On numbers of Davenport-Schinzel sequences

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Abstract

One class of Davenport-Schinzel sequences consists of finite sequences over n symbols without immediate repetitions and without any subsequence of the type *abab*. We present a bijective encoding of such sequences by rooted plane trees with distinguished nonleaves and we give a combinatorial proof of the formula

$$\frac{1}{k-n+1}\binom{2k-2n}{k-n}\binom{k-1}{2n-k-1}$$

for the number of such normalized sequences of length k. The formula was found by Gardy and Gouyou-Beauchamps by means of generating functions. We survey previous results concerning counting of DS sequences and mention several equivalent enumerative problems.

1 Introduction

The set $DS(n)$ of *Davenport-Schinzel sequences* over n symbols is formed by finite sequences $u = a_1a_2\dots a_k$ satisfying

1. $a_i \in [n] = \{1, 2, \dots, n\}$ for all i, each integer $j \in [n]$ appears in u.
2. For each pair $i < j$ of $[n]$ the first appearance of i in u precedes that of j.
3. $a_i \neq a_{i+1}$ for all $i = 1, 2, \dots, k-1$.

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4. $a_{i_1} = a_{i_3} = a \neq b = a_{i_2} = a_{i_4}$ holds for no four indices $1 \leq i_1 < \ldots < i_4 \leq k$.

Condition 3 forbids immediate repetitions while condition 4 does not allow any subsequence of the type $\ldots a \ldots b \ldots a \ldots b \ldots$ where a and b are two distinct numbers. Conditions 1 and 2 normalize sequences for purposes of enumeration.

One can consider *maximal* $DS(n)$ sequences, denoted as $MDS(n)$, which end with 1. For instance,

$$DS(3) = \{123, 1231, 1232, 12321, 1213, 12131\}$$

and

$$MDS(3) = \{1231, 12321, 12131\}.$$

The number of $MDS(n)$ sequences of length k is denoted by $f_{n,k}$ and their total number by f_n. Similarly, $b_{n,k}$ is the number of $DS(n)$ sequences of length k and $b_n = |DS(n)|$. Clearly, $b_1 = f_1 = 1$. The mapping $u \to u1$ is a bijection between $DS(n) \backslash MDS(n)$ and $MDS(n)$, $n > 1$. We see that

$$b_n = 2f_n \text{ and } b_{n,k} = f_{n,k} + f_{n,k+1}. \tag{1}$$

The minimum length of a $DS(n)$ sequence is n and the maximum length is $2n - 1$ (see [4]).

Our aim is to give a combinatorial proof of the formula

$$b_{n,k} = C_{k-n} \cdot \binom{k-1}{2n-k-1} = \frac{\binom{2k-2n}{k-n}\binom{k-1}{2n-k-1}}{k-n+1} \tag{2}$$

established by Gardy and Gouyou-Beauchamps in [6] by means of generating functions. Here $C_n = \binom{2n}{n}/(n+1)$ stands for the n-th *Catalan number* that counts, among other structures, the number of rooted plane trees on $n+1$ vertices.

The paper is organized as follows. In the next section we list several (classical) enumerative problems which are equivalent to counting of $MDS(n)$. In the third section a combinatorial proof of (2) is given. We introduce a new representation of $DS(n)$ by rooted plane trees on n vertices with distinguished nonleaves. To count such trees we encode them bijectively by another tree structure. The bijection is described in the fourth section.

We recall briefly some basic features of a *rooted plane tree* $T = (V, E)$, shortly an *rp tree*. It is a finite rooted tree with edges directed away from the *root* $r \in V$. For an edge $(u, v) \in E$ of T we call u the *parent* of v while v is a *child* of u. The order of children of u matters, we think of T as drawn in the plane with r at the lowest and all edges drawn as straight segments directed up. The number of children of $u \in V$ is denoted by $deg(u)$. A *leaf* is a vertex with no child. The number of leaves of T is denoted by $l(T)$. *Principal subtrees* of T are the trees which arise by deleting the root of T.

To conclude the present section we should say that Davenport-Schinzel sequences were introduced by Davenport and Schinzel [4] in a more general context where alternating subsequences $ababab\ldots$ of length d were excluded. The most important results of the theory of Davenport-Schinzel sequences are upper and lower bounds on their maximum length when d is fixed — [20], [8], and [2]. Applications include both computational and combinatorial geometry. From the enumerative point of view cases $d > 4$ have proven so far intractable. Surveys can be found in [1], [18], [13], and also in [9].

2 The Schröder family

There is an old *Schröder family* of mutually equivalent enumerative problems and the sequence of finite sets $\{MDS(n)\}_{n\geq 1}$ is a relatively new and less known member of it. As such $MDS(n)$ sequences had been enumerated and the generating function had been found well before they were defined. Since this is not articulated in other enumerative papers about $DS(n)$ sequences, it appears useful here to give a brief description of these problems bearing in mind $DS(n)$ sequences. Our list of references is by no means exhaustive.

The sequence of numbers $\{f_n\}_{n\geq 1}$ is the enumerator of the family. There is no closed formula for f_n but it can be computed by a recurrence relation, by a generating function, by sums with positive terms or by alternating sums. We list some of these expressions below.

Special rooted plane maps. The first enumerative paper about $DS(n)$ sequences is due to Mullin and Stanton [11]. They proved, not mentioning so, the membership of the problem to the Schröder family. We describe briefly their bijection between $MDS(n)$ and the set of special rooted plane maps which we will call *fences*.

By a *plane* multigraph we mean a planar multigraph with a specific embedding in the plane. We say it is *totally outerplane* if all edges lie on the boundary of the outer face. A *cut* edge in a connected multigraph G is an edge whose removal disconnects G. A *fence* (F, r, e) is a connected totally outerplane multigraph with no cut edges, with distinguished edge e and vertex r. The vertex r is incident with e and for an observer on r the outer face lies to the left of e.

Note that in a fence no two vertices are connected by three or more edges and that any fence arises from a connected totally outerplane graph by doubling the cut edges.

In F there is a unique closed Eulerian walk C which goes around F clockwise, starts at r, and uses e as its first edge. C produces an $MDS(n)$ sequence. We label r as 1 and we write down the labels of vertices in the order of C. Whenever an unlabeled vertex is encountered, it is given the least unused label.

Counting $MDS(n)$ or fences on n vertices is therefore equivalent. Mullin and Stanton proved the formula

$$b_{n,2n-1} = f_{n,2n-1} = C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1} \tag{3}$$

by observing that fences on n vertices with maximum number of edges are rp trees on n vertices with all edges doubled. They also proved that

$$(n+1)f_{n+1} - (6n-3)f_n + (n-2)f_{n-1} = 0 \quad (n \geq 3), \tag{4}$$

using the generating function

$$\sum_{n=1}^{\infty} f_n x^n = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4}. \tag{5}$$

They derived, for $n \geq 2$, the formula

$$f_n = \sum_{0 \leq k \leq n/2-1} 3^{n-2-2k}\, 2^k \binom{n-2}{2k} C_k. \tag{6}$$

Equation (5) together with the first ten values of f_n appear already in [17]. Interestingly, numbers f_n and equation (4) can also be found (without any combinatorial interpretation) in [15], p. 168.

Dissections of a convex polygon. A *dissection* of a convex polygon P with labeled vertices is a set of diagonals, no two of them crossing. Dissections with various restrictions on the face sizes were enumerated by Etherington [5]. Etherington pointed out that the case when there is no restriction at all is equivalent to Schröder's bracketing problem. Similar problems were investigated by Motzkin [10].

Roselle [16] gave the following bijection that matches dissections of a convex $(n+1)$-gon and $MDS(n)$ sequences. Let D be a dissection of P with vertices labeled by $1, 2, \dots, n+1$ clockwise. Start with the sequence $12 \dots n1$. Then insert between $j-1$ and j in the decreasing order the numbers k where $k < j$ and kj is a diagonal of D. Similarly insert between n and 1 the decreasing list of numbers k joined by a diagonal to $n+1$. What you get is an $MDS(n)$ sequence.

In fact, Roselle described this bijection only for the case of triangulations and $MDS(n)$ sequences with maximum length. It is well known that triangulations are counted by Catalan numbers and Roselle gave this way an alternative proof of (3). However, it is easy to see that the bijection works in general and that it matches the elements of $MDS(n)$ of length k with dissections of a convex $(n+1)$-gon with $k-n-1$ diagonals. And this implies already (2) because as early as 1866 Prouhet [14] (see [3], p. 75) counted the number, $r(n,d)$, of dissections of a convex n-gon by d diagonals:

$$r(n,d) = \frac{1}{d+1}\binom{n-3}{d}\binom{n+d-1}{d}. \tag{7}$$

Thus $f_{n,k} = r(n+1, k-n-1)$, and (7) combined with (1) give (2). Since this combination leading to a combinatorial proof of (2) went unnoticed, we take the freedom to present another combinatorial proof.

Bracketings of a product. Schröder [17] discovered the family in 1870 by solving the following problem. Given a noncomutative product of n terms, in how many ways can one bracket them so that each bracket contains at least two factors? The outer bracket is not allowed. The answer is again given by the numbers f_n.

A nice exposition of (4) and (5) is in Comtet [3] on p. 56 who gives the expression, $n > 2$,

$$f_n = \sum_{0 \le k \le n/2} (-1)^k \, \frac{(2n-2k-3)!!}{k!(n-2k)!} \, 3^{n-2k} \, 2^{-k-2}. \tag{8}$$

Here $(2n-2k-3)!!$ denotes the odd factorial $1 \cdot 3 \cdot 5 \cdots (2n-2k-3)$. Standard Lagrange inversion (see Goulden and Jackson [7], problem 2.7.12) yields a simpler alternating expression

$$f_n = \frac{1}{n}\sum_{i=0}^{n-1}(-1)^i 2^{n-1-i}\binom{n}{i}\binom{2n-2-i}{n-1}. \tag{9}$$

Other disguises. There is an obvious tree disguise of the problem. It was noticed already by Etherington that bracketings of n terms can be visualized by rooted plane trees having n leaves and no vertex with degree 1. Two other, less obvious, tree disguises are given in the next two sections.

Besides (2) Gardy and Gouyou-Beauchamps in [6] determined the average length and average number of symbols of a $DS(n)$ sequence and found the bivariate generating function for $b_{n,k}$'s. They gave also a bijection between $DS(n)$ and Schröder words of length $2n-2$. These are words over the alphabet $\{x, \bar{x}, y\}$ given by the language equation

$$X = 1 + yyX + xX\bar{x}X.$$

3 Coding and counting

The first step in our combinatorial proof of (2) is an encoding of $DS(n)$ by the set $CT(n)$ of pairs $\mathcal{T} = (T, S)$, where T is an rp tree on n vertices and S is a subset of nonleaves of T. We call them *circled rooted plane trees*, or shortly *crp trees*, since we visualize the distinguished nonleaves as being circled. See Figure 1. The encoding is easier to describe recursively but the nonrecursive version is easier to perform.

Recursive version. Suppose $u = a_1a_2\dots a_k$ is a $DS(n)$ sequence. If $k = 1$ then u is encoded by a single uncircled vertex. Otherwise we use the decomposition $u = 1u_11u_2\dots 1u_l$ of u by all appearances of 1. A moment of thought reveals that the segments u_i are nonempty, except possibly for u_l, they do not share symbols, and each u_i satisfies conditions 3 and 4 of the definition of $DS(n)$. We rename the symbols so that u_i complies with conditions 1 and 2 as well and we encode u_i by $\mathcal{T}_i$. The sequence u is encoded by the crp tree $\mathcal{T}$ with principal subtrees from left to right $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_l$, the

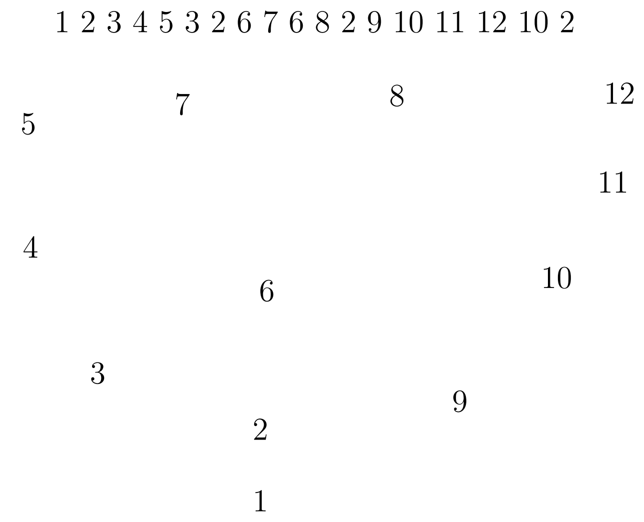

Figure 1: Encoding by crp trees

root of $\mathcal{T}$ is circled iff u_l is empty. We leave the inverse decoding to the reader.

Nonrecursive version. Suppose $u = a_1a_2\dots a_k$ is a $DS(n)$ sequence. A crp tree (T, S) on n vertices is generated, the algorithm uses three auxiliary variables: i is the index of the currently processed term of u, v denotes the currently processed vertex, and C is either empty or a singleton set containing a candidate for an element of the set S.

We initialize the variables by setting $i := 1, v := p$, and $S := C := \emptyset$, where p, the root, is an arbitrary point in the plane labeled by $a_1 = 1$. In the general step if $i = k$ we are done. If $i < k$ then there is to distinguish two cases.

1. a_{i+1} has appeared earlier in the sequence. We denote by q the unique vertex on the path joining the root and v which is labeled by a_{i+1}. We put

$$i := i + 1,\ v := q,\ S := S \cup C, \text{ and } C := \{v\} = \{q\}.$$

In the case that now $i = k$ (we did the last step) we add q to S.

2. a_{i+1} is a new symbol. We join to v, above v and to the right of the

children of v, a new child q and give it the label a_{i+1}. Then we put

$$i := i + 1,\ v := q,\ S := S, \text{ and } C := \emptyset.$$

So S consists of vertices which were reached by a jump from above, and from which we jumped down again or for which the procedure terminated. In the end we can discard the labels. Even so it is easy to reconstruct u from the crp tree (T, S). We describe it now.

If (T, S) is a crp tree then the corresponding $DS(n)$ sequence $u = a_1 a_2 \dots a_k$ arises by climbing up and jumping down around T clockwise and writing down the labels of vertices. On the beginning the vertices are unlabeled. We start at the root r and give it the label 1. Whenever an unlabeled vertex is encountered it is given the least unused label. We go up without jumps to the leftmost leaf z. For the crp tree on Figure 1 we produce 12345. Then we jump down on the r-z path P in jumps following elements of $P \cap S$ until we reach a vertex $v \in P$ that has a child to the right of P. In our example we perform the jumps 53 and 32. It is irrelevant now that 2 is circled, we would end in it anyway. From v we continue in consecutive steps upward to the second leftmost leaf and so on. For the rightmost leaf w, which is the last one to be visited, there is no such vertex v and we finish jumping at the lowest element of $Q \cap S$ where Q is the r-w path. If $Q \cap S = \emptyset$ then we finish at w. In our example we finish at 2 and only now it matters that 2 is circled.

We recall that $l(T)$ is the number of leaves in T. The following theorem summarizes the above encoding procedures.

Theorem 3.1 *The above encodings give a bijection between the sets $DS(n)$ and $CT(n)$. It follows that $b_{n,k}$ equals to the number of crp trees (T, S) on n vertices with $2n - k - 1$ uncircled nonleaves, i. e. crp trees (T, S) with $|V(T)| = n$ and $n - l(T) - |S| = 2n - k - 1$.*

Proof. Using our recursive version we can easily prove the bijectivity. If $u \in DS(n)$ has length k then it is encoded by a crp tree (T, S) on n vertices such that $k = n + l(T) + |S| - 1$. So the set of circled nonleaves S has $k - n - l(T) + 1$ elements and the complement S^c (complement in the set of nonleaves) has $n - l(T) - |S| = 2n - k - 1$ elements. □

It is easier to count the pairs (T, S^c) than the pairs (T, S) because the cardinality $|S^c|$ is independent of the structure of T. Therefore (formally we switch between circled and uncircled nonleaves) it suffices to count crp trees with a fixed number of vertices and circles. The next step is an encoding of crp trees by *rooted plane trees with dots*, shortly *drp trees*. We need few definitions.

Consider an rp tree T with n vertices drawn as a picture in the plane. Let v be a vertex with $d = deg(v)$ children. The $d+1$ edges incident with v, which are drawn as straight segments, split the neighborhood of v into $d+1$ wedge-shaped areas which we call *gaps* of v. For the root of T there is no difference, we imagine an edge joining it to a virtual parent. The set $g(T)$ of all gaps in T has $\sum_V(deg(v)+1) = 2n-1$ elements. A *drp tree* is a pair (T, D) where T is an rp tree and D is a finite multisubset of $g(T)$. This means that we distinguish, possibly with repetitions, some gaps of T. We visualize a drp tree (T, D) as an rp tree T with D determined by dots distributed in the gaps of T. The number of dots in a gap g is then the multiplicity of g in D. Look at the picture on Figure 2.

There is a bijection between crp trees with n vertices and m circles and drp trees with $n-m$ vertices and m dots, the proof is given in the next section. Since it is easy to count drp trees with a given number of vertices and dots, we are done.

Theorem 3.2 *The number of crp trees with n vertices and m circles is*

$$C_{n-m-1}\cdot\binom{2n-m-2}{m}.$$

Proof. From Lemma 4.2 of the next section we know that the number of crp trees with n vertices and m circles is the same as the number of drp trees with $n-m$ vertices and m dots. But this is equal to the number of rp trees on $n-m$ vertices times the number of m element multisubsets of a $2n-2m-1$ element set. □

The proof of (2) is finished, (2) follows immediately from Theorems 3.1 and 3.2 by setting $m = 2n-k-1$.

The total number b_n of $DS(n)$ sequences can be counted in two ways. One can sum (2) for all $k = n, n+1, \ldots, 2n-1$. Changing the summation range the expression found in [6] follows:

$$b_n = \sum_{j=0}^{n-1} \frac{1}{j+1}\binom{2j}{j}\binom{j+n-1}{2j}. \qquad (10)$$

The other way is to form groups of crp trees on n vertices with the same number of leaves. The number, $p(n,l)$, of rooted plane trees on n vertices with l leaves is given by the well known formula (first appearing implicitly in [12])

$$p(n,l) = \frac{1}{n-l}\binom{n-1}{l}\binom{n-2}{l-1}.$$

Note that $p(n,l) = p(n,n-l)$. The number of crp trees with the same underlying rp tree is 2^{n-l}. Hence

$$b_n = \sum_{l=1}^{n-1} p(n,l)\cdot 2^{n-l} = \sum_{l=1}^{n-1} \frac{2^l}{n-l}\binom{n-1}{l}\binom{n-2}{l-1}. \tag{11}$$

Well, how many $MDS(n)$ sequences are there then? From either (4), (6), (8), (9), (10) or (11), taking (1) into account, we get

$$\{f_n\}_{n\geq 1} = \{1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, \ldots\}.$$

This is the 1163-rd sequence in the phenomenal Sloane's handbook [19].

4 Contractions and expansions

We show that there is a natural bijection between crp trees with n vertices and m circles and drp trees with $n-m$ vertices and m dots. As an example to illustrate our idea we consider first crp and drp trees with one circle and one dot. Let $(T,\{v\})$ be such a crp tree, let e join v to its leftmost child. We put one dot d in the gap of v lying to the right of e and contract e. The drp tree obtained is denoted by $(T^*,\{d\})$. It is easy to see how to recover $(T,\{v\})$ from $(T^*,\{d\})$. Hence the mapping $(T,\{v\}) \to (T^*,\{d\})$ is the desired bijection in the case $m=1$.

To generalize this to $m>1$ we need to define a more general tree structure with both circles and dots and we need to define an order to perform the contractions. First we recall the standard linear order $(V,\prec)$ on the vertex set of an rp tree T. For two distinct vertices $u,v\in V$ one considers the paths P_u and P_v joining the root to u and v. Two cases arise.

1. One path — say P_u — is an initial segment of the other path. Then $u \prec v$.

2. Otherwise there is a branching point and one path — say P_u — branches to the right. Then again $u \prec v$.

Suppose (T,S,D) is a triple where (T,S), resp. (T,D), is a crp tree, resp. a drp tree. We define a partial ordering $(S \cup D, \prec)$. If $x \in S \cup D$ then x is either a circled vertex v or a dot in a gap of a vertex v, in both cases the expression the *vertex of* x refers to v. Let $x, y \in S \cup D$ be two distinct elements, let u be the vertex of x, and let v be the vertex of y.

1. $u \neq v$. We set $x \prec y$ iff $u \prec v$.

2. $u = v$. If x is a dot in a gap g and y is a dot in a gap h, g and h belong to the same vertex, we set $x \prec y$ iff g lies to the right of h. In the two remaining cases — both x and y are dots in the same gap or one of them is a dot and the other is a circled vertex — x and y are set to be incomparable.

A *circled rooted plane tree with dots*, shortly a *cdrp tree*, is a triple $\mathcal{T} = (T,S,D)$ where (T,S), resp. (T,D), is a crp tree, resp. a drp tree, and such that $S \prec D$. In other words, $v \prec d$ for any $v \in S$ and any $d \in D$. In particular, each gap of a circled vertex is empty. We define two mutually inverse operations on $\mathcal{T}$ with an example to illustrate them on Figure 2. The operations preserve the sum $|S| + |D|$. Let v be the largest, with respect to $\prec$, vertex of S and w be its leftmost child. Let d be one of the minimal dots.

Contraction of $\mathcal{T}$ contracts the edge $e = \{v, w\}$, i.e. e is deleted and v and w are identified. The new vertex z created by the identification is not circled. All other circles are preserved. The dots of the leftmost gap of w appear now in the leftmost gap of z and the dots of the rightmost gap of w appear now in what was the second leftmost gap of v. Furthermore we add to the latter one more dot. The distribution of dots in other gaps is preserved. Resulting cdrp tree is denoted by $C(\mathcal{T})$.

Expansion of $\mathcal{T}$ expands d. Suppose d is located in a gap g of a vertex z. We delete d and split z into two vertices w and v. The vertex w is slightly to the left of v and is joined only to those children of z which were to the left of g. Vertex v is joined to the remaining children and to the parent of z. Now w is moved upward a bit with all the dots it bears and is joined to v as its new leftmost child. The dots of g appear now in the rightmost gap of w. All gaps of v are empty. Vertex v is circled, vertex w is not circled. Dots in other gaps and other circles are preserved. Resulting cdrp tree is denoted by $E(\mathcal{T})$.

Lemma 4.1 *$C(\mathcal{T})$ and $E(\mathcal{T})$ are cdrp trees again. Also* $C(E(\mathcal{T}))$ =

$E(C(\mathcal{T})) = \mathcal{T}$ whenever the operations involved are defined.

Proof. The lemma can be easily proved by an inspection of the above definitions. The proof is left to an interested reader. $\square$

Let $\mathcal{T} = (T, S)$ be a crp tree with n vertices and m circles. We assign to $\mathcal{T}$ a drp tree $\mathcal{U} = C^m(\mathcal{T})$ which arises by m iterations of the contraction operation on $\mathcal{T}$.

Lemma 4.2 *The above assignment is a bijection between crp trees with n vertices and m circles and drp trees with $n - m$ vertices and m dots.*

Proof. It follows immediately from the previous lemma that the mappings $\mathcal{T} \to \mathcal{U} = C^m(\mathcal{T})$ and $\mathcal{U} \to \mathcal{T} = E^m(\mathcal{U})$ are inverses of one another. $\square$

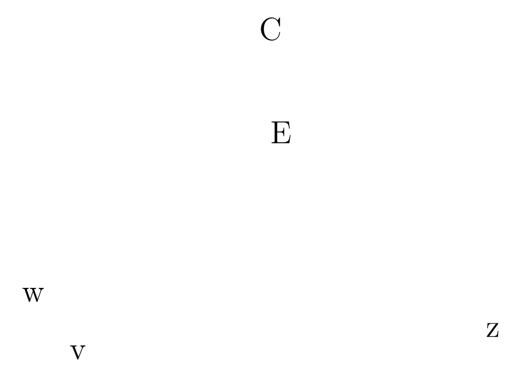

Figure 2: A contraction and an expansion

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References

[1] P. K. Agarwal, *Intersection and decomposition algorithms for planar arrangements*, Cambridge University Press, 1991.

[2] P. K. Agarwal, M. Sharir and P. Shor, Sharp upper and lower bounds on the lengths of general Davenport-Schinzel sequences, *J. Combin. Theory A* **52** (1989), 228–274.

[3] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Company, 1974.

[4] H. Davenport and A. Schinzel, A combinatorial problem connected with differential equations, *Amer. J. Math.* **87** (1965), 684–694.

[5] I. M. H. Etherington, Some problems of non-associative combinatorics, *The Edinburgh Math. Notes* **32** (1940), 1–6.

[6] D. Gardy and D. Gouyou-Beauchamps, Enumerating Davenport-Schinzel sequences, *Informatique théorique et Applications / Theoretical Informatics and Applications* **26** (1992), 387-402.

[7] I. P. Goulden and D. M. Jackson, *Combinatorial Enumeration*, J. Wiley, 1983.

[8] S. Hart and M. Sharir, Nonlinearity of Davenport-Schinzel sequences and of generalized path compression schemes, *Combinatorica* **6** (1986), 151–177.

[9] M. Klazar, *Combinatorial aspects of Davenport-Schinzel sequences*, thesis, Charles University, Prague 1995.

[10] Th. Motzkin, Relations between hypersurfaces crossratio, and a combinatorial formula for partitions of a polygon, for a permanent preponderance and for nonassociative products, *Bull. of the American Math. Soc.* **54** (1948), 362–370.

[11] R. C. Mullin and R. G. Stanton, A map-theoretic approach to Davenport-Schinzel sequences, *Pacific J. Math.* **40** (1972), 167–172.

[12] V. T. Narayana, A partial order and its application to probability, *Sankhyá* **21** (1959), 91–98.

[13] J. Pach (Editor), *New Trends in Discrete and Computational Geometry*, Springer, 1993.

[14] E. Prouhet, *Nouvelles Annales Mathematiques* **5** (1866), 384

[15] J. Riordan, *Combinatorial Identities*, John Wiley, 1968.

[16] D. P. Roselle, An algorithmic approach to Davenport-Schinzel sequences, *Utilitas Math.* **6** (1974), 91–93.

[17] E. Schröder, Vier combinatorische Probleme, *Zeitschrift für Mathematik und Physik* **15** (1870), 361–376.

[18] M. Sharir and P. K. Agarwal, *Davenport-Schinzel sequences and their geometric applications*, Cambridge University Press, 1995.

[19] N. J. A. Sloane, *A Handbook of Integer Sequences*, Academic Press, 1973. (new updated edition in Academic Press, 1995)

[20] E. Szemerédi, On a problem by Davenport and Schinzel, *Acta Arith.* **25** (1974), 213–224.

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