A bijection between nonnegative words and sparse *abba*-free partitions

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Abstract

We construct a bijection proving that the following two sets have the same cardinality: (i) the set of words over $\{-1,0,1\}$ of length $m-2$ which have every initial sum nonnegative, and (ii) the set of partitions of $\{1,2,\ldots,m\}$ such that no two consecutive numbers lie in the same block and for no four numbers the middle two are in one block and the end two are in another block. The words were considered by Gouyou-Beauchamps and Viennot who enumerated by means of them certain animals. The identity connecting (i) and (ii) was observed by Klazar who proved it by generating functions.

Keywords: set partition; bijection; nonnegative prefix

Let us denote, for $m>0$, $[m]=\{1,2,\ldots,m\}$. A sequence $a=a_1a_2\ldots a_k$ is a *nonnegative word* if $a_i\in\{-1,0,1\}$ for each i and for each initial segment of a the sum of its elements is nonnegative. Recall that $\mathcal{A}=\{A_1,A_2,\ldots,A_n\}$ is a partition of $[m]$ if the A_is (called *blocks*) are nonempty disjoint subsets of $[m]$ and their union is $[m]$. We say that $\mathcal{A}$ is *sparse* if for every $i\in[m-1]$ the elements i and $i+1$ lie in two distinct blocks. $\mathcal{A}$ is called *abba-free* if it does not happen for any four elements $i<j<k<l$ of $[m]$ that i,l lie in a common block and j,k in another common block. For example, $\{\{1,5,7\},\{2,4\},\{3,6\}\}$ is a sparse partition that is not *abba*-free.

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The partition $\{\{1,2,5,7\},\{4\},\{3\},\{6,8\}\}$ is *abba*-free but it is not sparse. We give a direct proof, without using generating functions, for the following theorem originally due to Klazar [2].

Theorem. For every $m \geq 3$ there exists a bijection G between the set of sparse *abba*-free partitions of $[m]$ and the set of nonnegative words of length $m-2$.

Gouyou-Beauchamps and Viennot [1] were interested in counting certain animals (certain sets of plane lattice points) and showed that their animal problem is equivalent to enumeration of nonnegative words (they use slightly different terminology). Klazar [2] was interested in counting set partitions subject to structural restrictions and obtained as a byproduct the above identity. His derivation uses substantially generating functions. Indeed, if r_m is the number of sparse *abba*-free partitions of $[m]$, then ([2])

$$\sum_{m=0}^{\infty} r_m x^m = 1 + \frac{x}{2}\sqrt{\frac{1+x}{1-3x}}.$$

Analogous formula for nonnegative words was derived before in [1]. The sequence

$$(r_m)_{m\geq 2} = (1,2,5,13,35,96,267,750,2123,6046,17303,49721,\ldots)$$

is sequence A005773 of Sloane [3]. Stanley [4, Problem 6.46] and [3] give further information and references on these numbers. Our aim is to avoid the use of generating functions and to give a bijection proving the identity.

We need few more definitions. A nonnegative word is a *correct word* if the first letter is 1, the last letter is -1, the sum of all letters is zero, and each proper initial segment has a positive sum. We say that the letter a_j in a word over $\{-1,0,1\}$ is *dominant* if $a_j = 1$ and the sum of letters in every interval beginning in a_j is positive. For a a correct word of length at least three, a' is obtained from a by deleting the first and the last letter. Obviously, a' is a nonnegative word. For a partition $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ of $[m]$ we denote $|\mathcal{A}| = m$. Similarly, for a sequence a we denote $|a|$ its length. We say that $j \in [m]$ is *covered* in $\mathcal{A}$ if there exist $i, k \in [m]$ and $A_r \in \mathcal{A}$ so that $i < j < k$, $i, k \in A_r$, and $j \notin A_r$. If every element of $\{2,\ldots,m-1\}$ is covered in $\mathcal{A}$, we say that $\mathcal{A}$ is a *connected partition*. Any partition $\mathcal{A}$, $|\mathcal{A}| = m$, can be written in a *sequential form*. This is a sequence $b = b_1 b_2 \ldots b_m$ of length m

over some alphabet such that $b_i = b_j$ if and only if i, j lie in the same block of $\mathcal{A}$. A partition has many sequential forms. One of them is the *canonical sequential form* in which the alphabet is $[n]$ (n is the number of blocks in $\mathcal{A}$) and the first occurrence of every $i \in [n]$, $i > 1$, in b is preceded by the first occurrence of $i - 1$. In particular, b starts with 1. Each partition has a unique canonical sequential form. It is convenient to write specific partitions in (canonical) sequential form. For example, the canonical sequential form of

$$\{\{1,5,7\},\{2,4\},\{3,6\}\} \text{ is } 1232131$$

(we will omit commas in the sequential forms of partitions).

Lemma 1. Each block of a connected sparse *abba*-free partition of $[m]$, $m \geq 3$, has at most two elements. Moreover, the block containing 1 and the block containing m have exactly two elements.

Proof. Suppose that $j < k < l$ belong to the same block, say B, of $\mathcal{A}$. Since k is covered, there exist s and t, $s < k < t$, belonging to the same block A that is different from B. It is easy to check that each of the four positions of s, j and t, l leads to the forbidden pattern *abba*. For example, if $j < s$ and $t < l$ then $j < s < t < l$ form the *abba* pattern. If $\{1\}$ were a block, 2 would not be covered. Similarly $\{m\}$ cannot be a block. □

We consider the following mapping F from the set of partitions of $[m]$ with no block with more than two elements to words over $\{-1, 0, 1\}$ with length m. $F(\mathcal{A}) = a_1a_2 \ldots a_m$ where $a_i = 0$ if $\{i\} \in \mathcal{A}$, $a_i = 1$ if i is the first element of the two-element block containing i, and $a_i = -1$ if i is the second element. For example, $F(1234153) = 1, 0, 1, 0, -1, 0, -1$.

Lemma 2. For every $m \geq 3$, F is a bijection between the set of connected sparse *abba*-free partitions of $[m]$ and the set of correct words of length m.

Proof. By the previous lemma, if $\mathcal{A}$ is a connected sparse *abba*-free partition, $F(\mathcal{A})$ is defined and is a word beginning with 1 and ending with -1. Every initial sum of $F(\mathcal{A})$ is nonnegative for else we would have in the corresponding initial segment of $\mathcal{A}$ more second elements of two-element blocks than the first elements, which is impossible. Moreover, for no i, $1 < i < m$, the sum of the first i letters is zero because then i would not be covered. Thus $F(\mathcal{A})$ is a correct word.

We define the inverse mapping F^{-1}. Let $a = a_1a_2 \ldots a_m$ be a correct word and let the partition $F^{-1}(a) = \mathcal{A}$ be defined in the following way. If

$a_i = 0$ then $\{i\}$ is a (singleton) block of $\mathcal{A}$ and if a_i is the kth occurrence of 1 in a and a_j is the kth occurrence of -1, then $\{i, j\}$ is a block of $\mathcal{A}$. Note that always $i < j$ and that the second elements of two-element blocks come in the same order as the first elements. Thus $\mathcal{A}$ is *abba*-free. $\mathcal{A}$ is connected because if an inner element i were not covered, then the sum of the first $i-1$ letters of a would be zero. $\mathcal{A}$ is sparse because $\{i, i+1\} \in \mathcal{A}$ implies that $a_1 + a_2 + \cdots + a_{i-1} = 0$ and $a_1 + a_2 + \cdots + a_{i+1} = 0$. Finally, it is easy to check that F and F^{-1} are inverses of one another and thus F is a bijection. □

For a sparse *abba*-free partition $\mathcal{A}$ of $[m]$, $m \geq 3$, consider the collection $\mathcal{A}^*$ of maximal subintervals $I \subset [m]$ of length at least three for which the induced partition $\mathcal{A}|I$ is connected.

Lemma 3. Every two distinct intervals $I_1, I_2 \in \mathcal{A}^*$ are disjoint or they overlap in one element only.

Proof. Any other position of I_1 and I_2 means that every inner element of $I = I_1 \cup I_2$ is inner in I_1 or in I_2 and thus $\mathcal{A}|I$ is connected. This contradicts the maximality of I_1 or of I_2. □

Thus we can order $\mathcal{A}^*$ as $\mathcal{A}^* = \{I_1, I_2, \ldots, I_n\}_<$ where $I_i = [u_i, v_i]$ and $1 \leq u_1 < v_1 \leq u_2 < v_2 \leq u_3 < \ldots \leq u_n < v_n \leq m$. We define the numbers a_i, $0 \leq i \leq n$, by $a_i = u_{i+1} - v_i - 1$ where we set $v_0 = 0$ and $u_{n+1} = m + 1$. Clearly, $a_i \geq -1$ and a_i is the number of elements strictly between I_i and I_{i+1}, where $a_i = -1$ means that the intervals overlap. Note that every element between I_i and I_{i+1} forms a singleton block.

Now we can define the desired bijection G:

$$G(\mathcal{A}) = 1^{a_0} F(\mathcal{A}_1)' 1^{a_1+2} F(\mathcal{A}_2)' 1^{a_2+2} \ldots 1^{a_{n-1}+2} F(\mathcal{A}_n)' 1^{a_n}.$$

Here $\mathcal{A}$ is a sparse *abba*-free partition of $[m]$, $m \geq 3$, 1^i abbreviates the sequence $1, 1, \ldots, 1$ of i 1s, a_i are the above defined numbers, $\mathcal{A}_i$ is the restriction of $\mathcal{A}$ to I_i (where $\mathcal{A}^* = \{I_1, I_2, \ldots, I_n\}_<$) normalized so that the ground set equals $[|I_i|] = [v_i - u_i + 1]$, F is the mapping of Lemma 2, and $'$ means the deletion of the first and last letter. If $n = 0$, that is if $\mathcal{A}^* = \emptyset$ and $\mathcal{A}$ has only singleton blocks, we set

$$G(\mathcal{A}) = 1^{a_0-2} = 1^{m-2}.$$

We prove that G is indeed a bijection between all sparse *abba*-free partitions of $[m]$ and all nonnegative words of length $m-2$. By Lemma 2, $F(\mathcal{A}_i)$ is a correct word. Hence $F(\mathcal{A}_i)'$ is a nonnegative word and the whole $G(\mathcal{A})$ is a nonnegative word. Its length is $m-2$ if $\mathcal{A}^* = \emptyset$ and

$$\sum_{i=1}^{n}(a_{i-1}+|F(\mathcal{A}_i)|-2)+a_n+2(n-1)=\sum_{i=1}^{n}(a_{i-1}+|I_i|)+a_n-2=m-2$$

if $\mathcal{A}^* \neq \emptyset$.

We define the inverse mapping G^{-1}. Let $b = b_1b_2\dots b_{m-2}$, $m \geq 3$, be a nonnegative word. There is a unique decomposition of b into intervals

$$b = c_0d_1c_1d_2\dots c_{n-1}d_nc_n$$

such that c_0 is the longest initial interval in which every element is dominant, d_1 is the longest interval starting immediately after c_0 whose elements sum up to zero, c_1 is the longest interval starting immediately after d_1 in which every element is dominant and so on. Note that c_0 and c_n may be empty but the other intervals are nonempty, $c_i = 1^{e_i}$ where e_i is a nonnegative integer, and every d_i is a nonnegative word. If $b = c_0$, b consists only of 1s, and we set $G^{-1}(b)$ to be the partition of $[m]$ having just the singleton blocks $\{1\}, \{2\}, \dots, \{m\}$. If $n > 0$, we define $\mathcal{A}_i = F^{-1}(1, d_i, -1)$ where F^{-1} is the inverse mapping to F of Lemma 2, defined in its proof. The word $1, d_i, -1$ is a correct word and $\mathcal{A}_i$ is a connected sparse *abba*-free partition of some initial interval of positive integers. We define the numbers a_i as $a_0 = e_0$, $a_n = e_n$, and $a_i = e_i - 2$ for $0 < i < n$. Finally, we set

$$G^{-1}(b) = \mathcal{B}_0\mathcal{A}_1\mathcal{B}_1\mathcal{A}_2\dots\mathcal{B}_{n-1}\mathcal{A}_n\mathcal{B}_n$$

where $\mathcal{B}_i$ is, for $a_i > 0$, a partition consisting of a_i singleton blocks. If $a_i = 0$, $\mathcal{B}_i = \emptyset$ and $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ are neighbours. If $a_i = -1$, $\mathcal{B}_i = \emptyset$ and $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ are made to overlap in the last element of $\mathcal{A}_i$ and the first element of $\mathcal{A}_{i+1}$. The two blocks which now intersect merge into one block. We have

$$|G^{-1}(b)| = \sum_{i=0}^{n} a_i + \sum_{i=1}^{n} |\mathcal{A}_i| = \sum_{i=0}^{n} |c_i| - 2(n-1) + \sum_{i=1}^{n} |d_i| + 2n = |b| + 2 = m.$$

The operation of concatenation includes, of course, the appropriate shifting of the ground sets of $\mathcal{A}_i$ and $\mathcal{B}_i$ so that the ground set of the resulting partition $G^{-1}(b)$ equals $[m]$.

It is easy to check that the resulting partition $G^{-1}(b)$ is a sparse *abba*-free partition of $[m]$ and that for every $\mathcal{A}$ and b we have $G^{-1}(G(\mathcal{A})) = \mathcal{A}$ and $G(G^{-1}(b)) = b$. Thus G and G^{-1} are bijections. The theorem is proved.

As an example, we list in the lexicographical order all 13 sparse *abba*-free partitions of [5] in their canonical sequential form and the corresponding nonnegative words with length 3:

$$\begin{aligned}
G(12123) &= F(1212)', 1 = (1,1,-1,-1)', 1 = 1,-1,1.\\
G(12131) &= F(121)', 1, F(131)' = (1,0,-1)', 1, (1,0,-1)' = 0,1,0.\\
G(12132) &= F(12132)' = (1,1,-1,0,-1)' = 1,-1,0.\\
G(12134) &= F(121)', 1^2 = (1,0,-1)', 1, 1 = 0,1,1.\\
G(12312) &= F(12312)' = (1,1,0,-1,-1)' = 1,0,-1.\\
G(12313) &= F(12313)' = (1,0,1,-1,-1)' = 0,1,-1.\\
G(12314) &= F(1231)', 1 = (1,0,0,-1)', 1 = 0,0,1.\\
G(12323) &= 1, F(2323)' = 1, (1,1,-1,-1)' = 1,1,-1.\\
G(12324) &= 1, F(232)', 1 = 1, (1,0,-1)', 1 = 1,0,1.\\
G(12341) &= F(12341)' = (1,0,0,0,-1)' = 0,0,0.\\
G(12342) &= 1, F(2342)' = 1, (1,0,0,-1)' = 1,0,0.\\
G(12343) &= 1^2, F(343)' = 1,1,(1,0,-1)' = 1,1,0.\\
G(12345) &= 1^3 = 1,1,1.
\end{aligned}$$

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References

[1] D. Gouyou-Beauchamps and G. Viennot, Equivalence of the two-dimensional directed animal problem to a one-dimensional path problem, Adv. Appl. Math. 9 (1988) 334–357.

[2] M. Klazar, On *abab*-free and *abba*-free set partitions, Europ. J. Comb. 17 (1996) 53–68.

[3] N. J. A. Sloane (2001), The On-Line Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/~njas/sequences/

[4] R. P. Stanley, Enumerative Combinatorics, Volume 2 (Cambridge University Press, Cambridge, UK, 1999).