On integrality and periodicity of the Motzkin numbers

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Abstract

In this note we investigate arithmetic properties of the Motzkin numbers. We prove that among all fractional sequences of Motzkin type, the only integral ones are integral multiples of the sequence of Motzkin numbers; in fact, we prove a stronger result. We show that the sequence of Motzkin numbers is nonperiodic modulo any prime.

Introduction

Let n be any positive integer. The *n-th Motzkin number*, denoted from here on by m_n, counts the number of lattice paths in the cartesian plane starting at $(0,\ 0)$, ending at $(n,\ 0)$, and which use line steps equal to either $(1,\ 0)$ (level step), or to $(1,\ 1)$ (up step), or to $(1,\ -1)$ (down step), and which never pass below the x-axis. Clearly, $m_1 = 1,\ m_2 = 2$, and it is known that the three-term recurrence

$$(n+2)m_n = (2n+1)m_{n-1} + 3(n-1)m_{n-2}$$

holds for all $n \geq 3$. It is convenient to set $m_0 := 1$, and the recurrence is then valid for all $n \geq 2$. It is also known (and easy to derive from the combinatorial interpretation of m_n) that the generating function $m = m(x) = \sum_{n\geq 0} m_n x^n = 1 + x + 2x^2 + \cdots$ satisfies the equation

$$x^2m^2 + (x-1)m + 1 = 0 \tag{1}$$

and therefore that

$$m(x) = \frac{1 - x - \sqrt{1-2x-3x^2}}{2x^2}. \tag{2}$$

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Information and references on Motzkin numbers can be found in Bernhart [1], Donaghey and Shapiro [3], EIS [7], and Stanley [8]. The sequence of Motzkin numbers begins

$$(m_n)_{n\geq 0} = (1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, \ldots)$$

and is listed as sequence A001006 in EIS [7].

From a number theoretical perspective, it is natural to pose the following question. Let $(M_n)_{n\geq 0}$ be any sequence of rational numbers starting with two initial values M_0 and M_1 and such that the recurrence

$$(n+2)M_n = (2n+1)M_{n-1} + 3(n-1)M_{n-2} \tag{3}$$

holds for all $n \geq 2$. *What are the odds that all the numbers M_n which arise in this way are integers?* That is, what are the odds that with the integers M_{n-1} and M_{n-2} already constructed, the linear combination $(2n+1)M_{n-1} + 3(n-1)M_{n-2}$ is always a multiple of $n+2$? If $M_0 = M_1 = 1$ then we know, apriorically, that M_n is always a positive integer as being the cardinality of a finite set, but what about if we start with values $M_0 \neq M_1$?

From now on, we use the notation

$$M(\mu, \lambda) = (M_n(\mu, \lambda))_{n\geq 0} = (M_n)_{n\geq 0}$$

for any sequence of rational numbers which satisfies recurrence (3) for all $n \geq 2$ and $M_0 = \mu, M_1 = \lambda$. We call such sequences $M(\mu, \lambda)$ *Motzkin-type sequences*. So $m_n = M_n(1, 1)$ for every $n \geq 0$. For any integer k we write $P(k)$ for the largest prime factor of k with the convention that $P(0) = P(\pm 1) = 1$.

We have the following result.

Theorem 1.

Let $M(\mu, \lambda) = (M_n)_{n\geq 0}$ be any Motzkin-type sequence of rational numbers with $M_0 = \mu \neq \lambda = M_1$. For any n write $M_n = a_n/b_n$, where a_n and b_n are coprime integers with $b_n \geq 1$. Then $\limsup_{n\to\infty} P(b_n) = \infty$.

In particular, the only Motzkin-type sequences of integers are the integral multiples of the sequence $(m_n)_{n\geq 0}$, i.e., the sequences $M(a, a)$ where a is an integer.

Let $T > 1$ be any integer and let $(u_n)_{n\geq 0}$ be any sequence of integers. We say that $(u_n)_{n\geq 0}$ is *eventually periodic modulo* T if there exists positive integers n_0 and n_1 such that the congruence $u_n \equiv u_{n+n_1} \pmod{T}$ is satisfied for all $n \geq n_0$.

Theorem 2.

For any prime number p, the Motzkin sequence $M(1,1) = (m_n)_{n\geq 0}$ is not eventually periodic modulo p.

The cases $p = 2$ and $p > 2$ are in the proof handled in different ways. Interestingly, $(m_n)_{n\geq 0}$ modulo 2 is related to the famous Thue–Morse sequence (we discuss this relation in the concluding comments).

The Proofs

The Proof of Theorem 1. We start by noticing that to prove the assertion of Theorem 1 it suffices to prove:

There exist rational numbers $\alpha \neq \beta$ such that the sequence $M(\alpha, \beta)$ contains rational numbers whose denominators are divisible by arbitrarily large primes.

Indeed, if $M(\mu, \lambda)$ is any Motzkin-type sequence with $\mu \neq \lambda$, then

$$M(\mu, \lambda) = \tfrac{\alpha\lambda - \beta\mu}{\alpha - \beta} \cdot M(1,1) + \tfrac{\mu - \lambda}{\alpha - \beta} \cdot M(\alpha, \beta).$$

Since $M(1,1)$ is integral and $\frac{\mu-\lambda}{\alpha-\beta} \neq 0$, we conclude that also the denominators in $M(\mu, \lambda)$ are divisible by arbitrarily large primes.

Now we find such α and β. For a power series $F(x) = a_0 + a_1 x + \cdots$ and an integer $n \geq 0$ we denote by $[x^n]F$ the coefficient a_n of x^n. Let $(M_n)_{n\geq 0}$ be a Motzkin-type sequence and $M = M(x) = \sum_{n\geq 0} M_n x^{n+2}$ be its modified generating function. It is easily checked that the relation (3) for $n \geq 2$ is equivalent to

$$[x^{n+1}]\ (1 - 2x - 3x^2)M' + (3x + 1)M = 0 \quad \text{for } n \geq 2.$$

Denoting $g = g(x) = 1 - 2x - 3x^2$ and completing $M(x)$ to $S = S(x) = a + bx + M(x)$, we have for every a and b the relation

$$gS' - \frac{1}{2}g'S = (a + b) + (2M_0 + 3a - b)x + (3M_1 - 3M_0)x^2.$$

We want to select a, b, M_0, and M_1 so that $S(0) = 0$ and the right hand side of the last identity equals $g(x)$. This gives a simple system of linear equations whose solution is

$$(a,\ b,\ M_0,\ M_1) = (0,\ 1,\ -1/2,\ -3/2).$$

We show that $\alpha = M_0 = -\frac{1}{2}$ and $\beta = M_1 = -\frac{3}{2}$ have the stated property; the sequence begins

$$M(-\tfrac{1}{2}, -\tfrac{3}{2}) = (-\tfrac{1}{2}, -\tfrac{3}{2}, -\tfrac{9}{4}, -\tfrac{99}{20}, -\tfrac{54}{5}, -\tfrac{891}{35}, -\tfrac{17253}{280}, -\tfrac{43011}{280}, -\tfrac{10935}{28}, \ldots).$$

With this choice the differential equation for S becomes

$$gS' - \frac{1}{2}g'S = g$$

and can be rewritten as

$$\frac{d}{dx}\left(\frac{S}{\sqrt{g}}\right) = \frac{1}{\sqrt{g}}.$$

The solution (in the ring of formal power series) is

$$S(x) = \sqrt{g} \cdot \int \frac{dx}{\sqrt{g}}$$

where the formal integral has constant term 0 because $S(0) = 0$. Expanding

$$\sqrt{g} = \sqrt{1 - 2x - 3x^2} = \sum_{n \geq 0} c_n x^n \quad \text{and} \quad \frac{1}{\sqrt{g}} = \frac{1}{\sqrt{1 - 2x - 3x^2}} = \sum_{n \geq 0} d_n x^n,$$

we get

$$S(x) = \Big(\sum_{n \geq 0} c_n x^n\Big) \cdot \Big(\sum_{n \geq 1} \frac{d_{n-1}}{n} x^n\Big).$$

Since $[x^m]S = M_{m-2}$, we have for $m \geq 2$ the formula

$$M_{m-2} = \sum_{n=0}^{m-1} \frac{c_n d_{m-n-1}}{m-n} = \frac{d_{m-1}}{m} + \sum_{n=1}^{m-1} \frac{c_n d_{m-n-1}}{m-n}. \tag{4}$$

From (2) we have $\sqrt{g(x)} = x - 1 + 2x^2 \cdot m(x) = -1 - x + \cdots$ and thus all c_n are integers. This implies that also all d_n, as coefficients in the reciprocal series, are integers.

We shall show that if $m = p > 3$ is a prime number, then d_{p-1} is not a multiple of p. Assume that we have proved this. Then $\frac{d_{p-1}}{p}$ is in lowest terms and its denominator is a multiple of p. Clearly, $\sum_{n=1}^{p-1} \frac{c_n d_{p-n-1}}{p-n}$ brought to lowest terms is a rational number whose denominator is not divisible by p. By (4), the sum M_{p-2} is a rational number whose denominator is a multiple of p. Since $p > 3$ is an arbitrary prime, Theorem 1 follows.

It remains to prove that d_{p-1} is not a multiple of p for any prime $p > 3$. We have

$$\begin{aligned} d_n &= [x^n](1-2x-3x^2)^{-1/2} = [x^n](1+x)^{-1/2}(1-3x)^{-1/2} \\ &= \sum_{i=0}^{n}(-3)^i \binom{-1/2}{i}\binom{-1/2}{n-i} \\ &= \frac{(-1)^n}{4^n}\sum_{i=0}^{n}(-3)^i \binom{2i}{i}\binom{2n-2i}{n-i}. \end{aligned} \tag{5}$$

For any prime $p > 2$ and $i = 0, 1, \ldots, p-1$ it is easy to see that

$$\binom{2i}{i} \not\equiv 0 \bmod p \text{ if and only if } i \leq \frac{p-1}{2}.$$

Thus in (5) for $n = p-1$ all products of the two binomial coefficients are divisible by p except for the single product corresponding to $i = (p-1)/2$. Indeed, $d_{p-1} \not\equiv 0 \bmod p$ for any prime $p > 3$. Theorem 1 is therefore proved.

Remark. We leave for the interested reader to check as an exercise that we proved more than what is asserted in Theorem 1: For every Motzkin-type sequence $M(\lambda, \mu)$ with $\mu \neq \lambda$ there is an n_0 such that for every prime $p > n_0$ the fraction $M_{p-2}(\lambda, \mu)$ has denominator divisible by p. In fact, a quick look at formula (3) reveals that if one wants to create a "large" prime in the denominator of M_n, then probably the best "guess" is to take $n+2$ (i.e., the coefficient of M_n in formula (3)) to be a prime number. Thus, all we have showed is that this "guess" is indeed correct for all sufficiently large prime numbers p.

The Proof of Theorem 2. We have to distinguish two cases depending on whether p is odd or even.

1. $p > 2$. In the ring of formal power series with integral coefficients we call two power series f and g to be congruent modulo p if $f - g$ is a power series

all whose coefficients are multiples of p. Assume now that $p > 2$ and that there exist positive integers n_0 and n_1 such that $m_n \equiv m_{n+n_1} \pmod{p}$ for all $n \geq n_0$. Using (2) and the periodicity assumption we have, modulo p,

$$\begin{aligned}\sqrt{1-2x-3x^2} &= 1-x-\sum_{n\geq 0} 2m_n x^{n+2} \\ &\equiv 1-x-2\sum_{n=0}^{n_0-1} m_n x^{n+2} - 2\sum_{n=n_0}^{n_0+n_1-1} m_n x^{n+2}\cdot\sum_{n\geq 0} x^{nn_1}.\end{aligned}$$

Multiplying both sides of the above congruence with $1-x^{n_1}$ we get

$$(1-x^{n_1})\sqrt{1-2x-3x^2} \equiv P(x) \pmod{p}, \tag{6}$$

where $P(x)$ is the polynomial

$$(1-x^{n_1})(1-x-2\sum_{n=0}^{n_0-1} m_n x^{n+2}) - 2\sum_{n=n_0}^{n_0+n_1-1} m_n x^{n+2}.$$

Squaring both sides of (6) we get

$$(1-x^{n_1})^2(1-2x-3x^2) \equiv P(x)^2 \pmod{p}.$$

This congruence shows that in $\mathbf{F}_p[x]$, where $\mathbf{F}_p$ is the finite field with p elements, we have

$$(1-x^{n_1})^2(1-2x-3x^2) = P(x)^2.$$

Since $\mathbf{F}_p[x]$ is an UFD, it follows that $1-x^{n_1}$ divides $P(x)$. Writing $P(x) = (1-x^{n_1})R(x)$ with $R \in \mathbf{F}_p[x]$, we get that $1-2x-3x^2 = R(x)^2$. If $p=3$, then $1-2x-3x^2 = 1+x$ cannot be the square of a polynomial in $\mathbf{F}_3[x]$ (because it has odd degree). If $p > 3$, then $1-2x-3x^2$ is a polynomial of degree 2 whose discriminant $16 = 2^4$ is not a multiple of p. Thus $1-2x-3x^2$ has two distinct roots (in the algebraic closure of $\mathbf{F}_p$) and cannot be the square of some other polynomial either. Theorem 2 is therefore proved in the case $p > 2$.

2. $p = 2$. It folows from (1) that for $n > 0$ we have

$$m_n = m_{n-1} + \sum_{i=0}^{n-2} m_i m_{n-2-i}.$$

Modulo $p = 2$ this gives

$$m_n \equiv \begin{cases} m_{n-1} & \text{for odd } n \\ m_{n-1} + m_{n/2-1} & \text{for even } n. \end{cases}$$

It suffices to look only on the parity of m_{2n} and hence for $n \geq 0$ we define $k_n \in \{0, 1\}$ to be the parity of m_{2n}. Then we have in $\mathbf{F}_2$ the recurrence $k_0 = 1$ and, for $n > 0$,

$$k_n = k_{n-1} + k_{\lfloor (n-1)/2 \rfloor}. \tag{7}$$

Thus

$$(k_n)_{n\geq 0} = (1, 0, 1, 1, 1, 0, 1, 0, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, \ldots). \tag{8}$$

For integers $n \geq 1$ we denote $\alpha_2(n)$ the maximum integer $m \geq 0$ such that 2^m divides n. From (7) it follows by an easy induction that for every $n \geq 0$

$$k_n \equiv 1 + \alpha_2(n+1) \pmod 2. \tag{9}$$

Hence m_n is eventually periodic modulo 2 if and only if $\alpha_2(n)$ is. But it is easy to prove that $\alpha_2(n)$ is not eventually periodic modulo 2. Suppose, for the contradiction, that there are positive integers n_0 and n_1 such that $\alpha_2(n + n_1) \equiv \alpha_2(n) \pmod 2$ for every $n \geq n_0$. Let $n_1 = 2^a b$ with b odd. We take an integer c such that $c \geq a$ and $2^c \geq n_0$. For every $n \geq 0$ we should have $c = \alpha_2(2^c) \equiv \alpha_2(2^c + nn_1) \pmod 2$. But $\alpha_2(2^c + 2^{c-a}bn_1) = \alpha_2(2^c(1+b^2)) = c+1$ because $1 + b^2 \equiv 2 \pmod 4$. Thus m_n is not eventually periodic modulo 2 and the proof of Theorem 2 is complete.

Comments

One can prove that the binary sequence $K = (k_n)_{n\geq 0}$, see (8), can be generated instead of (7) or (9) more effectively by starting with the one-term sequence 1 and then repeatedly replacing in one step all 0's and all 1's according to the rules $1 \mapsto 10$ and $0 \mapsto 11$. We get

1
10
1011
10111010
1011101010111011
⋮

Hence K is closely related to the classical *Thue–Morse sequence* $T = (t_n)_{n\geq 0} = (1, 0, 0, 1, 0, 1, 1, 0, \ldots)$ that is generated from 1 by the rules $1 \mapsto 10$ and $0 \mapsto 01$. It is known (see Lothaire [5] and Berstel [2] for more information on T) that T contains no three consecutive repetitions of any interval. In particular, T is very far from being eventually periodic. K has similar properties — we can prove that K contains no four consecutive repetitions of any interval.

The results from the present note extend easily to other sequences which naturally arise in enumerative combinatorics. One of such sequences is the Schröder sequence $(s_n)_{n\geq 1}$ (A001003 of EIS [7]). For $n \geq 1$, the number s_n counts the number of lattice paths in the cartesian plane starting at $(0,\ 0)$, ending at $(2n,\ 0)$, and which use line steps equal to either $(2,\ 0)$ (level step), or to $(1,\ 1)$ (up step), or to $(1,\ -1)$ (down step), and which never pass below the x-axis. It is well-known ([8]) that the Schröder numbers satisfy $s_1 = s_2 = 1$ and $ns_n = 3(2n-3)s_{n-1} - (n-3)s_{n-2}$ for all $n \geq 3$ and that

$$\sum_{n\geq 1} s_n x^n = \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{2x}.$$

We can prove in a way similar to the proof of Theorem 2 that $(s_n)_{n\geq 1}$ is not eventually periodic modulo p for any prime number p. An argument similar to the one used in the proof of Theorem 1 can be used to yield that if $S := (S_n)_{n\geq 1}$ is any sequence of rational numbers satisfying the Schröder recurrence $nS_n = 3(2n-3)S_{n-1} - (n-3)S_{n-2}$ for all $n \geq 3$ but with $S_1 \neq S_2$, then the sequence S contains rational numbers whose denominators are divisible by arbitrarily large primes. This suggests that an analysis of the arithmetical properties of sequences of rational numbers $(U_n)_{n\geq 0}$ which obey a recurrence relation of the type $f(n)U_n = g(n)U_{n-1} + h(n)U_{n-2}$ for all $n \geq 2$, where f, g, and h are linear (or higher degree) polynomials with rational coefficients in the variable n, might be worthwhile, but we shall treat this problem with a different occasion.

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