom{n-j}{k-j}[n+2a-2j]_{k-j}[n-2a]_{n-k}$$
$$-2[n+2a]_{2j}(n-j)\sum_{k=j+1}^{n}\binom{n-j-1}{k-j-1}[n+2a-2j]_{k-j}[n-2a]_{n-k}=$$
$$=[n+2a]_{2j}(n+2a-2j)[2n-2j]_{n-j}$$
$$-2[n+2a]_{2j}(n-j)(n+2a-2j)[2n-2j-1]_{n-j-1}$$

where we have used Chu–Vandermonde (8.3) on the two sums. Now, this vanishes except for $j=n$, in which case the whole sum becomes

$$(-1)^n[n-b-c-1]_n[n+2a]_{2n}(2a-n)=[b+c]_n[n+2a]_{2n+1}$$

which proves the statement. □

There exist a family of balanced formulas too, namely:

Theorem 11.4. *For any $n\in\mathbb{N}$ and $a,b,c,d\in\mathbb{C}$ satisfying the condition,*

$$p=b+c+d-\tfrac{n}{2}+1\in\mathbb{N}_0 \tag{11.10}$$

we have

(11.11)

$$\sum_{k=0}^{n}\binom{n}{k}[n+2a]_k[b+a]_k[c+a]_k[d+a]_k$$
$$[n-2a]_{n-k}[b-a]_{n-k}[c-a]_{n-k}[d-a]_{n-k}(-1)^k$$
$$=4^n\left[\tfrac{n-1}{2}+a\right]_n\left[b+\tfrac{n}{2}-p\right]_{n-p}[b+d-p]_{n-p}\left[d+\tfrac{n}{2}-p\right]_{n-p}\cdot$$
$$\sum_{j=0}^{p}\binom{p}{j}[n]_j[c+a]_j[c-a]_j[b+d-n]_{p-j}\left[b-\tfrac{n}{2}\right]_{p-j}\left[d-\tfrac{n}{2}\right]_{p-j}$$
$$=4^n\left[\tfrac{n-1}{2}+a\right]_n$$
$$\sum_{j=0}^{p}\binom{p}{j}[n]_j[c+a]_j[c-a]_j[b+d-p]_{n-j}\left[b+\tfrac{n}{2}-p\right]_{n-j}\left[d+\tfrac{n}{2}-p\right]_{n-j}$$

Proof. This theorem is a corollary to the generalization of Dougall's theorem in chapter 12. We choose $e=\frac{n}{2}$ in (12.11), and remark that

$$\left[\tfrac{n}{2}+a\right]_k\left[\tfrac{n}{2}-a\right]_{n-k}(n+2a-2k)=(-1)^{n-k}2\left[\tfrac{n}{2}+a\right]_{n+1}$$

Dividing (12.11) with this constant yields (11.11). □

The special case of (11.11) for $p=0$ was found by S. Ramanujan 1911–12, cf. G. H. Hardy, [74], p. 493.

CHAPTER 12. SUMS OF TYPE II$(6,6,\pm 1)$

There is a single well–balanced formula of this type and three others. But we have a kind of general transformation available from type II$(6,6,-1)$ to type II$(3,3,1)$.

Theorem 12.1. *For any $a,b,c,d \in \mathbb{C}$ and $n \in \mathbb{N}$ we have*

(12.1)

$$\sum_{k=0}^{n} \binom{n}{k} [n+2a]_k [b+a]_k [c+a]_k [d+a]_k \cdot$$
$$[n-2a]_{n-k} [b-a]_{n-k} [c-a]_{n-k} [d-a]_{n-k} (n+2a-2k)$$
$$= [n+2a]_{2n+1} \sum_{j=0}^{n} \binom{n}{j} [b+a]_j [n-c-d-1]_j [c-a]_{n-j} [d-a]_{n-j} (-1)^j$$

Proof. We apply the Pfaff–Saalschütz formula (9.1) to write

(12.2)

$$[c+a]_k [d+a]_k = \sum_{j=0}^{k} \binom{k}{j} [c-a-n+k]_{k-j} [d-a-n+k]_{k-j} [n+2a-k]_j [n-c-d-1]_j (-1)^j$$

Then we shall join the factorials,

(12.3)

$$[c-a]_{n-k} [c-a-n+k]_{k-j} = [c-a]_{n-j}$$

(12.4)

$$[d-a]_{n-k} [d-a-n+k]_{k-j} = [d-a]_{n-j}$$

(12.5)

$$[n+2a]_k [n+2a-k]_j = [n+2a]_{k+j} = [n+2a]_{2j} [n+2a-2j]_{k-j}$$

to write the left sum of (12.1) after changing the order of summation as

(12.6)

$$\sum_{j=0}^{n} \binom{n}{j} [n+2a]_{2j} [c-a]_{n-j} [d-a]_{n-j} [n-c-d-1]_j [b+a]_j (-1)^j$$
$$\sum_{k=j}^{n} \binom{n-j}{k-j} [n+2a-2j]_{k-j} [b+a-j]_{k-j} \cdot$$
$$[n-2a]_{n-k} [b-a]_{n-k} (n+2a-2k)$$

The inner sum has the form of (10.6), if we replace n with $n-j$, a with $a-\frac{j}{2}$ and b with $b-\frac{j}{2}$. Then theorem 10.3 gives us

$$\sum_{k=j}^{n} \binom{n-j}{k-j} [n+2a-2j]_{k-j} [b+a-j]_{k-j} \cdot \tag{12.7}$$
$$[n-2a]_{n-k} [b-a]_{n-k} (n+2a-2k) =$$
$$= [n-j+2a-j]_{2n-2j+1}$$

which inserted in (12.6) gives

$$[n+2a]_{2n+1}\sum_{j=0}^{n}\binom{n}{j}[b+a]_j[n-c-d-1]_j[c-a]_{n-j}[d-a]_{n-j}(-1)^j \tag{12.8}$$

□

This transformation allows no nice formulas of type II(6, 6, −1).

The well–balanced formula of type II(6, 6, 1):

Theorem 12.2. *For any $n \in \mathbb{N}$ and $a, b, c, d \in \mathbb{C}$ satisfying the condition, $b+c+d = \frac{n-1}{2}$, we have*

(12.9)

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(-1)^k[n+2a]_k[b+a]_k[c+a]_k[d+a]_k\cdot\\&\qquad[n-2a]_{n-k}[b-a]_{n-k}[c-a]_{n-k}[d-a]_{n-k}(n+2a-2k)\\&=(-1)^n2^{2n+1}[a+\tfrac{n}{2}]_{n+1}[b+\tfrac{n-1}{2}]_n[c+\tfrac{n-1}{2}]_n[d+\tfrac{n-1}{2}]_n\end{aligned}$$

Proof. This follows as a corollary to theorem 13.2 choosing $e = \frac{n-1}{2}$.

In [48] I. M. Gessel proves three formulas of this type, no. 19.1a, 19.2a and 19.3a.

(12.10)

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}[-b-1]_k\left[-a-b-\tfrac{1}{2}\right]_k[a+2b-n-1]_k[b]_k\left[n+\tfrac{1}{2}\right]_{n-k}\cdot\\&\qquad[n-2a-3b]_{n-k}[3n+1-b]_{n-k}[2a+3b-n-1]_{n-k}(b-1-3k)(-1)^k=\\&\qquad=[a+b-1]_n[b-1-n]_{2n+1}\left[a+2b-\tfrac{1}{2}\right]_n(-1)^n\end{aligned}$$

(12.11)

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}[1-b-a]_k[1-a-2b-n]_k\left[b-\tfrac{1}{2}\right]_k[b+a-1]_k\left[n-\tfrac{1}{2}\right]_{n-k}\cdot\\&\qquad[3n-1+a+b]_{n-k}[n-2+3b+a]_{n-k}[1-a-3b-n]_{n-k}\\&\qquad(a+b-1+3k)(-1)^k=\\&\qquad=(a+b-1)\left[\tfrac{1}{2}-a-2b\right]_n[-b]_n[-a-b-n]_{2n}(-1)^n\end{aligned}$$

(12.12)

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}[-b]_k[1-2a]_k[-b-a-n]_k\left[\tfrac{1}{2}-b-a\right]_k\cdot\\&\qquad[-n-1]_{n-k}[2(b+a)+3n-1]_{n-k}\left[n-1+\tfrac{b}{2}+a\right]_{n-k}\\&\qquad\left[n-\tfrac{1}{2}+\tfrac{b}{2}+a\right]_{n-k}(2(b+a)-1+3k)(-1)^k=\\&\qquad=-2[-a]_n[2n+b]_n[-2(b+a+n)]_n\left[\tfrac{1}{2}-b-a\right]_{n+1}\end{aligned}$$

CHAPTER 13. SUMS OF TYPE II$(7,7,1)$

There are but a few well–balanced formulas of type II$(7,7,1)$, and a couple of others. But we have a kind of general transformation available, which e.g. may be used to prove the famous formula due to J. Dougall from 1907, [34].

Theorem 13.1. *For any $a,b,c,d,e\in\mathbb{C}$ and $n\in\mathbb{N}$ we have*

(13.1)
$$\sum_{k=0}^{n}\binom{n}{k}[n+2a]_k[b+a]_k[c+a]_k[d+a]_k[e+a]_k\cdot$$
$$[n-2a]_{n-k}[b-a]_{n-k}[c-a]_{n-k}[d-a]_{n-k}[e-a]_{n-k}(n+2a-2k)$$
$$=[n+2a]_{2n+1}(-1)^n\sum_{j=0}^{n}\binom{n}{j}[b+a]_j[c+a]_j[n-d-e-1]_j\cdot$$
$$[d-a]_{n-j}[e-a]_{n-j}[n-b-c-1]_{n-j}$$

Proof. We apply the Pfaff–Saalschütz formula (9.1) to write
(13.2)
$$[d+a]_k[e+a]_k=\sum_{j=0}^{k}\binom{k}{j}[d-a-n+k]_{k-j}[e-a-n+k]_{k-j}[n+2a-k]_j[n-d-e-1]_j(-1)^j$$

Then we shall join the factorials,

(13.3)
$$[d-a]_{n-k}[d-a-n+k]_{k-j}=[d-a]_{n-j}$$
(13.4)
$$[e-a]_{n-k}[e-a-n+k]_{k-j}=[e-a]_{n-j}$$
(13.5)
$$[n+2a]_k[n+2a-k]_j=[n+2a]_{k+j}=[n+2a]_{2j}[n+2a-2j]_{k-j}$$

to write the left sum of (13.1) after changing the order of summation as

(13.6)
$$\sum_{j=0}^{n}\binom{n}{j}[n+2a]_{2j}[d-a]_{n-j}[e-a]_{n-j}[n-d-e-1]_j[b+a]_j[c+a]_j(-1)^j\cdot$$
$$\sum_{k=j}^{n}\binom{n-j}{k-j}[n+2a-2j]_{k-j}[b+a-j]_{k-j}[c+a-j]_{k-j}\cdot$$
$$[n-2a]_{n-k}[b-a]_{n-k}[c-a]_{n-k}(n+2a-2k)$$

The inner sum has the form of (11.2), if we replace n with $n-j$, a with $a-\frac{j}{2}$, b

with $b - \frac{j}{2}$ and c with $c - \frac{c}{2}$. Then theorem 11.2 gives us

(13.7)
$$\sum_{k=j}^{n} \binom{n-j}{k-j} [n+2a-2j]_{k-j} [b+a-j]_{k-j} [c+a-j]_{k-j} \cdot$$
$$[n-2a]_{n-k} [b-a]_{n-k} [c-a]_{n-k} (n+2a-2k) =$$
$$= [n-j+2a-j]_{2n-2j+1} [b+c-j]_{n-j}$$

which inserted in (13.6) gives
(13.8)
$$[n+2a]_{2n+1}(-1)^n \sum_{j=0}^{n} \binom{n}{j} [b+a]_j [c+a]_j [n-d-e-1]_j [d-a]_{n-j} [e-a]_{n-j} [n-b-c-1]_{n-j}$$

□

This theorem transferring a sum of type II(7, 7, 1) to a certain sum of type II(4, 4, 1) proves particularly useful to prove the identity of J. Dougall, [34], from 1907, cf. [64], formulas (71.1) and (97.1).

Theorem 13.2. *For any $n \in \mathbb{N}$ and $a, b, c, d, e \in \mathbb{C}$ satisfying the condition, $b + c + d + e = n - 1$, we have*

(13.9)
$$\sum_{k=0}^{n} \binom{n}{k} [n+2a]_k [b+a]_k [c+a]_k [d+a]_k [e+a]_k \cdot$$
$$[n-2a]_{n-k} [b-a]_{n-k} [c-a]_{n-k} [d-a]_{n-k} [e-a]_{n-k} (n+2a-2k)$$
$$= [n+2a]_{2n+1} [b+e]_n [c+e]_n [d+e]_n$$
$$= (-1)^n [n+2a]_{2n+1} [b+c]_n [c+d]_n [d+b]_n$$

Proof. The condition implies that we may write
$$[n-b-c-1]_{n-j} = [d+e]_{n-j}$$
and as we have
$$[n-d-e-1]_j = [d+e-n+j]_j (-1)^j$$
the product of these two factors from the right sum of (13.1) becomes
$$[n-b-c-1]_{n-j} [n-d-e-1]_j = [d+e]_{n-j} [d+e-n+j]_j (-1)^j = [d+e]_n (-1)^j$$
The right sum therefore becomes
$$[d+e]_n \sum_{j=0}^{n} \binom{n}{j} [b+a]_j [c+a]_j [d-a]_{n-j} [e-a]_{n-j} (-1)^j$$
But the condition now implies that the Pfaff–Saalschütz identity, (9.1), tells us that this sum equals $(-1)^n [b+e]_n [c+e]_n$. This proves the theorem. □

Remark. *If we express e in (13.9) by the condition, and consider the identity as a polynomial identity in the variable, d, then the identity between the coefficients to* d^n *becomes (11.2).*

This theorem allows a generalization analogous to the generalization of the Pfaff–Saalschütz formula given in (9.15).

Theorem 13.3. *For any* $n \in \mathbb{N}$ *and* $a, b, c, d, e \in \mathbb{C}$ *satisfying the condition,*

$$p = b + c + d + e - n + 1 \in \mathbb{N}_0 \tag{13.10}$$

we have

(13.11)

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} [n+2a]_k [b+a]_k [c+a]_k [d+a]_k [e+a]_k \cdot \\
&\quad [n-2a]_{n-k} [b-a]_{n-k} [c-a]_{n-k} [d-a]_{n-k} [e-a]_{n-k} (n+2a-2k) \\
&=(-1)^n \cdot [n+2a]_{2n+1} [b+e-p]_{n-p} [b+d-p]_{n-p} [d+e-p]_{n-p} \cdot \\
&\quad \sum_{j=0}^{p} \binom{p}{j} [n]_j [c+a]_j [c-a]_j [b+e-n]_{p-j} [b+d-n]_{p-j} [d+e-n]_{p-j} \\
&=(-1)^n \cdot [n+2a]_{2n+1} \cdot \\
&\quad \sum_{j=0}^{p} \binom{p}{j} [n]_j [c+a]_j [c-a]_j [b+e-p]_{n-j} [b+d-p]_{n-j} [d+e-p]_{n-j}
\end{aligned}$$

Proof. The theorem 13.1 yields

$$(-1)^n \cdot [n+2a]_{2n+1} \cdot S$$

where

$$S = \sum_{j=0}^{n} \binom{n}{j} [b+a]_j [c+a]_j [n-d-e-1]_j [d-a]_{n-j} [e-a]_{n-j} [n-b-c-1]_{n-j}$$

Using the condition (13.10) we can write the product

$$[n-d-e-1]_j [n-b-c-1]_{n-j} = (-1)^j [d+e-p]_{n-p} [j+d+e-n]_p$$

Next we apply the Chu–Vandermonde formula (8.3) to write

$$[j+d+e-n]_p = \sum_{i=0}^{p} \binom{p}{i} [j]_i [d+e-n]_{p-i}$$

and then the product as

$$[n-d-e-1]_j[n-b-c-1]_{n-j} = (-1)^j[d+e-p]_{n-p}\sum_{i=0}^{p}\binom{p}{i}[j]_i[d+e-n]_{p-i}$$
$$= (-1)^j\sum_{i=0}^{p}\binom{p}{i}[j]_i[d+e-p]_{n-i}$$

Changing the order of summation we get

$$S = \sum_{i=0}^{p}\binom{p}{i}[d+e-p]_{n-i}\sum_{j=0}^{n}\binom{n}{j}[j]_i[b+a]_j[c+a]_j[d-a]_{n-j}[e-a]_{n-j}(-1)^j$$

Using $\binom{n}{j}[j]_i = [n]_i\binom{n-i}{j-i}$ and $[x]_j = [x]_i[x-i]_{j-i}$ for $x = b+a$ and $x = c+a$ we obtain

$$S = \sum_{i=0}^{p}\binom{p}{i}[d+e-p]_{n-i}[n]_i[b+a]_i[c+a]_i\cdot$$
$$\sum_{j=0}^{n}\binom{n-i}{j-i}[b+a-i]_{j-i}[c+a-i]_{j-i}[d-a]_{n-j}[e-a]_{n-j}(-1)^j$$

Now the condition (13.10) assures that we may apply the generalized Pfaff–Saalschütz formula, (9.15), with the number (9.14) equal to $p-i$. Therefore the sum becomes

$$S = \sum_{i=0}^{p}\binom{p}{i}[d+e-p]_{n-i}[n]_i[b+a]_i[c+a]_i\cdot$$
$$\sum_{\ell=0}^{p-i}\binom{p-i}{\ell}[n-i]_\ell[c+a-i]_\ell[b+d-p]_{n-i-\ell}[b+e-p]_{n-i-\ell}(-1)^{\ell+i}$$

In this sum we may join two pairs, $[c+a]_i[c+a-i]_\ell = [c+a]_{i+\ell}$ and $[n]_i[n-i]_\ell = [n]_{i+\ell}$. Then we substitute $\ell - i$ for ℓ and interchange the order of summation to get

$$S = \sum_{\ell=0}^{p}\binom{p}{\ell}[n]_\ell[b+d-p]_{n-\ell}[b+e-p]_{n-\ell}[c+a]_\ell(-1)^\ell\cdot$$
$$\sum_{i=0}^{\ell}\binom{\ell}{i}[d+e-p]_{n-i}[b+a]_i$$

After writing the product $[d+e-p]_{n-i} = [d+e-p]_{n-\ell}[d+e-p-n+\ell]_{\ell-i}$ we can apply the Chu–Vandermonde formula (8.3) to get for the inner sum

$$[d+e-p]_{n-\ell}[a+b+d+e-p-n+\ell]_\ell = [d+e-p]_{n-\ell}[c-a]_\ell(-1)^\ell$$

and hence the very sum as

$$S=\sum_{\ell=0}^{p}\binom{p}{\ell}[n]_\ell[b+d-p]_{n-\ell}[b+e-p]_{n-\ell}[c+a]_\ell[d+e-p]_{n-\ell}[c-a]_\ell$$

Applying the factorization $[x-p]_{n-\ell}=[x-p]_{n-p}[x-n]_{p-\ell}$ we get the form of (13.11). □

Furthermore we have from J. L. Slater, [105], formula III.19:

Theorem 13.4. *For any $a,d\in\mathbb{C}$ and any $n\in\mathbb{N}$ we have*

(13.12)

$$\sum_{k=0}^{n}\binom{n}{k}[n+2a]_k[d+a]_k\left[d+a-\tfrac{1}{2}\right]_k[n-2d]_k[n-1+2a-2d]_k(n+2a-2k)\cdot$$
$$[n-2a]_{n-k}[d-a]_{n-k}\left[d-a-\tfrac{1}{2}\right]_{n-k}[n-2d-2a]_{n-k}[n-1-2d]_{n-k}=$$
$$=\left(\tfrac{1}{4}\right)^n[n+2a]_{2n+1}[4d-n]_n[2d-2a]_n[2d+2a-1]_n$$

Proof. We apply the transformation theorem, theorem 13.1, with the arguments (a,b,c,d,e) replaced by $(a,d,d-\frac{1}{2},n-2d-a,n-1+a-2d)$. Then we get

(13.13)

$$(-1)^n[n+2a]_{2n+1}\sum_{j=0}^{n}\binom{n}{j}[d+a]_j\left[d+a-\tfrac{1}{2}\right]_j\cdot$$
$$[4d-n]_j[n-2d-2a]_{n-j}[n-1-2d]_{n-j}[n-2d-\tfrac{1}{2}]_{n-j}$$

Now we can apply theorem 10.5 for the arguments (a,b) replaced with $(2n-1-4d,2d+2a)$ and obtain for the sum (reversed)

$$\left(\tfrac{1}{4}\right)^n[2n-1-4d]_n[2d-2a-1]_n[2d-2a]_n \tag{13.14}$$

Now, by changing the signs of the first factor, we obtain the result in the theorem.

□

There are three formulas to be derived from the theorems 10.5 and 10.6. They follow from the following lemma:

Lemma 13.1. *For any $a,b\in\mathbb{C}$ we have*

(13.15)

$$\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{n}{2}\rfloor}{k}\left[\lceil\tfrac{n}{2}\rceil-\tfrac{1}{2}\right]_k\left[\lfloor\tfrac{n}{2}\rfloor-\tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor-k}\left[\tfrac{n}{2}+a\right]_k\left[\tfrac{n-1}{2}+a\right]_k\left[\tfrac{n}{2}-a\right]_{\lceil\frac{n}{2}\rceil-k}\cdot$$
$$\left[\tfrac{n-1}{2}-a\right]_{\lfloor\frac{n}{2}\rfloor-k}\left[\tfrac{b+a}{2}\right]_k\left[\tfrac{b+a-1}{2}\right]_k\left[\tfrac{b-a}{2}\right]_{\lceil\frac{n}{2}\rceil-k}\left[\tfrac{b-a-1}{2}\right]_{\lfloor\frac{n}{2}\rfloor-k}(n+2a-4k)=$$
$$=\left[\lfloor\tfrac{n}{2}\rfloor-\tfrac{1}{2}\right]_{\lfloor\frac{n}{2}\rfloor}\left[\tfrac{n}{2}+a\right]_{n+1}\left(\left[\tfrac{n-1}{2}+a\right]_n+(-1)^n\left[\tfrac{n-1}{2}+b\right]_n\right)$$

(13.16)
$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor}{k} \left[\lceil \tfrac{n}{2} \rceil - \tfrac{1}{2}\right]_k \left[\lfloor \tfrac{n}{2} \rfloor + \tfrac{1}{2}\right]_{\lfloor \frac{n}{2} \rfloor - k} \left[\tfrac{n}{2} + a\right]_k \left[\tfrac{n-1}{2} + a\right]_k \left[\tfrac{n+1}{2} - a\right]_{\lceil \frac{n}{2} \rceil - k} \cdot$$
$$\left[\tfrac{n}{2} - a\right]_{\lfloor \frac{n}{2} \rfloor - k} \left[\tfrac{b+a-2}{2}\right]_k \left[\tfrac{b+a-1}{2}\right]_k \left[\tfrac{b-a}{2}\right]_{\lceil \frac{n}{2} \rceil - k} \left[\tfrac{b-a-1}{2}\right]_{\lfloor \frac{n}{2} \rfloor - k} (n + 2a - 4k - 1) =$$
$$= \frac{\left[\lfloor \frac{n}{2} \rfloor + \frac{1}{2}\right]_{\lfloor \frac{n}{2} \rfloor} \left[\frac{n-1}{2} + a\right]_{n+1}}{2(n+1)(a+b)} \left(\left[\tfrac{n}{2} + a\right]_{n+1} + (-1)^n \left[\tfrac{n}{2} + b\right]_{n+1} \right)$$

Proof. The formulas are obtained by adding and subtracting the formulas 10.9 and 10.10 and then applying the expansions 2.33–34. □

Theorem 13.5. *For any $a, b \in \mathbb{C}$ and any $n \in \mathbb{N}$ we have*

(13.17)
$$\sum_{k=0}^{n} \binom{n}{k} \left[n - \tfrac{1}{2}\right]_k \left[n - \tfrac{1}{2}\right]_{n-k} [n + a]_k \left[n - \tfrac{1}{2} + a\right]_k [n - a]_{n-k} \cdot$$
$$\left[n - \tfrac{1}{2} - a\right]_{n-k} \left[\tfrac{b+a}{2}\right]_k \left[\tfrac{b+a-1}{2}\right]_k \left[\tfrac{b-a}{2}\right]_{n-k} \left[\tfrac{b-a-1}{2}\right]_{n-k} (n + a - 2k) =$$
$$= \frac{1}{2} \left[n - \tfrac{1}{2}\right]_n [n + a]_{2n+1} \left(\left[n - \tfrac{1}{2} + a\right]_{2n} + \left[n - \tfrac{1}{2} + b\right]_{2n} \right)$$

(13.18)
$$\sum_{k=0}^{n} \binom{n}{k} \left[n - \tfrac{1}{2}\right]_k \left[n + \tfrac{1}{2}\right]_{n-k} \left[n + a + \tfrac{1}{2}\right]_k [n + a]_k [n - a]_{n-k} \cdot$$
$$\left[n - a - \tfrac{1}{2}\right]_{n-k} \left[\frac{b+a-\frac{3}{2}}{2}\right]_k \left[\frac{b+a-\frac{1}{2}}{2}\right]_k \left[\frac{b-a-\frac{1}{2}}{2}\right]_{n-k} \left[\frac{b-a-\frac{3}{2}}{2}\right]_{n-k} (n + a - 2k) =$$
$$= \frac{\left[n - \frac{1}{2}\right]_n [n + a]_{2n+1}}{a + b + \frac{1}{2}} \left(\left[n + a + \tfrac{1}{2}\right]_{2n+1} + [n + b]_{2n+1} \right)$$

(13.19)
$$\sum_{k=0}^{n} \binom{n}{k} \left[n + \tfrac{1}{2}\right]_k \left[n + \tfrac{1}{2}\right]_{n-k} \left[n + \tfrac{1}{2} + a\right]_k [n + a]_k [n - a]_{n-k} \cdot$$
$$\left[n + \tfrac{1}{2} - a\right]_{n-k} \left[\tfrac{b+a-2}{2}\right]_k \left[\tfrac{b+a-1}{2}\right]_k \left[\tfrac{b-a-2}{2}\right]_{n-k} \left[\tfrac{b-a-1}{2}\right]_{n-k} (n + a - 2k) =$$
$$= \frac{\left[n + \frac{1}{2}\right]_n [n + a]_{2n+1}}{(n+1)(a+b)(a-b)} \left(\left[n + \tfrac{1}{2} + a\right]_{2n+2} - \left[n + \tfrac{1}{2} + b\right]_{2n+2} \right)$$

Proof. The formulas follow immediately from lemmas 13.1 and 13.2 by replacing n by $2n$ and $2n + 1$. □

CHAPTER 14. SUMS OF TYPE II$(8,8,1)$

There is a single formula known to us of this type.

Theorem 14.1. *For any $a,b,c\in\mathbb{C}$ and $n,p\in\mathbb{N}_0$ we have that if the arguments satisfies*

$$p=2(a+b+c+1-n)+\tfrac{1}{2} \tag{14.1}$$

then we have the following balanced formula

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\left[n+\tfrac{1}{2}\right]_k\left[n-\tfrac{1}{2}\right]_{n-k}\left[a+\tfrac{1}{2}\right]_k\left[b+\tfrac{1}{2}\right]_k\left[c+\tfrac{1}{2}\right]_k[a]_k[b]_k[c]_k\cdot\\&\qquad[a]_{n-k}[b]_{n-k}[c]_{n-k}\left[a-\tfrac{1}{2}\right]_{n-k}\left[b-\tfrac{1}{2}\right]_{n-k}\left[c-\tfrac{1}{2}\right]_{n-k}=\\&(-1)^p\left(\tfrac{1}{64}\right)^n[2n]_n[2a]_n[2b]_n[2c]_n\left[2a+\tfrac{1}{2}-p\right]_{n-p}\left[2b+\tfrac{1}{2}-p\right]_{n-p}\left[2c+\tfrac{1}{2}-p\right]_{n-p}\cdot\\&\sum_{i=0}^{p}\binom{p}{i}[n]_i[2c+1]_i[2c-n]_i\left[p-2c-\tfrac{3}{2}\right]_{p-i}\left[2a+\tfrac{1}{2}-n\right]_{p-i}\left[2b+\tfrac{1}{2}-n\right]_{p-i}\end{aligned} \tag{14.2}$$

Proof. We apply theorem (10.1) to $2n+1$, $2a+1$, $2b+1$ and $2c+1$, and the remark, that the corresponding symmetric sum of type II$(4,4,-1)$ must vanish. Adding the two sums we may apply formula (2.31) to the binomial coefficient and split the 6 factorials in two each. □

For $p=0$ the formula has a much nicer right side:

Corollary 14.1. *For any $a,b,c\in\mathbb{C}$ and $n\in\mathbb{N}_0$ we have that if the arguments satisfies*

$$a+b+c+1-n+\tfrac{1}{4}=0 \tag{14.3}$$

then we have the following balanced formula

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\left[n+\tfrac{1}{2}\right]_k\left[n-\tfrac{1}{2}\right]_{n-k}\left[a+\tfrac{1}{2}\right]_k\left[b+\tfrac{1}{2}\right]_k\left[c+\tfrac{1}{2}\right]_k[a]_k[b]_k[c]_k\cdot\\&\qquad[a]_{n-k}[b]_{n-k}[c]_{n-k}\left[a-\tfrac{1}{2}\right]_{n-k}\left[b-\tfrac{1}{2}\right]_{n-k}\left[c-\tfrac{1}{2}\right]_{n-k}=\\&\qquad\left(\tfrac{1}{4096}\right)^n[2n]_n[4a+1]_{2n}[4b+1]_{2n}[4c+1]_{2n}\end{aligned} \tag{14.4}$$

Proof. Obvious. □

CHAPTER 15. SUMS OF TYPES II(p,p,z)

Sums of types II(p,p,z). The general formula for the family is the following, cf. Gould's table, [64], formula no. 1.53:

Theorem 15.1. *For $p,n \in \mathbb{N}$, $y \in \mathbb{C}$ and $q,r \in \mathbb{N}_0$ satisfying $r,q < p$, $r+q \le p$ we have with $\rho = e^{\frac{2\pi i}{p}}$ and $z = y^p$ the equivalent identities*

(15.1)
$$\sum_{k=0}^{n} \binom{pn+r+q}{pk+r} z^k = \frac{y^{-r}}{p} \sum_{j=0}^{p-1} \left(1+\rho^j y\right)^{pn+r+q} \rho^{-jr}$$

$$\sum_{k=0}^{n} \binom{n}{k} \prod_{j=1}^{q} \left[n+\tfrac{j}{p}\right]_k \prod_{j=1}^{p-q-1} \left[n-\tfrac{j}{p}\right]_k \prod_{j=1}^{r} \left[n+\tfrac{j}{p}\right]_{n-k} \prod_{j=1}^{p-r-1} \left[n-\tfrac{j}{p}\right]_{n-k} z^k =$$

(15.2)
$$\frac{\prod_{j=1}^{r} \left[n+\frac{j}{p}\right]_n \prod_{j=1}^{p-r-1} \left[n-\frac{j}{p}\right]_n}{p\binom{pn+r+q}{r} y^r} \sum_{j=0}^{p-1} \left(1+\rho^j y\right)^{pn+r+q} \rho^{-jr}$$

Proof. We consider the sum on the right sides of (15.1–15.2), and apply the binomial formula to the power:

(15.3)
$$S = \sum_{j=0}^{p-1} \left(1+\rho^j y\right)^{pn+r+q} \rho^{-jr} = \sum_{j=0}^{p-1} \sum_{i=0}^{pn+r+q} \binom{pn+r+q}{i} \rho^{ij} y^i \rho^{-rj} =$$
$$\sum_{i=0}^{pn+r+q} \binom{pn+r+q}{i} y^i \sum_{j=0}^{p-1} \rho^{(i-r)j}$$

As we have

$$(15.4) \qquad \sum_{j=0}^{p-1} \rho^{(i-r)j} = \begin{cases} p & \text{if } i \equiv r \ (p) \\ 0 & \text{if } i \not\equiv r \ (p) \end{cases}$$

we may sum only the nonzero terms by changing the summation variable to k with $i = r + kp$ in (15.3) to get

$$(15.5) \qquad S = \sum_{k=0}^{n} \binom{pn+r+q}{pk+r} y^{pk+r} p = py^r \sum_{k=0}^{n} \binom{pn+r+q}{pk+r} z^k$$

This yields (15.1).

To obtain (15.2) it only remains to write the binomial coefficient in the appropriate way, i.e.,

(15.6)

$$\binom{pn+r+q}{pk+r} = \frac{[pn+r+q]_{pk+r}}{[pk+r]_{pk+r}} = \frac{[pn+r+q]_r\,[pn+q]_{pk}}{[pk+r]_{pk}\,[r]_r} =$$

$$= \binom{pn+r+q}{r} \frac{\prod_{j=0}^{p-1} [pn+q-j,p]_k}{\prod_{j=0}^{p-1} [pk+r-j,p]_k} =$$

$$= \binom{pn+r+q}{r} \binom{n}{k} \frac{\prod_{j=1}^{q} \left[n+\frac{j}{p}\right]_k \prod_{j=1}^{p-q-1} \left[n-\frac{j}{p}\right]_k}{\prod_{j=1}^{r} \left[k+\frac{j}{p}\right]_k \prod_{j=1}^{p-r-1} \left[k-\frac{j}{p}\right]_k} =$$

$$= \frac{\binom{pn+r+q}{r}}{\prod_{j=1}^{r} \left[n+\frac{j}{p}\right]_n \prod_{j=1}^{p-r-1} \left[n-\frac{j}{p}\right]_n} \cdot$$

$$\binom{n}{k} \cdot \prod_{j=1}^{q} \left[n+\frac{j}{p}\right]_k \prod_{j=1}^{p-q-1} \left[n-\frac{j}{p}\right]_k \prod_{j=1}^{r} \left[n+\frac{j}{p}\right]_{n-k} \prod_{j=1}^{p-r-1} \left[n-\frac{j}{p}\right]_{n-k}$$

Substitution of (15.6) in (15.5) yields (15.2). □

The special choice of $p = 2$ gives for $(r,q) = (0,0)$ the formula (8.29), for $(r,q) = (0,1)$ the formula (8.30) and for $(r,q) = (1,1)$ the formula (8.31).

CHAPTER 16. ZEILBERGER'S ALGORITHM

Zeilberger's algorithm. In 1990 D. Zeilberger, [119], and with H. Wilf, [117], cf. T. H. Kornwinder, [83], and GKP, [70], and later with Marco Petkovšek [96], used Gosper's algorithm to prove formulas of the form, where a is independent of n and k, and the limits are natural in the sense that the terms vanish outside the interval of summation,

$$T(a,n) = \sum_{k=0}^{n} t(a,n,k) \tag{16.1}$$

It is required that the quotients,

$$q_t(a,n,k) = \frac{t(a,n,k+1)}{t(a,n,k)} \tag{16.2}$$

are rational not only as a function of k, but as a function of n as well, and furthermore, that the quotients

$$r_t(a,n,k) = \frac{t(a,n+1,k)}{t(a,n,k)} \tag{16.3}$$

are rational as function of both k and n.

In [117] and [96], cf. also GKP, [70], p. 239 ff., they proved that the method works in all cases where the terms are *proper,* by which notation they understand that the terms consist of products of polynomials in n and k, powers to the degrees k and n and factorials of the form $[a + pk + qn]_m$, where $a \in \mathbb{C}$ and $p, q, m \in \mathbb{Z}$. And of course that the terms vanish outside the interval of summation.

In the simplest cases the method not only provides us with a tool to prove known – or guessed – formulas, but also allows the finding of one.

The aim is to find a difference equation satisfied by the sums, $T(a,n)$, with coefficients that are rational in n, i.e., to find an equation

$$\beta_0(n)T(a,n) + \beta_1(n)T(a,n+1) + \cdots + \beta_\ell(n)T(a,n+\ell) = 0 \tag{16.4}$$

To do that we seek to sum the corresponding terms,

$$\beta_0(n)t(a,n,k) + \beta_1(n)t(a,n+1,k) + \cdots + \beta_\ell(n)t(a,n+\ell,k) \tag{16.5}$$

by Gosper's algorithm on the form $\Delta S(a,n,k)$, because, with the terms the sums $S(a,n,k)$ must also vanish outside a finite interval. Hence we shall have

$$\sum_{k=-\infty}^{\infty} \Delta S(a,n,k) = \sum_{k=-\infty}^{\infty} S(a,n,k+1) - \sum_{k=-\infty}^{\infty} S(a,n,k) = 0 \tag{16.6}$$

We want of course the order of the difference equation as small as possible, so we start with $\ell = 1$ and try to find the coefficients, such that Gosper's algorithm work. If not, we try again with $\ell = 2$, etc. If we succeed with a first order equation, we can always find the sum. Otherwise we may use the difference equation to prove a known formula, unless we are able to solve the equation. But, this is more often than not – not the case, cf. chapter 4.

To make the ideas more transparent, let's consider the case $\ell = 1$. Then we shall handle the quotient of

$$s(a,n,k) = \beta_0(n)t(a,n,k) + \beta_1(n)t(a,n+1,k) \tag{16.7}$$

namely

$$q_s(a,n,k) = \frac{\beta_0(n)t(a,n,k+1) + \beta_1(n)t(a,n+1,k+1)}{\beta_0(n)t(a,n,k) + \beta_1(n)t(a,n+1,k)} \tag{16.8}$$

Now we may apply (16.3) to write this quotient as

$$q_s(a,n,k) = \frac{\beta_0(n) + \beta_1(n)r_t(a,n,k+1)}{\beta_0(n) + \beta_1(n)r_t(a,n,k)} \cdot \frac{t(a,n,k+1)}{t(a,n,k)} \tag{16.9}$$

If the function r_t behaves nicely, we may rewrite this quotient as

$$q_s(a,n,k) = \frac{f(a,n,k+1)}{f(a,n,k)} \cdot \frac{g(a,n,k)}{h(a,n,k+1)} \tag{16.10}$$

with f, g and h polynomials in k with coefficients which are rational functions of n. If this is the case, we proceed with Gosper forming

$$G(a,n,k) = g(a,n,k) - h(a,n,k) \tag{16.11}$$

$$H(a,n,k) = g(a,n,k) + h(a,n,k) \tag{16.12}$$

to estimate the degree, d, of the polynomial, s, in k solving

$$f(a,n,k) = s(a,n,k+1)g(a,n,k) - s(a,n,k)h(a,n,k) \tag{16.13}$$

This polynomial identity must be solved in the $d+1$ unknown coefficients in s and the $\ell + 1 = 2$ unknown coefficients in f as rational functions of n. As soon as the existence of a solution is verified, we are satisfied with the two functions, β, or even less, namely their ratio,

$$\frac{\beta_0(n)}{\beta_1(n)} \tag{16.14}$$

because we then find immediately the wanted sum as

$$T(a,n) = T(a,0) \cdot \prod_{\nu=0}^{n-1} -\frac{\beta_0(\nu)}{\beta_1(\nu)} \tag{16.15}$$

A simple example of Zeilberger's algorithm. Let us consider the sum on standard form

$$T(n) = \sum_k \binom{n}{k} [a]_k [b]_{n-k} x^k \tag{16.16}$$

and look at the terms (16.7) with $\ell = 1$. It may be written as

(16.17)
$$\begin{aligned} s(n,k) &= \beta_0(n)\binom{n}{k}[a]_k[b]_{n-k}x^k + \beta_1(n)\binom{n+1}{k}[a]_k[b]_{n+1-k}x^k \\ &= \left(\frac{n+1-k}{n+1}\beta_0(n) + (b-n+k)\beta_1(n)\right)\cdot\binom{n+1}{k}[a]_k[b]_{n-k}x^k \end{aligned}$$

The quotient hence becomes

(16.18)
$$q_s(n,k) = \frac{(n-k)\beta_0(n) + (b-n+k+1)(n+1)\beta_1(n)}{(n+1-k)\beta_0(n) + (b-n+k)(n+1)\beta_1(n)} \cdot \frac{(n+1-k)(a-k)x}{(-1-k)(n-1-b-k)}$$

and it is obvious to obtain the polynomials as

(16.19)
$$f(n,k) = (n+1-k)\beta_0(n) + (b-n+k)(n+1)\beta_1(n)$$

(16.20)
$$g(n,k) = xk^2 - x(n+1+a)k + xa(n+1)$$

(16.21)
$$h(n,k) = k^2 + (b-n)k$$

(16.22)
$$G(n,k) = (x-1)k^2 - (x(n+1+a) - n + b)k + xa(n+1)$$

(16.23)
$$H(n,k) = (x+1)k^2 - (x(n+1+a) + n - b)k + xa(n+1)$$

Now, if $x \neq 1$, the degrees of G and H are both 2, while the degree of f is 1 leaving us with no solution. Nevertheless, if we allow $\ell = 2$, there is a solution, namely – found by computer algebra using MAPLE – (the program supplied by D. Zeilberger), cf. (8.138):

$$T(n+2) + ((1+x)(n+1) - xa - b)\,T(n+1) + x(n-a-b)(n+1)T(n) = 0 \tag{16.24}$$

But, for $x = 1$, we get the reduction

$$G(n,k) = (-1-a-b)k + a(n+1) \tag{16.25}$$

Hence the degree from (6.24) becomes 0, so we just have one rational function of n, $\alpha(n)$.

The equation (16.13) takes the simple form

$$f(n,k)=\alpha(n)G(n,k) \tag{16.26}$$

or

$$(n+1-k)\beta_0(n)+(b-n+k)(n+1)\beta_1(n)=\alpha(n)((-1-a-b)k+a(n+1)) \tag{16.27}$$

or, comparing coefficients of the two polynomials, we get the couple of equations

$$-\beta_0(n)+(n+1)\beta_1(n)=\alpha(n)(-1-a-b) \tag{16.28}$$

(16.29)

$$(n+1)\beta_0(n)+(b-n)(n+1)\beta_1(n)=\alpha(n)a(n+1))$$

having the wanted fraction (16.14) to be

$$\frac{\beta_0(n)}{\beta_1(n)}=-(a+b-n) \tag{16.30}$$

which yields the solution

$$T(n)=[a+b]_n \tag{16.31}$$

the well–known Chu–Vandermonde formula, cf. (8.3).

A less simple example of Zeilberger's algorithm. Let us consider the sum on standard form

$$T(n)=\sum_k\binom{n}{k}[a]_k[b]_k[c]_{n-k}[n-1-a-b-c]_{n-k}(-1)^k \tag{16.32}$$

and look at the terms (16.7) with $\ell=1$. It may be written as

(16.33)

$$\begin{aligned}s(n,k)=&\beta_0(n)\binom{n}{k}[a]_k[b]_k[c]_{n-k}[n-1-a-b-c]_{n-k}(-1)^k+\\&+\beta_1(n)\binom{n+1}{k}[a]_k[b]_k[c]_{n+1-k}[n-a-b-c]_{n+1-k}(-1)^k\\&=\left(\frac{n+1-k}{n+1}\beta_0(n)+(c-n+k)(n-a-b-c)\beta_1(n)\right)\cdot\\&\cdot\binom{n+1}{k}[a]_k[b]_k[c]_{n-k}[n-1-a-b-c]_{n-k}(-1)^k\end{aligned}$$

The quotient hence becomes

(16.34)

$$\begin{aligned}q_s(n,k)=&\frac{(n-k)\beta_0(n)+(c-n+k+1)(n-a-b-c)(n+1)\beta_1(n)}{(n+1-k)\beta_0(n)+(c-n+k)(n-a-b-c)(n+1)\beta_1(n)}\cdot\\&\cdot\frac{(n+1-k)(a-k)(b-k)}{(-1-k)(n-1-c-k)(a+b+c-k)}\end{aligned}$$

and it is obvious to obtain the polynomials as

(16.35)
$$f(n,k) = (n+1-k)\beta_0(n) + (c-n+k)(n-a-b-c)(n+1)\beta_1(n)$$
(16.36)
$$g(n,k) = k^3 - (n+1+a+b)k^2 + ((a+b)(n+1)+ab)k - (n+1)ab$$
(16.37)
$$h(n,k) = k^3 - (n+1+a+b)k^2 + (n-c)(a+b+c+1)k$$
(16.38)
$$G(n,k) = ((c+1)(a+b+c-n)+ab)k - (n+1)ab$$
(16.39)
$$\begin{aligned} H(n,k) = &2k^3 - 2(n+1+a+b)k^2 + \\ &+ ((n+1)(a+b) + ab + (n-c)(a+b+c+1))k - (n+1)ab \end{aligned}$$

We get the degree from (6.24) to be 0, so we just have one rational function of n, $\alpha(n)$.

The equation (16.13) takes the simple form

$$f(n,k) = \alpha(n)G(n,k) \tag{16.40}$$

or

(16.41)
$$\begin{aligned} &(n+1-k)\beta_0(n) + (c-n+k)(n-a-b-c)(n+1)\beta_1(n) = \\ &= \alpha(n)(((c+1)(a+b+c-n)+ab)k - (n+1)ab) \end{aligned}$$

or, comparing coefficients of the two polynomials, we get the couple of equations

(16.42)
$$-\beta_0(n) + (n-a-b-c)(n+1)\beta_1(n) = \alpha(n)((c+1)(a+b+c-n)+ab)$$
(16.43)
$$(n+1)\beta_0(n) + (c-n)(n-a-b-c)(n+1)\beta_1(n) = -\alpha(n)(n+1)ab$$

having the wanted fraction (16.14) to be

$$\frac{\beta_0(n)}{\beta_1(n)} = (a+c-n)(b+c-n) \tag{16.44}$$

which yields the solution

$$T(n) = [a+c]_n[b+c]_n(-1)^n \tag{16.45}$$

the well–known Pfaff–Saalschütz formula, cf. (9.1).

Sporadic formulas of types $II(2,2,z)$. Besides (16.45) there are a lot of formulas with the limit of summation as the only free parameter. In each case, the transformation group gives up to six equivalent formulas with different factors. We shall prefer to choose the biggest of the six factors in the following presentation of such formulas.

The factor 2. Besides the theorems of Gauß and Bailey, there are a few other formulas known with a factor of 2.

(16.46)
$$\sum_{k=0}^{n}\binom{n}{k}\left[4n+\tfrac{1}{2}\right]_k\left[-n-\tfrac{1}{2}\right]_{n-k}2^k=\frac{\left[-\frac{3}{8}\right]_n\left[-\frac{5}{8}\right]_n}{\left[-\frac{1}{2}\right]_n}(-16)^n$$

(16.47)
$$\sum_{k=0}^{n}\binom{n}{k}\left[4n+\tfrac{5}{2}\right]_k\left[-n-\tfrac{3}{2}\right]_{n-k}2^k=\frac{\left[-\frac{7}{8}\right]_n\left[-\frac{9}{8}\right]_n}{\left[-\frac{3}{2}\right]_n}(-16)^n$$

The factor 4. The following seven are known

(16.48)
$$\sum_{k=0}^{n}\binom{n}{k}\left[3n+\tfrac{1}{2}\right]_k\left[-\tfrac{1}{2}\right]_{n-k}4^k=\left[-\tfrac{1}{2}\right]_n(-27)^n$$

(16.49)
$$\sum_{k=0}^{n}\binom{n}{k}\left[3n+\tfrac{3}{2}\right]_k\left[-\tfrac{1}{2}\right]_{n-k}4^k=\frac{\left[-\frac{5}{6}\right]_n\left[-\frac{7}{6}\right]_n}{\left[-\frac{3}{2}\right]_n}(-27)^n=\frac{[6n+1,2]_{2n}}{[2n]_n}$$

(16.50)
$$\sum_{k=0}^{n}\binom{n}{k}\left[3n+\tfrac{1}{2}\right]_k\left[-3n-1\right]_{n-k}4^k=\frac{\left[-\frac{5}{6}\right]_n\left[-\frac{1}{2}\right]_n}{\left[-\frac{2}{3}\right]_n}(-16)^n$$

(16.51)
$$\sum_{k=0}^{n}\binom{n}{k}\left[3n+\tfrac{3}{2}\right]_k\left[-3n-3\right]_{n-k}4^k=\frac{\left[-\frac{5}{6}\right]_n\left[-\frac{3}{2}\right]_n}{\left[-\frac{5}{3}\right]_n}(-16)^n$$

(16.52)
$$\sum_{k=0}^{n}\binom{n}{k}\left[-\tfrac{1}{2}\right]_k\left[\tfrac{3n}{2}\right]_{n-k}4^k=\left[\tfrac{3n}{2}\right]_n\frac{2(-1)^n+1}{3}$$

(16.53)
$$\sum_{k=0}^{n}\binom{n}{k}\left[-\tfrac{1}{2}\right]_k\left[\tfrac{3n+1}{2}\right]_{n-k}4^k=\begin{cases}0 & \text{if } n \text{ odd}\\ [n]_m^2\left(\frac{27}{16}\right)^m & \text{if } n=2m\end{cases}$$

The formulas (16.48) and (16.53) are due to Ira M. Gessel, [46], formula 5.22 and 5.25.

The formula (16.52) allows for even values of the limit of summation a slight generalization,

$$\sum_{k=0}^{2n} \binom{2n}{k} \left[-\tfrac{1}{2}-p\right]_k [3n+p]_{2n-k}\, 4^k = \frac{[3n+p]_{2n+p}}{[p]_p} \qquad p=0,1,2 \tag{16.54}$$

The formula (16.54) for $p=0$ is due to Ira M. Gessel, [48], formula no. 28.1a, and for $p=1$ is from Gould's table, [64], no. 7.8.

The factor 5. The following four are known

(16.55)
$$\sum_{k=0}^{n} \binom{n}{k} \left[5n+\tfrac{1}{2}\right]_k \left[n-\tfrac{1}{2}\right]_{n-k} 5^k = \frac{\left[-\frac{1}{2}\right]_n \left[-\frac{3}{10}\right]_n \left[-\frac{7}{10}\right]_n}{\left[-\frac{2}{5}\right]_n \left[-\frac{3}{5}\right]_n} (-64)^n$$

(16.56)
$$\sum_{k=0}^{n} \binom{n}{k} \left[5n+\tfrac{3}{2}\right]_k \left[n-\tfrac{1}{2}\right]_{n-k} 5^k = \frac{\left[-\frac{1}{2}\right]_n \left[-\frac{9}{10}\right]_n \left[-\frac{11}{10}\right]_n}{\left[-\frac{4}{5}\right]_n \left[-\frac{6}{5}\right]_n} (-64)^n$$

(16.57)
$$\sum_{k=0}^{n} \binom{n}{k} \left[5n+\tfrac{5}{2}\right]_k \left[n+\tfrac{1}{2}\right]_{n-k} 5^k = \frac{\left[-\frac{3}{2}\right]_n \left[-\frac{7}{10}\right]_n \left[-\frac{13}{10}\right]_n}{\left[-\frac{7}{5}\right]_n \left[-\frac{8}{5}\right]_n} (-64)^n$$

(16.58)
$$\sum_{k=0}^{n} \binom{n}{k} \left[5n+\tfrac{7}{2}\right]_k \left[n+\tfrac{1}{2}\right]_{n-k} 5^k = \frac{\left[-\frac{3}{2}\right]_n \left[-\frac{9}{10}\right]_n \left[-\frac{11}{10}\right]_n}{\left[-\frac{6}{5}\right]_n \left[-\frac{9}{5}\right]_n} (-64)^n$$

The factor 9. The following eighteen are known

(16.59 $n>1$)
$$\sum_{k=0}^{n} \binom{n}{k} \left[2n-\tfrac{8}{3}\right]_k [-2n+3]_{n-k}\, 9^k = \frac{[-2n+3]_n}{\left[\frac{2}{3}\right]_n} \left(\left[\tfrac{5}{6}\right]_n + \left(\tfrac{3}{4}-\tfrac{n}{3}\right) \left[\tfrac{1}{2}\right]_{n-1} \right) (-4)^n$$

(16.60)
$$\sum_{k=0}^{n} \binom{n}{k} \left[2n-\tfrac{4}{3}\right]_k [-2n+1]_{n-k}\, 9^k = \frac{[3n-2]_n}{\left[\frac{1}{3}\right]_n} \left(\left[\tfrac{1}{6}\right]_n + \tfrac{1}{2} \left[\tfrac{1}{2}\right]_n \right) 4^n$$

(16.61)
$$\sum_{k=0}^{n} \binom{n}{k} \left[2n-\tfrac{2}{3}\right]_k [-2n]_{n-k}\, 9^k = \frac{[3n-1]_n}{\left[-\frac{1}{3}\right]_n} \left(\left[-\tfrac{1}{6}\right]_n + \tfrac{1}{2} \left[-\tfrac{1}{2}\right]_n \right) 4^n$$

(16.62)
$$\sum_{k=0}^{n} \binom{n}{k} \left[2n+\tfrac{1}{3}\right]_k [-2n-1]_{n-k}\, 9^k = \left[-\tfrac{2}{3}\right]_n (-27)^n$$

(16.63)
$$\sum_{k=0}^{n} \binom{n}{k} \left[2n+\tfrac{2}{3}\right]_k [-2n-2]_{n-k}\, 9^k = \frac{\left[-\frac{5}{6}\right]_n \left[-\frac{4}{3}\right]_n}{\left[-\frac{3}{2}\right]_n} (-27)^n$$

(16.64)
$$\sum_{k=0}^{n}\binom{n}{k}\left[2n+\tfrac{4}{3}\right]_k\left[-2n-3\right]_{n-k}9^k = 6\cdot\frac{\left[-\frac{5}{3}\right]_n}{\left[2n+2\right]_{n+2}}\left(\left[-\tfrac{1}{6}\right]_{n+1}-\left[-\tfrac{1}{2}\right]_{n+1}\right)108^n$$

(16.65)
$$\sum_{k=0}^{n}\binom{n}{k}\left[3n+\tfrac{1}{2}\right]_k\left[n-\tfrac{1}{2}\right]_{n-k}9^k = \left[-\tfrac{1}{2}\right]_n(-64)^n$$

(16.66)
$$\sum_{k=0}^{n}\binom{n}{k}\left[3n+\tfrac{3}{2}\right]_k\left[n+\tfrac{1}{2}\right]_{n-k}9^k = \frac{\left[-\frac{3}{2}\right]_n\left[-\frac{5}{6}\right]_n\left[-\frac{7}{6}\right]_n}{\left[-\frac{4}{3}\right]_n\left[-\frac{5}{3}\right]_n}(-64)^n$$

(16.67)
$$\sum_{k=0}^{n}\binom{n}{k}\left[3n+\tfrac{1}{4}\right]_k\left[n-\tfrac{1}{4}\right]_{n-k}9^k = \frac{\left[-\frac{3}{4}\right]_n\left[-\frac{5}{12}\right]_n}{\left[-\frac{2}{3}\right]_n}(-64)^n$$

(16.68)
$$\sum_{k=0}^{n}\binom{n}{k}\left[3n+\tfrac{3}{4}\right]_k\left[n+\tfrac{1}{4}\right]_{n-k}9^k = \frac{\left[-\frac{5}{4}\right]_n\left[-\frac{7}{12}\right]_n}{\left[-\frac{4}{3}\right]_n}(-64)^n$$

(16.69)
$$\sum_{k=0}^{n}\binom{n}{k}\left[3n+\tfrac{5}{4}\right]_k\left[n-\tfrac{1}{4}\right]_{n-k}9^k = \frac{\left[-\frac{3}{4}\right]_n\left[-\frac{13}{12}\right]_n}{\left[-\frac{4}{3}\right]_n}(-64)^n$$

(16.70)
$$\sum_{k=0}^{n}\binom{n}{k}\left[3n+\tfrac{7}{4}\right]_k\left[n+\tfrac{1}{4}\right]_{n-k}9^k = \frac{\left[-\frac{5}{4}\right]_n\left[-\frac{11}{12}\right]_n}{\left[-\frac{5}{3}\right]_n}(-64)^n$$

(16.71)
$$\sum_{k=0}^{n}\binom{n}{k}\left[\tfrac{n}{2}-\tfrac{1}{3}\right]_k\left[-\tfrac{n}{2}-1\right]_{n-k}9^k = \begin{cases} 0 & \text{for } n \text{ odd} \\ \left[-\frac{1}{2}\right]_m\left[-\frac{2}{3}\right]_m(-108)^m & \text{for } n=2m \end{cases}$$

(16.72)
$$\sum_{k=0}^{n}\binom{n}{k}\left[\tfrac{n}{2}-\tfrac{1}{3}\right]_k\left[-\tfrac{n}{2}\right]_{n-k}9^k = \begin{cases} \left[\frac{1}{3}\right]_n\cdot(-1)^{m+1}3^{n+m} & \text{for } n=2m+1 \\ \left(\left[\frac{1}{3}\right]_n+2\left[-\frac{1}{2}\right]_m\left[-\frac{2}{3}\right]_m4^m\right)\cdot(-1)^m3^{n+m-1} & \text{for } n=2m \end{cases}$$

(16.73)
$$\sum_{k=0}^{n}\binom{n}{k}\left[\tfrac{n}{2}-\tfrac{1}{6}\right]_k\left[-\tfrac{n}{2}-\tfrac{1}{2}\right]_{n-k}9^k = \left[-\tfrac{2}{3}\right]_n(-1)^{\lfloor\frac{n}{2}\rfloor}3^{n+\lfloor\frac{n}{2}\rfloor}$$

(16.74)
$$\sum_{k=0}^{n}\binom{n}{k}\left[\tfrac{n}{2}+\tfrac{1}{6}\right]_k\left[-\tfrac{n}{2}-\tfrac{3}{2}\right]_{n-k}9^k = \begin{cases} \frac{1}{n+1}\left[-\frac{1}{3}\right]_{n+1}(-1)^{m+1}3^{n+m+1} & \text{f. } n=2m \\ \frac{1}{n+1}\left(\left[-\frac{1}{3}\right]_{n+1}(-1)^{m+1}3^{n+m+1}-\frac{1}{3}\left[-\frac{1}{2}\right]_{m+1}\left[-\frac{2}{3}\right]_{m+1}(-108)^{m+1}\right) & \text{f. } n=2m+1 \end{cases}$$

The formulas (16.59), (16.60), (16.61), (16.62), (16.63), (16.71), (16.72), (16.73) and (16.74) only for n even are found i Gessel, [46], formulas 3.10, 3.9, 3.8, 5.23, 3.7, 5.24, 3.13, 3.12 and 3.14 respectively. The formulas (16.67) and (16.69) are mentioned as conjectures of Gosper in [46], formulas 6.5 and 6.6, respectively.

Two similar formulas are known only for n even, namely

(16.75)
$$\sum_{k=0}^{n} \binom{n}{k} \left[\tfrac{n}{2}+\tfrac{1}{3}\right]_k \left[-\tfrac{n}{2}-2\right]_{n-k} 9^k = \frac{\left[-\frac{5}{3}\right]_n (-1)^{\frac{n}{2}} 3^{n+\frac{n}{2}+1}}{n+2}$$

(16.76)
$$\sum_{k=0}^{n} \binom{n}{k} \left[\tfrac{n}{2}+\tfrac{2}{3}\right]_k \left[-\tfrac{n}{2}-3\right]_{n-k} 9^k = \tfrac{5}{(n+2)(n+4)} \cdot \left[-\tfrac{7}{3}\right]_n (-1)^{\frac{n}{2}} 3^{n+\frac{n}{2}+1}$$

The formulas (16.75) and (16.76) are found in [46] as formulas 3.15 and 3.16.

Sporadic formulas of types $II(4,4,z)$. A few formulas are known with $z \neq \pm 1$, all of them proven by the Zeilberger method by I. M. Gessel in [48]. They have respectively z equal to 4, $\frac{32}{27}$, $\frac{2}{27}$, $-\frac{9}{16}$ and -8.

The formulas with $z = -8$ are in [48] as no. 11.1a, 11.2a, 21.1a, 21.3a and 30.3a.

(16.77)
$$\sum_{k=0}^{n} \binom{n}{k} [n+a]_k \left[a-\tfrac{1}{2}\right]_k [-n-1]_{n-k}[n-2a]_{n-k}(2a+2n-3k)8^k =$$
$$= [2a]_{n+1}[-a-1]_n(-4)^n$$

(16.78)
$$\sum_{k=0}^{n} \binom{n}{k} [a-1]_k \left[n-a+\tfrac{1}{2}\right]_k [n+1-2a]_{n-k}[2a-n-2]_{n-k}(2n-3k)8^k = 0$$

(16.79)
$$\sum_{k=0}^{n} \binom{n}{k} [n-b]_k \left[-b-\tfrac{1}{2}\right]_k [-n-1]_{n-k}[2b+n]_{n-k}\cdot$$
$$(2(n-b)-3k)8^k = [b-1]_n[-2b]_{n+1}$$

(16.80)
$$\sum_{k=0}^{n} \binom{n}{k} [n-b]_k \left[-b-\tfrac{1}{2}\right]_k [-n-1]_{n-k}[2b+n]_{n-k}\cdot$$
$$(2(n-b)-3k)8^k = [b-1]_n[-2b]_{n+1}$$

(16.81)
$$\sum_{k=0}^{n} \binom{n}{k} [-a-1/2]_k [n+a]_k [n+2a]_{n-k} [-1-2a-n]_{n-k} (2n-3k)8^k = 0$$

The formula with $z=-\frac{9}{16}$ is in [48] as no. 28.7a.

(16.82)
$$\sum_{k=0}^{n}\binom{n}{k}\left[-\tfrac{1}{2}\right]_k\left[2n+\tfrac{1}{2}\right]_k\left[-\tfrac{n}{2}-1\right]_{n-k}\cdot$$
$$\left[-\tfrac{n}{2}-\tfrac{1}{2}\right]_{n-k}(3(4n+1)-10k)\left(\frac{9}{16}\right)^k=3[4n+1]_{2n+1}\left(\frac{27}{256}\right)^n$$

The formulas with $z=\frac{2}{27}$ are in [48] as no. 30.4a, 30.5a, 30.6a and 30.7a.

(16.83)
$$\sum_{k=0}^{n}\binom{n}{k}\left[-\tfrac{2}{3}\right]_k[-2n-2]_k\left[2n+\tfrac{5}{6}\right]_{n-k}\left[\tfrac{1}{3}+n\right]_{n-k}(2n+5+10k)\left(-\frac{2}{27}\right)^k=$$
$$=5\frac{[n+1/3]_n\,[2n+5/6]_{2n}}{[2n+1]_{n+1}}4^n$$

(16.84)
$$\sum_{k=0}^{n}\binom{n}{k}\left[-\tfrac{1}{6}\right]_k[-2n]_k\left[n-\tfrac{2}{3}\right]_{n-k}\left[-\tfrac{2}{3}+2n\right]_{n-k}(n+5k)\left(-\frac{2}{27}\right)^k=$$
$$=\frac{n\,[2n-2/3]_{2n}\,[n-2/3]_n}{[2n]_n}4^n$$

(16.85)
$$\sum_{k=0}^{n}\binom{n}{k}\left[-\tfrac{1}{3}\right]_k[-2n-1]_k\left[n-\tfrac{1}{3}\right]_{n-k}\left[\tfrac{1}{6}+2n\right]_{n-k}(2n+1+10k)\left(-\frac{2}{27}\right)^k=$$
$$=\frac{[n-1/3]_n\,[2n+1/6]_{2n}}{[2n]_n}4^n$$

(16.86)
$$\sum_{k=0}^{n}\binom{n}{k}\left[-\tfrac{5}{6}\right]_k[-2-2n]_k\left[n+\tfrac{2}{3}\right]_{n-k}\left[2n+\tfrac{2}{3}\right]_{n-k}(n+4+5k)\left(-\frac{2}{27}\right)^k=$$
$$=\frac{\left[n+\frac{2}{3}\right]_n\left[2n+\frac{2}{3}\right]_{2n}}{[2n+1]_{n+1}}4^{n+1}$$

The formulas with $z=\frac{32}{27}$ are in [48] as no. 30.10a.

(16.87)
$$\sum_{k=0}^{n}\binom{n}{k}\left[-\tfrac{1}{3}\right]_k\left[n-\tfrac{1}{2}\right]_k\left[n-\tfrac{1}{3}\right]_{n-k}[-3n]_{n-k}(4n+5k)\left(\frac{32}{27}\right)^k=$$
$$=3\frac{\left[n-\frac{1}{3}\right]_n\left[2n-\frac{5}{6}\right]_{2n}}{[3n-1]_{n-1}}(-16)^n$$

(16.88)
$$\sum_{k=0}^{n}\binom{n}{k}\left[-\tfrac{1}{3}\right]_k\left[n+\tfrac{1}{2}\right]_k\left[n-\tfrac{1}{3}\right]_{n-k}[-3n-2]_{n-k}(4n+2+5k)\left(\frac{32}{27}\right)^k=$$
$$=12\frac{\left[n-\tfrac{1}{3}\right]_n\left[2n+\tfrac{1}{6}\right]_{2n+1}}{[3n+1]_n}(-16)^n$$

The formulas with $z=4$ are in [48] as no. 22.1a, 27.3a, 29.1a, 29.2a, 29.4a and 29.5a.

$$\sum_{k=0}^{n}\binom{n}{k}\left[b+\tfrac{n}{2}\right]_k\left[b+\tfrac{n-1}{2}\right]_k[n-b]_{n-k}[-2b]_{n-k}(2b+n-3k)(-4)^k=$$
(16.89)
$$=-[b-1]_n[-2b]_{n+1}$$

(16.90)
$$\sum_{k=0}^{n}\binom{n}{k}\left[n-\tfrac{1}{2}\right]_k[3n+a]_k[a]_{n-k}[-a]_{n-k}(3n+a-3k)(-4)^k=(-1)^n a[3n+a]_{2n}$$

(16.91)
$$\sum_{k=0}^{n}\binom{n}{k}\left[b-\tfrac{1}{2}\right]_k[n+b]_k[n-b]_{n-k}[n-2b]_{n-k}\cdot$$
$$(2(n+b)-3k)(-4)^k=-2[2b-1]_n[-b]_{n+1}$$

(16.92)
$$\sum_{k=0}^{n}\binom{n}{k}\left[n-\tfrac{1}{2}\right]_k[2b]_k[b]_{n-k}[-n]_{n-k}(2n-3k)(-4)^k=-(-1)^n[2n]_{n+1}[b]_n$$

(16.93)
$$\sum_{k=0}^{n}\binom{n}{k}\left[\tfrac{n-3}{2}-a\right]_k\left[\tfrac{n-2}{2}-a\right]_k[a+n+1]_{n-k}[2(a+1)]_{n-k}\cdot$$
$$(2(a+1)-n+3k)(-4)^k=-2[2a+1]_n[-a-1]_{n+1}$$

(16.94)
$$\sum_{k=0}^{n}\binom{n}{k}\left[n-\tfrac{1}{2}\right]_k[a-1]_k[-n]_{n-k}\left[-\tfrac{1}{2}+\tfrac{a}{2}-n\right]_{n-k}(4n-3k)(-4)^k=$$
$$=[2n]_{n+1}\left[\tfrac{a}{2}-1\right]_n(-4)^n$$

(16.95)
$$\sum_{k=0}^{n}\binom{n}{k}\left[n+\tfrac{1}{2}\right]_k[a-1]_k[-n-1]_{n-k}\left[\tfrac{a}{2}-\tfrac{3}{2}-n\right]_{n-k}(4n+2-3k)(-4)^k=$$
$$=2[2n+1]_{n+1}\left[\tfrac{a}{2}-1\right]_n(-4)^n$$

CHAPTER 17. SUMS OF TYPE III–IV

The Abel, Hagen–Rothe, Cauchy and Jensen formulas. The classical formulas of type III are the Hagen–Rothe formula and the Jensen formula.

The Hagen–Rothe formula generalizes the Chu–Vandermonde identity and was established 1793 in H. A. Rothe (1773–1841), [100], and mentioned in J. G. Hagen (1847–1930) [73] in 1891. It may be found in H. W. Gould, [64], as formula 3.142 or 3.146, or in J. Kaucký, [81], formula 6.4, or J. Riordan, [99], p. 169, or of course in our papers, [7], p. 19, and [14], formula 6.

(17.1)

$$\sum_{k=0}^{n} \frac{x}{x+kz}\binom{x+kz}{k}\frac{y}{y+(n-k)z}\binom{y+(n-k)z}{n-k} = \frac{x+y}{x+y+nz}\binom{x+y+nz}{n}$$

The Jensen formula generalizes the binomial formula and was established 1902 in J. L. W. V. Jensen (1859–1925) [80], formula 10, see H. W. Gould [64], formula 1.125, or J. Kaucký, [81], formula 6.6.5, or of course ours, [7], p. 19, and [14], formula 4.

$$\sum_{k=0}^{n}\binom{n}{k}(x+kz)^{k-1}(y-kz)^{n-k-1} = \frac{x+y-nz}{x(y-nz)}(x+y)^{n-1} \tag{17.2}$$

Both formulas are valid for any complex number, z.

To treat these two formulas as one, we introduce in analogy with J. Riordan's treatment of the identities due to N. H. Abel (1802–1829), see [1], [99], p. 18–23:

(17.3)

$$S = S(x,y,z,d;p,q,n) = \sum_{k=0}^{n}\binom{n}{k}[x+kz-pd,d]_{k-p}[y+(n-k)z-qd,d]_{n-k-q}$$

where $x,y,z,d \in \mathbb{C},\ p,q \in \mathbb{Z}$ and $n \in \mathbb{N}_0$.

Remark that by changing the direction of summation we get the corresponding sum with x exchanged with y and p with q:

$$S(y,x,z,d;q,p,n) = S(x,y,z,d;p,q,n) \tag{17.4}$$

For $p=q=1$ the sum becomes for $d=1$ after division by $[n]_n$ equal to the left side of (17.1), and for $d=0$ after the substitution of $y-nz$ for y equal to the left side of (17.2).

For $d=0$ the addition of the terms kz was studied by A.–L. Cauchy (1789–1857) in 1826 see [27] in the case of $p=q=0$, and the same year by N. H. Abel, see [1], for $p=1, q=0$, see also [64] formula 1.124, J. Riordan [99], p. 18, [7], p. 20, while J. L. W. V. Jensen was the first one to study the case of $p=q=1$ in 1902 [80].

For $d=1$ the story took the opposite direction. The case of $p=q=1$ was studied already 1793 by H. A. Rothe in [100] and again by J. G. Hagen in 1891 in [73], while the cases $p = q = 0$ and $p = 1, q = 0$ was postponed until J. L. W. V. Jensen in 1902, [80], formulas 17 and 18. See also H. W. Gould, [64] formula 3.144 and J. Kaucký, [81], formula 6.4.2, and [7], p. 19.

We shall follow our explanation in the paper, [14] from 1994. We shall allow d to be arbitrary, and then prove the formulas for $(p, q) = (0, 0)$, then $(p, q) = (1, 0)$ (and hereby from (17.4) also for $(p, q) = (0, 1)$) and eventually for $(p, q) = (1, 1)$. For technical reasons we shall replace y with $y + nz$ in (17.3).

Theorem 17.1. *The generalized Cauchy–Jensen identity for $x, y, z, d \in \mathbb{C}$ and $n \in \mathbb{N}_0$*

$$\sum_{k=0}^{n} \binom{n}{k} [x + kz, d]_k [y - kz, d]_{n-k} = \sum_{j=0}^{n} [n]_j z^j [x + y - jd, d]_{n-j} \tag{17.5}$$

Proof. Let

$$S = S(x, y - nz, z, d; 0, 0, n) = \sum_{k=0}^{n} \binom{n}{k} [x + kz, d]_k [y - kz, d]_{n-k} \tag{17.6}$$

We apply (8.4) to rewrite the first factor as

$$[x + kz, d]_k = \sum_{i=0}^{k} \binom{k}{i} [x, d]_i [kz, d]_{k-i}$$

Hereby we may write S as a double sum

$$S = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} \binom{k}{i} [x, d]_i [kz, d]_{k-i} [y - kz, d]_{n-k}$$

Now we apply (2.1) to the last factor, and then (8.4):

$$(-1)^{n-k} [y - kz, d]_{n-k} = [-y + kz + (n - k - 1)d, d]_{n-k} =$$
$$= \sum_{j=i}^{n-k+i} \binom{n-k}{j-i} [-y + (n - i - 1)d, d]_{j-i} [kz + (i - k)d, d]_{n-k-j+i}$$

We the get S written as a triple sum:

$$S = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} \binom{k}{i} \sum_{j=i}^{n-k+i} \binom{n-k}{j-i} [x, d]_i [kz, d]_{k-i} (-1)^{n-k} \cdot$$
$$\cdot [-y + (n - i - 1)d, d]_{j-i} [kz + (i - k)d, d]_{n-k-j+i}$$

Now we apply (2.10) and (2.13) several times on the product of the three binomial coefficients and get by that the rewriting

$$\binom{n}{k}\binom{k}{i}\binom{n-k}{j-i}=\binom{n}{i}\binom{n-i}{k-i}\binom{n-k}{j-i}=\binom{n}{i}\binom{n-i}{n-k}\binom{n-k}{j-i}=$$
$$=\binom{n}{i}\binom{n-i}{j-i}\binom{n-j}{n-k-j+i}=\binom{n}{j}\binom{j}{i}\binom{n-j}{k-i}$$

After this, the sum (17.6) may be rearranged to

$$S=\sum_{j=0}^{n}\binom{n}{j}\sum_{i=0}^{j}\binom{j}{i}[x,d]_i[-y+(n-i-1)d,d]_{j-i}(-1)^{n-i}T_{ji}(z,d;n) \tag{17.7}$$

where

$$T_{ji}(z,d;n)=\sum_{k=i}^{n-j+i}\binom{n-j}{k-i}[kz,d]_{k-i}[kz+(i-k)d,d]_{n-k-j+i}(-1)^{k-i} \tag{17.8}$$

Now we apply (2.2) on the sum (17.8) and obtain

$$T_{ji}(z,d;n)=\sum_{k=i}^{n-j+i}\binom{n-j}{k-i}[kz,d]_{n-j}(-1)^{k-i} \tag{17.9}$$

By the translation of the variable of summation with the amount of i we get

$$T_{ji}(z,d;n)=\sum_{k=0}^{n-j}\binom{n-j}{k}[(k+i)z,d]_{n-j}(-1)^{k} \tag{17.10}$$

Now the factorial in (17.10) is a polynomial in k of the form:

$$[(k+i)z,d]_{n-j}=\prod_{\ell=0}^{n-j-1}(kz+iz-\ell d)=z^{n-j}k^{n-j}+\cdots .$$

Hence when we apply (8.17) on (17.10) we only get one term, which, as $\mathfrak{S}_n^{(n)}=1$, becomes

$$T_{ji}(z,d;n)=z^{n-j}[n-j]_{n-j}(-1)^{n-j} \tag{17.11}$$

When we imbed (17.11) in (17.7), then we get the sum

$$S=\sum_{j=0}^{n}\binom{n}{j}\sum_{i=0}^{j}\binom{j}{i}[x,d]_i[-y+(n-i-1)d,d]_{j-i}(-1)^{j-i}z^{n-j}[n-j]_{n-j}$$

Now we may place the terms which are independent of i outside the inner sum, apply (2.13) and (2.9) outside the inner sum, and (2.1) inside it, to get

$$S = \sum_{j=0}^{n} [n]_{n-j} z^{n-j} \sum_{i=0}^{j} \binom{j}{i} [x,d]_i [y-(n-j)d,d]_{j-i}$$

Then we may apply (8.4) to the inner sum to get rid of it

$$S = \sum_{j=0}^{n} [n]_{n-j} z^{n-j} [x+y-(n-j)d,d]_j$$

When we eventually change the direction of summation, then we get

$$S = \sum_{j=0}^{n} [n]_j z^j [x+y-jd,d]_{n-j}$$

which gives (17.5). □

If we replace y with $y+nz$ in (17.5), then we get the expression

$$\sum_{k=0}^{n} \binom{n}{k} [x+kz,d]_k [y+(n-k)z,d]_{n-k} = \sum_{j=0}^{n} [n]_j z^j [x+y+nz-jd,d]_{n-j} \tag{17.12}$$

Now, let again p and q be arbitrary integers. We want to prove a recursion, which allows us to find a generalization of the formulas due to N. H. Abel and J. L. W. V. Jensen for $p=1$ and $q=0$ by the help of (17.5) and (17.12).

Recursion formula in p and n. *For all $p,q \in \mathbb{Z}, n \in \mathbb{N}$ we have*

(17.13)

$$\begin{aligned} &S(x,y,z,d;p,q,n) = \\ &\quad = (x-pd)S(x,y,z,d;p+1,q,n) + nzS(x-d+z,y,z,d;p,q,n-1) \end{aligned}$$

Proof. We apply (2.2) with $m=1$ to the first factor in (17.3) to write it as

$$[x+kz-pd,d]_{k-p} = (x+kz-pd)[x+kz-(p+1)d,d]_{k-(p+1)}$$

We split the factor $(x+kz-pd)$ in the two parts, $(x-pd)$ and kz, by which S is split into two sums, and apply (2.9) on the second sum:

$$\begin{aligned} &S(x,y,z,d;p,q,n) = \\ &\quad = \sum_{k=0}^{n} \binom{n}{k} (x-pd)[x+kz-(p+1)d,d]_{k-p-1} [y+(n-k)z-qd,d]_{n-k-q} \\ &\qquad + \sum_{k=1}^{n} \binom{n}{k} kz[x+kz-(p+1)d,d]_{k-p-1} [y+(n-k)z-qd,d]_{n-k-q} = \\ &\quad = (x-pd)S(x,y,z,d;p+1,q,n) \\ &\qquad + nz \sum_{k=1}^{n} \binom{n-1}{k-1} [x-d+kz-pd,d]_{k-1-p} [y+(n-k)z-qd,d]_{n-k-q} = \\ &\quad = (x-pd)S(x,y,z,d;p+1,q,n) + nzS(x-d+z,y,z,d;p,q,n-1) \qquad \square \end{aligned}$$

With this we have a two–dimensional recursion formula in the variables, p and n, relating the three pairs, (p,n), $(p+1,n)$ and $(p,n-1)$. As we know the sum for $p=q=0$ and all n in (17.12), we may use the recursion formula from here.

Choosing $p=q=0$ in (17.13) we get applying (17.12):

$$\begin{aligned}
&S(x,y,z,d;1,0,n)=\\
&=\frac{1}{x}\left(S(x,y,z,d;0,0,n)-nzS(x-d+z,y,z,d;0,0,n-1)\right)=\\
&=\frac{1}{x}\left(\sum_{j=0}^{n}[n]_j z^j[x+y+nz-jd,d]_{n-j}-\right.\\
&\qquad\left.-nz\sum_{j=0}^{n-1}[n-1]_j z^j[x-d+z+y+(n-1)z-jd,d]_{n-1-j}\right)=\\
&=\frac{1}{x}\left(\sum_{j=0}^{n}[n]_j z^j[x+y+nz-jd,d]_{n-j}-\right.\\
&\qquad\left.-\sum_{j=0}^{n-1}[n]_{j+1} z^{j+1}[x+y+nz-(j+1)d,d]_{n-(j+1)}\right)=
\end{aligned}$$

$$\begin{aligned}
&=\frac{1}{x}\left(\sum_{j=0}^{n}[n]_j z^j[x+y+nz-jd,d]_{n-j}-\sum_{j=1}^{n}[n]_j z^j[x+y+nz-jd,d]_{n-j}\right)=\\
&=\frac{1}{x}[x+y+nz,d]_n
\end{aligned}$$

We have proved the formulas:

Theorem 17.2 (The generalized Abel–Jensen Identity I).

For $n\in\mathbb{N}_0$ and $x,y,z,d\in\mathbb{C}$ we have the formula:

$$\sum_{k=0}^{n}\binom{n}{k}[x+kz-d,d]_{k-1}[y+(n-k)z,d]_{n-k}=\frac{1}{x}[x+y+nz,d]_n \tag{17.14}$$

and the formula we get by exchanging y with $y-nz$, i.e.:

Corollary 17.3 (The generalized Abel–Jensen Identity II).

For $n\in\mathbb{N}_0$ and $x,y,z,d\in\mathbb{C}$ we have the formula:

$$\sum_{k=0}^{n}\binom{n}{k}[x+kz-d,d]_{k-1}[y-kz,d]_{n-k}=\frac{1}{x}[x+y,d]_n \tag{17.15}$$

For $d = 0$ the formula (17.15) becomes the formula of N. H. Abel from 1826, [1], and for $d = 1$ after division by $[n]_n$, it becomes the formula of J. L. W. V. Jensen no. 17 from 1902, [80].

The surprise is that for $p = 1$ and $q = 0$ the sum is reduced to a single term, while the sum for $p = q = 0$ only is rewritten as another sum. Even more surprising is the fact, that a single term also is sufficient for $p = q = 1$.

For this purpose we shall apply the symmetry (17.4) to evaluate the sum for $p = 0$ and $q = 1$. Then we shall apply the following recursion in p, q and n:

Recursion formula in p, q and n.

For all $p, q \in \mathbb{Z}, n \in \mathbb{N}$ we have

(17.16)

$$S(x, y, z, d; p, q, n) = S(x+z-d, y, z, d; p-1, q, n-1)+S(x, y+z-d, z, d; p, q-1, n-1)$$

Proof. We split the sum (17.3) by using (2.8) into

$$\begin{aligned}
S &= \sum_{k=0}^{n} \binom{n}{k} [x + kz - pd, d]_{k-p}[y + (n-k)z - qd, d]_{n-k-q} = \\
&= \sum_k \binom{n-1}{k-1} [x + kz - pd, d]_{k-p}[y + (n-k)z - qd, d]_{n-k-q}+ \\
&\qquad + \sum_k \binom{n-1}{k} [x + kz - pd, d]_{k-p}[y + (n-k)z - qd, d]_{n-k-q} = \\
&= \sum_k \binom{n-1}{k} [x + z - d + kz - (p-1)d, d]_{k-(p-1)} \cdot \\
&\qquad \cdot [y + (n-1-k)z - qd, d]_{n-1-k-q}+ \\
&\qquad + \sum_k \binom{n-1}{k} [x + kz - pd, d]_{k-p} \cdot \\
&\qquad \cdot [y + z - d + (n-1-k)z - (q-1)d, d]_{n-1-k-(q-1)} = \\
&= S(x + z - d, y, z, d; p-1, q, n-1) + S(x, y + z - d, z, d; p, q-1, n-1)
\end{aligned}$$

□

When we put $p = q = 1$ in (17.16) and apply (17.14) and its symmetric companion onto the first and second of the two sums, respectively, then we get

$$\begin{aligned}
&S(x, y, z, d; 1, 1, n) = \\
&\quad = S(x + z - d, y, z, d; 0, 1, n-1) + S(x, y + z - d, z, d; 1, 0, n-1) = \\
&\quad = \frac{1}{y}[x + z - d + y + (n-1)z, d]_{n-1} + \frac{1}{x}[y + z - d + x + (n-1)z, d]_{n-1} = \\
&\quad = \frac{x+y}{xy}[x + y + nz - d, d]_{n-1}
\end{aligned}$$

Hence we have proven the formulas

Theorem 17.3 (The generalized Hagen–Rothe–Jensen Identity I).

For $n \in \mathbb{N}_0$ and $x, y, z, d \in \mathbb{C}$ we have the formula

(17.17)
$$\sum_{k=0}^{n} \binom{n}{k} [x+kz-d, d]_{k-1} [y+(n-k)z-d, d]_{n-k-1} = \frac{x+y}{xy} [x+y+nz-d, d]_{n-1}$$

As above we may write y for $y + nz$ and get the equivalent formula

Corollary (The generalized Hagen–Rothe–Jensen Identity II).

For $n \in \mathbb{N}_0$ and $x, y, z, d \in \mathbb{C}$ we have the formula

$$\sum_{k=0}^{n} \binom{n}{k} [x+kz-d, d]_{k-1} [y-kz-d, d]_{n-k-1} = \frac{x+y-nz}{x(y-nz)} [x+y-d, d]_{n-1} \tag{17.18}$$

For $d = 1$ the formula (17.17) is identical to (17.1), when this is multiplied with $[n]_n$, while $d = 0$ makes (17.18) identical to (17.2).

A Polynomial Identity. In 1995 we, cf. [15], posed the problem:

For $x \in \mathbb{C}$ and $n \in \mathbb{N}$, prove the following identities between the polynomials.

$$(-4)^n \sum_{j=0}^{n} \binom{x+\frac{1}{2}}{j} \binom{n-1-x}{2n-j} = \binom{2n}{n} \sum_{j=0}^{n} \binom{x+j}{2j} \binom{x-j}{2n-2j}. \tag{17.19}$$

For all $m \in \mathbb{N}$ with $0 \le m \le 2n$ generalize (17.19) to

(17.20)
$$(-4)^n \sum_{j=0}^{n} \binom{x+\frac{1}{2}}{j} \binom{n-1-x}{2n-j} = \binom{2n}{n} \sum_{j=-\left[\frac{m}{2}\right]}^{n-\left[\frac{m}{2}\right]} \binom{x+j}{2j+m} \binom{x-j}{2n-m-2j}.$$

Proof. We give the proof of (17.19), because the proof of (17.20) is similar, but with some technical complications.

The two sides of (17.19) represent polynomials in x of degree $2n$, so it is sufficient to establish the identity for $2n+1$ different values of x.

The points $x = 0, 1, \ldots, n-1$ shall be proven zeros of the two polynomials.

The left hand side is zero for $x = 0, 1, \ldots, n-1$, because $0 \le n-1-x < n \le 2n-j$.

The right hand side is zero for $x = 0, 1, \ldots, n-1$, because, we have $x+j \ge 0$, so to get a nonzero term, we must have $x+j \ge 2j$, i.e. $x-j \ge 0$. Hence the other factor can only be nonzero, if $x-j \geq 2n-2j$, i.e., $x+j \geq 2n$, and hence $2x \geq 2n$ contrary to the assumption.

Now consider $x = y - \frac{1}{2}$, $y = 0, 1, \ldots, n$. The polynomials shall be proven to take the same constant value for all of these points.

The left hand side becomes

$$(-4)^n \sum_{j=0}^{n} \binom{y}{j} \binom{n-\frac{1}{2}-y}{2n-j} = (-4)^n \sum_{j=0}^{n} \binom{y}{j} \frac{[n-\frac{1}{2}-y]_{2n-j}}{[2n-j]_{2n-j}} \tag{17.21}$$

As long as $y \leq n$, the limit of summation becomes y. So we get

$$\frac{(-4)^n [n-\frac{1}{2}-y]_{2n-y}}{[2n]_{2n}} \sum_{j=0}^{y} \binom{y}{j} [-n-\tfrac{1}{2}]_{y-j} [2n]_j \tag{17.22}$$

But this is a Chu–Vandermonde, (8.4) or (20.4), so we get

(17.23)

$$\frac{(-4)^n \left[n-\frac{1}{2}-y\right]_{2n-y}}{[2n]_{2n}} \left[n-\tfrac{1}{2}\right]_y = \frac{(-4)^n \left[n-\frac{1}{2}\right]_{2n}}{[2n]_{2n}} = \frac{(-4)^n \left[n-\frac{1}{2}\right]_n \left[-\frac{1}{2}\right]_n}{[2n]_{2n}} =$$

$$= \frac{[2n-1,2]_n^2 [2n,2]_n^2}{[2n]_{2n} [n]_n^2} \left(\tfrac{1}{2}\right)^{2n} = \frac{[2n]_{2n}}{[n]_n^2} \left(\tfrac{1}{2}\right)^{2n} = \binom{2n}{n} \left(\tfrac{1}{2}\right)^{2n}$$

The right hand side of (17.19) may be written for $x = y - \frac{1}{2}$ as

$$\frac{1}{[n]_n^2} \sum_{k \equiv 0(2)} \binom{2n}{k} \left[y-\tfrac{1}{2}+\tfrac{k}{2}\right]_k \left[y-\tfrac{1}{2}-\tfrac{k}{2}\right]_{2n-k} \tag{17.24}$$

If we in the form of the sum, (17.24), consider an odd k, $0 < k < 2n$, then the terms are zero for $y = 0, 1, \ldots, n$. If $k \leq y - \frac{1}{2} + \frac{k}{2}$ and $2n - k \leq y - \frac{1}{2} - \frac{k}{2}$, then we get $2n \leq 2y - 1$ i.e., $n < y$. And if we have $y - \frac{1}{2} + \frac{k}{2} < 0$ and $y - \frac{1}{2} + \frac{k}{2} \geq 2n - k$, then we must have $2n < 0$, and if $y - \frac{1}{2} - \frac{k}{2} < 0$ and then $y - \frac{1}{2} + \frac{k}{2} \geq k$, then we must have $y - \frac{1}{2} - \frac{k}{2} \geq 0$ too. Eventually, if both $y - \frac{1}{2} - \frac{k}{2} < 0$ and $y - \frac{1}{2} + \frac{k}{2} < 0$, then we get $2y - 1 < 0$, so that we must have $y = 0$. But then $-\frac{1}{2} + \frac{k}{2} < 0$, so that $k < 1$,which is impossible for an odd integer, $0 < k < 2n$. So, one of the factors must vanish.

This means that we might as well take the sum for all indices, $k = 0, \ldots 2n$. The sum in (17.24) equals

$$\frac{1}{[n]_n^2} \sum_{k=0}^{2n} \binom{2n}{k} \left[y-\tfrac{1}{2}+\tfrac{k}{2}\right]_k \left[y-\tfrac{1}{2}-\tfrac{k}{2}\right]_{2n-k} \tag{17.25}$$

We may now apply J. L. W. V. Jensen's formula (17.5).

With $n = 2n$, $x = y = y - \frac{1}{2}$, $d = 1$ and $z = \frac{1}{2}$ substituted in (17.25), we get it to be written as

$$\frac{1}{[n]_n^2} \sum_{j=0}^{2n} [2n]_j \left(\tfrac{1}{2}\right)^j [2y - 1 - j]_{2n-j} \tag{17.26}$$

Now we shall rewrite

$$[2y - 1 - j]_{2n-j} = (-1)^j [2n - 2y]_{2n-j} = (-1)^j [2n - j]_{2n-j} \binom{2n - 2y}{2n - j} \tag{17.27}$$

Now, two factors join, $[2n]_j [2n - j]_{2n-j} = [2n]_{2n}$, so that the sum takes the form after substitution of (17.27) in (17.26)

(17.28)

$$\frac{[2n]_{2n}}{[n]_n^2} \sum_{j=0}^{2n} \binom{2n - 2y}{2n - j} \left(-\tfrac{1}{2}\right)^j = \binom{2n}{n} \left(\tfrac{1}{2}\right)^{2y} \sum_{j=0}^{2n} \binom{2n - 2y}{j - 2y} \left(-\tfrac{1}{2}\right)^{j-2y} = \binom{2n}{n} \left(\tfrac{1}{2}\right)^{2n}$$

Where we have applied the binomial theorem, (7.1).

As this is the same result as in (17.23), the identity, (17.19), is established.

□

The generalization, (17.20), is proved the same way without serious trouble.

Joseph Sinyor and Ted Speevak's problem.

The problem is to prove a combinatorial identity, conjectured by Joseph Sinyor and Ted Speevak in Discrete Mathematics [104]. The identity is in their formulation, that the double sum,

(17.29)

$$\sum_{\ell' \le \ell} \sum_{j' \le j} \frac{1}{(m - \ell')} \binom{m + \ell' - j'}{2\ell' - j' + 1} \binom{m - \ell' + j' - 1}{j'} \binom{m - \ell'}{2(\ell - \ell') - (j - j')} \binom{m - \ell'}{(j - j')}$$

for $m > \ell \ge 0 \le j \le 2\ell + 1$ is equal to the following expressions:

$$\frac{1}{2(m - \ell)} \binom{2m}{2\ell + 1} \binom{2\ell + 1}{j} = \frac{1}{2\ell + 1} \binom{2m}{2\ell} \binom{2\ell + 1}{j} \tag{17.30}$$

Sinyor and Speevak support the conjecture with proofs of four special cases. In the case of $j = 0$, cf. remark 1, they reduce the formula to the Pfaff–Saalschütz identity, cf. (9.1). In the case of $j = 1$, cf. remark 3, they give a sophisticated combinatorial argument applying partition, and the cases of $j = 2\ell$ and $j = 2\ell + 1$, cf. remark 4, are reduced to the cases $j = 1$ and $j = 0$ respectively.

We have of course computed both sides of the formula for different integral values of j and ℓ and complex values of m, and could not find any counterexample. But, remind the formula

$$\sum_j \sum_k \binom{n}{j - k} \binom{m}{j + k} = 2^{n+m-1} \tag{17.31}$$

where the summation is over all integer values of j and k such that the binomial coefficients do not vanish. This is true for all natural numbers m and n, which are not both zero, but fails for $n = m = 0$. So, even convinced by experiments, a proof is needed.

Changing the problem to a polynomial identity.

We shall prefer to change the notations of the names of the variables, x for m, n for j, k for j' and i for ℓ'. Then (17.29) looks like:
(17.32)

$$\sum_{i\le\ell}\sum_{k\le n}\frac{1}{x-i}\binom{x+i-k}{2i-k+1}\binom{x-i+k-1}{k}\binom{x-i}{2(\ell-i)-(n-k)}\binom{x-i}{n-k}$$

And (17.30) looks like:

$$\frac{1}{2(x-\ell)}\binom{2x}{2\ell+1}\binom{2\ell+1}{n}=\frac{1}{2\ell+1}\binom{2x}{2\ell}\binom{2\ell+1}{n} \tag{17.33}$$

Now we want to multiply the expressions (17.32–17.33) with $n!(2\ell+1-n)!$ which is a "constant", i.e. independent of the summation indices i and k, to get the formula:
(17.34)

$$\sum_i\sum_k\binom{2\ell+1-n}{2i-k+1}\binom{n}{k}[x+i-k]_{2i-k+1}[x-i+k-1]_k[x-i-1]_{2\ell-1-2i+k-n}[x-i]_{n-k}$$

At the same time the multiplication of the result, (17.33), gives the simplification, that only a single factorial is left:

$$[2x]_{2\ell} \tag{17.35}$$

In the forms, (17.34) and (17.35), the number, ℓ only appear with the factor 2. Hence it is tempting to write $m = 2\ell$, and ask if the formula is true for all and not only even values of $m \ge 0$? Doing so, we ask for a proof of the equality of the sum, (17.36),
(17.36)

$$\sum_i\sum_k\binom{m+1-n}{2i-k+1}\binom{n}{k}[x+i-k]_{2i-k+1}[x-i+k-1]_k[x-i-1]_{m-1-2i+k-n}[x-i]_{n-k}$$

with the factorial, (17.37):

$$[2x]_m \tag{17.37}$$

The sum in (17.36) is a polynomial in x of degree m, so we may as well establish the identity of (17.36) with (17.37) for all $x \in \mathbb{C}$, but we must keep the condition, $0 \le n \le m+1$.

We know the zeros of the polynomial (17.37), the values $\frac{j}{2}, j = 0, \ldots, m-1$. Hence we shall prove that these numbers are zeros of the polynomial (17.36). When this is done we only need to establish that the two polynomials coincide in one point further.

Fortunately, the factorials in (17.36) join:

$$[x+i-k]_{2i-k+1}[x-i-1]_{m-1-2i+k-n} = [x+i-k]_{m-n} \tag{17.38}$$

$$[x-i+k-1]_k[x-i]_{n-k} = (x-i)[x-i+k-1]_{n-1} \tag{17.39}$$

where we have applied the formula (2.2).

Hence we may rewrite (17.36) as

$$\sum_i \sum_k \binom{m+1-n}{2i-k+1}\binom{n}{k}(x-i)[x+i-k]_{m-n}[x-i+k-1]_{n-1} \tag{17.40}$$

Rewriting the double sum as two single sums.

Now it is time to change variables in (17.40). We write $i = j+k$ and change the summation indices to j and k and receive

$$\sum_j [x+j]_{m-n}[x-1-j]_{n-1} \sum_k \binom{m+1-n}{2j+k+1}\binom{n}{k}(x-j-k) \tag{17.41}$$

The inner sum may be written as the difference of two sums,

$$\sum_k \binom{m+1-n}{2j+k+1}\binom{n}{k}(x-j) - \sum_k \binom{m+1-n}{2j+k+1}\binom{n}{k}k \tag{17.42}$$

Both sums are Chu–Vandermonde convolutions, cf. (8.1), i.e.

$$\begin{aligned}\sum_k \binom{m+1-n}{2j+k+1}\binom{n}{k}(x-j) &= (x-j)\sum_k \binom{m+1-n}{m-2j-n-k}\binom{n}{k} = \\ &= (x-j)\binom{m+1}{m-2j-n}\end{aligned} \tag{17.43}$$

$$\begin{aligned}\sum_k \binom{m+1-n}{2j+k+1}\binom{n}{k}k &= n\sum_k \binom{m+1-n}{m-2j-n-k}\binom{n-1}{k-1} = \\ &= n\binom{m}{m-2j-n-1}\end{aligned} \tag{17.44}$$

This means that the polynomial in (17.41) may be written as the difference of two,

(17.45)
$$\begin{aligned}&\sum_j \binom{m+1}{m-2j-n}(x-j)[x+j]_{m-n}[x-j-1]_{n-1}-\\ -n&\sum_j \binom{m}{m-2j-n-1}[x+j]_{m-n}[x-j-1]_{n-1}=\\ =&\sum_j \binom{m+1}{2j+n+1}[x+j]_{m-n}[x-j]_n-\\ -&\sum_j \binom{m}{2j+n+1}[x+j]_{m-n}\, n[x-j-1]_{n-1}\end{aligned}$$

where we have applied (2.2) and (2.13)

Now, we have, cf. (2.4),

(17.46)
$$n[x-j-1]_{n-1} = [x-j]_n - [x-j-1]_n$$

and, cf. (2.8)

(17.47)
$$\binom{m+1}{2j+n+1} = \binom{m}{2j+n+1} + \binom{m}{2j+n}$$

so we may write the difference (17.45) as a sum, namely

(17.48)
$$\sum_j \binom{m}{2j+n}[x+j]_{m-n}[x-j]_n + \sum_j \binom{m}{2j+n+1}[x+j]_{m-n}[x-j-1]_n$$

The integral zeros of (17.48).

The polynomial (17.48) is zero for integral values of x, $x = 0, \ldots, \left[\frac{m-1}{2}\right]$, because it is two sums of zeros. The numbers $x+j$ and $x-j$ cannot both be negative, and if $x+j \geq m-n$ and $x-j \geq n$, then $2x \geq m$. And if $x+j \geq m-n$, but $x-j(-1) < 0$, then $2j+n(+1) > m-x+x = m$, such that the binomial coefficient is zero. And if $x+j < 0$ and $x-j(-1) \geq n$, then $2j+n(+1) < x-x = 0$, such that the binomial coefficient is zero.

The evaluation of the polynomial (17.48) for $x = -1$.

For $x = -1$ we may compute the value of the polynomial.

First for $n \leq m$: In the first sum of (17.48) only the term corresponding to $j = 0$ is different from zero, and in the second sum, only the terms corresponding to $j = 0, -1$ are different from zero. Hence the sum of the sums is the sum of the three terms:

(17.49)
$$\binom{m}{n}[-1]_{m-n}[-1]_n = (-1)^m[m]_n[m-n]_{m-n} = (-1)^m[m]_m$$

(17.50)
$$\binom{m}{n+1}[-1]_{m-n}[-2]_n = (-1)^m[m]_{n+1}[m-n]_{m-n} = (m-n)(-1)^m[m]_m$$

(17.51)
$$\binom{m}{n-1}[-2]_{m-n}[-1]_n = (-1)^m[m]_{n-1}[m-n+1]_{m-n+1}\,n = n(-1)^m[m]_m$$

where we have applied (2.1) and (2.2). The sum of the three results, (17.49–17.51), gives

$$(17.52)\quad (1+m-n+n)(-1)^m[m]_m = (-1)^m[m+1]_{m+1} = -[-1]_{m+1} = [-2]_m$$

which is the expected value according to (17.37).

If $n = m+1$, then the first sum of (17.48) is zero, and the second sum may have only the term of $j = -1$ different from zero. So, the value of (17.48) equals the value of (17.51), i.e., $(m+1)(-1)^m[m]_m = [-2]_m$ as in (17.52).

The non–integral zeros of (17.48).

In order to find the remaining zeros of the polynomial (17.48), we shall change variables in the sums to $k = 2j+n$ and $k = 2j+n+1$ respectively. Then we may write the polynomial as

(17.53)
$$\sum_{k\equiv n(2)} \binom{m}{k}\left[x+\frac{k-n}{2}\right]_{m-n}\left[x-\frac{k-n}{2}\right]_n +$$
$$+\sum_{k\not\equiv n(2)} \binom{m}{k}\left[x-\frac{1}{2}+\frac{k-n}{2}\right]_{m-n}\left[x-\frac{1}{2}-\frac{k-n}{2}\right]_n$$

Now, let $x = \frac{y}{2}$ for y an odd integer, $-1 \le y \le m-1$. Then we get

(17.54)
$$\sum_{k\equiv n(2)} \binom{m}{k}\left[\frac{y-n}{2}+\frac{k}{2}\right]_{m-n}\left[\frac{y+n}{2}-\frac{k}{2}\right]_n +$$
$$+\sum_{k\not\equiv n(2)} \binom{m}{k}\left[\frac{y-n-1}{2}+\frac{k}{2}\right]_{m-n}\left[\frac{y+n-1}{2}-\frac{k}{2}\right]_n$$

The missing terms for the rest of the k's are all zeros, provided y is an integer of the prescribed size. Hence we may omit the restrictions on the summations and consider the function:

$$P(y, n, m) = \sum_{k=0}^{m} \binom{m}{k} \left[\frac{y-n}{2} + \frac{k}{2}\right]_{m-n} \left[\frac{y+n}{2} - \frac{k}{2}\right]_{n} \tag{17.55}$$

with which definition the polynomial in (17.54) for $0 \le y \le m-1$ must coincide with the sum of consecutive values (for y) of the polynomial (17.55), i.e:

$$P(y, n, m) + P(y-1, n, m) \tag{17.56}$$

To evaluate (17.55), we shall rewrite the factorials using (2.2) a couple of times:

(17.57)

$$\left[\frac{y-n}{2} + \frac{k}{2}\right]_{m-n} \left[\frac{y+n}{2} - \frac{k}{2}\right]_{n} =$$

$$= \left[\frac{y-n}{2} + \frac{k}{2}\right]_{k} \left[\frac{y-n}{2} - \frac{k}{2}\right]_{m-n-k} \left[\frac{y+n}{2} - \frac{k}{2}\right]_{m-k} \left[\frac{y+n}{2} + \frac{k}{2} - m\right]_{n-m+k} =$$

$$= \left[\frac{y-n}{2} + \frac{k}{2}\right]_{k} \left[\frac{y+n}{2} - \frac{k}{2}\right]_{m-k}$$

With this rewriting we may write the polynomial P in (17.55) as

$$P(y, n, m) = \sum_{k=0}^{m} \binom{m}{k} \left[\frac{y-n}{2} + \frac{k}{2}\right]_{k} \left[\frac{y+n}{2} - \frac{k}{2}\right]_{m-k} \tag{17.58}$$

In the form (17.58) the polynomial is suitable for applying the famous formula of J. L. W. V. Jensen, (17.5). Doing this we get:

$$P(y, n, m) = \sum_{j=0}^{m} [m]_j \left(\tfrac{1}{2}\right)^j [y-j]_{m-j} \tag{17.59}$$

We apply (2.1) to write

$$[y-j]_{m-j} = (-1)^{m+j}[m-y-1]_{m-j} \tag{17.60}$$

Then we multiply and divide by $[m-j]_{m-j}$ to get

$$[m-y-1]_{m-j} = [m-j]_{m-j} \binom{m-y-1}{m-j} \tag{17.61}$$

Now we apply (2.2) and (2.3) to get for $-1 \le y \le m-1$:

(17.62)

$$P(y,n,m) = (-1)^m \sum_{j=0}^{m} [m]_j [m-j]_{m-j} \binom{m-y-1}{m-j} \left(-\tfrac{1}{2}\right)^j =$$
$$= (-1)^m [m]_m \left(-\tfrac{1}{2}\right)^{y+1} \sum_{j=0}^{m} \binom{m-y-1}{j-y-1} \left(-\tfrac{1}{2}\right)^{j-y-1} =$$
$$= (-1)^m [m]_m \left(-\tfrac{1}{2}\right)^{y+1} \left(1-\tfrac{1}{2}\right)^{m-y-1} = (-1)^{y+m+1} [m]_m \left(\tfrac{1}{2}\right)^m$$

where we have applied the binomial formula (7.1).

From (17.62) follows immediately, that we have

(17.63)
$$P(y,n,m) + P(y-1,n,m) = 0$$

Hence the polynomial (17.34) has the same values as the polynomial (17.35) in the $m+1$ points, $-1, 0, \frac{1}{2}, 1, \ldots, \frac{m-1}{2}$, so that the polynomials must be identical.

This ends the proof.

Remark 1.

We have now proven that the polynomial (17.40) is in fact equal to (17.37) independent of n. As soon as this fact is known, it is possible to prove the formula by evaluating the polynomial for $n=0$, in which case (17.40) with $m = 2\ell$ reduces to

(17.64)

$$\sum_{i=0}^{\ell} \sum_{k=0}^{0} \binom{2\ell+1}{2i-k+1} (x-i)[x+i-k]_{2\ell-0}[x-i+k-1]_{-1} =$$
$$= \sum_{i=0}^{\ell} \binom{2\ell+1}{2i+1} [x-i]_1 [x+i]_{2\ell} [x-i-1]_{-1} = \sum_{i=0}^{\ell} \binom{2\ell+1}{2i+1} [x+i]_{2\ell}$$

where we have used (2.2). Now we shall apply (2.2), (2.3) and (2.5) to write using (17.34) with $d=2$

(17.65)

$$\binom{2\ell+1}{2i+1} = \frac{[2\ell+1]_{2i+1}}{[2i+1]_{2i+1}} = \frac{[2\ell+1,2]_{i+1}\,[2\ell,2]_i}{[2i+1,2]_{i+1}[2i,2]_i} = \frac{\left[\ell+\frac{1}{2}\right]_{i+1}[\ell]_i}{\left[i+\frac{1}{2}\right]_{i+1}[i]_i} =$$
$$= \binom{\ell}{i} \frac{\left[\ell+\frac{1}{2}\right]_{i+1}\left[\ell+\frac{1}{2}\right]_{\ell+1}}{\left[i+\frac{1}{2}\right]_{i+1}\left[\ell+\frac{1}{2}\right]_{\ell+1}} = \frac{1}{\left[\ell-\frac{1}{2}\right]_\ell} \binom{\ell}{i} \left[\ell-\tfrac{1}{2}\right]_i \left[\ell+\tfrac{1}{2}\right]_{\ell-i}$$

and (2.1) and (2.2) to write

(17.66)
$$[x+i]_{2\ell} = [x+i]_i [x]_{2\ell-i} = [-x-1]_i (-1)^i [x]_\ell [x-\ell]_{\ell-i}$$

Hence we may write (17.64) by substitution of (17.65) and (17.66) as

$$\frac{[x]_\ell}{\left[\ell-\frac{1}{2}\right]_\ell}\sum_{i=0}^{\ell}\binom{\ell}{i}\left[\ell-\tfrac{1}{2}\right]_i[-x-1]_i\left[\ell+\tfrac{1}{2}\right]_{\ell-i}[x-\ell]_{\ell-i}(-1)^i \tag{17.67}$$

The sum,(17.67), satisfies the Pfaff–Saalschütz condition

$$\ell-\tfrac{1}{2}-x-1+\ell+\tfrac{1}{2}+x-\ell-\ell+1=0 \tag{17.68}$$

so we may apply the Pfaff–Saalschütz identity, cf. (9.1), to obtain

(17.69)

$$\frac{[x]_\ell}{\left[\ell-\frac{1}{2}\right]_\ell}[2\ell]_\ell\left[x-\tfrac{1}{2}\right]_\ell=\frac{[2x,2]_\ell[2x-1,2]_\ell[2\ell]_\ell[\ell]_\ell}{[2\ell-1,2]_\ell[2\ell,2]_\ell}=\frac{[2x]_{2\ell}[2\ell]_{2\ell}}{[2\ell]_{2\ell}}=[2x]_{2\ell}$$

as already observed by Joseph Sinyor and Ted Speevak, cf. [104].

Remark 2.

This form of Pfaff–Saalschütz is also interesting in itself, combining (17.64) with (17.69) to the formula

$$\sum_{i=0}^{\ell}\binom{2\ell+1}{2i+1}[x+i]_{2\ell}=[2x]_{2\ell} \tag{17.70}$$

This raises two questions about generalizations, one, is the number 2 crucial, or do we have a formula:

$$\sum_{i=0}^{\left[\frac{m}{2}\right]}\binom{m+1}{2i+1}[x+i]_m=[2x]_m \tag{17.71}$$

and what about the formula

$$\sum_{i=0}^{\left[\frac{m+1}{2}\right]}\binom{m+1}{2i}[x+i]_m=[2x+1]_m \tag{17.72}$$

They both prove valid. The formulas (17.71) and (17.72) are reduced to Pfaff–Saalschütz similarly to the proof of (17.70).

Remark 3.

In [104] Joseph Sinyor and Ted Speevak give a beautiful combinatorial proof of the formula for $n=1$. With the help of the formulas in remark 2, we may compute this result algebraically from (17.40) with $m=2\ell$. We get

(17.73)

$$\sum_{i=0}^{\ell}\sum_{k=0}^{1}\binom{2\ell}{2i-k+1}(x-i)[x+i-k]_{2\ell-1}[x-i+k-1]_0=$$
$$=\sum_{i=0}^{\ell-1}\binom{2\ell}{2i+1}(x-i)[x+i]_{2\ell-1}+\sum_{i=0}^{\ell}\binom{2\ell}{2i}(x-i)[x+i-1]_{2\ell-1}$$

We may write $x-i=(2x-2\ell+1)-(x-2\ell+1-i)$ in the first sum, and $x-i=2x-(x-i)$ in the second to rewrite the sums in (17.73) conveniently as

(17.74)

$$\sum_{i=0}^{\ell-1}\binom{2\ell}{2i+1}((2x-2\ell+1)-(x-2\ell+1+i))[x+i]_{2\ell-1}+$$

$$+\sum_{i=0}^{\ell}\binom{2\ell}{2i}(2x-(x+i))[x+i-1]_{2\ell-1}=$$

$$=(2x-2\ell+1)\sum_{i=0}^{\ell-1}\binom{2\ell}{2i+1}[x+i]_{2\ell-1}-\sum_{i=0}^{\ell-1}\binom{2\ell}{2i+1}(x-2\ell+1+i)[x+i]_{2\ell-1}+$$

$$+2x\sum_{i=0}^{\ell}\binom{2\ell}{2i}[x+i-1]_{2\ell-1}-\sum_{i=0}^{\ell}\binom{2\ell}{2i}(x+i)[x+i-1]_{2\ell-1}=$$

$$=(2x-2\ell+1)[2x]_{2\ell-1}-\sum_{i=0}^{\ell-1}\binom{2\ell}{2i+1}[x+i]_{2\ell}+$$

$$+2x[2(x-1)+1]_{2\ell-1}-\sum_{i=0}^{\ell}\binom{2\ell}{2i}[x+i]_{2\ell}=$$

$$=[2x]_{2\ell}-\sum_{i=0}^{\ell}\binom{2\ell+1}{2i+1}[x+i]_{2\ell}+[2x]_{2\ell}=[2x]_{2\ell}$$

where we have joined the second and fourth sums by the formulas (2.4), (2.2) and eventually (17.71) and (17.72) from remark 2.

Remark 4.

Also the cases of $n=2\ell$ and $n=2\ell+1$ as considered in [104] by Joseph Sinyor and Ted Speevak are easy to handle from (17.40) with $m=2\ell$. We obtain for $n=2\ell$:

(17.75)

$$\sum_i\sum_{k=2i}^{2i+1}\binom{1}{2i-k+1}\binom{2\ell}{k}(x-i)[x+i-k]_0[x-i+k-1]_{2\ell-1}=$$

$$=\sum_i\binom{2\ell}{2i}(x-i)[x-i+2i-1]_{2\ell-1}+\sum_i\binom{2\ell}{2i+1}(x-i)[x-i+2i]_{2\ell-1}=$$

$$=\sum_i\binom{2\ell}{2i}(x-i)[x+i-1]_{2\ell-1}+\sum_i\binom{2\ell}{2i+1}(x-i)[x+i]_{2\ell-1}$$

exactly the same as in (17.73).

And for $n=2\ell+1$ even easier, because, as $2i-k+1=0$, the sum (17.40) with $m=2\ell$ reduces to

$$(17.76)\quad \sum_i\binom{2\ell+1}{2i+1}[x-i]_1[x-i-1]_{-1}[x+i]_{2\ell}=\sum_i\binom{2\ell+1}{2i+1}[x+i]_{2\ell}=[2x]_{2\ell}$$

using (17.71).

Remark 5.

We also get an interesting interpolation formula. If we apply (17.57) directly to the first sum in (17.48) with $m = 2\ell > 0$, then we may write it as

$$\sum_j \binom{2\ell}{2j+n} [x+j]_{2j+n} [x-j]_{2\ell-n-2j} \tag{17.77}$$

and we know its value for $x = -\frac{1}{2}, 0, \frac{1}{2}, 1, ..., \ell - \frac{1}{2}$.

If we divide by $(2\ell)!$ and multiply with $2^{2\ell}$, we may write this result, for $0 \le n \le 2\ell + 1$, as

$$2^{2\ell} \sum_j \binom{x+j}{2j+n} \binom{x-j}{2\ell-n-2j} = \begin{cases} 0 & \text{for } x = 0, \dots, \ell \\ 1 & \text{for } x = -\frac{1}{2}, \dots, \ell - \frac{1}{2} \end{cases} \tag{17.78}$$

From (17.49) we know that this polynomial takes the value $2^{2\ell}$ for $x = -1$.

This sum can not be reduced to the Pfaff–Saalschütz formula.

Remark 6.

For comparison the Lagrange interpolation formula gives the polynomial in the form

$$\binom{2x+1}{2\ell+1} (2\ell+1) \sum_{j=0}^{\ell} \binom{2\ell}{2j} \frac{1}{2x+1-2j} \tag{17.79}$$

The sum in (17.79) may be considered as a generalized Pfaff–Saalschütz sum by the changes

(17.80)

$$\binom{2\ell}{2j} = \frac{[2\ell]_{2j}}{[2j]_{2j}} = \frac{[2\ell,2]_j [2\ell-1,2]_j}{[2j,2]_j [2j-1,2]_j} = \frac{[\ell]_j \left[\ell-\frac{1}{2}\right]_j}{[j]_j \left[j-\frac{1}{2}\right]_j} = \binom{\ell}{j} \frac{\left[\ell-\frac{1}{2}\right]_j \left[\ell-\frac{1}{2}\right]_\ell}{\left[j-\frac{1}{2}\right]_j \left[\ell-\frac{1}{2}\right]_\ell} =$$
$$= \frac{1}{\left[\ell-\frac{1}{2}\right]_\ell} \binom{\ell}{j} \left[\ell-\frac{1}{2}\right]_j \left[\ell-\frac{1}{2}\right]_{\ell-j}$$

similar to the changes in (17.65) and

(17.81)

$$\frac{1}{2x+1-2j} = \frac{1}{2}\left[x-\frac{1}{2}-j\right]_{-1} = -\frac{1}{2}\left[-x+\frac{1}{2}+j-2\right]_{-1} =$$
$$= -\frac{1}{2}\left[-x-\frac{3}{2}+j\right]_j \left[-x-\frac{3}{2}\right]_{-j-1} =$$
$$= -\frac{(-1)^j}{2}\left[x+\frac{3}{2}-j+j-1\right]_j \left[-x-\frac{3}{2}\right]_{-\ell-1} \left[-x-\frac{3}{2}+\ell+1\right]_{\ell-j} =$$
$$= \frac{(-1)^{j+1}}{2\left[-x-\frac{3}{2}+\ell+1\right]_{\ell+1}} \left[x+\frac{1}{2}\right]_j \left[\ell-x-\frac{1}{2}\right]_{\ell-j}$$

where we have applied (2.1)–(2.2).

Then the sum (17.79) becomes

(17.82)
$$\frac{-\binom{2x+1}{2\ell+1}(2\ell+1)}{2\left[\ell-\frac{1}{2}\right]_\ell\left[\ell-\frac{1}{2}-x\right]_{\ell+1}}\sum_{j=0}^{\ell}\binom{\ell}{j}\left[\ell-\tfrac{1}{2}\right]_j\left[x+\tfrac{1}{2}\right]_j\left[\ell-\tfrac{1}{2}\right]_{\ell-j}\left[\ell-\tfrac{1}{2}-x\right]_{\ell-j}(-1)^j$$

The factor in (17.82) may be rewritten as

(17.83)
$$-\frac{[2x+1]_{2\ell+1}(2\ell+1)}{[2\ell+1]_{2\ell+1}\,2\left[\ell-\frac{1}{2}\right]_\ell\left[\ell-\frac{1}{2}-x\right]_{\ell+1}}=-\frac{[2x+1,2]_{\ell+1}[2x,2]_\ell\,2^{2\ell}[\ell]_\ell}{[2\ell]_{2\ell}\,[2\ell-1,2]_\ell\,[2\ell-1-2x,2]_{\ell+1}\,[\ell]_\ell}=$$
$$-\frac{[2x+1,2]_{\ell+1}[x]_\ell\,2^{4\ell}[\ell]_\ell}{[2\ell]_{2\ell}\,[2\ell-1,2]_\ell\,[2\ell,2]_\ell(-1)^{\ell+1}\,[1+2x,2]_{\ell+1}}=\frac{(-16)^\ell[x]_\ell}{[2\ell]_\ell[2\ell]_{2\ell}}=\frac{16^\ell[\ell-1-x]_\ell}{[2\ell]_\ell[2\ell]_{2\ell}}$$

The sum in (17.82) with the factor replaced by (17.83), may be rewritten as a generalized Pfaff–Saalschütz sum, cf. (9.16):

(17.84)
$$\frac{16^\ell[\ell-1-x]_\ell}{[2\ell]_\ell[2\ell]_{2\ell}}\sum_{j=0}^{2\ell}\binom{2\ell}{j}[\ell]_j\left[x+\tfrac{1}{2}\right]_j[-1]_{\ell-j}[-1-x]_{\ell-j}(-1)^j$$

but this rewriting may not seem much of an improvement. Nevertheless, it is possible to simplify it. First denote, that $[\ell]_j=0$ for $j>\ell$, so we may change the upper limit of summation to ℓ. Next, two of the factors in the sum join, as long as $j\le\ell$

(17.85)
$$[\ell]_j[-1]_{\ell-j}=[\ell]_j[\ell-j]_{\ell-j}(-1)^{\ell-j}=[\ell]_\ell(-1)^{\ell-j}$$

and two other factors join too,

(17.86)
$$[\ell-1-x]_\ell[-1-x]_{\ell-j}=[\ell-1-x]_{2\ell-j}$$

Substitution of (17.85) and (17.86) in (17.84) yields

(17.87)
$$\frac{(-16)^\ell}{[2\ell]_{2\ell}\binom{2\ell}{\ell}}\sum_{j=0}^{\ell}\binom{2\ell}{j}\left[x+\tfrac{1}{2}\right]_j[\ell-1-x]_{2\ell-j}=\frac{(-16)^\ell}{\binom{2\ell}{\ell}}\sum_{j=0}^{\ell}\binom{x+\frac{1}{2}}{j}\binom{\ell-1-x}{2\ell-j}$$

In the case of Lagrange interpolation, the Pfaff–Saalschütz formula proved useful to change the formula to some extent. The result is of Chu–Vandermonde type, but the limits of summation are not natural. We may say, it is a half Chu–Vandermonde.

The coincidence between the two expressions of this polynomial, (17.78) and (17.87) is a surprising identity.

CHAPTER 18. SUMS OF TYPE V, HARMONIC SUMS

Harmonic sums. Harmonic sums are such sums which terms include a generalized harmonic factor, cf. (1.20), i.e., a factor of the form

$$H_{c,n}^{(m)} := \sum_{k=1}^{n} \frac{1}{(c+k)^m} \tag{18.1}$$

Remark that we consider the harmonic numbers well defined for $n = 0$ as the empty sum, $H_{c,0}^{(m)} = 0$.

Harmonic sums of power $m = 1$. Only the simplest allow $c \neq 0$.

The first indefinite harmonic sum is the sum of harmonic numbers with factorial factors.

Theorem 18.1. *For all $m, n \in \mathbb{Z}$, $m \neq -1$, we have*

$$\sum [c+n+k]_m H_{c,n+k}^{(1)} \delta k = \frac{[c+n+k]_{m+1}}{m+1} \left(H_{c,n+k}^{(1)} - \frac{1}{m+1} \right) \tag{18.2}$$

Proof. Straightforward taking the difference of the right side. □

The omission is covered by the result

Theorem 18.2. *For all $n \in \mathbb{Z}$, we have*

$$\sum [c+n+k]_{-1} H_{c,n+k}^{(1)} \delta k = \tfrac{1}{2} \left(\left(H_{c,n+k}^{(1)} \right)^2 - H_{c,n+k}^{(2)} \right) \tag{18.3}$$

Proof. Straightforward taking the difference of the right side. □

Corollary 18.2. *We have for $c = 0$*

$$\sum_{k=1}^{n} \frac{H_k}{k} = \tfrac{1}{2} \left(H_n^2 + H_{0,n}^{(2)} \right) \tag{18.4}$$

This is the form preferred by GKP, cf. [70], formula 6.71.

Remark. *The formulas (18.2–18.3) allow the summation of sums of the form*

$$\sum \frac{p(k)}{q(k)} H_{c,k}^{(1)} \delta k \tag{18.5}$$

with a rational factor, provided the denominator has different roots. We just have to apply (5.4).

Furthermore we shall mention

Theorem 18.3. *We have the indefinite summation formula for $c \in \mathbb{C}$ and $n \in \mathbb{Z}$*

(18.6)
$$\sum \left(H^{(1)}_{c,n+k}\right)^2 \delta k = (c+n+k)\left(H^{(1)}_{c,n+k}\right)^2 - (2(c+n+k)+1)H^{(1)}_{c,n+k} + 2(c+n+k)$$

Proof. Straightforward. □

Theorem 18.4. *For all $x \in \mathbb{C} \setminus \{0\}$ we have*

$$\sum (-1)^{k-1}\binom{x}{k} H_k \delta k = (-1)^k \left(\binom{x-1}{k-1} H_k - \frac{1}{x}\binom{x-1}{k}\right) \tag{18.7}$$

Proof. Straightforward. □

This is actually a special case of the more general formula:

Theorem 18.5. *For all $x, y \in \mathbb{C}$, $y \notin \mathbb{N}$ such that $x - y - 1 \neq 0$ we have*

$$\sum \frac{[x]_k}{[y]_k} H^{(1)}_{-y-1,k} \delta k = \frac{1}{x-y-1} \frac{[x]_k}{[y]_{k-1}} \left(H^{(1)}_{-y-1,k-1} + \frac{1}{x-y-1}\right) \tag{18.8}$$

Proof. Straightforward. □

For $x \in \mathbb{N}$ we may generalize the harmonic factor,

Theorem 18.6. *For all $n \in \mathbb{N}$, $m \in \mathbb{Z}$ and $c \in \mathbb{C}$ we have the formula*

$$\sum_{k=0}^{n} (-1)^{k-1}\binom{n}{k} H^{(1)}_{c,m+k} = \frac{(n-1)!}{[c+m+n]_n} = \frac{1}{n\binom{c+m+n}{n}} \tag{18.9}$$

Proof. Let $m = 0$. We apply the formula (2.16) and summation by parts, (2.24), to write the sum as

$$-\sum_{k=0}^{n} (-1)^{k-1}\binom{n-1}{k} \frac{1}{c+k+1} = \sum_{k=0}^{n} (-1)^k \binom{n-1}{k} [c+k]_{-1}$$

This sum is a Chu–Vandermonde sum, (8.20), so it equals

$$\frac{(n-1)!}{[c+n]_n}$$

□

The special case of $c = 0$ is found in [81], formula 6.7.1.

Inversion, cf. (2.29), of the formula (18.9) for $m = 0$ yields a formula of type II$(2, 2, 1)$ which we could have known already:

Corollary 18.6. *For all $n \in \mathbb{N}$ and $c \in \mathbb{C}$ we have the formula*

$$\sum_{k=1}^{n} \frac{1}{k} \cdot \frac{[n]_k}{[-c-1]_k} = H_{c,n}^{(1)} \tag{18.10}$$

Proof. Let us define

$$f(n) = \sum_{k=1}^{n} \frac{1}{k} \cdot \frac{[n]_k}{[-c-1]_k}$$

Then we enjoy that the upper limit is natural, so the difference

$$\begin{aligned} f(n) - f(n-1) = \sum_{k=1}^{n} \frac{[n-1]_{k-1}}{[-c-1]_k} &= \\ = \frac{1}{n} \cdot \frac{1}{c+n} \left[\frac{[n]_k}{[-c-1]_{k-1}} \right]_1^{n+1} &= \\ = \frac{1}{n} \cdot \frac{1}{c+n}(0-n) = \frac{1}{c+n} & \end{aligned}$$

where we have used the formula (7.4). □

Theorem 18.7. *For any $x \in \mathbb{C}$ we have the indefinite summation formula, but in case $x \in \mathbb{N}$, we must require $k \leq n$,*

$$\sum (-1)^k \binom{x}{k}^{-1} H_k \delta k = \frac{(-1)^k k}{x+2} \binom{x}{k-1}^{-1} \left(\frac{1}{x+2} - H_k \right) \tag{18.11}$$

Proof. Straightforward. □

The following formulas are found in [81], formulas 6.7.6 and 6.7.7. They are very special cases of theorem 18.6, for the choice of $x = 2n$ and $x = 2n-1$, respectively, and the limits 1 and $2n-1$.

Theorem 18.8. *For any integer, $n \in \mathbb{N}$, we have the sums*

$$\sum_{k=1}^{2n} (-1)^{k-1} \binom{2n}{k}^{-1} H_k = \frac{n}{2(n+1)^2} + \frac{1}{2n+2} H_{2n} \tag{18.12}$$

$$\sum_{k=1}^{2n-1} (-1)^{k-1} \binom{2n-1}{k}^{-1} H_k = \frac{2n}{2n+1} H_{2n} \tag{18.13}$$

Theorem (18.7) may be generalized to:

Theorem 18.9. *For all $x, y \in \mathbb{C}$, such that $x - y - 1 \neq 0$, $x \notin \mathbb{N}$, we have*

$$\sum \frac{[x]_k}{[y]_k} H^{(1)}_{-x-1,k} \delta k = \frac{1}{x-y-1} \frac{[x]_k}{[y]_{k-1}} \left(H^{(1)}_{-x-1,k} + \frac{1}{x-y-1} \right) \tag{18.14}$$

Proof. Straightforward. □

The following formula is found in [81], formula 6.7.3.

Theorem 18.10. *For all $n \in \mathbb{N}$, we have the sum*

$$\sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} H_{2k} = \frac{1}{2n} + \frac{[2n-2,2]_{n-1}}{[2n-1,2]_n} \tag{18.15}$$

Proof. We have from (2.15)

$$\Delta (-1)^k \binom{n-1}{k-1} = (-1)^{k-1} \binom{n}{k}$$

Summation by parts therefore may yield

$$\begin{aligned}
& -\sum_{k=0}^{n} (-1)^{k+1} \binom{n-1}{k} \left(\frac{1}{2k+2} + \frac{1}{2k+1} \right) = \\
& \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} [k]_{-1} + \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \left[-\tfrac{1}{2} + k \right]_{-1} \\
& = \frac{(-1)^{n-1}}{2} \left([0]_{-n} [-1]_{n-1} + \left[-\tfrac{1}{2} \right]_{-n} [-1]_{n-1} \right) = \frac{1}{2n} + \frac{[n-1]_{n-1}}{2 \left[n - \frac{1}{2} \right]_n}
\end{aligned}$$

where we have applied the Chu–Vandermonde convolution, (8.20). The formula follows by multiplication with the right power of 2. □

The following formula is a generalization of a strange formula due to W. Ljunggren from 1947, [90], which has only been proved by analytic methods, by J. E. Fjeldstad, [41] and J. Kvamsdal, [86] in 1948, cf. [81], formula 6.7.5.

Theorem 18.11. *For any complex number, $x \in \mathbb{C}$, and any integers, $n \in \mathbb{N}$ and $p \in \mathbb{N}_0$, we have the sum*

$$\sum_{k=p}^{n} \binom{n-p}{k-p} \binom{x+p}{k} H_k = \sum_{k=0}^{n-p} \binom{n-p}{k} \binom{x+p}{k+p} H_{k+p} = \binom{x+n}{n} \left(H_n + H^{(1)}_{x,p} - H^{(1)}_{x,n} \right) \tag{18.16}$$

From this theorem Ljunggren's formula follows from the choices $x = n$ and $p = 0$:

Corollary 18.11. *For any integer, $n \in \mathbb{N}$, we have the sum*

$$\sum_{k=1}^{n} \binom{n}{k}^2 H_k = \binom{2n}{n}\left(H_n - H_{n,n}^{(1)}\right) = \binom{2n}{n}(2H_n - H_{2n}) \tag{18.17}$$

Proof of theorem 18.10. We consider the function

$$f(n,x,p) = \sum_{k=p}^{n} \binom{n-p}{k-p}\binom{x+p}{k} H_k \tag{18.18}$$

which we shall compute. Now we remark that

$$\Delta_k (-1)^k \binom{x}{k} H_k = (-1)^{k+1}\binom{x+1}{k+1}\left(H_{k+1} - \frac{1}{x+1}\right) \tag{18.19}$$

Using (18.19) summation by parts, (2.24), yields

$$f(n,x,p) = -\sum_{k=p}^{n}(-1)^k \binom{n-p-1}{k-p}(-1)^{k+1}\binom{x+p+1}{k+1}\left(H_{k+1} - \frac{1}{x+p+1}\right) =$$

$$\sum_{k=p+1}^{n-1}\binom{n-p-1}{k-p-1}\binom{x+p+1}{k} H_k - \frac{1}{x+p+1}\sum_{k=p}^{n-1}\binom{n-p-1}{n-1-k}\binom{x+p+1}{k+1}$$

The first sum is just $f(n,x,p+1)$ as defined in (18.18), while the second sum is a Chu–Vandermonde convolution as of (8.1), so it yields

$$\frac{1}{x+p+1}\binom{x+n}{n}$$

Hence we have derived the formula with the difference taken in the variable p,

$$\Delta_p f(n,x,p) = \frac{1}{x+p+1}\binom{x+n}{n}$$

Summation gives us

$$f(n,x,p) = f(n,x,n) - \binom{x+n}{n}\sum_{k=p}^{n-1}\frac{1}{x+k+1} =$$

$$= \binom{x+n}{n} H_n - \binom{x+n}{n} H_{x+p,n-p}^{(1)} = \binom{x+n}{n}\left(H_n + H_{x,p}^{(1)} - H_{x,n}^{(1)}\right)$$

proving the formula. □

A more general result is the following formula,

Theorem 18.12. *For any complex number, $x \in \mathbb{C}$ we have the sum*

$$\sum_{k=1}^{n} \binom{n}{k} [-x]_k [x]_{n-k} H_k = (-1)^n (n-1)! - \frac{1}{n} [x-1]_n \tag{18.20}$$

Proof. From corollary (6.1) we have the difference formula

$$\binom{n}{k} [-x]_k [x]_{n-k} = \Delta \binom{n-1}{k-1} [-x-1]_{k-1} [x]_{n-k+1} \tag{18.21}$$

Summation by parts, (2.24), now yields

$$\begin{aligned} & -[x]_n - \sum_{k=1}^{n} \binom{n-1}{k} [-x-1]_k [x]_{n-k} \frac{1}{k+1} = \\ & = -[x]_n + \tfrac{1}{n} \sum_{k=1}^{n} \binom{n}{k+1} [-x]_{k+1} [x-1]_{n-k-1} \end{aligned}$$

This sum is a Chu–Vandermonde sum, missing the first two terms, corresponding to $k = 0$ and $k = -1$. Hence we get

$$-[x]_n + \frac{1}{n} \left([-1]_n - n(-x)[x-1]_{n-1} - [x-1]_n \right) = (-1)^n (n-1)! - \tfrac{1}{n} [x-1]_n$$

This ends the proof. □

Remark. *If $x \in \mathbb{N}_0$, $x \leq n$, the second term vanish, so the right side becomes just*

$$(-1)^n (n-1)!$$

independent of x. The case of $x = n$ is found in [81], formula 6.7.2.

A similar, but more specialized formula is the following:

Theorem 18.13. *For all $p \in \mathbb{N}$, we have the sum*

$$\sum_{k=0}^{n} \binom{n}{k} [-p]_k [p]_{n-k} H_{p+k-1} = (-1)^n (n-1)! \tag{18.22}$$

Proof. We write the sum as

$$[p]_n H_{p-1} + \sum_{k=1}^{n} \binom{n}{k} [-p]_k [p]_{n-k} H_{p+k-1}$$

We have from (18.21) above

$$\binom{n}{k}[-p]_k[p]_{n-k} = \Delta\binom{n-1}{k-1}[-p-1]_{k-1}[p]_{n-k+1}$$

Hence summation by parts yields

$$\begin{aligned}
&[p]_n H_{p-1} - [p]_n H_p - \sum_{k=1}^{n}\binom{n-1}{k}[-p-1]_k[p]_{n-k}\cdot\frac{1}{p+k} = \\
&= -[p-1]_{n-1} - \sum_{k=1}^{n}\binom{n-1}{k}[-p-1]_{k-1}[p]_{n-k} = \\
&= -[p-1]_{n-1} + \sum_{k=1}^{n}\binom{n-1}{k}[-p]_k[p-1]_{n-1-k} = \\
&= -[p-1]_{n-1} + [-1]_{n-1} - [p-1]_{n-1} = (-1)^n(n-1)!
\end{aligned}$$

where we have moved a factor p and applied the Chu–Vandermonde convolution, (8.4). □

The special case of $p = n$ is similar to the remark above,

Remark. *For all $n \in \mathbb{N}$ we have the formula*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{-n}{k}H_{n+k-1} = \frac{(-1)^n}{n} \tag{18.23}$$

Proof. Chose $p = n$ in (18.22) and divide by $n!$. □

A similar formula not containing any of the three above as special cases is found in [64] as formula 7.15.

Theorem 18.14. *For any $x, y \in \mathbb{C}$ except certain negative integers, we have*

$$\sum_{k=0}^{n}\binom{n}{k}[x]_k[y+n]_{n-k}H^{(1)}_{y,k} = [x+y+n]_n\left(H^{(1)}_{y,n} - H^{(1)}_{x+y,n}\right) \tag{18.24}$$

Proof. By induction after n. For $n = 1$ the result is obvious. Let us denote the left hand side of (18.24) by S,

$$S(n,x,y) = \sum_{k=0}^{n}\binom{n}{k}[x]_k[y+n]_{n-k}H^{(1)}_{y,k}$$

Splitting the binomial coefficient by (2.8) we get the sum as a sum of two,

$$(1) = \sum_{k=0}^{n-1} \binom{n-1}{k} [x]_k [y+n]_{n-k} H^{(1)}_{y,k}$$
$$(2) = \sum_{k=1}^{n} \binom{n-1}{k-1} [x]_k [y+n]_{n-k} H^{(1)}_{y,k}$$

The first is

$$(1) = (y+n)S(n-1,x,y) = (y+n)[x+y+n-1]_{n-1}\left(H^{(1)}_{y,n-1} - H^{(1)}_{x+y,n-1}\right)$$

In the second we translate the index by 1 to get

$$(2) = \sum_{k=0}^{n-1} \binom{n-1}{k} [x]_{k+1} [y+1+n-1]_{n-1-k} H^{(1)}_{y,k+1}$$

We remark, that $H^{(1)}_{y,k+1} = \frac{1}{y+1} + H^{(1)}_{y+1,k}$, so we may write

$$\begin{aligned}
(2) = {} & x\sum_{k=0}^{n-1} \binom{n-1}{k} [x-1]_k [y+1+n-1]_{n-1-k} H^{(1)}_{y+1,k} + \\
& + x\sum_{k=0}^{n-1} \binom{n-1}{k} [x-1]_k [y+1+n-1]_{n-1-k} \frac{1}{y+1} = \\
& = xS(n-1,x-1,y+1) + \frac{x}{y+1}[x+y+n-1]_{n-1} = \\
& = x[x+y+n-1]_{n-1}\left(H^{(1)}_{y+1,n-1} - H^{(1)}_{x+y,n-1} + \frac{1}{y+1}\right) = \\
& = x[x+y+n-1]_{n-1}\left(H^{(1)}_{y,n-1} - H^{(1)}_{x+y,n-1} + \frac{1}{y+n}\right)
\end{aligned}$$

using (18.24) to the first sum and the Chu–Vandermonde convolution (8.4) to the second sum.

Adding (1) and (2) we get

$$\begin{aligned}
& (x+y+n)[x+y+n-1]_{n-1}\left(H^{(1)}_{y,n-1} - H^{(1)}_{x+y,n-1}\right) + \frac{x}{y+n}[x+y+n-1]_{n-1} = \\
= & [x+y+n]_n\left(H^{(1)}_{y,n-1} - H^{(1)}_{x+y,n-1}\right) + [x+y+n]_n\left(\frac{1}{y+n} - \frac{1}{x+y+n}\right) = \\
= & [x+y+n]_n\left(H^{(1)}_{y,n} - H^{(1)}_{x+y,n}\right)
\end{aligned}$$

Remark. *The analog of the harmonic numbers with alternating signs in the sum do not provide anything new. Actually we have the formula,*

$$\sum_{k=1}^{n}\frac{(-1)^k}{k}=H_{\lfloor\frac{n}{2}\rfloor}-H_n \tag{18.25}$$

Harmonic sums of power $m>1$. Some formulas containing the generalized harmonic numbers, (1.20), of power $m=2$ are known,

Theorem 18.15. *For all $x\in\mathbb{C}$ we have the formula*

$$\sum\binom{x}{k}\binom{-x}{k}H^{(2)}_{0,k}\delta k=\binom{x-1}{k-1}\binom{-x-1}{k-1}H^{(2)}_{0,k}+\frac{1}{x^2}\binom{x-1}{k}\binom{-x-1}{k} \tag{18.26}$$

Proof. Straightforward. □

Corollary 18.15. *For all $n\in\mathbb{N}$ we have the formula*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{-n}{k}H^{(2)}_{0,k}=-\frac{1}{n^2} \tag{18.27}$$

Proof. Obvious. □

The sums with a factorial with nonnegative index are also possible to find. The first three are:

Theorem 18.16. *We have the following indefinite sums for $m\in\mathbb{N}_0$*
(18.28)

$$\sum[c+k]_m H^{(2)}_{c,k}\delta k=\frac{[c+k]_{m+1}}{m+1}H^{(2)}_{c,k}-\frac{(-1)^m m!}{m+1}H^{(1)}_{c,k}-\sum_{j=1}^{m}\frac{(-1)^j[m]_{m-j}[c+k]_j}{j(m+1)}$$

Proof. Straightforward. □

Seung–Jin Bang's problem. In 1995 Seung–Jin Bang, Ajou University, Suwon, Korea, posed the problem, [23]:

Show that

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k}\sum_{j=1}^{k}\frac{1}{j}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{j}\right)=\sum_{k=1}^{n}\frac{1}{k^3} \tag{18.29}$$

for all positive integers n.

This is the special case $p=2$ of the following

Theorem 18.17. *For $n \in \mathbb{N}$ and $p \in \mathbb{N}_0$ we define the multiple sum*

$$S(n,p) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} \sum_{j_1=1}^{k} \frac{1}{j_1} \sum_{j_2=1}^{j_1} \frac{1}{j_2} \sum_{j_3=1}^{j_2} \cdots \frac{1}{j_{p-1}} \sum_{j_p=1}^{j_{p-1}} \frac{1}{j_p} \tag{18.30}$$

Then we have

$$S(n,p) = \sum_{k=1}^{n} \frac{1}{k^{p+1}} = H_{0,n}^{(p+1)} \tag{18.31}$$

Proof. We consider the difference $D(n,p) = S(n,p) - S(n-1,p)$,

$$D(n,p) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left(\binom{n}{k} - \binom{n-1}{k} \right) \sum_{j_1=1}^{k} \frac{1}{j_1} \sum_{j_2=1}^{j_1} \frac{1}{j_2} \sum_{j_3=1}^{j_2} \cdots \sum_{j_p=1}^{j_{p-1}} \frac{1}{j_p} \tag{18.32}$$

and want to prove that it satisfies

$$D(n,p) = \frac{1}{n^{p+1}} \tag{18.33}$$

Since

$$\frac{1}{k} \left(\binom{n}{k} - \binom{n-1}{k} \right) = \frac{1}{k} \binom{n-1}{k-1} = \frac{1}{n} \binom{n}{k} \tag{18.34}$$

the difference may be written (where we for convenience have added the zero term for $k = 0$)

$$D(n,p) = \frac{1}{n} \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \sum_{j_1=1}^{k} \frac{1}{j_1} \sum_{j_2=1}^{j_1} \frac{1}{j_2} \sum_{j_3=1}^{j_2} \cdots \frac{1}{j_{p-1}} \sum_{j_p=1}^{j_{p-1}} \frac{1}{j_p} \tag{18.35}$$

Now summation by parts or Abelian summation (2.24) yields – as the constant terms vanish – that this is

(18.36)

$$\begin{aligned}
D(n,p) &= -\frac{1}{n} \sum_{k=0}^{n} (-1)^{k+1} \binom{n-1}{k} \frac{1}{k+1} \sum_{j_2=1}^{k+1} \frac{1}{j_2} \sum_{j_3=1}^{j_2} \cdots \frac{1}{j_{p-1}} \sum_{j_p=1}^{j_{p-1}} \frac{1}{j_p} \\
&= \frac{1}{n^2} \sum_{k=0}^{n-1} (-1)^{k} \binom{n}{k+1} \sum_{j_2=1}^{k+1} \frac{1}{j_2} \sum_{j_3=1}^{j_2} \cdots \frac{1}{j_{p-1}} \sum_{j_p=1}^{j_{p-1}} \frac{1}{j_p} \\
&= \frac{1}{n^2} \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \sum_{j_2=1}^{k} \frac{1}{j_2} \sum_{j_3=1}^{j_2} \cdots \frac{1}{j_{p-1}} \sum_{j_p=1}^{j_{p-1}} \frac{1}{j_p} = \frac{1}{n} D(n,p-1)
\end{aligned}$$

I.e., the difference satisfies the recursion

$$D(n,p) = \frac{1}{n} D(n, p-1) \tag{18.37}$$

and hence we get

$$D(n,p) = \frac{1}{n^p} D(n,0) = \frac{1}{n^{p+1}} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \tag{18.38}$$

where we must change the limit to 1, because the zero term no longer vanishes.

But from (2.15) we have

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} = (-1)^k \binom{n-1}{k-1} \Big]_1^{n+1} = 0 - (-1) = 1 \tag{18.39}$$

or, if you please, it follows from the binomial theorem (7.1) as

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = (1-1)^n = 0 \tag{18.40}$$

This proves the theorem. □

The Larcombe identities. In our papers [87, 88] we prove a family of harmonic identities.

Let us define the sum for non–negative integers, n and p, and a complex number z, not a negative integer, $-1, -2, \cdots, -n$:

$$S(n,p,z) = z \binom{z+n}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(z+k)^p} \tag{18.41}$$

Theorem 18.18.

$$S(n,0,z) = z \cdot 0^n \tag{18.42}$$

Proof. The binomial formula for $(1-1)^n$.

Theorem 18.19.

$$S(n,1,z) = 1 \tag{18.43}$$

Proof.

$$\begin{aligned} S(n,1,z) &= \frac{[z+n]_{n+1}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (-1)^k \frac{1}{z+k} \\ &= \sum_{k=0}^{n} \frac{1}{k!(n-k)!} (-1)^k [z+n]_{n-k} [z+k-1]_k \\ &= \sum_{k=0}^{n} \binom{-z}{k} \binom{n+z}{n-k} = \binom{n}{n} = 1 \end{aligned}$$

using Chu–Vandermonde (8.1).

Theorem 18.20. *For $p > 0$ we have the recursion*

$$S(n,p,z) = \sum_{k=0}^{n} \frac{1}{z+k} S(k,p-1,z) \tag{18.44}$$

Corollary 18.20. *For $p > 0$ we have the recursion*

$$\Delta_k S(k-1,p,z) = \frac{1}{z+k} S(k,p-1,z) \tag{18.45}$$

Proof of the theorem.

$$\begin{aligned} S(n,p,z) &= \frac{[z+n]_{n+1}}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1}{(z+k)^p} \\ &= \sum_{k=0}^{n} \binom{-z}{k} \binom{z+n}{n-k} \frac{1}{(z+k)^{p-1}} \\ &= \sum_{k=0}^{n} \binom{-z}{k} \binom{-z-k-1}{n-k} (-1)^{n-k} \frac{1}{(z+k)^{p-1}} \end{aligned}$$

Now we apply Chu–Vandermonde to write

$$\begin{aligned} \binom{-z-k-1}{n-k} &= \sum_{j=k}^{n} \binom{-z-k}{j-k} \binom{-1}{n-j} \\ &= \sum_{j=k}^{n} \binom{-z-k}{j-k} (-1)^{n-j} \end{aligned}$$

Appreciating (2.10)

$$\binom{-z}{k} \binom{-z-k}{j-k} = \binom{-z}{j} \binom{j}{k}$$

and changing the order of summation we obtain using (2.12), (2.11) and (18.40)

$$\begin{aligned} S(n,p,z) &= \sum_{j=0}^{n} \binom{-z}{j} (-1)^j \sum_{k=0}^{j} \binom{j}{k} (-1)^k \frac{1}{(z+k)^{p-1}} \\ &= \sum_{j=0}^{n} \binom{z+j-1}{j} \sum_{k=0}^{j} \binom{j}{k} (-1)^k \frac{1}{(z+k)^{p-1}} \\ &= \sum_{j=0}^{n} \frac{1}{z+j} z \binom{z+j}{j} \sum_{k=0}^{j} \binom{j}{k} (-1)^k \frac{1}{(z+k)^{p-1}} \\ &= \sum_{j=0}^{n} \frac{1}{z+j} S(j,p-1,z) \end{aligned}$$

Theorem 18.21. *For $p > 1$*

$$S(n,p,z) = \frac{1}{p-1}\sum_{j=1}^{p-1} H_{z,n}^{(j)} S(n,p-j,z) \tag{18.46}$$

Proof. By induction after p.

For $p = 2$ we have

$$S(n,2,z) = \sum_{k=0}^{n} \frac{1}{z+k} S(k,1,z) = \sum_{k=0}^{n} \frac{1}{z+k} = H_{z,n}^{(1)} = \frac{1}{2-1}\sum_{j=1}^{2-1} H_{z,n}^{(j)} S(n,2-j,z)$$

by theorem (18.19), then theorem (18.18), and eventually the definition of harmonic numbers, (1.20).

The induction step goes

$$\begin{aligned}
&\Delta_k \frac{1}{p-1}\sum_{j=1}^{p-1} H_{z,k-1}^{(j)} S(k-1,p-j,z)\\
=&\frac{1}{p-1}\sum_{j=1}^{p-1} \frac{1}{(z+k)^j} S(k,p-j,z) + \frac{1}{p-1}\sum_{j=1}^{p-2} H_{z,k}^{(j)} \frac{1}{z+k} S(k,p-1-j,z)\\
&-\frac{1}{p-1}\sum_{j=1}^{p-2} \frac{1}{(z+k)^{j+1}} S(k,p-1-j,z)\\
=&\frac{1}{p-1}\frac{1}{z+k} S(k,p-1,z) + \frac{1}{p-1}\sum_{j=2}^{p-1} \frac{1}{(z+k)^j} S(k,p-j,z)\\
&+\frac{1}{p-1}(p-2)\frac{1}{z+k} S(k,p-1,z) - \frac{1}{p-1}\sum_{j=2}^{p-1} \frac{1}{(z+k)^j} S(k,p-j,z)\\
=&\frac{1}{z+k} S(k,p-1,z)
\end{aligned}$$

Use the formula (2.19) on each term of the sum and corollary 18.19 for ΔS. Take outside the first term of the first sum. Apply the induction assumption for $p-1$ on the second sum. Change the summation index by 1 in the last sum. Then add the first and third expression while second and last sums cancel.

Corollary 18.21.

(18.47)
$$S(n,1,z) = 1$$
$$S(n,2,z) = H^{(1)}_{z,n}$$
$$S(n,3,z) = \frac{1}{2}\left(\left(H^{(1)}_{z,n}\right)^2 + H^{(2)}_{z,n}\right)$$
$$S(n,4,z) = \frac{1}{6}\left(\left(H^{(1)}_{z,n}\right)^3 + 3H^{(1)}_{z,n}H^{(2)}_{z,n} + 2H^{(3)}_{z,n}\right)$$
$$S(n,5,z) = \frac{1}{24}\left(\left(H^{(1)}_{z,n}\right)^4 + 6\left(H^{(1)}_{z,n}\right)^2 H^{(2)}_{z,n} + 3\left(H^{(2)}_{z,n}\right)^2 + 8H^{(1)}_{z,n}H^{(3)}_{z,n} + 6H^{(4)}_{z,n}\right)$$
$$S(n,6,z) = \frac{1}{120}\left(\left(H^{(1)}_{z,n}\right)^5 + 10\left(H^{(1)}_{z,n}\right)^3 H^{(2)}_{z,n} + 15H^{(1)}_{z,n}\left(H^{(2)}_{z,n}\right)^2\right.$$
$$\left. +20\left(H^{(1)}_{z,n}\right)^2 H^{(3)}_{z,n} + 20H^{(2)}_{z,n}H^{(3)}_{z,n} + 30H^{(1)}_{z,n}H^{(4)}_{z,n} + 24H^{(5)}_{z,n}\right)$$
$$S(n,7,z) = \frac{1}{720}\left(\left(H^{(1)}_{z,n}\right)^6 + 15\left(H^{(1)}_{z,n}\right)^4 H^{(2)}_{z,n} + 45\left(H^{(1)}_{z,n}\right)^2\left(H^{(2)}_{z,n}\right)^2\right.$$
$$+40\left(H^{(1)}_{z,n}\right)^3 H^{(3)}_{z,n} + 120H^{(1)}_{z,n}H^{(2)}_{z,n}H^{(3)}_{z,n} + 90\left(H^{(1)}_{z,n}\right)^2 H^{(4)}_{z,n} +$$
$$\left. +144H^{(1)}_{z,n}H^{(5)}_{z,n} + 15\left(H^{(2)}_{z,n}\right)^3 + 90H^{(2)}_{z,n}H^{(4)}_{z,n} + 40\left(H^{(3)}_{z,n}\right)^2 + 120H^{(6)}_{z,n}\right)$$

Proof. Let's do $S(n,5,z)$. Take
$$H^{(1)}_{z,n}S(n,4,z)/4 = \frac{1}{24}\left(\left(H^{(1)}_{z,n}\right)^4 + 3\left(H^{(1)}_{z,n}\right)^2 H^{(2)}_{z,n} + 2H^{(1)}_{z,n}H^{(3)}_{z,n}\right)$$
Then take
$$H^{(2)}_{z,n}S(n,3,z)/4 = \frac{1}{8}\left(\left(H^{(1)}_{z,n}\right)^2 H^{(2)}_{z,n} + \left(H^{(2)}_{z,n}\right)^2\right) = \frac{1}{24}\left(3\left(H^{(1)}_{z,n}\right)^2 H^{(2)}_{z,n} + 3\left(H^{(2)}_{z,n}\right)^2\right)$$
So take
$$H^{(3)}_{z,n}S(n,2,z)/4 = \frac{1}{24}6H^{(1)}_{z,n}H^{(3)}_{z,n}$$
And eventually
$$H^{(4)}_{z,n}S(n,1,z)/4 = \frac{1}{24}6H^{(4)}_{z,n}$$
Adding we get the coefficients 1, 3+3=6, 3, 2+6=8, and 6.

Theorem 18.22. *The coefficients in the corollary are:*

$$\frac{(p-1)!}{\prod_i i^{n_i} n_i!}\prod_i\left(H^{(i)}_{z,n}\right)^{n_i} \tag{18.48}$$

with $\sum_i i n_i = p-1$.

Proof. Guessing the coefficients as class orders from symmetric groups, cf. [109].

CHAPTER 19. APPENDIX ON INDEFINITE SUMS

Rational functions. Rational functions may always be summed indefinitely in principle. The function may be split into a polynomial part and a sum of principal fractions. All we need are the formulas cf. (3.5 ff.), (1.20) and (3.47)

$$\sum [c+k]_m \delta k = \begin{cases} \frac{[c+k]_{m+1}}{m+1} & m \neq -1 \\ H_{c,k}^{(1)} & m = -1 \end{cases} \tag{19.1}$$

$$\sum \frac{1}{(c+k)^m} \delta k = H_{c,k}^{(m)} \tag{19.2}$$

$$\sum k^m \delta k = \frac{B_{m+1}(k)}{m+1} \tag{19.3}$$

Rational functions may take the convenient form, cf. (7.4),

$$\sum \frac{[a]_k}{[b]_k} \delta k = \frac{1}{a-b-1} \frac{[a]_k}{[b]_{k-1}} \tag{19.4}$$

to be recognized as a sum of type I with quotient

$$q(k) = \frac{a-k}{b-k} \tag{19.5}$$

A more complicated form with condition on the arguments is, cf. (8.70), if $p = a+b-c-d \in \mathbb{N}_0$, we have

$$\sum \frac{[a]_k [b]_k}{[c-1]_k [d-1]_k} \delta k = \frac{[a]_k [b]_k}{[c-1]_{k-1} [d-1]_{k-1}} \sum_{j=0}^{p} \frac{[p]_j [k-c-1]_j}{[a-c]_{j+1} [b-c]_{j+1}} \tag{19.6}$$

with quotient

$$q(k) = \frac{(a-k)(b-k)}{(c-1-k)(d-1-k)} \tag{19.7}$$

A similar, but more special, formula is

$$\sum \binom{x}{k} (x-2k) \delta k = k \binom{x}{k} \tag{19.8}$$

with quotient

$$q(k) = \frac{(x-k)\left(\frac{x}{2}-1-k\right)}{(-1-k)\left(\frac{x}{2}-k\right)} \cdot (-1) \tag{19.9}$$

Furthermore, we have the surprising formula, cf. (11.1),

$$\sum \frac{[a+b+c]_k [a]_k [b]_k [c]_k (a+b+c-2k)(-1)^k}{k! [b+c-1]_k [a+c-1]_k [a+b-1]_k} \delta k = \tag{19.10}$$
$$= \frac{[a+b+c]_k [a-1]_{k-1} [b-1]_{k-1} [c-1]_{k-1} (-1)^k}{k! [b+c-1]_{k-1} [a+c-1]_{k-1} [a+b-1]_{k-1}}$$

Other indefinite sums. We know a few indefinite sums involving harmonic numbers, cf. (18.2), (18.3), (18.6), (18.7), (18.8), (18.14), (18.11), (18.28), seven with first order harmonic numbers,

(19.11)
$$\sum [c+k]_m H_{c,k}^{(1)} \delta k = \frac{[c+k]_{m+1}}{m+1} \left(H_{c,k}^{(1)} - \frac{1}{m+1} \right)$$

(19.12)
$$\sum [c+k]_{-1} H_{c,k}^{(1)} \delta k = \tfrac{1}{2} \left(\left(H_{c,k}^{(1)} \right)^2 - H_{c,k}^{(2)} \right)$$

(19.13)
$$\sum \left(H_{c,k}^{(1)} \right)^2 \delta k = (c+k) \left(H_{c,k}^{(1)} \right)^2 - (2(c+k)+1) H_{c,k}^{(1)} + 2(c+k)$$

(19.14)
$$\sum (-1)^{k-1} \binom{x}{k} H_k \delta k = (-1)^k \left(\binom{x-1}{k-1} H_k - \frac{1}{x} \binom{x-1}{k} \right)$$

(19.15)
$$\sum \frac{[x]_k}{[y]_k} H_{-y-1,k}^{(1)} \delta k = \frac{1}{x-y-1} \frac{[x]_k}{[y]_{k-1}} \left(H_{-y-1,k-1}^{(1)} + \frac{1}{x-y-1} \right)$$

(19.16)
$$\sum \frac{[x]_k}{[y]_k} H_{-x-1,k}^{(1)} \delta k = \frac{1}{x-y-1} \frac{[x]_k}{[y]_{k-1}} \left(H_{-x-1,k}^{(1)} + \frac{1}{x-y-1} \right)$$

(19.17)
$$\sum (-1)^k \binom{x}{k}^{-1} H_k \delta k = \frac{(-1)^k k}{x+2} \binom{x}{k-1}^{-1} \left(\frac{1}{x+2} - H_k \right)$$

and with second order harmonic numbers

(19.18)
$$\sum H_{c,k}^{(2)} \delta k = (c+k) H_{c,k}^{(2)} - H_{c,k}^{(1)}$$

(19.19)
$$\sum (c+k) H_{c,k}^{(2)} \delta k = \frac{1}{2} \left([c+k]_2 H_{c,k}^{(2)} + H_{c,k}^{(1)} - (c+k) \right)$$

(19.20)
$$\sum [c+k]_2 H_{c,k}^{(2)} \delta k = \frac{1}{3} \left([c+k]_3 H_{c,k}^{(2)} - 2 H_{c,k}^{(1)} + 2(c+k) - \frac{[c+k]_2}{2} \right)$$

The general form for $m \in \mathbb{N}_0$ is

(19.21)
$$\sum [c+k]_m H_{c,k}^{(2)} \delta k = \frac{[c+k]_{m+1}}{m+1} H_{c,k}^{(2)} - \frac{(-1)^m m!}{m+1} H_{c,k}^{(1)} - \sum_{j=1}^{m} \frac{(-1)^j [m]_{m-j} [c+k]_j}{j(m+1)}$$

CHAPTER 20. APPENDIX ON BASIC IDENTITIES

Introduction.

In this chapter we shall present basic generalizations of the simplest ordinary combinatorial identities. We prove the generalizations from which the ordinary cases are deduced simply by substitution of $Q = R = 1$ in the generalizations.

Our definition of a basic number is slightly more general than the usual definition. The main reason for the generalization is that it makes many formulas more symmetric in their appearance.

We shall assume the existence of two "universal constants" Q and R. We define the basic transformation $\{x\}$ of x by:

$$\{x\} = \begin{cases} x \cdot Q^{x-1} & \text{if } Q = R \\ (Q^x - R^x)/(Q - R) & \text{if } Q \neq R\,. \end{cases} \tag{20.1}$$

The following properties of $\{x\}$ are easily verified:

$$\{0\} = 0 \quad \text{and} \quad \{1\} = 1\,, \tag{20.2}$$

$$\{-x\} = -\{x\} \cdot (Q \cdot R)^{-x}\,, \tag{20.3}$$

$$\{x+y\} = \{x\} \cdot Q^y + \{y\} \cdot R^x = \{x\} \cdot R^y + \{y\} \cdot Q^x\,, \tag{20.4}$$

$$\begin{aligned} \{x-y\} &= \{x\} \cdot R^{-y} - \{y\} \cdot R^{-y} \cdot Q^{x-y} \\ &= \{x\} \cdot Q^{-y} - \{y\} \cdot Q^{-y} \cdot R^{x-y}\,, \end{aligned} \tag{20.5}$$

$$\begin{cases} \text{if } n \text{ is a positive integer then} \\ \{n\} = Q^{n-1} + Q^{n-2} \cdot R + \cdots + R^{n-1} \\ \{-n\} = -(Q^{-1} \cdot R^{-n} + Q^{-2} \cdot R^{-n+1} + \ldots Q^{-n} \cdot R^{-1})\,. \end{cases} \tag{20.6}$$

Remark. *Because of the symmetry in Q and R in the definition (20.1) Q and R must be interchangeable in all the following formulas.*

Remark 2. *The expression (20.6) shows our definition as a generalization of the definition of $[k]_{p,q}$ due to Wachs and White [113] 1991.*

In his original introduction of basic numbers in 1847, [75], *Untersuchungen über die Reihe ...,*

$$1 + \frac{(1-q^\alpha)(1-q^\beta)}{(1-q)(1-q^\gamma)} \cdot x + \frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})}{(1-q)(1-q^2)(1-q^\gamma)(1-q^{\gamma+1})} \cdot x^2 + \cdots$$

E. Heine replaced α, β, etc. in a Gaußian hypergeometric series by $\frac{1-q^\alpha}{1-q}$, $\frac{1-q^\beta}{1-q}$, etc. in order to obtain a continuous generalization from α as $\frac{1-q^\alpha}{1-q} \to \alpha$ for $q \to 1$. He also noticed the similarity between q and $r = \frac{1}{q}$.

We have replaced q by Q/R to obtain

$$\frac{1-q^{\alpha}}{1-q}=\frac{R^{\alpha}-Q^{\alpha}}{R-Q}\cdot R^{1-\alpha}=\{\alpha\}\cdot R^{1-\alpha} \tag{20.7}$$

– hardly a generalization at all. Nevertheless, some of our formulas contains two of Heine's by choosing $R=q$ and $Q=1$ or $R=1$ and $Q=q$.

It is sometimes customary to omit the denominator $1-q$, because it often cancels out anyway. We have chosen not to do so to keep the continuity. It is further customary to replace q^{α} with a, and further $q^{\alpha+1}$ with aq, etc., but we shall not do so, because it makes no difference, as mentioned by E. Heine 1878, [76], p. 98. One just has to apply the usual formulas to q^{α} for comparison.

Factorials and binomial coefficients.

We introduce a generalized basic factorial:

$$\{x,d\}_n=\begin{cases}\prod_{j=0}^{n-1}\{x-j\cdot d\} & \text{if } n\in\mathbb{N}\\ 1 & \text{if } n=0\,.\\ \prod_{j=1}^{-n}\frac{1}{\{x+j\cdot d\}} & \text{if } -n\in\mathbb{N}\end{cases} \tag{20.8}$$

Since descending factorials are the most important in combinatorics we shall use the notation $\{x\}_n$ for $\{x,1\}_n$. If $Q=R=1$ then we have the ordinary descending factorial $[x]_n$. The most important properties of the generalized factorial are:

(20.9)
$$\{x,d\}_n=\{x-d,d\}_n\cdot Q^{n\cdot d}+\{x-d,d\}_{n-1}\cdot\{n\cdot d\}\cdot R^{x-n\cdot d}$$

(20.10)
$$\{x,d\}_n=\{x-d,d\}_n\cdot Q^{n\cdot d}-\{x-d,d\}_{n-1}\cdot\{-n\cdot d\}\cdot R^{x}\cdot Q^{n\cdot d}$$

(20.11)
$$\{x,d\}_n=\{-x+(n-1)\cdot d,d\}_n\cdot(-1)^n\cdot(Q\cdot R)^{n\cdot x-\binom{n}{2}\cdot d}$$

(20.12)
$$\{x,d\}_n=\{x-(n-1)\cdot d,-d\}_n$$

(20.13)
$$\{x,d\}_n=\{-x,-d\}_n\cdot(-1)^n\cdot(Q\cdot R)^{n\cdot x-\binom{n}{2}\cdot d}$$

(20.14)
$$\{x,d\}_{m+n}=\{x,d\}_m\cdot\{x-m\cdot d,d\}_n$$

(20.15)
$$\{x,d\}_{-n}=1/\{x+n\cdot d,d\}_n$$

(20.16)
$$\frac{\{2\cdot x, 2\cdot d\}_n}{\{x,d\}_n} = \prod_{j=0}^{n-1}(Q^{x-j\cdot d}+R^{x-j\cdot d})$$

(20.17)
$$\{2\cdot n\cdot d, d\}_n = \{2\cdot n\cdot d - d, 2\cdot d\}_n\cdot\frac{\{(2\cdot n\cdot d, 2\cdot d\}_n}{\{n\cdot d, d\}_n} =$$
$$= \{-d, 2\cdot d\}_n\cdot(-1)^n\cdot(Q\cdot R)^{n^2\cdot d}\cdot\prod_{j=1}^{n}(Q^{j\cdot d}+R^{j\cdot d})$$

The basic binomial coefficients are defined by:

(20.18)
$$\begin{Bmatrix} x \\ n \end{Bmatrix} = \begin{cases} \{x\}_n/\{n\}_n & \text{if } n\in\mathbb{N}_0 \\ 0 & \text{if } n\notin\mathbb{N}_0\,. \end{cases}$$

We note the following properties of the basic binomial coefficient:

(20.19)
$$\begin{Bmatrix} n \\ 0 \end{Bmatrix} = \begin{Bmatrix} n \\ n \end{Bmatrix} = 1$$

(20.20)
$$\begin{Bmatrix} x \\ n \end{Bmatrix} = \begin{Bmatrix} x-1 \\ n-1 \end{Bmatrix}\cdot Q^{x-n} + \begin{Bmatrix} x-1 \\ n \end{Bmatrix}\cdot R^n$$

(20.21)
$$\begin{Bmatrix} x \\ n \end{Bmatrix}\cdot\begin{Bmatrix} n \\ m \end{Bmatrix} = \begin{Bmatrix} x \\ m \end{Bmatrix}\cdot\begin{Bmatrix} x-m \\ n-m \end{Bmatrix}$$

(20.22)
$$\begin{Bmatrix} x \\ n \end{Bmatrix}\cdot\{n\}_m = \{x\}_m\cdot\begin{Bmatrix} x-m \\ n-m \end{Bmatrix}$$

If $x\in\mathbb{N}_0$ and $x\geq n$ then $\left\{\begin{smallmatrix} x \\ n \end{smallmatrix}\right\} = \left\{\begin{smallmatrix} x \\ x-n \end{smallmatrix}\right\}$, and from (20.20) follows that $\left\{\begin{smallmatrix} x \\ n \end{smallmatrix}\right\}$ is a homogeneous polynomial in Q and R of degree $(x-n)\cdot n$.

Two basic binomial theorems.

Theorem 20.1.

For arbitrary x and y and integers m and n $(\geq m)$ and arbitrary α and β:

(20.23)
$$\sum_{k=m}^{n}\begin{Bmatrix} n-m \\ k-m \end{Bmatrix}\cdot x^{k-m}\cdot y^{n-k}\cdot Q^{\binom{k-m}{2}}\cdot R^{\binom{n-k}{2}} = \prod_{j=0}^{n-m-1}(x\cdot Q^j + y\cdot R^j)\,.$$

(20.24)
$$\sum_{k=m}^{n}\begin{Bmatrix} n-m \\ k-m \end{Bmatrix}\cdot x^{k-m}\cdot y^{n-k}\cdot Q^{\binom{k-m}{2}+\alpha(k-m)}\cdot R^{\binom{n-k}{2}+\beta(n-k)} =$$
$$\prod_{j=0}^{n-m-1}(x\cdot Q^{j+\alpha}+y\cdot R^{j+\beta})\,.$$

Proofs.

It is sufficient to prove (20.23) for $m = 0$, because the general case follows by substitution of $k+m$ for k and $n+m$ for n. Let

$$S(n,x,y) = \sum_k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \cdot x^k \cdot y^{n-k} \cdot Q^{\binom{k}{2}} \cdot R^{\binom{n-k}{2}}$$

Using

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} \cdot Q^{n-k} + \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} \cdot R^k ,$$

we split S into two sums. In the sum with $\left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\}$ we may substitute $k+1$ for k. When we take advantage of common factors in the two sums, this gives us:

$$\begin{aligned} S(n,x,y) &= \sum_k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} \cdot x^k \cdot y^{n-1-k} \cdot Q^{\binom{k}{2}} \cdot R^{\binom{n-k}{2}} \cdot (x \cdot Q^{n-1} + y \cdot R^{n-1}) \\ &= S(n-1,x,y) \cdot (x \cdot Q^{n-1} + y \cdot R^{n-1}) . \end{aligned}$$

Iteration now yields the formulae (20.23).

The more general formula (20.24) follows from (20.23) by substitution of $x \cdot Q^\alpha$ for x and $y \cdot R^\beta$ for y in (20.23). □

Two basic Chu–Vandermonde convolutions.

Theorem 20.2. *For arbitrary a and b and arbitrary integers $0 \le m \le n$*

$$\begin{aligned} &\sum_{k=m}^{n} \left\{ \begin{matrix} n-m \\ k-m \end{matrix} \right\} \cdot \{a\}_{k-m} \cdot \{b\}_{n-k} \cdot (QR)^{-(k-m)\cdot(n-k)} \cdot Q^{a\cdot(n-k)} \cdot R^{b\cdot(k-m)} \\ &\qquad = \{a+b\}_{n-m} , \end{aligned} \tag{20.25}$$

$$\begin{aligned} &\sum_{k=m}^{n} \left\{ \begin{matrix} n-m \\ k-m \end{matrix} \right\} \cdot \{a,-1\}_{k-m} \cdot \{b,-1\}_{n-k} \cdot Q^{a\cdot(n-k)} \cdot R^{b\cdot(k-m)} \\ &\qquad = \{a+b,-1\}_{n-m} . \end{aligned} \tag{20.26}$$

Proofs.

It is sufficient to prove (20.25) and (20.26) for $m = 0$. We shall prove (20.26) from which (20.25) follows.

Let

$$S(n,a,b) = \sum_k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \cdot \{a,-1\}_k \cdot \{b,-1\}_{n-k} \cdot Q^{a\cdot(n-k)} \cdot R^{b\cdot k} .$$

Using (20.20),

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \cdot R^{n-k} + \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \cdot Q^k ,$$

we split S into two sums. In the sum with $\left\{{n-1 \atop k-1}\right\}$ we substitute $k+1$ for k. When we take advantage of common factors in the two sums, this gives us:

$$S(n,a,b) = \sum_k \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \cdot \{a,-1\}_k \cdot \{b,-1\}_{n-1-k} \cdot Q^{a\cdot(n-1-k)} \cdot R^{b\cdot k} \cdot (\{a+k\} \cdot R^{b+n-1-k} + \{b+n-1-k\} \cdot Q^{a+k})$$

Using (20.4) this gives

$$S(n,a,b) = S(n-1,a,b) \cdot \{a+b+n-1\})$$

Iteration now yields the formula (20.26).

If we apply (20.26) to $-a$ and $-b$ using (20.6) to change the sign of d, we get (20.25). □

Two special cases of the basic Chu–Vandermonde convolutions.

Theorem 20.3.

For arbitrary c and p an arbitrary integer and integers m and n $(\geq m)$:

(20.27)
$$\sum_{k=m}^{n} \begin{Bmatrix} n-m \\ k-m \end{Bmatrix} \cdot (-1)^{n-k} \cdot \{c-m+k\}_p \cdot Q^{(n-m)\cdot(p-c-n+m)+\binom{n-k}{2}} \cdot R^{(n-k)\cdot(p+m-n+1)+\binom{n-k}{2}} = \{p\}_{n-m} \cdot \{c\}_{p-n+m} \cdot$$

(20.28)
$$\sum_{k=m}^{n} \begin{Bmatrix} n-m \\ k-m \end{Bmatrix} \cdot (-1)^{k-m} \cdot \{c+n-k\}_p \cdot Q^{(n-m)\cdot(p-c-n+m)+\binom{k-m}{2}} \cdot R^{(k-m)\cdot(p+m-n+1)+\binom{k-m}{2}} = \{p\}_{n-m} \cdot \{c\}_{p-n+m} \cdot$$

Proofs.

In (20.27) we substitute $-1-c$ for a and $c-m+n-p$ for b and obtain using (20.11) on the right hand side:

(20.29)
$$\sum_k \begin{Bmatrix} n-m \\ k-m \end{Bmatrix} \cdot \{-1-c\}_{k-m} \cdot \{c-m+n-p\}_{n-k} \cdot Q^{(-1-c-k+m)\cdot(n-k)} \cdot R^{(c-m-p+k)\cdot(k-m)} = \{n-m-p-1\}_{n-m} = \{p\}_{n-m} \cdot (-1)^{n-m} \cdot (Q \cdot R)^{(n-m-p-1)\cdot(n-m)-\binom{n-m}{2}} .$$

In the sum we use formula (20.11) on the factor $\{-1-c\}_{k-m}$ and observe that

(20.30) $$\{c-m+k\}_{k-m}\cdot\{c-m+n-p\}_{n-k}=\{c-m+k\}_p\cdot\{c-m+n-p\}_{n-m-p}$$

since all factors occur with the same multiplicity on both sides. We obtain

(20.31)
$$\sum_k \begin{Bmatrix} n-m \\ k-m \end{Bmatrix}\cdot(-1)^{k-m}\cdot\{c-m+k\}_p\cdot\{c-m+n-p\}_{n-m-p}\cdot$$
$$\cdot Q^{(-1-c-k+m)\cdot(n-k)}\cdot R^{(c-m-p+k)\cdot(k-m)}\cdot(Q\cdot R)^{(-1-c)\cdot(k-m)-\binom{k-m}{2}}=$$
$$\{p\}_{n-m}\cdot(-1)^{n-m}\cdot(Q\cdot R)^{(n-m-p-1)\cdot(n-m)-\binom{n-m}{2}}.$$

We now multiply the equation (20.31) by $\{c\}_{p-n+m}=1/\{c-m+n-p\}_{n-m-p}$ and by $(-1)^{n-m}\cdot(Q\cdot R)^{-(n-m-p-1)\cdot(n-m)+\binom{n-m}{2}}$. Simplifying the exponents of Q and R yields the equation (20.27).

The equation (20.28) is obtained from (20.27) by substitution of $n+m-k$ for k i.e. by reversal of the direction of summation. □

The formula (20.27) for $p=-1$ has a nice interpretation as a generalization of a useful partition of the quotient into partial fractions.

Corollary 20.1. *For arbitrary c and arbitrary nonnegative integer n*

(20.32) $$\frac{1}{\{c+n\}_{n+1}}=\frac{1}{\{n\}_n}\sum_{k=0}^{n}\begin{Bmatrix} n \\ k \end{Bmatrix}\frac{(-1)^k}{\{c+k\}}(QR)^{\binom{k+1}{2}}Q^{-n\cdot(c+k)}$$

Proof. Substitution of $m=0$, $p=-1$ and $c=c-1$ gives the formula. □

The symmetric Kummer identity.

Theorem 20.4. *For arbitrary a and arbitrary nonnegative integer n*

(20.33)
$$\sum_{k=0}^{2n}\begin{Bmatrix} 2n \\ k \end{Bmatrix}\{a\}_k\{a\}_{2n-k}(-1)^k(QR)^{-k(2n-k)}=\{2a,2\}_n\{-1,2\}_n$$
$$=(-1)^n\{2a,2\}_n\{2n-1,2\}_n(QR)^{-n^2}$$

(20.34)
$$\sum_{k=0}^{2n+1}\begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}\{a\}_k\{a\}_{2n+1-k}(-1)^k(QR)^{-k(2n+1-k)}=0$$

Proofs. Let
$$S(n,a)=\sum_k\begin{Bmatrix} n \\ k \end{Bmatrix}\{a\}_k\{a\}_{n-k}(-1)^k(QR)^{-k(n-k)}$$

Then remark, that

$$\begin{aligned} S(2n+1,a) &= \sum_k \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix} \{a\}_k \{a\}_{2n+1-k} (-1)^k (QR)^{-k(2n+1-k)} \\ &= -S(2n+1,a) = 0 \end{aligned}$$

by reversing the order of summation.

Now,

$$\begin{aligned} S(2n,a) = &\sum_k \begin{Bmatrix} 2n-1 \\ k \end{Bmatrix} \{a\}_k \{a\}_{2n-k} (-1)^k Q^{-k(2n-1-k)} R^{-k(2n-k)} \\ &+ \sum_k \begin{Bmatrix} 2n-1 \\ k-1 \end{Bmatrix} \{a\}_k \{a\}_{2n-k} (-1)^k Q^{-k(2n-k)} R^{-(k-1)(2n-k)} \end{aligned}$$

by (20.20). By replacing k with $k+1$ the second sum becomes

$$-\sum_k \begin{Bmatrix} 2n-1 \\ k \end{Bmatrix} \{a\}_{k+1} \{a\}_{2n-1-k} (-1)^k Q^{-(k+1)(2n-1-k)} R^{-k(2n-1-k)}$$

Application of (20.14) with $m = 1$ to $\{a\}_{k+1}$ and $\{a\}_{2n-k}$ yields

$$\begin{aligned} S(2n,a) = \{a\} &\sum_k \begin{Bmatrix} 2n-1 \\ k \end{Bmatrix} \{a\}_k \{a-1\}_{2n-1-k} (-1)^k Q^{-k(2n-1-k)} R^{-k(2n-k)} \\ - \{a\} &\sum_k \begin{Bmatrix} 2n-1 \\ k \end{Bmatrix} \{a-1\}_k \{a\}_{2n-1-k} (-1)^k Q^{-(k+1)(2n-1-k)} R^{-k(2n-1-k)} \end{aligned}$$

We apply the formula (20.9) in the forms

$$\begin{aligned} \{a\}_k &= \{a-1\}_k R^k + \{a-1\}_{k-1} \{k\} Q^{a-k} \\ \{a\}_{2n-1-k} &= \{a-1\}_{2n-1-k} Q^{2n-1-k} \\ &\quad + \{a-1\}_{2n-2-k} \{2n-1-k\} R^{a-2n+1+k} \end{aligned}$$

and get

$$\begin{aligned} &S(2n,a) = \\ &\{a\} \Bigg(\sum_k \begin{Bmatrix} 2n-1 \\ k \end{Bmatrix} \{a-1\}_k \{a-1\}_{2n-1-k} (-1)^k (QR)^{-k(2n-1-k)} \\ &+ \sum_k \begin{Bmatrix} 2n-1 \\ k \end{Bmatrix} \{a-1\}_{k-1} \{a-1\}_{2n-1-k} \{k\} (-1)^k Q^a (QR)^{-k(2n-k)} \\ &- \sum_k \begin{Bmatrix} 2n-1 \\ k \end{Bmatrix} \{a-1\}_k \{a-1\}_{2n-1-k} (-1)^k (QR)^{-k(2n-1-k)} \\ &- \sum_k \begin{Bmatrix} 2n-1 \\ k \end{Bmatrix} \{a-1\}_k \{a-1\}_{2n-2-k} \{2n-1-k\} (-1)^k R^a (QR)^{-(k+1)(2n-1-k)} \Bigg) \end{aligned}$$

The first and the third sums cancel. On the remaining two sums we apply the formula (20.22) to get

$$
\begin{aligned}
&S(2n,a) = \\
&\{a\}\{2n-1\}\left(\sum_k \begin{Bmatrix} 2n-2 \\ k-1 \end{Bmatrix}\{a-1\}_{k-1}\{a-1\}_{2n-1-k}(-1)^k Q^a (QR)^{-k(2n-k)}\right. \\
&\qquad \left. - \sum_k \begin{Bmatrix} 2n-2 \\ k \end{Bmatrix}\{a-1\}_k\{a-1\}_{2n-2-k}(-1)^k R^a (QR)^{-(k+1)(2n-1-k)}\right) \\
&\qquad = -\{a\}\{2n-1\}\left(Q^a + R^a\right) \cdot \\
&\qquad \cdot \sum_k \begin{Bmatrix} 2n-2 \\ k \end{Bmatrix}\{a-1\}_k\{a-1\}_{2n-2-k}(-1)^k (QR)^{-k(2n-2-k)+1-2n} \\
&\qquad = -\{a\}\{2n-1\}\left(Q^a + R^a\right) S(2n-2, a-1)(QR)^{1-2n}
\end{aligned}
$$

by replacing k with $k+1$ in the first sum. From the definition (20.1) we have

$$\{a\}\left(Q^a + R^a\right) = \{2a\}$$

Iteration of the reduction yields because $S(0,x) = 1$

$$S(2n,a) = (-1)^n\{2a,2\}_n\{2n-1,2\}_n(QR)^{-n^2} = \{2a,2\}_n\{-1,2\}_n$$

The last equality due to (20.11). □

The symmetry of the formula (20.33) is emphasized by the following change of summation variable.

Corollary 20.2. *For arbitrary a and arbitrary nonnegative integer n*

(20.35)

$$
\begin{aligned}
\sum_{k=-n}^{n} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}\{a\}_{n+k}\{a\}_{n-k}(-1)^k (QR)^{k^2} &= \{2a,2\}_n\{2n-1,2\}_n \\
&= (-1)^n\{2a,2\}_n\{-1,2\}_n(QR)^{n^2}
\end{aligned}
$$

The quasi–symmetric Kummer identity.

Theorem 20.5. *For arbitrary a, arbitrary integer p and arbitrary nonnegative integers $m \le n$*

(20.36)

$$
\begin{aligned}
&\sum_{k=m}^{n} \begin{Bmatrix} n-m \\ k-m \end{Bmatrix}\{a\}_{k-m}\{a+p\}_{n-k}(-1)^{k-m}(QR)^{-(k-m)\cdot(n-k)}R^{(k-m)\cdot p} = \\
&(\sigma(p))^{n-m}\sum_{j=\lceil \frac{n-m-|p|}{2} \rceil}^{\lfloor \frac{n-m}{2} \rfloor} \begin{Bmatrix} |p| \\ n-m-2j \end{Bmatrix}\{n-m\}_{n-m-2j}\{2(a+(p\wedge 0)),2\}_j\{-1,2\}_j \cdot \\
&\qquad Q^{(a+(p\wedge 0)-2j)\cdot(n-m-2j)}R^{(|p|-n+m+2j)\cdot 2j+(n-m)\cdot(p\wedge 0)}
\end{aligned}
$$

Proof. It is enough to prove the theorem for $m=0$. We apply the Chu–Vandermonde convolution (20.25) on $\{a+p\}_{n-k}$ in the form

$$\{a+p\}_{n-k} = \sum_j \begin{Bmatrix} n-k \\ j-k \end{Bmatrix} \{a\}_{j-k}\{p\}_{n-j} Q^{(a-j+k)\cdot(n-j)} R^{(p-n+j)\cdot(j-k)}$$

to get

$$\sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} \{a\}_k (-1)^k \sum_j \begin{Bmatrix} n-k \\ j-k \end{Bmatrix} \{a\}_{j-k}\{p\}_{n-j} Q^{(a-j+k)\cdot(n-j)} R^{(p-n+j)\cdot(j-k)} =$$

$$= \sum_j \{p\}_{n-j} \sum_k (-1)^k \begin{Bmatrix} n \\ k \end{Bmatrix} \begin{Bmatrix} n-k \\ j-k \end{Bmatrix} \{a\}_k \{a\}_{j-k} Q^{(a-j+k)\cdot(n-j)} R^{(p-n+j)\cdot(j-k)}$$

Next (20.21) allows the transformation

$$\begin{Bmatrix} n \\ k \end{Bmatrix} \begin{Bmatrix} n-k \\ j-k \end{Bmatrix} = \begin{Bmatrix} n \\ n-k \end{Bmatrix} \begin{Bmatrix} n-k \\ n-j \end{Bmatrix} = \begin{Bmatrix} n \\ n-j \end{Bmatrix} \begin{Bmatrix} j \\ j-k \end{Bmatrix} = \begin{Bmatrix} n \\ j \end{Bmatrix} \begin{Bmatrix} j \\ k \end{Bmatrix}$$

Hence we obtain from (20.33)

$$\sum_j \{p\}_{n-j} \begin{Bmatrix} n \\ j \end{Bmatrix} Q^{(a-j)\cdot(n-j)} R^{(p-n+j)\cdot j} \sum_k (-1)^k \begin{Bmatrix} j \\ k \end{Bmatrix} \{a\}_k \{a\}_{j-k} (QR)^{-k\cdot(j-k)} =$$

$$\sum_j \{p\}_{n-2j} \begin{Bmatrix} n \\ 2j \end{Bmatrix} \{2a,2\}_j \{-1,2\}_j Q^{(a-2j)\cdot(n-2j)} R^{(p-n+2j)\cdot 2j}$$

In the case of $p \geq 0$, we can write

$$\{p\}_{n-2j} \begin{Bmatrix} n \\ 2j \end{Bmatrix} = \begin{Bmatrix} p \\ n-2j \end{Bmatrix} \{n\}_{n-2j}$$

and note, that

$$\begin{Bmatrix} p \\ n-2j \end{Bmatrix} = 0 \text{ for } j < \frac{n-p}{2}$$

In the case of negative p, we apply the above formula for $a+p$ and $-p$. □

The balanced Kummer identity.

Theorem 20.6.

For arbitrary a and arbitrary integer $n \geq 0$

(20.37)

$$\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} \{n+a\}_k \{n-a\}_{n-k} (-1)^k (QR)^{k\cdot(k-n-a)} = \{n-a-1,2\}_n \cdot \frac{\{2n,2\}_n}{\{n\}_n}$$

In order to prove this theorem we need a 2–step version of the corollary (20.27).

Lemma 20.1. *For arbitrary c and arbitrary nonnegative integer n*

$$\frac{\{2n,2\}_n}{\{2(c+n),2\}_{n+1}}=\sum_{k=0}^{n}\frac{\{2n,2\}_k}{\{2k,2\}_k}\cdot\frac{(-1)^k}{\{2(c+k)\}}\cdot(QR)^{k\cdot(k+1)}Q^{-2n\cdot(c+k)} \tag{20.38}$$

Proof. We consider the formula (20.32) for the universal constants Q^2 and R^2, in which case we shall denote the basic transformation with $\{\{\cdot\}\}$, i.e. e.g.

$$\{\{x\}\}=(Q^{2x}-R^{2x})/(Q^2-R^2)=\{2x\}/(Q+R) \tag{20.39}$$

Then we have

$$\{\{x\}\}_n=\{2x,2\}_n/(Q+R)^n \tag{20.40}$$

and hence,

$$\left\{\begin{Bmatrix}x\\n\end{Bmatrix}\right\}=\frac{\{\{x\}\}_n}{\{\{n\}\}_n}=\frac{\{2x,2\}_n}{\{2n,2\}_n} \tag{20.41}$$

Therefore we get

$$\frac{1}{\{\{c+n\}\}_{n+1}}=\frac{1}{\{\{n\}\}_n}\sum_{k=0}^{n}\left\{\begin{Bmatrix}n\\k\end{Bmatrix}\right\}\frac{(-1)^k}{\{\{c+k\}\}}(QR)^{k\cdot(k+1)}Q^{-2n\cdot(c+k)} \tag{20.42}$$

Substitution of (20.39–41) gives (20.38). □

Proof of theorem 20.6. Consider the sum

$$S:=\sum_k\begin{Bmatrix}n\\k\end{Bmatrix}\{n+a\}_k\{n-a\}_{n-k}(-1)^k(QR)^{k\cdot(k-n-a)} \tag{20.43}$$

We apply (20.13) and (20.12) to write

$$\{n+a\}_k(-1)^k=\{-a-n,-1\}_k\cdot(QR)^{k\cdot(n+a)-\binom{k}{2}}=\{-a-n-1+k\}_k\cdot(QR)^{k\cdot(n+a)-\binom{k}{2}}$$

the sum then looks

$$S=\sum_k\begin{Bmatrix}n\\k\end{Bmatrix}\{n-a\}_{n-k}\{-a-n-1+k\}_k\cdot(QR)^{\binom{k+1}{2}}$$

Now we apply (20.14) to write

$$\{n-a\}_{2n+1}=\{n-a\}_{n-k}\{-a+k\}_{n+1}\{-a-n-1+k\}_k$$

The sum is then written

$$S = \{n-a\}_{2n+1} \sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} \frac{(QR)^{\binom{k+1}{2}}}{\{-a+k\}_{n+1}}$$

Now we apply corollary (20.1) on

$$\frac{1}{\{-a+k\}_{n+1}} = \frac{1}{\{n\}_n} \sum_j \begin{Bmatrix} n \\ j \end{Bmatrix} \frac{(-1)^j}{\{-a-n+k+j\}} \cdot (QR)^{\binom{j+1}{2}} \cdot Q^{-n(-a-n+k+j)}$$

Substitution of $j = i - k$ in the sum, and the sum in the sum above, yields

$$S = \{n-a\}_{2n+1} \sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} \frac{1}{\{n\}_n} \sum_i \begin{Bmatrix} n \\ i-k \end{Bmatrix} \frac{(-1)^{i-k}}{\{-a-n+i\}} \cdot$$
$$(QR)^{\binom{k+1}{2}+\binom{i-k+1}{2}} \cdot Q^{-n(-a-n+i)}$$

Interchanging the order of summation gives

$$S = \frac{\{n-a\}_{2n+1}}{\{n\}_n} \sum_i \frac{(-1)^i}{\{-a-n+i\}} \cdot Q^{-n(-a-n+i)} \cdot (QR)^{\binom{i+1}{2}} \cdot$$
$$\sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} \begin{Bmatrix} n \\ i-k \end{Bmatrix} (-1)^k \cdot (QR)^{k^2 - i\cdot k}$$

We use the definition of the binomial coefficients, (20.18), to recognize the second sum as an example of the symmetric Kummer expression, (20.33–34). Hence the odd sums vanishes, and the even sums can be written with $2j$ substituted for i according to (20.33).

$$\begin{aligned}
&\sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} \begin{Bmatrix} n \\ i-k \end{Bmatrix} (-1)^k \cdot (QR)^{k^2 - i\cdot k} = \\
&\frac{1}{\{i\}_i} \sum_k \begin{Bmatrix} i \\ k \end{Bmatrix} \{n\}_k \{n\}_{i-k} (-1)^k \cdot (QR)^{-k\cdot(i-k)} = \\
&\frac{1}{\{2j\}_{2j}} \sum_k \begin{Bmatrix} 2j \\ k \end{Bmatrix} \{n\}_k \{n\}_{2j-k} (-1)^k \cdot (QR)^{-k\cdot(2j-k)} = \\
&\frac{1}{\{2j\}_{2j}} (-1)^j \{2n,2\}_j \{2j-1,2\}_j \cdot (QR)^{-j^2} = \\
&\frac{\{2n,2\}_j}{\{2j,2\}_j} \cdot (-1)^j \cdot (QR)^{-j^2}
\end{aligned}$$

With this reduction we obtain

$$S = \frac{\{n-a\}_{2n+1}}{\{n\}_n} \sum_j \frac{\{2n,2\}_j}{\{2j,2\}_j} \frac{(-1)^j}{\{-a-n+2j\}} \cdot (QR)^{j^2+j} \cdot Q^{-n(-a-n+2j)}$$

On this sum we may apply lemma (20.1) with $c = \frac{-a-n}{2}$. That proves the theorem.
$\square$

The quasi–balanced Kummer identity.

Theorem 20.7.

For arbitrary a and arbitrary integers $n \geq m \geq 0$ and p

(20.44)

$$\sum_{k=m}^{n} \begin{Bmatrix} n-m \\ k-m \end{Bmatrix} \{n-m+a-p\}_{k-m}\{n-m-a\}_{n-k}(-1)^{k-m} \cdot (QR)^{(k-m)\cdot(k-n-a)} Q^{(k-m)\cdot p} =$$

$$= \begin{cases} \dfrac{\{2n-2m-2p,2\}_{n-m-p}}{\{n-m-p\}_{n-m-p}} \cdot & \text{for } p \geq 0 \\ \cdot \displaystyle\sum_{j=0}^{p} \begin{Bmatrix} p \\ j \end{Bmatrix} \{n-m-a-1+j,2\}_{n-m} \cdot Q^{(p-j)\cdot(n-m-p)+\binom{p+1-j}{2}} R^{-p\cdot j+\binom{j+1}{2}} & \\ \dfrac{\{2n-2m,2\}_{n-m}}{\{n-m-p\}_{n-m-p}} R^{-p\cdot(a+n-m)+\binom{1-p}{2}} \cdot & \text{for } p < 0 \\ \cdot \displaystyle\sum_{j=0}^{-p} \begin{Bmatrix} -p \\ j \end{Bmatrix} \cdot (-1)^j \cdot \{n-m-p-a-1-j,2\}_{n-m-p} \cdot Q^{j\cdot(n-m)+\binom{j+1}{2}} \cdot R^{\binom{j}{2}} & \end{cases}$$

Proof. It is enough to prove the theorem for $m = 0$.

1) $p \geq 0$:

Let

$$S(n,p,a) = \sum_{k} \begin{Bmatrix} n \\ k \end{Bmatrix} \{n+a-p\}_k \{n-a\}_{n-k} (-1)^k Q^{k\cdot(k-n-a+p)} R^{k\cdot(k-n-a)} \tag{20.45}$$

We split this sum in two by (4.2) applied to

$$\{n-a\}_{n-k} = \{n-a-1\}_{n-k} R^{n-k} + \{n-k\}\{n-a-1\}_{n-k-1} Q^{k-a} \tag{20.46}$$

Then we get the following two sums

(20.47)

$$S(n,p,a) = \sum_{k} \begin{Bmatrix} n \\ k \end{Bmatrix} \{n-p+a\}_k \{n-a-1\}_{n-k} (-1)^k \cdot Q^{k\cdot(k-n+p-a)} R^{k\cdot(k-n-a-1)} R^n$$

$$+\{n\} \sum_{k} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \{n-p+a\}_k \{n-a-1\}_{n-k-1} (-1)^k \cdot Q^{k\cdot(k-n+p-a+1)-a} R^{k\cdot(k-n-a)}$$

And we split the sum of $S(n,p,a+1)$ by applying (20.9) to

$$\{n-p+a+1\}_k = \{n-p+a\}_k Q^k + \{k\}\{n-p+a\}_{k-1} R^{n-p+a+1-k} \tag{20.48}$$

Then we get the following two sums

(20.49)

$$\begin{aligned}
S(n,p,a+1) = &\sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} \{n-p+a\}_k \{n-a-1\}_{n-k} (-1)^k \cdot \\
&Q^{k\cdot(k-n+p-a)} R^{k\cdot(k-n-a-1)} \\
&+ \{n\} \sum_k \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \{n-p+a\}_{k-1} \{n-a-1\}_{n-k} (-1)^k \cdot \\
&Q^{k\cdot(k-n+p-a-1)} R^{(k-1)\cdot(k-n-a-1)-p} \\
&= S(n,p+1,a+1) + \\
&- \{n\} \sum_k \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \{n-p+a\}_k \{n-a-1\}_{n-k-1} (-1)^k \cdot \\
&Q^{k\cdot(k-n+p-a+1)-a} R^{k\cdot(k-n-a)} Q^{p-n} R^{-p}
\end{aligned}$$

The two second sums are proportional by the factor $Q^{p-n}R^{-p}$, and the two first sums are both proportional to $S(n,p+1,a+1)$.

By eliminating the two second sums we obtain the formula

$$(Q^{n-p} + R^{n-p}) S(n,p+1,a+1) = R^{-p} S(n,p,a) + Q^{n-p} S(n,p,a+1) \tag{20.50}$$

Iteration of this formula m times yields the expression

(20.51)

$$\begin{aligned}
&\prod_{j=1}^{m} \left(Q^{n-p+j} + R^{n-p+j}\right) \cdot S(n,p,a) = \\
&\sum_i \begin{Bmatrix} m \\ i \end{Bmatrix} Q^{\sum_{j=1}^{i}(n+j-p)} R^{\sum_{j=1}^{m-i}(j-p)} S(n,p-m,a-m+i)
\end{aligned}$$

For $m=0$ we get a trivial identity, and for $m=1$ we get (20.50) for $p-1$ and $a-1$.

We prove (20.51) by induction after m. We apply (20.50) to $S(n,p-m,a-m+i)$ and obtain

(20.52)

$$\prod_{j=1}^{m+1}\left(Q^{n-p+j}+R^{n-p+j}\right)\cdot S(n,p,a)=$$

$$\sum_i \left\{\begin{matrix} m \\ i \end{matrix}\right\} Q^{\sum_{j=1}^{i}(n+j-p)} R^{\sum_{j=1}^{m-i}(j-p)} R^{m+1-p} S(n,p-m-1,a-m-1+i)$$

$$+\sum_i \left\{\begin{matrix} m \\ i \end{matrix}\right\} Q^{\sum_{j=1}^{i}(n+j-p)} R^{\sum_{j=1}^{m-i}(j-p)} Q^{n+m+1-p} S(n,p-m-1,a-m+i)=$$

$$\sum_i \left\{\begin{matrix} m \\ i \end{matrix}\right\} Q^{\sum_{j=1}^{i}(n+j-p)} R^{\sum_{j=1}^{m-i}(j-p)} R^{m+1-p} S(n,p-m-1,a-m-1+i)$$

$$+\sum_i \left\{\begin{matrix} m \\ i-1 \end{matrix}\right\} Q^{\sum_{j=1}^{i-1}(n+j-p)} R^{\sum_{j=1}^{m-i+1}(j-p)} Q^{n+m+1-p}.$$

$$S(n,p-m-1,a-m-1+i)=$$

$$\left(\sum_i Q^{\sum_{j=1}^{i}(n+j-p)} R^{\sum_{j=1}^{m+1-i}(j-p)} S(n,p-m-1,a-m-1+i)\right)\cdot$$

$$\left(\left\{\begin{matrix} m \\ i \end{matrix}\right\} R^i+\left\{\begin{matrix} m \\ i-1 \end{matrix}\right\} Q^{m+1-i}\right)=$$

$$\sum_i \left\{\begin{matrix} m+1 \\ i \end{matrix}\right\} Q^{\sum_{j=1}^{i}(n+j-p)} R^{\sum_{j=1}^{m+1-i}(j-p)} S(n,p-m-1,a-m-1+i)$$

by substitution of $i+1$ for i and using (20.20).

Now we apply (20.52) to $m=p$, and rewrite the product using (20.16) as

$$\prod_{j=1}^{p}\left(Q^{n-p+j}+R^{n-p+j}\right)=\prod_{j=0}^{p-1}\left(Q^{n-j}+R^{n-j}\right)=\frac{\{2n,2\}_p}{\{n\}_p} \tag{20.53}$$

Furthermore we reverse the order of summation and apply (20.37) to $S(n,0,a-i)$. This yields

(20.54)

$$S(n,p,a)=\frac{\{n\}_p}{\{2n,2\}_p}\cdot\sum_i \left\{\begin{matrix} p \\ i \end{matrix}\right\} Q^{(p-i)\cdot(n-p)+\binom{p+1-i}{2}} R^{-p\cdot i+\binom{i+1}{2}}.$$
$$\{n-a+i-1,2\}_n\frac{\{2n,2\}_n}{\{n\}_n}$$

Canceling common factors proves (20.44).

2) $p<0$:

Let

$$S(n,a,p,x) := \sum_k \left\{ {n \atop k} \right\} \{n+a-p\}_k \cdot \{n-a\}_{n-k} \cdot x^k \cdot (QR)^{k\cdot(k-n-a)} \cdot Q^{k\cdot p} \tag{20.55}$$

We split the sum $S(n+1,a,p+1,x)$ in two by (20.9) applied to

$$\{n+1-a\}_{n-k+1} = \{n-a\}_{n-k+1} \cdot Q^{n-k+1} + \{n-k+1\} \cdot \{n-a\}_{n-k} \cdot R^{k-a} \tag{20.56}$$

Then we get

(20.57)

$$\begin{aligned} &S(n+1,a,p+1,x) = \\ &\sum_k \left\{ {n+1 \atop k} \right\} \cdot \{n-p+a\}_k \cdot \{n-a\}_{n-k+1} \cdot (x \cdot R/Q)^k \cdot \\ &\qquad (QR)^{k\cdot(k-n-a-2)} \cdot Q^{k\cdot(p+2)+n+1} + \\ &\{n+1\} \cdot \sum_k \left\{ {n \atop k} \right\} \cdot \{n-p+a\}_k \cdot \{n-a\}_{n-k} \cdot x^k \cdot (QR)^{k\cdot(k-n-a)} \cdot Q^{k\cdot p} \cdot R^{-a} \\ &= S(n+1,a+1,p+2,x \cdot R/Q) \cdot Q^{n+1} + S(n,a,p,x) \cdot \{n+1\} \cdot R^{-a} \end{aligned}$$

And we split $S(n+1,a+1,p+1,x)$ by applying (20.9) to

$$\{n-p+a+1\}_k = \{n-p+a\}_k \cdot R^k + \{k\} \cdot \{n-p+a\}_{k-1} \cdot Q^{k\cdot p} \tag{20.58}$$

Then we get

(20.59)

$$\begin{aligned} &S(n+1,a+1,p+1,x) = \\ &\sum_k \left\{ {n+1 \atop k} \right\} \cdot \{n-p-a\}_k \cdot \{n-a\}_{n-k+1} \cdot x^k \cdot (QR)^{k\cdot(k-n-a-1)} \cdot Q^{k\cdot p} + \\ &\{n+1\} \cdot \sum_k \left\{ {n \atop k-1} \right\} \cdot \{n-p+a\}_{k-1} \cdot \{n-a\}_{n-k+1} \cdot x^k \cdot \\ &\qquad (QR)^{k\cdot(k-n-a-2)} \cdot Q^{k\cdot p+n-p+a+1} \\ &= S(n+1,a+1,p+2,x \cdot R/Q) + \\ &\{n+1\} \cdot \sum_k \left\{ {n \atop k} \right\} \cdot \{n-p+a\}_k \cdot \{n-a\}_{n-k} \cdot x^{k+1} \cdot \\ &\qquad (QR)^{k\cdot(k-n-a)} \cdot Q^{k\cdot p} \cdot R^{-n-a-1} \\ &= S(n+1,a+1,p+2,x \cdot R/Q) + S(n,a,p,x) \cdot x \cdot \{n+1\} \cdot R^{-n-a-1} \end{aligned}$$

By elimination of $S(n+1,a+1,p+2,x \cdot R/Q)$ we obtain the formula

(20.60)

$$\begin{aligned} &(R^{-a} \cdot \{n+1\} - x \cdot Q^{n+1} \cdot R^{-a-n-1} \cdot \{n+1\}) \cdot S(n,a,p,x) \\ &= S(n+1,a,p+1,x) - Q^{n+1} \cdot S(n+1,a+1,p+1,x) \end{aligned}$$

which can be rewritten as
(20.61)
$$S(n,a,p,x)=\frac{S(n+1,a,p+1,x)-Q^{n+1}\cdot S(n+1,a+1,p+1,x)}{\{n+1\}\cdot(R^{n+1}-x\cdot Q^{n+1})}\cdot R^{a+n+1}$$

For $x=-1$ we have $\{n+1\}\cdot(Q^{n+1}+R^{n+1})=\{2\cdot(n+1)\}$ so we obtain
(20.62)
$$S(n,a,p,-1)=\frac{S(n+1,a,p+1,-1)-Q^{n+1}\cdot S(n+1,a+1,p+1,-1)}{\{2\cdot n+2\}}\cdot R^{a+n+1}$$

Iteration of this formula m times yields the expression

(20.63)
$$S(n,a,p,-1)=$$
$$\frac{R^{m\cdot(a+m)+\binom{m+1}{2}}}{\{2\cdot(n+m),2\}_m}\sum_i \left\{\begin{matrix} m\\ i\end{matrix}\right\}\cdot(-1)^i\cdot Q^{i\cdot n+\binom{i+1}{2}}\cdot R^{\binom{i}{2}}\cdot S(n+m,a+i,p+m,-1)$$

For $m=0$ (20.63) is a trivial identity, and for $m=1$ we get (20.62).

We prove (20.63) by induction with respect to m. We apply (20.62) to $S(n+m,a+i,p+m,-1)$ and obtain

(20.64)
$$S(n,a,p,-1)=\frac{R^{m\cdot(a+n)+\binom{m+1}{2}}}{\{2\cdot(n+m),2\}_m}\sum_i \left\{\begin{matrix} m\\ i\end{matrix}\right\}\cdot(-1)^i\cdot Q^{i\cdot n+\binom{i+1}{2}}\cdot R^{\binom{i}{2}+a+n+m+i+1}\cdot$$
$$\frac{S(n+m+1,a+i,p+m+1,-1)-Q^{n+m+1}\cdot S(n+m+1,a+i+1,p+m+1,-1)}{\{2\cdot(n+m),2\}}$$
$$=\frac{R^{(m+1)\cdot(a+n)+\binom{m+2}{2}}}{\{2\cdot(n+m+1),2\}_{m+1}}\cdot\sum_i S(n+m+1,a+i,p+m+1,-1)\cdot(-1)^i\cdot$$
$$\left(\left\{\begin{matrix} m\\ i\end{matrix}\right\}\cdot Q^{i\cdot n+\binom{i+1}{2}}\cdot R^{\binom{i}{2}+i}+\left\{\begin{matrix} m\\ i-1\end{matrix}\right\}\cdot Q^{i\cdot n+\binom{i}{2}+m+1}\cdot R^{\binom{i}{2}}\right)$$
$$=\frac{R^{(m+1)\cdot(a+n)+\binom{m+2}{2}}}{\{2\cdot(n+m+1),2\}_{m+1}}\cdot\sum_i S(n+m+1,a+i,p+m+1,-1)\cdot(-1)^i\cdot$$
$$Q^{i\cdot n+\binom{i+1}{2}}\cdot R^{\binom{i}{2}}\cdot\left\{\begin{matrix} m+1\\ i\end{matrix}\right\}$$

Choosing $m = -p$ in (20.63) and using the formula (20.37) we obtain

(20.65)

$$
\begin{aligned}
&S(n,a,p,-1) = \\
&\frac{R^{-p\cdot(a+n)+\binom{1-p}{2}}}{\{2\cdot(n-p),2\}_{-p}}\cdot\sum_i \begin{Bmatrix}-p\\ i\end{Bmatrix}\cdot(-1)^i\cdot Q^{i\cdot n+\binom{i+1}{2}}\cdot R^{\binom{i}{2}}\cdot S(n-p,a+i,0,-1) \\
&= \frac{R^{-p\cdot(a+n)+\binom{1-p}{2}}}{\{2\cdot(n-p),2\}_{-p}}\cdot\sum_i \begin{Bmatrix}-p\\ i\end{Bmatrix}\cdot(-1)^i\cdot Q^{i\cdot n+\binom{i+1}{2}}\cdot R^{\binom{i}{2}}. \\
&\{n-p-a-1-i,2\}_{n-p}\cdot\frac{\{2\cdot(n-p),2\}_{n-p}}{\{n-p\}_{n-p}} \\
&= \frac{R^{-p\cdot(a+n)+\binom{1-p}{2}}\cdot\{2\cdot n,2\}_n}{\{n-p\}_{n-p}}\cdot \\
&\sum_i \begin{Bmatrix}-p\\ i\end{Bmatrix}\cdot(-1)^i\cdot Q^{i\cdot n+\binom{i+1}{2}}\cdot R^{\binom{i}{2}}\cdot\{n-p-a-1-i,2\}_{n-p}
\end{aligned}
$$

This completes the proof. □

A basic transformation of a II$(2,2,z)$ sum.

Theorem 20.8. *For arbitrary a, b, x and y and nonnegative integers n and m, we have*

(20.66)

$$
\begin{aligned}
&\sum_{k=m}^{n}\begin{Bmatrix}n-m\\ k-m\end{Bmatrix}\{a\}_{k-m}\{b\}_{n-k}x^{k-m}y^{n-k}(QR)^{-(k-m)\cdot(n-k)} \\
&= \sum_{k=m}^{n}\begin{Bmatrix}n-m\\ k-m\end{Bmatrix}\{n-m-a-b-1\}_{k-m}\{b\}_{n-k}(-x)^{k-m} \\
&\cdot Q^{(k-m)\cdot(a-n+m+1)+\binom{k-m}{2}}R^{(k-m)\cdot(b-n+k)-\binom{n-m}{2}+a\cdot(n-m)}\cdot\prod_{j=0}^{n-k-1}(y\cdot R^{j-a}-x\cdot Q^{j-b})
\end{aligned}
$$

(20.67)

$$
\begin{aligned}
&\sum_{k=m}^{n}\begin{Bmatrix}n-m\\ k-m\end{Bmatrix}\{a\}_{k-m}\{b\}_{n-k}x^{k-m}y^{n-k}(QR)^{-(k-m)\cdot(n-k)} \\
&= \sum_{k=m}^{n}\begin{Bmatrix}n-m\\ k-m\end{Bmatrix}\{a\}_{k-m}\{n-m-a-b-1\}_{n-k}(-y)^{n-k} \\
&\cdot Q^{(n-k)\cdot(b-n+m+1)+\binom{n-k}{2}}R^{(n-k)\cdot(a-k+m)-\binom{n-m}{2}+b\cdot(n-m)}\cdot\prod_{j=0}^{k-m-1}(x\cdot R^{j-b}-y\cdot Q^{j-a})
\end{aligned}
$$

or, equivalently,

(20.68)

$$\sum_{k=m}^{n}\left\{{n-m \atop k-m}\right\}\{a\}_{k-m}\{b\}_{n-k}x^{k-m}y^{n-k}\cdot(QR)^{-(k-m)\cdot(n-k)}\cdot Q^{b\cdot(k-m)}\cdot R^{a\cdot(n-k)}$$

$$=\sum_{k=m}^{n}\left\{{n-m \atop k-m}\right\}\{n-m-a-b-1\}_{k-m}\{b\}_{n-k}(-x)^{k-m}$$

$$\cdot(QR)^{-(k-m)\cdot(n-k-b)}\cdot Q^{a\cdot(k-m)-\binom{k-m}{2}}\cdot R^{a\cdot(n-m)-\binom{n-m}{2}}\cdot\prod_{j=0}^{n-k-1}(y\cdot R^{j}-x\cdot Q^{j})$$

The form (20.66) is symmetric in Q and R, and reversal of the direction of summation corresponds to the exchange of a with b and x with y.

Proof. It is sufficient to prove (20.66) for $m=0$. In the left hand expression we substitute the formula (20.11) in the form

$$\{a\}_{k}=(-1)^{k}\cdot\{-a+k-1\}_{k}(QR)^{k\cdot a-\binom{k}{2}}$$

Next we apply the Chu–Vandermonde (6.1) on the factorial to get

$$\{-a+k-1\}_{k}=\sum_{j}\left\{{k \atop j}\right\}\{n-a-b-1\}_{j}\{b-n+k\}_{k-j}Q^{(n-a-b-1-j)(k-j)}R^{(b-n+j)j}$$

Substitution in the left hand expression yields after exchanging the order of summation

$$\sum_{j}\{n-a-b-1\}_{j}(-x)^{j}\sum_{k}\left\{{n \atop k}\right\}\left\{{k \atop j}\right\}\{b\}_{n-k}\{b-n+k\}_{k-j}\cdot(-x)^{k-j}y^{n-k}$$

$$\cdot\, Q^{-k(n-k)+ka-\binom{k}{2}+(n-a-b-1-j)(k-j)}R^{-k(n-k)+ka-\binom{k}{2}+(b-n+j)j}$$

Now we apply the formula (20.21) on

$$\left\{{n \atop k}\right\}\left\{{k \atop j}\right\}=\left\{{n \atop j}\right\}\left\{{n-j \atop k-j}\right\}$$

and the formula (20.14) on

$$\{b\}_{n-k}\{b-n+k\}_{k-j}=\{b\}_{n-j}$$

to obtain the expression

$$\sum_{j}\left\{{n \atop j}\right\}\{n-a-b-1\}_{j}\{b\}_{n-j}Q^{(a-n+1)j+\binom{j}{2}}R^{(b-n+j)j-\binom{n}{2}+a\cdot n}.$$

$$\cdot(-x)^{j}\cdot\sum_{k}\left\{{n-j \atop k-j}\right\}\cdot(-x)^{k-j}y^{n-k}\cdot Q^{\binom{k-j}{2}-b\cdot(k-j)}R^{\binom{n-k}{2}-a\cdot(n-k)}$$

Now we apply the basic binomial theorem (20.24) to obtain (20.66).

The formula (20.67) follows from (20.66) by reversing the order of summation, exchanging a with b and x with y.

Substitution of $x := x \cdot Q^b/R^a$ in (20.66) and multiplication with $R^{a\cdot(n-m)}$ gives (20.68). □

A basic Gauß' theorem.

Theorem 20.9. *For arbitrary a and integer p, and nonnegative integers $m \le n$ we have*

(20.69)

$$\sum_{k=m}^{n} \left\{ {n-m \atop k-m} \right\} \{2a,2\}_{k-m}\{n-m-p-2a-1\}_{n-k}\cdot$$

$$\cdot\, (QR)^{k^2+(n-2k)\cdot(a+m)+n\cdot p-\binom{n}{2}} Q^{-k\cdot(p+1)+m\cdot a+\binom{m+1}{2}} R^{-n\cdot k+\binom{k}{2}+n\cdot(a+m)-p\cdot m} =$$

$$(-\sigma(p))^{n-m} \sum_{j=\lceil \frac{n-m-|p|}{2}\rceil}^{\lfloor \frac{n-m}{2}\rfloor} \left\{ {|p| \atop n-m-2j} \right\} \{n-m\}_{n-m-2j}\{2(a+(p\wedge 0)),2\}_j\{-1,2\}_j\cdot$$

$$Q^{(a+(p\wedge 0)-2j)\cdot(n-m-2j)} R^{(|p|-n+m+2j)\cdot 2j+(n-m)\cdot(p\wedge 0)}$$

Proof. We substitute $b := a+p$ in the right side of (20.67). Substitution of $x := -R^p$ and $y := 1$ gives the left side of (20.36), so we can apply the quasi–symmetric Kummer formula. Eventually we apply (20.16) to the product.

□

A basic Bailey's theorem.

Theorem 20.10. *For arbitrary a and integer p, and nonnegative integers $m \le n$ we have*

(20.70)

$$\sum_{k=m}^{n} \frac{\{2(n-m),2\}_{k-m}}{\{k-m\}_{k-m}} \{a\}_{k-m}\{m-n+p-1\}_{n-k}\cdot$$

$$\cdot\, (QR)^{(n-m)\cdot(n-a-k)-n\cdot(p+k)+\binom{k-m}{2}+\binom{n+1}{2}} Q^{k\cdot p+\binom{k}{2}} R^{k\cdot(k-m)+a\cdot(n-k)+m\cdot p+\binom{m}{2}} =$$

$$\begin{cases} (-1)^{n-m}\dfrac{\{2(n-m-p),2\}_{n-m-p}}{\{n-m-p\}_{n-m-p}}\cdot & \textit{for } p\ge 0 \\ \cdot\displaystyle\sum_{j=0}^{p}\left\{ {p \atop j} \right\}\{2(n-m)-p-a-1+j,2\}_{n-m}\cdot Q^{(p-j)\cdot(n-m-p)+\binom{p+1-j}{2}} R^{-p\cdot j+\binom{j+1}{2}} & \\ (-1)^{n-m}\dfrac{\{2(n-m),2\}_{n-m}}{\{n-m-p\}_{n-m-p}} R^{-p\cdot(a+p)+\binom{1-p}{2}}\,\cdot & \textit{for } p<0 \\ \cdot\displaystyle\sum_{j=0}^{-p}\left\{ {-p \atop j} \right\}\cdot(-1)^j\cdot\{2(n-m-p)-a-p-1-j,2\}_{n-m-p}\cdot Q^{j\cdot(n-m)+\binom{j+1}{2}}\cdot R^{\binom{j}{2}} & \end{cases}$$

Proof. We substitute $b := 2(n-m) - a - p$ in the right side of (20.67) and chose $x := -(QR)^{n-m-a} \cdot R^{-p}$ and $y := 1$. Then we apply (20.16) on the product, and we apply the quasi–balanced Kummer identity (20.44) with $a := a + p - n + m$.

□

Notes.

In the ordinary – non–basic – case the Binomial theorem and the Chu–Vandermonde convolution are very well known results [29,112]. Theorem 8.1 – the Chu–Vandermonde convolution with arbitrary step length – is also a generalization of the Binomial theorem. We have neither seen this nor the generalization to commutative rings before.

The symmetric Kummer identity and the balanced Kummer identity are special cases of Kummer's theorem for Gaußian hypergeometric series [112], theorem 8.3,

$$
\begin{aligned}
{}_2F_1[a, b; 1 + a - b; -1] &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (-1)^n}{n!(1 + a - b)_n} \\
&= 2^{-a} \frac{\Gamma(\frac{1}{2})\Gamma(1 + a - b)}{\Gamma(\frac{1+a}{2})\Gamma(1 + \frac{a}{2} - b)} \\
&= \frac{\Gamma(1 + a - b)\Gamma(1 + \frac{a}{2})}{\Gamma(1 + a)\Gamma(1 + \frac{a}{2} - b)}
\end{aligned}
$$

The first of the two evaluations of ${}_2F_1[a, b; 1 + a - b; -1]$ yields the symmetric Kummer identity if we substitute $-n$ for a. The second evaluation yields the balanced Kummer identity if we substitute $-n$ for b.

We believe that the quasi–symmetric and the quasi–balanced Kummer identities are new, although special cases with numerically small values of the parameter p may be found e.g. in H. W. Gould's table of Combinatorial Identities, [64].

Special cases of the transformation formula in section 13 can be found in Gould [64]. The general form, however, we have not seen before.

Using this transformation formula we have the Gauß identity and the Bailey identity almost as corollaries of the quasi–symmetric and the quasi–balanced Kummer identity.

When the parameter p vanishes the Gauß identity is a special case of Gauß' evaluation of

$$
{}_2F_1[a, b; \tfrac{1+a+b}{2}; \tfrac{1}{2}] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1+a+b}{2})}{\Gamma(\frac{1+a}{2})\Gamma(\frac{1+b}{2})}
$$

and Bailey's identity is a special case of Bailey's evaluation of

$$
{}_2F_1[a, 1 - a; b; \tfrac{1}{2}] = \frac{\Gamma(\frac{b}{2})\Gamma(\frac{1+b}{2})}{\Gamma(\frac{a+b}{2})\Gamma(\frac{1-a+b}{2})}
$$

The basic binomial theorem is due to Cauchy [26] and Heine [75]. Heine also proved the basic version of Gauß' summation formula for hypergeometric series [75].

The basic version of Kummer's theorem is due to Bailey [21] and Daum [30].

The basic version of Gauß' second theorem and Bailey's theorem are due to Andrews [19].

Further references may be found in G. Gasper and M. Rahman, [44].

In 1921 F. H. Jackson, [78], [105], p. 94, proved a basic version of Dougall's theorem.

INDEX

Abel, N. H. 148, 152,
Abel–Jensen identity 152
Abelian summation 15
Alkan, E. 43
Al–Karajī 61
Al–Salam, W. A. 123
Andrews, G. 203
anti–difference 13
arbitrary limits 49
argument 46
Bailey, W. N. 77, 201 ff
Bailey's formula 77, 83, 201, 203
balanced 51, 99, 120, 191
Bang, S.–J. 175
basic 183
Bernoulli 18 ff., 37
binomial coefficient 5, 11, 184
binomial formula 61, 185
canonical 47
Cauchy, A.–L. 148, 149, 203
Cauchy–Jensen formula 149
Cayley–Hamilton 35
cealing 11
Chen, K.–W. 37
Chu, S.–S. 64
Chu–Vandermonde 64, 186 ff.
classification 46
Daum, J. A. 203
Davis, H. T. 68
definite sum 13, 15, 46
difference equations 29
Dixon, A. C. 93
Dixon's formula 93, 98, 100, 175
Dougall, J. 123, 128, 203
Dougall's formula 128
Doster, D. 40
Doyle, A. C. 68
Ekhad, S. B. 40
excess 75, 96, 102, 104, 111, 114
factorial 11, 13, 184
Faulhaber 17
Fjeldstad, J. E. 170
floor 11
Gauß C. F. 16, 78, 183, 201 ff.
Gauß ́formula 78 ff., 201
generalized harmonic 12
generating function 36
Gessel, I. M. 42, 95, 127 142 ff.
GKP, Graham, Knuth etc. 137, 167
Golombek, R. 25
Gosper, R. W. 5, 51, 55, 58, 95, 138 ff.
Gosper's algorithm 51, 55
Gould, H. W. 68 ff., 111, 117, 123, 148 ff., 202
Grosjean, C. C. 108
Hagen, J. G. 5, 148
Hagen–Rothe formula 148
Hagen–Rothe–Jensen formula 154
Hardy, G. H. 125
harmonic 12, 167 ff., 182
Heine, E. 183 ff, 203
homogeneous equations 29
hypergeometric 50
identity 12
indefinite sum 12, 46, 182
inhomogeneous equations 31
Jackson, F. H. 203

Jensen, J. L. W. V. 148 ff.
Kaucký, J. 166
Koornwinder, T. H. 137
Krall, H. L. 117
Kummer, E. E. 77, 188 ff.
Kummer's formula 77, 188 ff.
Kvamsdal, J. 170
Laguerre polynomials 67
Larcombe, P. 110, 177
length 11
Ljunggren, W. 170
maximum 11
mechanical summation 55 ff., 137 ff.
minimum 11
Moriarty 68
natural limit 48
parameter 47
Pfaff, J. F. 92
Pfaff–Saalschütz formula 92 ff.
polynomial coefficients 38
polynomial factors 54
quasi 52
Ramanujan, S. 125
Riordan, J. 166
Rothe, H. A. 148 ff.
Saalschütz, L. 92
Santmyer, J. M. 73
shift 12
sign 11
Slater, L. J. 95, 115 ff., 123, 132
Stanton, D. 95
Staver, T. B. 61
stepsize 11
Stirling 23
symmetric 51
systems of equations 33
Tian–Ming, W. 104
transformations of type II$(2, 2, z)$ 75
transformations of type II$(3, 3, 1)$ 92
type 46
Vandermonde, A. T. 64, 186
Watson, G. N. 92
Watson's formula 99, 102
well–balanced 5, 52
Whipple, F. J. W. 92
Whipple's formula 103
Wilf, H. 137
Wronskian 29
Xin–Rong, M. 104
Zeilberger, D. 137
Zeilberger's algorithm 137 ff

REFERENCES

1. Niels Henrik Abel, *Démonstration d'une expression de laquelle la formule binome est un cas particulier, dans: Œuvres complètes Rédigées par ordre du roi, par B. Holmboe, Tome Premier,* Chr. Gröndahl, Christiania, 1839, 31–32.
2. Milton Abramowitz and Irene A. Stegun (ed.), *Handbook of Mathematical Functions,* Dover Publications, Inc., New York, 1965, 1046 p.
3. Emre Alkan, problem no. 10473, *Amer. Math. Monthly* no. 102, 8, October (1995) 745.
4. Gert Almkvist, Zeilbergers bevismaskin, *Elementa* 72 (1989) 137–142.
5. Gert Almkvist and Doron Zeilberger, The Method of Differentiating under the Integral Sign, *J. Symbolic Computation* 10 (1990) 571–591.
6. W. A. Al–Salam, Note on a q–Identity, *Math. Scand.* 5 (1957) 202–204.
7. Erik Sparre Andersen, Classification of combinatorial identities, *KUMI Preprint Series* 26 (1989) 21.
8. Erik Sparre Andersen and Mogens Esrom Larsen, Algebraic Summation Identities I, *KUMI Preprint Series* 5 (1991) 29.
9. Erik Sparre Andersen and Mogens Esrom Larsen, Algebraic Summation Identities II, *KUMI Preprint Series* 10 (1991) 11.
10. Erik Sparre Andersen and Mogens Esrom Larsen, A generalization of Kummer's formula for hypergeometric series and its q–basic analogue, *KUMI Preprint Series* 13 (1991) 7.
11. Erik Sparre Andersen and Mogens Esrom Larsen, Combinatorial Summation Identities, *Proceedings from the 21. Nordic congress in Luleå 1992* (1994) 1–23 p.
12. Erik Sparre Andersen and Mogens Esrom Larsen, A Finite Sum of Products of Binomial Coefficients, Problem 92–18, by C. C. Grosjean, Solution, *SIAM Review* 35, 4 (1993) 645–646.
13. Erik Sparre Andersen og Mogens Esrom Larsen, Den nøgne sandhed om J. L. W. V. Jensens Identiteter, *Famøs* 7,3 (1994) 9–20.
14. Erik Sparre Andersen og Mogens Esrom Larsen, Rothe–Abel–Jensen identiteter, *Normat* 42, 3 (1994) 116–128.
15. Erik Sparre Andersen and Mogens Esrom Larsen, Problem no. 10466, *Am. Math. Monthly* 102,7 (1995) 654.
16. Erik Sparre Andersen and Mogens Esrom Larsen, Combinatorial Identities: A Generalization of Dougall's Identity, *Advances in Combinatorial Methods and Applications to Probability and Statistics In Honor of Sri Gopal Mohanty, Birkhäuser, Boston* (1997) 77–88.
17. Erik Sparre Andersen and Mogens Esrom Larsen, A Surprisingly Simple Summation Solution, Problem no. 10424, *Am. Math. Monthly* 104 (1997) 466–67.
18. George E. Andrews, On the q–analog of Kummer's theorem and applications, *Duke Math. J.* 40 (1973) 525–528.

19. George E. Andrews, Applications of basic hypergeometric functions, *SIAM Review* 16 (1974) 442–484.

20. W. N. Bailey, *Generalized Hypergeometric Series,* Cambridge University Press, London, 1935, 11.

21. W. N. Bailey, A note on certain q–identities, *Quart. J. Math.* 12 (1941) 173–175.

22. W. N. Bailey, Identities of the Rogers–Ramanujan type, *J. London Math. Soc. (2)* 50 (1949) 1–10.

23. Seung–Jin Bang, problem no. 10490, *Amer. Math. Monthly* 102 (1995) 930.

24. A. F. Beardon, Sums of Powers of Integers, *Amer. Math. Monthly* 103 (1996) 201–213.

25. L. Carlitz, Summation of a special ${}_4F_3$, *Boll. Un. Mat. Ital.* 18 (1963) 90–93.

26. Augustin–Louis Cauchy, Mémoire sur les fonctions dont plusieurs valeurs sont liées entre elles par une équation linéaire, et sur diverses transformations de produits composés d'un nombre indéfini de facteurs, *C. R. Acad. Sci. Paris* 17 (1843) 523–31.

27. Augustin–Louis Cauchy, *Application du Calcul des Résidus a la Sommation de Plusieurs Suites, dans: Œuvres complètes II^E Série Tome VI,* Gauthier–Villars, Paris, 1887, 72–73.

28. Kwang–Wu Chen, problem no. 10416, *Amer. Math. Monthly* 101, 9, November (1994) 912.

29. Shih–Chieh Chu, *Ssu Yuan Yü Chien (Precious Mirror of the Four Elements),* , China, 1303.

30. J. A. Daum, The basic analogue of Kummer's theorem, *Bull. Amer. Math. Soc.* 48 (1942) 711–713.

31. H. T. Davis, *The Summation of Series,* Principia Press, San Antonio, Texas, 1962.

32. A. C. Dixon, Summation of a certain series, *Proc. London Math. Soc.* 35 (1903) 285–289.

33. David Doster, problem no. 10403, *Amer. Math. Monthly* 101, October (1994) 792.

34. J. Dougall, On Vandermonde's theorem and some more general expansions, *Proc. Edin. Math. Soc.* 25 (1907) 114–132.

35. A. Conan Doyle, The adventure of the final problem, *The Strand Magazine* 6 (1883) 558–570.

36. A. W. F. Edwards, *Pascal's Arithmetical Triangle,* Charles Griffin & Co., Ltd., London, 1987, 51–52.

37. Shalosh B. Ekhad, problem no. 10356, *Amer. Math. Monthly* 101, January (1994) 75.

38. Saber N. Elaydi, *An Introduction to Difference Equations,* Springer Verlag, New York, 1995, 389 p.

39. Andreas v. Ettingshausen, *Die combinatorische Analysis als Vorbereitungslehre zum Studium der theoretischen höhern Mathematik,* I. B. Wallishausser, Wien, 1826, 350 p.

40. Jonas Ekman Fjeldstad, A generalization of Dixon's formula, *Math. Scand.* 2 (1954) 46–48.

41. Jonas Ekman Fjeldstad, Løste Oppgaver 4, *Norsk Mat. Tidss.* 30 (1948) 94–95.

42. D. Foata, Une démonstration combinatoire de l'identité de Pfaff–Saalschütz, *C.R. Acad. Sci. Paris Sér. I Math.* 297, no.4 (1983) 221–224.

43. Gregory Galperin and Hillel Gauchman, Problem no. 11103, *Amer. Math. Monthly* 111,8 (2004) 725.

44. George Gasper and Mizan Rahman, *Basic Hypergeometric Series, Encyclopedia of Mathematics and its Applications, vol. 35,* Cambridge University Press, Cambridge, 1990, 249–276.

45. C. F. Gauß, Disquisitiones generales circa seriem infinitam..., *Comm. soc. reg. sci. Gött. rec.* II (1813) 123–162.

46. Ira M. Gessel and Dennis Stanton, Strange Evaluations of Hypergeometric Series, *SIAM J. Math. Anal.* 13 (1982) 295–308.

47. Ira M. Gessel, problem no. 10424, *Amer. Math. Monthly* no. 102, January (1995) 70.

48. Ira M. Gessel, Finding Identities with the WZ Method, *J. Symbolic Comput.* 20 (1995) 537–566.

49. R. William Gosper, Jr., Decision procedure for indefinite hypergeometric summations, *Proc. Nat. Acad. Sci. U.S.A.* 75 (1978) 40–42.

50. Renate Golombek, Problem no. 1088, *Elemente der Mathematik* 49,3 (1994) 126–127.

51. Henry W. Gould, A Note on a Paper of Grosswald, *Amer. Math. Monthly* 61 (1953) 251–253.

52. Henry W. Gould, Some Generalizations of Vandermonde's Convolution, *Amer. Math. Monthly* 63 (1956) 84–91.

53. Henry W. Gould, Final Analysis of Vandermonde's Convolution, *Amer. Math. Monthly* 64 (1957) 409–415.

54. Henry W. Gould, Note on a paper of Sparre Andersen, *Math. Scand.* 6 (1958) 226–230.

55. Henry W. Gould, Dixon's Series Expressed as a Convolution, *Nord. Mat. Tidskrift* 7 (1959) 73–76.

56. Henry W. Gould, Fjeldstad's series and the q–analog of a series of Dougall, *Math. Scand.* 8 (1960) 198–200.

57. Henry W. Gould, Generalizations of a Theorem of Jensen Concerning Convolutions, *Duke Math. J.* 27 (1960) 71–76.

58. Henry W. Gould, The q–series generalization of a formula of Sparre Andersen, *Math. Scand.* 9 (1961) 90–94.

59. Henry W. Gould, Brill's series and some related expansions, *Technical Report West Virginia University, Morgantown* 3 (1961) 1–14.

60. Henry W. Gould, Exponential Binomial Coefficient Series, *Technical Report West Virginia University, Morgantown* 4 (1961) 1–37.

61. Henry W. Gould, Congruences involving sums of binomial coefficients and a formula of Jensen, *Amer. Math. Monthly* 69 (1962) 400–402.

62. Henry W. Gould and J. Kaucky, Evaluation of a Class of Binomial Coefficient Summations, *J. Comb. Th.* 1 (1966) 233–247.

63. Henry W. Gould, A Remarkable Combinatorial Formula of Heselden, *Proceeding of the West Virginia Academy of Scinces* 43 (1971) 198–201.

64. Henry W. Gould, *Combinatorial Identities,* Gould Publications, Morgantown, W. Va., 1972, 106 p.

65. Henry W. Gould, The Case of the Strange Binomial Identities of Professor Moriarty, *The Fibonacci Quarterly* 10 (1972) 381–391, 402.

66. Henry W. Gould, The Design of the four Binomial Identities: Moriarty Intervenes, *The Fibonacci Quarterly* 12 (1974) 300-308.

67. Henry W. Gould, Sherlock Holmes and the Godfather of Organised Crime, *The Baker Street Journal* 26 (1976) 34–36.

68. Henry W. Gould, Generalization of a Formula of Hilden and Fenger, *Nord. Mat. Tidskrift* 24 (1976) 109–110.

69. Henry W. Gould, A Class of Binomial Sums and a Series Transform, *Rocky Mountain J.* (1993) 1–13.

70. Ronald Lewis Graham, Donald Knuth and Oren Patashnik, *Concrete Mathematics, Second Edition,* Addison–Wesley Publishing Company, Reading, Mass., 1994, 657 p.

71. Daniel H. Greene and Donald E. Knuth, *Mathematics for the Analysis of Algorithms Second Edition,* Birkhäuser, Boston, 1982, 123 p.

72. C. C. Grosjean, problem no. 92–18, *SIAM REVIEW* 34, No. 4, December (1992) 649.

73. John George Hagen, *Synopsis der Höheren Mathematik, Bd. I,* Verlag von Felix L. Dames, Berlin, 1891, 67.

74. Godfrey Harold Hardy, A chapter from Ramanujan's note–book, *Proc. Cambridge Phil. Soc.* 21 (1923) 492–503.

75. E. Heine, Untersuchungen über die Reihe ...., *J. reine u. ang. Math.* 34 (1847) 285–328.

76. E. Heine, *Theorie der Kugelfunctionen und der verwandten Functionen,* G. Reimer, Berlin, 1878, 98 p.

77. John Hoe, *Les Systèmes d'Équations Polynômes dans le Siyuan Yujian (1303),* Collège de France, Institut des hautes Études Chinoises, Paris, 1977, 307–320.

78. F. H. Jackson, Summation of q-hypergeometric series, *Mess. Math.* 50 (1921) 101–112.

79. C. G. J. Jacobi, De usu legitimo formulae summatoriae Maclaurinianae, *J. reine u. ang. Math.* 12 (1834) 263–272.

80. Johan Ludvig William Valdemar Jensen, Sur une Identité d'Abel et sur d'autres Formules Analogues, *Acta Math.* 26 (1902) 307–318.

81. Josef Kaucký, *Kombinatorické identity,* Veda, Vydavatel'stvo Slovenskej Akadémie Vied, Bratislava, 1975, 475 p.

82. Donald E. Knuth, Johann Faulhaber and sums of powers, *Math. of Computation* 61 (1993) 277–294.

83. Tom H. Kornwinder, Handling Hypergeometric Series in Maple, *Report, Centre for Mathematics and Computer Science* (1990) 1–15.

84. H. L. Krall, Differential equations for Chebyshef polynomials, *Duke Math. J.* 4 (1938) 705–718.

85. E. E. Kummer, Über die hypergeometrische Reihe $F(\alpha, \beta, \gamma, x)$, *J. reine u. ang. Math.* 15 (1836) 39–83, 127–172.

86. Johannes Kvamsdal, Løste Oppgaver 4, *Norsk Mat. Tidss.* 30 (1948) 113–114.

87. Peter J. Larcombe, Mogens E. Larsen and Eric J. Fennessey, On Two Classes of Identities Involving Harmonic Numbers, *Util. Math.* 67 (2005) 65–80.

88. Peter J. Larcombe and Mogens E. Larsen, Some Binomial Coefficient Identities of Specific and General Type, *To Appear* – (2006?) –.

89. J. L. Lavoie, Some Summation Formulas for the Series ${}_3F_2(1)$, *Math. of Comp.* 49 (1987) 269–274.

90. Wilhelm Ljunggren, Oppgave 4, *Norsk Mat. Tidss.* 29 (1947) 122.

91. Ma Xin–Rong and Wang Tian–Ming, problem no. 95–1, *SIAM REVIEW* 37, No. 1, January (1995) 98.

92. Eugen Netto, *Lehrbuch der Combinatorik,* G. Teubner Verlag, Leipzig, 1927, 341 p.

93. N. E. Nörlund, *Vorlesungen über Differenzenrechnung,* Springer Verlag, Berlin, 1924, 551 p.

94. Peter Paule and Carsten Schneider, Computer proofs of a new family of harmonic number identities, *Advances in Applied Mathematics* 31 (2003) 359–378.

95. Marco Petkovšek, Hypergeometric Solutions of Linear Recurrences with Polynomial Coefficients, *J. Symbolic Comp.* 14 (1992) 243–264.

96. Marco Petkovšek, Herbert S. Wilf and Doron Zeilberger, *A = B,* A K Peters, Wellesley, Mass, 1996, 212 p.

97. J. F. Pfaff, Observationes analyticae ad L. Euleri Institutiones Calculi Integralis, *Nova acta acad. sci. Petropolitanae* IV supplem. II. et IV. (1797) 38–57.

98. John Riordan, *An Introduction to Combinatorial Analysis,* John Wiley & Sons, Inc., New York, 1958, 244 p.

99. John Riordan, *Combinatorial Identities,* John Wiley & Sons, Inc., New York, 1968, 253 p.

100. Heinrich August Rothe, *Formulae de serierum reversione demonstratio universalis signis localibus combinatorio–analyticorum vicariis exhibita,* Litteraturis Sommeriis, Leipzig, 1793, 36 p.

101. Heinrich August Rothe, *Theorema Binomiale ex simplicissimis analyseos finitorum fontibus universaliter demonstratum,* Christ. Theoph. Rabenhorst, Leipzig, 1796, 16 p.

102. L. Saalschütz, Eine Summationsformel, *Zeitschr. Math. Phys.* 35 (1890) 186–188.

103. Joseph M. Santmyer, Problem no. 10363, *Amer. Math. Monthly* 101, 2, February (1994) 175.

104. Joseph Sinyor and Ted Speevak, A new combinatorial identity, *Int. J. of Math. and Math. Sc.* (To appear) 9 p.

105. Lucy Joan Slater, *Generalized Hypergeometric Functions,* Cambridge University Press, Cambridge, 1966, 273 p.

106. N. J. A. Sloane, *A Handbook of Integer Sequences,* Academic Press, Inc., New York, 1995, 206 p.

107. N. J. A. Sloane, Simon Plouffe, *The Encyclopedia of Integer Sequences,* Academic Press, Inc., New York, 1995, 587 p.

108. H. M. Srivastava, An Extension of the q–Saalschützian Theorem, *Acta Math. Hung.* 53 (1989) 115–118.

109. Richard P. Stanley, *Enumerative Combinatorics, Volume 1,* Cambridge University Press, New York, 1997, 325 p.

110. Tor B. Staver, Om summasjon av potenser av binomialkoeffisientene, *Norsk Matematisk Tidsskrift* 29 (1947) 97–103.

111. Volker Strehl, Identities of Rothe–Abel–Schäfli–Hurwitz–type, *Discrete Math.* 99 (1992) 321–340.

112. A. T. Vandermonde, Mémoire sur des irrationnelles de différens ordres avec une application au cercle, *Mèm. Acad. Roy. Sci. Paris* (1772) 489–498.

113. Michelle Wachs and Dennis White, p, q–Stirling Numbers and Set Partition Statistics, *J. Comb. Th. Ser. A* 56 (1991) 27–46.

114. G. N. Watson, Dixon's theorem on generalized hypergeometric functions, *Proc. London Math. Soc. (2)* 23 (1924) xxxi–xxxiii.

115. G. N. Watson, A note on generalized hypergeometric series, *Proc. London Math. Soc. (2)* 23 (1925) xiii–xv.

116. F. J. W. Whipple, A group of generalized hypergeometric series; relations between 120 allied series of the type $F(a, b, c; e, f)$, *Proc. London Math. Soc. (2)* 23 (1925) 104–114.

117. Herbert S. Wilf and Doron Zeilberger, Rational functions certify combinatorial identities, *J. Amer. Math. Soc.* 3 (1990) 147–158.

118. James A. Wilson, Some Hypergeometric Orthogonal Polynomials, *SIAM J. Math. Anal.* 11 (1980) 690–701.

119. Doron Zeilberger, A holonomic systems approach to special function identities, *J. of Computational and Applied Math.* 32 (1990) 321–368.

120. Doron Zeilberger, The Method of Creative Telescoping, *J. Symbolic Computation* 11 (1991) 195–204.