Jacobsthal Numbers and Alternating Sign Matrices *

Darrin D. Frey and James A. Sellers
Department of Science and Mathematics
Cedarville College
Cedarville, OH 45314

freyd@cedarville.edu   sellersj@cedarville.edu

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Abstract

We let $A(n)$ equal the number of $n \times n$ alternating sign matrices and we denote the m^{th} Jacobsthal number by J_m. From the work of a variety of sources, we know that

$$A(n) = \prod_{\ell=0}^{n-1} \frac{(3\ell+1)!}{(n+\ell)!}.$$

The values of $A(n)$ are, in general, highly composite. The primary goal of this paper is to prove that $A(n)$ is odd if and only if n is a Jacobsthal number, thus showing that $A(n)$ is odd infinitely often.

1 Introduction

In this paper we relate two seemingly unrelated areas of mathematics: alternating sign matrices and Jacobsthal numbers. We begin with a brief discussion of alternating sign matrices.

An $n \times n$ alternating sign matrix is a $n \times n$ matrix of 1s, 0s, and -1s satisfying two very special properties:

- the sum of the entries of each row and column must be 1, and
- the signs of the nonzero entries in every row and column must alternate.

Indeed, these alternating sign matrices include the permutation matrices, in which each row and column contains only one nonzero entry, a 1.

As an example, we exhibit here the seven 3×3 alternating sign matrices.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Throughout this paper, we let $A(n)$ denote the number of $n \times n$ alternating sign matrices. The determination of a closed formula for $A(n)$ was undertaken by a variety of mathematicians over the last 25

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years or so. David Bressoud's recent text [1] chronicles these endeavors and discusses in a very readable way the underlying mathematics. The interested reader is encouraged to peruse this text. The reader is also encouraged to see the recent survey article [2] by Bressoud and Propp.

As noted in [1], the formula for $A(n)$ is given by the following:

$$A(n) = \prod_{\ell=0}^{n-1} \frac{(3\ell+1)!}{(n+\ell)!} \tag{1}$$

It is clear from this formula that, for most values of n, $A(n)$ will be highly composite. Indeed, the presence of factorials in the numerator whose input values are much larger than those in the denominator of the formula guarantees this phenomenon.

At this point, we display a small table of values for $A(n)$. Note the high degree of compositeness about which we alluded above.

n	$A(n)$	Prime Factorization of $A(n)$
1	1	1
2	2	2
3	7	7
4	42	$2 \cdot 3 \cdot 7$
5	429	$3 \cdot 11 \cdot 13$
6	7436	$2^2 \cdot 11 \cdot 13^2$
7	218348	$2^2 \cdot 13^2 \cdot 17 \cdot 19$
8	10850216	$2^3 \cdot 13 \cdot 17^2 \cdot 19^2$
9	911835460	$2^2 \cdot 5 \cdot 17^2 \cdot 19^3 \cdot 23$
10	129534272700	$2^2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 17 \cdot 19^3 \cdot 23^2$
11	31095744852375	$3^2 \cdot 5^3 \cdot 7 \cdot 19^2 \cdot 23^3 \cdot 29 \cdot 31$
12	12611311859677500	$2^2 \cdot 3^3 \cdot 5^4 \cdot 19 \cdot 23^3 \cdot 29^2 \cdot 31^2$
13	8639383518297652500	$2^2 \cdot 3^5 \cdot 5^4 \cdot 23^2 \cdot 29^3 \cdot 31^3 \cdot 37$
14	9995541355448167482000	$2^4 \cdot 3^5 \cdot 5^3 \cdot 23 \cdot 29^4 \cdot 31^4 \cdot 37^2$
15	19529076234661277104897200	$2^4 \cdot 3^3 \cdot 5^2 \cdot 29^4 \cdot 31^5 \cdot 37^3 \cdot 41 \cdot 43$
16	64427185703425689356896743840	$2^5 \cdot 3^2 \cdot 5 \cdot 11 \cdot 29^3 \cdot 31^5 \cdot 37^4 \cdot 41^2 \cdot 43^2$
17	358869201916137601447486156417296	$2^4 \cdot 3 \cdot 7^2 \cdot 11 \cdot 29^2 \cdot 31^4 \cdot 37^5 \cdot 41^3 \cdot 43^3 \cdot 47$
18	3374860639258750562269514491522925456	$2^4 \cdot 7^3 \cdot 13 \cdot 29 \cdot 31^3 \cdot 37^6 \cdot 41^4 \cdot 43^4 \cdot 47^2$
19	53580350833984348888878646149709092313244	$2^2 \cdot 7^3 \cdot 13^2 \cdot 31^2 \cdot 37^6 \cdot 41^5 \cdot 43^5 \cdot 47^3 \cdot 53$
20	1436038934715538200913155682637051204376827212	$2^2 \cdot 7^4 \cdot 13^2 \cdot 31 \cdot 37^5 \cdot 41^6 \cdot 43^6 \cdot 47^4 \cdot 53^2$
21	64971294999808427895847904380524143538858551437757	$7^5 \cdot 13 \cdot 37^4 \cdot 41^6 \cdot 43^7 \cdot 47^5 \cdot 53^3 \cdot 59 \cdot 61$
22	4962007838317808727469503296608693231827094217799731304	$2^3 \cdot 3 \cdot 7^6 \cdot 37^3 \cdot 41^5 \cdot 43^7 \cdot 47^6 \cdot 53^4 \cdot 59^2 \cdot 61^2$

Table 1: Values for $A(n)$

These values are found as sequence A005130 in Sloane's On–Line Encyclopedia of Integer Sequences [8]. Other sequences of values related to alternating sign matrices can be found at [8] as well.

A careful examination of this table reveals something quite interesting. Namely, the first few values of n for which $A(n)$ is odd are $1, 3, 5, 11$, and 21. When we utilize a computer algebra package to evaluate (1), we find that a longer version of the list above is

$$1, 3, 5, 11, 21, 43, 85, 171.$$

These numbers are certainly well–known, often called the *Jacobsthal numbers*, and are found as sequence A001045 in Sloane's On–Line Encyclopedia of Integer Sequences [8]. They are defined by the recurrence

$$J_{n+2} = J_{n+1} + 2J_n \tag{2}$$

with initial values $J_0 = 1$ and $J_1 = 1$. Here J_n denotes the n^{th} Jacobsthal number.

This sequence of numbers has a rich history, especially in view of its relationship to the Fibonacci numbers. For examples of recent work involving the Jacobsthal numbers, see [3], [4], [5], and [6].

The primary goal of this paper is to prove that if n is a positive integer, then $A(n)$ is odd if and only if n is a Jacobsthal number.

2 The Necessary Machinery

In order to show that $A(J_m)$ is odd for each positive integer m, we simply need to show that the number of factors of 2 in the prime decomposition of $A(J_m)$ is zero.

To accomplish this, we develop formulas for the number of factors of 2 in

$$N(n) = \prod_{\ell=0}^{n-1} (3\ell+1)! \qquad \text{and} \qquad D(n) = \prod_{\ell=0}^{n-1} (n+\ell)!$$

respectively. Once we prove that the number of factors of 2 is the same for both $N(J_m)$ and $D(J_m)$ and not the same for $N(n)$ and $D(n)$ if n is not a Jacobsthal number, we will have our result.

The most classic formula for counting the number of factors of a prime p in $N!$ is arguably the most powerful piece of machinery used in this paper. We state it here.

Lemma 2.1. *The number of factors of a prime p in $N!$ is equal to*

$$\sum_{k\geq 1} \left\lfloor \frac{N}{p^k} \right\rfloor .$$

Proof. The proof of this result can be found in a variety of elementary number theory books. For example, see [7, Theorem 2.29]. □

With Lemma 2.1 in hand, we now see that the number of factors of 2 in $N(n)$ is given by

$$\begin{aligned} N^{\#}(n) &= \sum_{\ell=0}^{n-1} \sum_{k\geq 1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor \\ &= \sum_{k\geq 1} \sum_{\ell=0}^{n-1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor \\ &= \sum_{k\geq 1} N_k^{\#}(n) \end{aligned}$$

where

$$N_k^{\#}(n) = \sum_{\ell=0}^{n-1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor . \tag{3}$$

Moreover, the number of factors of 2 in $D(n)$ is given by

$$\begin{aligned} D^{\#}(n) &= \sum_{\ell=0}^{n-1} \sum_{k\geq 1} \left\lfloor \frac{n+\ell}{2^k} \right\rfloor \\ &= \sum_{k\geq 1} \sum_{\ell=0}^{n-1} \left\lfloor \frac{n+\ell}{2^k} \right\rfloor \\ &= \sum_{k\geq 1} D_k^{\#}(n) \end{aligned}$$

where

$$D_k^{\#}(n) = \sum_{\ell=0}^{n-1} \left\lfloor \frac{n+\ell}{2^k} \right\rfloor . \tag{4}$$

Formulas (3) and (4) will be explored in great detail below.

Before closing this section of necessary machinery, we quickly develop a closed form formula for the m^{th} Jacobsthal number J_m. The technique used below is a common one, and can be found in [9].

Theorem 2.2. *The m^{th} Jacobsthal number J_m is given by*

$$J_m = \frac{2^{m+1} + (-1)^m}{3}.$$

Proof. We assume that $J_m = \alpha^m$ for some nonzero real number α. Then from (2) we know that

$$\alpha^{m+2} = \alpha^m + 2\alpha^m.$$

Dividing both sides of this equation by α^m yields:

$$\begin{aligned} &\alpha^2 = \alpha + 2 \\ &\alpha^2 - \alpha - 2 = 0 \\ &(\alpha - 2)(\alpha + 1) = 0 \end{aligned}$$

Hence, we know $\alpha = 2$ or $\alpha = -1$. At this point, we have $J_m = c_1(2)^m + c_2(-1)^m$ for some reals c_1 and c_2. We can use the facts that $J_0 = 1$ and $J_1 = 1$ to determine that $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$. This then implies the desired result. ☐

3 Formulas for $N_k^{\#}(n)$ and $D_k^{\#}(n)$

We now turn our attention to the development of "nice" formulas for $N_k^{\#}(n)$ and $D_k^{\#}(n)$.

Lemma 3.1. *The smallest value of ℓ for which*

$$\left\lfloor \frac{3\ell + 1}{2^k} \right\rfloor = m,$$

where m and k are positive integers and $k \geq 2$, is

$$\begin{cases} \frac{m}{3}2^k & \text{if} \quad m \equiv 0 \pmod 3 \\ \frac{m-1}{3}2^k + J_{k-1} & \text{if} \quad m \equiv 1 \pmod 3 \\ \frac{m-2}{3}2^k + J_k & \text{if} \quad m \equiv 2 \pmod 3. \end{cases}$$

Proof. Suppose $m \equiv 0 \pmod 3$ and $\ell = \dfrac{m}{3}2^k$. Then

$$\left\lfloor \frac{3\ell + 1}{2^k} \right\rfloor = \left\lfloor \frac{3\left(\frac{m}{3}2^k\right) + 1}{2^k} \right\rfloor = \left\lfloor \frac{m2^k}{2^k} + \frac{1}{2^k} \right\rfloor = m,$$

and no smaller value of ℓ yields m since the numerators differ by multiples of three.

If $m \equiv 1 \pmod 3$ and $\ell = \dfrac{m-1}{3}2^k + J_{k-1}$, then

$$\begin{aligned} \left\lfloor \frac{3\ell + 1}{2^k} \right\rfloor &= \left\lfloor \frac{3\left(\frac{m-1}{3}2^k + J_{k-1}\right) + 1}{2^k} \right\rfloor \\ &= \left\lfloor \frac{(m-1)2^k + 3\left(\frac{2^k + (-1)^{k-1}}{3}\right) + 1}{2^k} \right\rfloor \\ &= \left\lfloor \frac{(m-1)2^k + 2^k + (-1)^{k-1} + 1}{2^k} \right\rfloor \\ &= m, \quad \text{if } k \geq 2, \end{aligned}$$

and no smaller value of ℓ yields m.

If $m \equiv 2 \pmod 3$ and $\ell = \frac{m-2}{3}2^k + J_k$, then

$$\begin{aligned}
\left\lfloor \frac{3\ell+1}{2^k} \right\rfloor &= \left\lfloor \frac{3\left(\frac{m-2}{3}2^k + J_k\right)+1}{2^k} \right\rfloor \\
&= \left\lfloor \frac{(m-2)2^k + 3\left(\frac{2^{k+1}+(-1)^k}{3}\right)+1}{2^k} \right\rfloor \\
&= \left\lfloor \frac{(m-2)2^k + 2^{k+1} + (-1)^k + 1}{2^k} \right\rfloor \\
&= m,
\end{aligned}$$

and no smaller value of ℓ yields m.

□

Lemma 3.2. *For any positive integer k, $J_{k-1} + J_k = 2^k$.*

Proof.

$$\begin{aligned}
J_{k-1} + J_k &= \frac{2^k + (-1)^{k-1}}{3} + \frac{2^{k+1} + (-1)^k}{3} \\
&= \frac{2^k + 2^{k+1} + (-1)^{k-1} + (-1)^k}{3} \\
&= \frac{2^k(1+2)}{3} \\
&= 2^k
\end{aligned}$$

□

Lemma 3.3. *For any positive integer k,*

$$\sum_{v=0}^{2^k-1} \left\lfloor \frac{3v+1}{2^k} \right\rfloor = 2^k.$$

Proof. If $k = 1$, then the sum in the statement of the Lemma is

$$\sum_{v=0}^{1} \left\lfloor \frac{3v+1}{2} \right\rfloor$$

which is equal to $\left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{4}{2} \right\rfloor = 2$ and we have our result.

If $k \geq 2$, then by Lemma 3.1, J_{k-1} is the smallest value of v for which $\left\lfloor \frac{3v+1}{2^k} \right\rfloor = 1$ and J_k is the smallest value of v for which $\left\lfloor \frac{3v+1}{2^k} \right\rfloor = 2$. Thus,

$$\begin{aligned}
\sum_{v=0}^{2^k-1} \left\lfloor \frac{3v+1}{2^k} \right\rfloor &= 0 \times J_{k-1} + 1 \times [(J_k - 1) - (J_{k-1} - 1)] + 2 \times [(2^k - 1) - (J_k - 1)] \\
&= J_k - J_{k-1} + 2(2^k - J_k) \\
&= 2^{k+1} - J_k - J_{k-1} \\
&= 2^{k+1} - 2^k \quad \text{by Lemma 3.2} \\
&= 2^k.
\end{aligned}$$

$\square$

Theorem 3.4. *Let $n = 2^k q + r$ where q is a nonnegative integer and $0 \le r < 2^k$. Then we have*

$$N_k^{\#}(n) = \binom{n-r}{2^{k+1}} (3(n-r) - 2^k) + tail(n) \tag{5}$$

where

$$tail(n) = \begin{cases} 3qr & \text{if} \quad 0 \le r \le J_{k-1} \\ 3qr + (r - J_{k-1}) & \text{if} \quad J_{k-1} < r \le J_k \\ (3q+2)r - 2^k & \text{if} \quad J_k < r < 2^k. \end{cases} \tag{6}$$

Proof. To analyze the sum

$$N_k^{\#}(n) = \sum_{\ell=0}^{n-1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor$$

we let $\ell = 2^k u + v$, where $0 \le v < 2^k$. Then

$$\left\lfloor \frac{3\ell+1}{2^k} \right\rfloor = \left\lfloor \frac{3(2^k u + v)+1}{2^k} \right\rfloor = \left\lfloor \frac{2^k(3u)}{2^k} + \frac{3v+1}{2^k} \right\rfloor = 3u + \left\lfloor \frac{3v+1}{2^k} \right\rfloor.$$

Thus,

$$\begin{aligned}
\sum_{\ell=0}^{2^k q-1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor &= \sum_{u=0}^{q-1} \sum_{v=0}^{2^k-1} \left(3u + \left\lfloor \frac{3v+1}{2^k} \right\rfloor \right) \\
&= \sum_{u=0}^{q-1} \left((3u)2^k + \sum_{v=0}^{2^k-1} \left\lfloor \frac{3v+1}{2^k} \right\rfloor \right) \\
&= \sum_{u=0}^{q-1} ((3u)2^k + 2^k) \qquad \text{by Lemma 3.3} \\
&= 2^k \sum_{u=0}^{q-1} (3u+1) \\
&= 2^k (3(0 + 1 + \cdots + (q-1)) + q) \\
&= 2^k \left(3 \left(\frac{(q-1)q}{2} \right) + q \right) \\
&= 2^k q \left(3 \left(\frac{n - r - 2^k}{2^{k+1}} \right) + 1 \right) \\
&= q \left(\frac{3(n-r-2^k)}{2} + 2^k \right) \\
&= \binom{q}{2} (3(n - r - 2^k) + 2^{k+1}) \\
&= \binom{n-r}{2^{k+1}} (3(n-r) - 2^k).
\end{aligned}$$

If $r = 0$, we have our result. If $r > 0$ and $k = 1$, then $r = 1$ and we have one extra term in our sum, namely,

$$\left\lfloor \frac{3(2q)+1}{2} \right\rfloor = 3q$$

and again we have our result since $r = 1$. If $r > 0$ and $k \ge 2$, then by Lemma 3.1, $2^k q$ is the smallest value of ℓ for which $\left\lfloor \frac{3\ell+1}{2^k} \right\rfloor = 3q$, $2^k q + J_{k-1}$ is the smallest value of ℓ for which

$$\left\lfloor \frac{3\ell+1}{2^k} \right\rfloor = 3q + 1,$$

and $2^k q + J_k$ is the smallest value of ℓ for which

$$\left\lfloor \frac{3\ell+1}{2^k} \right\rfloor = 3q+2.$$

Hence,

$$\sum_{\ell=2^k q}^{2^k q+r-1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor = \begin{cases} 3qr & \text{if} \quad r \le J_{k-1} \\ 3qJ_{k-1} + (3q+1)(r - J_{k-1}) & \text{if } J_{k-1} < r \le J_k \\ 3qJ_{k-1} + (3q+1)(J_k - J_{k-1}) + (3q+2)(r - J_k) & \text{if } J_k < r < 2^k. \end{cases}$$

So, if $n = 2^k q + r$ where $0 \le r < 2^k$,

$$\begin{aligned} N_k^{\#}(n) &= \sum_{\ell=0}^{n-1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor \\ &= \sum_{\ell=0}^{2^k q-1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor + \sum_{\ell=2^k q}^{2^k q+r-1} \left\lfloor \frac{3\ell+1}{2^k} \right\rfloor \\ &= \left(\frac{n-r}{2^{k+1}} \right) (3(n-r) - 2^k) + tail(n), \end{aligned}$$

where

$$tail(n) = \begin{cases} 3qr & \text{if} \quad r \le J_{k-1} \\ 3qJ_{k-1} + (3q+1)(r - J_{k-1}) & \text{if } J_{k-1} < r \le J_k \\ 3qJ_{k-1} + (3q+1)(J_k - J_{k-1}) + (3q+2)(r - J_k) & \text{if } J_k < r < 2^k. \end{cases}$$

The second expression in $tail(n)$ is clearly equal to $3qr + r - J_{k-1}$. For the third expression, we have

$$\begin{aligned} 3qJ_{k-1} + (3q+1)(J_k - J_{k-1}) + (3q+2)(r - J_k) &= 3qr + J_k - J_{k-1} + 2r - 2J_k \\ &= (3q+2)r - 2^k \quad \text{by Lemma 3.2.} \end{aligned}$$

□

Theorem 3.5. *Let $n = 2^k q + r$ where q is a nonnegative integer and $0 \le r < 2^k$. Then we have*

$$D_k^{\#}(n) = \begin{cases} \left(\frac{n-r}{2^{k+1}} \right) (3(n+r) - 2^k) & \text{if} \quad 0 \le r \le 2^{k-1} \\ \left(\frac{n-(2^k-r)}{2^{k+1}} \right) (3(n-r) + 2^{k+1}) & \text{if} \quad 2^{k-1} < r < 2^k. \end{cases} \tag{7}$$

Proof. We begin with the definition

$$D_k^{\#}(n) = \sum_{\ell=0}^{n-1} \left\lfloor \frac{n+\ell}{2^k} \right\rfloor$$

and rewrite it as

$$D_k^{\#}(n) = \sum_{\ell=0}^{2n-1} \left\lfloor \frac{\ell}{2^k} \right\rfloor - \sum_{\ell=0}^{n-1} \left\lfloor \frac{\ell}{2^k} \right\rfloor.$$

In both sums,

$$\left\lfloor \frac{\ell}{2^k} \right\rfloor = s$$

if $2^k s \le \ell < 2^k(s+1)$, so if $n = 2^k q + r$, where $0 < r \le 2^k$, we have

$$
\begin{aligned}
\sum_{\ell=0}^{n-1} \left\lfloor \frac{\ell}{2^k} \right\rfloor &= 2^k\,[1+2+\cdots+q-1]+qr \\
&= 2^k \left[\frac{(q-1)q}{2}\right] + qr \\
&= q\left(\frac{2^k}{2}q - \frac{2^k}{2} + r\right) \\
&= q\left(\frac{2^k}{2}\left(\frac{n-r}{2^k}\right) - \frac{2^k}{2} + r\right) \\
&= q\left(\frac{n-r-2^k}{2} + r\right) \\
&= q\left(\frac{n+r-2^k}{2}\right).
\end{aligned}
$$

If $0 < r \le 2^{k-1}$, then $2n-1 = 2^k(2q) + (2r-1)$, which means

$$
\begin{aligned}
\sum_{\ell=0}^{2n-1} \left\lfloor \frac{\ell}{2^k} \right\rfloor &= 2^k[1+2+\cdots+(2q-1)] + (2r-1+1)(2q) \\
&= 2^k\left(\frac{(2q-1)(2q)}{2}\right) + 2r(2q) \\
&= 2q\left(\frac{2^k}{2}\left(2\left(\frac{n-r}{2^k}\right)-1\right)+2r\right) \\
&= 2q\left(\frac{2n-2r-2^k}{2}+2r\right) \\
&= 2q\left(\frac{2n+2r-2^k}{2}\right) \\
&= q(2n+2r-2^k).
\end{aligned}
$$

Hence, in this case,

$$
\begin{aligned}
D_k^{\#}(n) &= \sum_{\ell=0}^{n-1} \left\lfloor \frac{n+\ell}{2^k} \right\rfloor \\
&= \sum_{\ell=0}^{2n-1} \left\lfloor \frac{\ell}{2^k} \right\rfloor - \sum_{\ell=0}^{n-1} \left\lfloor \frac{\ell}{2^k} \right\rfloor \\
&= q(2n+2r-2^k) - q\left(\frac{n+r-2^k}{2}\right) \\
&= q\left(\frac{4n+4r-2^{k+1}-n-r+2^k}{2}\right) \\
&= \frac{q}{2}(3(n+r)-2^k) \\
&= \left(\frac{n-r}{2^{k+1}}\right)(3(n+r)-2^k).
\end{aligned}
$$

If $2^{k-1} < r \le 2^k$, say, $r = 2^{k-1}+s$ where $0 < s \le 2^{k-1}$, then

$$\begin{aligned}
2n-1 &= 2(2^k q + r) - 1 \\
&= 2^k(2q) + 2(2^{k-1} + s) - 1 \\
&= 2^k(2q+1) + 2s - 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{\ell=0}^{2n-1} \left\lfloor \frac{\ell}{2^k} \right\rfloor &= 2^k[1 + 2 + \cdots + 2q] + (2s - 1 + 1)(2q+1) \\
&= 2^k \left(\frac{(2q)(2q+1)}{2} \right) + 2s(2q+1) \\
&= (2q+1) \left(\frac{2 \cdot 2^k \left(\frac{n-r}{2^k}\right)}{2} + 2s \right) \\
&= (2q+1)\,(n - r + 2s) \\
&= (2q+1)(n - r + 2(r - 2^{k-1})) \\
&= (2q+1)(n + r - 2^k).
\end{aligned}$$

So, in this case,

$$\begin{aligned}
D_k^{\#}(n) &= \sum_{\ell=0}^{n-1} \left\lfloor \frac{n+\ell}{2^k} \right\rfloor \\
&= \sum_{\ell=0}^{2n-1} \left\lfloor \frac{\ell}{2^k} \right\rfloor - \sum_{\ell=0}^{n-1} \left\lfloor \frac{\ell}{2^k} \right\rfloor \\
&= (2q+1)(n + r - 2^k) - q\left(\frac{n + r - 2^k}{2} \right) \\
&= (n + r - 2^k)\left(2q + 1 - \frac{q}{2}\right) \\
&= (n + r - 2^k)\left(\frac{3q}{2} + 1\right) \\
&= \left(\frac{n + r - 2^k}{2^{k+1}} \right)(3(n-r) + 2^{k+1}).
\end{aligned}$$

The alert reader will note that in the statement of the Theorem, we have the cases separated as $0 \le r \le 2^{k-1}$ and $2^{k-1} < r < 2^k$ whereas in the proof, the cases are $0 < r \le 2^{k-1}$ and $2^{k-1} < r \le 2^k$. However, both are equivalent since $\frac{n-0}{2^{k+1}}(3(n+0) - 2^k) = \frac{n - (2^k - 2^k)}{2^{k+1}}(3(n - 2^k) + 2^{k+1})$. Thus, we have our result. $\square$

4 $A(J_m)$ is odd

Now that we have closed formulas for $N_k^{\#}(n)$ and $D_k^{\#}(n)$, we can proceed to prove that $A(J_m)$ is odd for all Jacobsthal numbers J_m.

Theorem 4.1. *For all positive integers m, $A(J_m)$ is odd.*

Proof. The proof of this theorem simply involves plugging J_m into the formulas (5) and (7) and showing that $N_k^{\#}(J_m) = D_k^{\#}(J_m)$ for all k. This implies that $N^{\#}(J_m) = D^{\#}(J_m)$, which means the number of factors of 2 in $A(J_m)$ is zero. Our theorem is then proven.

We now break the proof into two cases, based on whether or not the parity of k is equal to the parity of m.

- **Case 1:** The parity of m equals the parity of k.

  In this case we have

$$\begin{aligned} 2^k(J_{m-k}-1)+J_k &= 2^k\left(\frac{2^{m-k+1}+(-1)^{m-k}}{3}-1\right)+\frac{2^{k+1}+(-1)^k}{3} \\ &= \frac{2^{m+1}+2^k-3\cdot 2^k+2^{k+1}+(-1)^k}{3} \quad \text{since } (-1)^{m-k}=1 \\ &= \frac{2^{m+1}+(-1)^m}{3} \quad \text{since } (-1)^k=(-1)^m \\ &= J_m \end{aligned}$$

  Thus, in the notation of Theorems 3.4 and 3.5, $q = J_{m-k}-1$ and $r = J_k$. We now calculate $N_k^{\#}(J_m)$ and $D_k^{\#}(J_m)$ using Theorems 3.4 and 3.5.

$$\begin{aligned} N_k^{\#}(J_m) &= \left(\frac{J_m-J_k}{2^{k+1}}\right)\left(3(J_m-J_k)-2^k\right) \\ &\quad +3(J_{m-k}-1)J_k+(J_k-J_{k-1}) \\ &= \frac{1}{2^{k+1}}\left(\frac{2^{m+1}+(-1)^m}{3}-\frac{2^{k+1}+(-1)^k}{3}\right)\left(3\left(\frac{2^{m+1}+(-1)^m}{3}-\frac{2^{k+1}+(-1)^k}{3}\right)-2^k\right) \\ &\quad +(3J_{m-k}-1)J_k-2^k \quad \text{by Lemma 3.2} \\ &= \frac{1}{3\cdot 2^{k+1}}\left(2^{m+1}-2^{k+1}\right)\left(2^{m+1}-2^{k+1}-2^k\right) \\ &\quad +\left(3\left(\frac{2^{m-k+1}+(-1)^{m-k}}{3}\right)-1\right)\left(\frac{2^{k+1}+(-1)^k}{3}\right)-2^k \quad \text{since } (-1)^m=(-1)^k \\ &= \frac{1}{3}\left(2^{2m-k+1}-2^{m+2}+2^{k+1}-2^m+2^k\right) \\ &\quad +\frac{1}{3}(2^{m-k+1}(2^{k+1}+(-1)^k)-3\cdot 2^k) \quad \text{since } (-1)^{m-k}=1 \\ &= \frac{1}{3}\left(2^{2m-k+1}-2^m+(-1)^k 2^{m-k+1}\right) \end{aligned}$$

  after much simplification. Next, we calculate $D_k^{\#}(J_m)$, recalling that $2^{k-1} < r = J_k < 2^k$.

$$\begin{aligned} D_k^{\#}(J_m) &= \frac{\left(J_m-2^k+J_k\right)}{2^{k+1}}\left(3(J_m-J_k)+2^{k+1}\right) \\ &= \frac{1}{2^{k+1}}\left(\frac{2^{m+1}+(-1)^m}{3}+\frac{2^{k+1}+(-1)^k}{3}-2^k\right)\left(3\left(\frac{2^{m+1}+(-1)^m}{3}-\frac{2^{k+1}+(-1)^k}{3}\right)+2^{k+1}\right) \\ &= \frac{1}{3\cdot 2^{k+1}}\left(2^{m+1}+2^{k+1}+2(-1)^k-3\cdot 2^k\right)\left(2^{m+1}-2^{k+1}+2^{k+1}\right) \quad \text{since } (-1)^m=(-1)^k \\ &= \frac{1}{3}\left(2^{2m-k+1}+2^{m+1}+2^{m-k+1}(-1)^k-3\cdot 2^m\right) \\ &= \frac{1}{3}\left(2^{2m-k+1}-2^m+(-1)^k 2^{m-k+1}\right) \end{aligned}$$

after simplification. We see that $N_k^{\#}(J_m) = D_k^{\#}(J_m)$ in this case.

- **Case 2:** The parity of m is not equal to the parity of k.

  In this case we have

$$\begin{aligned} 2^k(J_{m-k}) + J_{k-1} &= 2^k\left(\frac{2^{m-k+1} + (-1)^{m-k}}{3}\right) + \frac{2^k + (-1)^{k-1}}{3} \\ &= \frac{2^{m+1} - 2^k + 2^k + (-1)^{k-1}}{3} \\ &= J_m. \end{aligned}$$

  Thus, in the notation of Theorems 3.4 and 3.5, $q = J_{m-k}$ and $r = J_{k-1}$. We now calculate $N_k^{\#}(J_m)$ and $D_k^{\#}(J_m)$ using Theorems 3.4 and 3.5.

$$\begin{aligned} N_k^{\#}(J_m) &= \left(\frac{J_m - J_{k-1}}{2^{k+1}}\right)\left(3(J_m - J_{k-1}) - 2^k\right) \\ &\quad + 3J_{m-k}J_{k-1} \\ &= \frac{1}{2^{k+1}}\left(\frac{2^{m+1}+(-1)^m}{3} - \frac{2^k+(-1)^{k-1}}{3}\right)\left(3\left(\frac{2^{m+1}+(-1)^m}{3} - \frac{2^k+(-1)^{k-1}}{3}\right) - 2^k\right) \\ &\quad + 3\left(\frac{2^{m-k+1}+(-1)^{m-k}}{3}\right)\left(\frac{2^k+(-1)^{k-1}}{3}\right) \\ &= \frac{1}{3\cdot 2^{k+1}}\left(2^{m+1} - 2^k\right)\left(2^{m+1} - 2\cdot 2^k\right) \\ &\quad + \frac{1}{3}((2^{m-k+1}-1)(2^k+(-1)^{k-1})) \ \text{ since } (-1)^m = (-1)^{k-1} \text{ and } (-1)^{m-k} = -1 \\ &= \frac{1}{3}(2^{2m-k+1} - 2^{m+1} - 2^m + 2^k + 2^{m+1} - 2^k + 2^{m-k+1}(-1)^{k-1} + (-1)^k) \\ &= \frac{1}{3}(2^{2m-k+1} - 2^m + 2^{m-k+1}(-1)^{k-1} + (-1)^k) \end{aligned}$$

  after much simplification.

  Now we calculate $D_k^{\#}(J_m)$, recalling that $0 < r < 2^{k-1}$.

$$\begin{aligned} D_k^{\#}(J_m) &= \frac{(J_m - J_{k-1})}{2^{k+1}}(3(J_m + J_{k-1}) - 2^k) \\ &= \frac{1}{2^{k+1}}\left(\frac{2^{m+1}+(-1)^m}{3} - \frac{2^k+(-1)^{k-1}}{3}\right)\left(3\left(\frac{2^{m+1}+(-1)^m}{3} + \frac{2^k+(-1)^{k-1}}{3}\right) - 2^k\right) \\ &= \frac{1}{3\cdot 2^{k+1}}(2^{m+1} - 2^k)(2^{m+1} + 2(-1)^{k-1}) \ \text{ since } (-1)^m = (-1)^{k-1} \\ &= \frac{1}{3}(2^{2m-k+1} - 2^m + 2^{m-k+1}(-1)^{k-1} + (-1)^k) \end{aligned}$$

  after simplification. We see that $N_k^{\#}(J_m) = D_k^{\#}(J_m)$ in this case.

This completes the proof that $A(J_m)$ is odd for all Jacobsthal numbers J_m.

□

5 The Converse

We now wish to prove the converse of Theorem 4.1. That is, we want to prove that $A(n)$ is even if n is not a Jacobsthal number. As a guide in how to proceed, we include a table of values for $N_k^{\#}(n)$ and $D_k^{\#}(n)$ for small values of n and k. This table suggests that $N_k^{\#}(n) \geq D_k^{\#}(n)$ for all positive integers n and k. It also suggests that for each value of n, there is at least one value of k for which $N_k^{\#}(n)$ is strictly greater than $D_k^{\#}(n)$ except when n is a Jacobsthal number. (To aid in readability, we have boldfaced the rows of values which begin with a Jacobsthal number.)

n	$N_1^{\#}(n)$	$D_1^{\#}(n)$	$N_2^{\#}(n)$	$D_2^{\#}(n)$	$N_3^{\#}(n)$	$D_3^{\#}(n)$	$N_4^{\#}(n)$	$D_4^{\#}(n)$	$N_5^{\#}(n)$	$D_5^{\#}(n)$	$N_6^{\#}(n)$	$D_6^{\#}(n)$
1	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	2	2	1	0	0	0	0	0	0	0	0	0
3	**5**	**5**	**2**	**2**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
4	10	10	4	4	1	0	0	0	0	0	0	0
5	**16**	**16**	**7**	**7**	**2**	**2**	**0**	**0**	**0**	**0**	**0**	**0**
6	24	24	11	10	4	4	1	0	0	0	0	0
7	33	33	15	15	6	6	2	0	0	0	0	0
8	44	44	20	20	8	8	3	0	0	0	0	0
9	56	56	26	26	11	11	4	2	0	0	0	0
10	70	70	33	32	14	14	5	4	0	0	0	0
11	**85**	**85**	**40**	**40**	**17**	**17**	**6**	**6**	**0**	**0**	**0**	**0**
12	102	102	48	48	21	20	8	8	1	0	0	0
13	120	120	57	57	25	25	10	10	2	0	0	0
14	140	140	67	66	30	30	12	12	3	0	0	0
15	161	161	77	77	35	35	14	14	4	0	0	0
16	184	184	88	88	40	40	16	16	5	0	0	0
17	208	208	100	100	46	46	19	19	6	2	0	0
18	234	234	113	112	52	52	22	22	7	4	0	0
19	261	261	126	126	58	58	25	25	8	6	0	0
20	290	290	140	140	65	64	28	28	9	8	0	0
21	**320**	**320**	**155**	**155**	**72**	**72**	**31**	**31**	**10**	**10**	**0**	**0**
22	352	352	171	170	80	80	35	34	12	12	1	0
23	385	385	187	187	88	88	39	37	14	14	2	0
24	420	420	204	204	96	96	43	40	16	16	3	0
25	456	456	222	222	105	105	47	45	18	18	4	0

Table 2: Values for $N_k^{\#}(n)$ and $D_k^{\#}(n)$

(We note in passing that the sequence of values given by $N_1^{\#}(n)$ is A001859 in Sloane's On–Line Encyclopedia of Integer Sequences [8].)

In order to prove the first assertion (that $N_k^{\#}(n) \geq D_k^{\#}(n)$), we separate the functions defined by the cases in equations 5 and 7 into individual functions denoted $N_k^{\#(1)}(n), N_k^{\#(2)}(n), \ldots, D_k^{\#(2)}(n)$. That is,

$$
\begin{aligned}
N_k^{\#(1)}(n) &:= \binom{\frac{n-r}{2^{k+1}}}{} (3(n-r)-2^k) + 3qr \\
N_k^{\#(2)}(n) &:= \binom{\frac{n-r}{2^{k+1}}}{} (3(n-r)-2^k) + 3qr + (r - J_{k-1}) \\
N_k^{\#(3)}(n) &:= \binom{\frac{n-r}{2^{k+1}}}{} (3(n-r)-2^k) + (3q+2)r - 2^k \\
D_k^{\#(1)}(n) &:= \binom{\frac{n-r}{2^{k+1}}}{} (3(n+r)-2^k) \\
D_k^{\#(2)}(n) &:= \binom{\frac{n-(2^k-r)}{2^{k+1}}}{} (3(n-r)+2^{k+1})
\end{aligned}
$$

For a given value of n, $N_k^{\#}(n)$ will equal $N_k^{\#(i)}(n)$ for some $i \in \{1,2,3\}$ and $D_k^{\#}(n)$ will be $D_k^{\#(j)}(n)$ for some $j \in \{1,2\}$ depending on the value of r. We should remember that not all combinations of i and j are possible (for example, there is no value of n such that $i = 1$ and $j = 2$). In Lemmas 5.1 through 5.4, we show that $N_k^{\#(i)}(n) \geq D_k^{\#(j)}(n)$ for all possible combinations of i and j (that correspond to some integer n) which implies that $N_k^{\#}(n) \geq D_k^{\#}(n)$ for all positive integers n.

Lemma 5.1. *For all integers n and k, $N_k^{\#(1)}(n) = D_k^{\#(1)}(n)$.*

Proof. We first note that, in the notation of Theorem 3.4, $\frac{n-r}{2^{k+1}} = \frac{2^k q}{2^{k+1}} = \frac{q}{2}$.

Then,

$$\begin{aligned}
N_k^{\#(1)}(n) &= \left(\frac{n-r}{2^{k+1}}\right)(3(n-r)-2^k) + 3qr \\
&= \left(\frac{n-r}{2^{k+1}}\right)\left(3(n-r)-2^k + 3qr\left(\frac{2}{q}\right)\right) \quad \text{since } \frac{n-r}{2^{k+1}} = \frac{q}{2} \\
&= \left(\frac{n-r}{2^{k+1}}\right)(3n-3r-2^k+6r) \\
&= \left(\frac{n-r}{2^{k+1}}\right)(3n+3r-2^k) \\
&= \left(\frac{n-r}{2^{k+1}}\right)(3(n+r)-2^k) \\
&= D_k^{\#(1)}(n).
\end{aligned}$$

□

Lemma 5.2. *For all integers k and all integers n such that $r > J_{k-1}$ (in the notation of Theorem 3.4),*

$$N_k^{\#(2)}(n) \geq D_k^{\#(1)}(n).$$

Proof.

$$\begin{aligned}
N_k^{\#(2)}(n) &= \left(\frac{n-r}{2^{k+1}}\right)(3(n-r)-2^k) + 3qr + (r - J_{k-1}) \\
&> \left(\frac{n-r}{2^{k+1}}\right)(3(n-r)-2^k) + 3qr \quad \text{since } r > J_{k-1} \\
&= N_k^{\#(1)}(n) \\
&= D_k^{\#(1)}(n) \quad \text{by Lemma 5.1.}
\end{aligned}$$

This proves our result. □

Lemma 5.3. *For all integers k and all integers n such that $r \leq J_k$ (in the notation of Theorem 3.4),*

$$N_k^{\#(2)}(n) \geq D_k^{\#(2)}(n).$$

Proof. We see that $r \leq J_k = 2^k - J_{k-1}$ by Lemma 3.2. Thus, $2^k q + r \leq 2^k(q+1) - J_{k-1}$. This implies $n \leq 2^k(q+1) - J_{k-1}$, so $2n - 2^k(q+1) \leq n - J_{k-1}$. Hence,

$$\begin{aligned}
N_k^{\#(2)}(n) &= \left(\frac{n-r}{2^{k+1}}\right)(3(n-r)-2^k) + 3qr + (r - J_{k-1}) \\
&= \frac{q}{2}(3(2^k q) - 2^k) + 3q(n - 2^k q) + n - 2^k q - J_{k-1} \\
&= 3(2^{k-1})q^2 - 2^{k-1}q + 3qn - 3(2^k)q^2 + n - 2^k q - J_{k-1} \\
&= q^2(-3(2^{k-1})) + q(-3(2^{k-1}) + 3n) + n - J_{k-1} \\
&\geq q^2(-3(2^{k-1})) + q(-3(2^{k-1}) + 3n) + 2n - 2^k(q+1) \quad \text{by the above argument}
\end{aligned}$$

$$\begin{aligned}
&= 3qn - 3(2^{k-1})q - 3(2^{k-1})q^2 + 2n - 2^k - 2^k q \\
&= \frac{2n - 2^k - 2^k q}{2^{k+1}}(3(2^k q) + 2^{k+1}) \\
&= D_k^{\#(2)}(n).
\end{aligned}$$

□

Lemma 5.4. *For all positive integers n and k, $N_k^{\#(3)}(n) = D_k^{\#(2)}(n)$.*

Proof.

$$\begin{aligned}
N_k^{\#(3)}(n) &= \left(\frac{n-r}{2^{k+1}}\right)(3(n-r) - 2^k) + (3q+2)r - 2^k \\
&= \left(\frac{q}{2}\right)(3(2^k q) - 2^k) + 3q(n - 2^k q) + 2(n - 2^k q) - 2^k \\
&= 3(2^{k-1})q^2 - 2^{k-1}q + 3qn - 3(2^k q^2) + 2n - 2^{k+1}q - 2^k \\
&= q^2(-3(2^{k-1})) + q(3n - 5(2^{k-1})) + 2n - 2^k \\
&= q^2(-3(2^{k-1})) + q(3n - 3(2^{k-1}) - 2^k) + 2n - 2^k \\
&= 3qn - 3(2^{k-1})q - 3(2^{k-1})q^2 + 2n - 2^k - 2^k q \\
&= \frac{n - 2^k + n - 2^k q}{2^k + 1}(3(2^k q) + 2^{k+1}) \\
&= D_k^{\#(2)}(n).
\end{aligned}$$

□

Remark 5.5. *To summarize, Lemmas 5.1 through 5.4 tell us that for any positive integer n,*

$$N_k^{\#}(n) \geq D_k^{\#}(n).$$

For Propositions 5.6 through 5.9 we make the assumption that $J_\ell < n < J_{\ell+1}$ for some positive integer ℓ.

Proposition 5.6. *For ℓ and n, as given above, $N_{\ell+1}^{\#}(n) = n - J_\ell$.*

Proof. By Lemma 3.1,

$$\begin{aligned}
\sum_{i=0}^{n-1} \left\lfloor \frac{3i+1}{2^{\ell+1}} \right\rfloor &= 0 \times (J_\ell) + 1 \times ((n-1) - (J_\ell - 1)) \\
&= n - J_\ell.
\end{aligned}$$

□

Proposition 5.7. *$D_k^{\#}(n) = 0$ if $n < 2^{k-1}$. In particular, $D_{\ell+1}^{\#}(n) = 0$ if $n < 2^\ell$.*

Proof. If $n < 2^k$ then, in the notation of Theorem 3.5, $n = r$ and $q = 0$, so by Theorem 3.5, $D_k^{\#}(n) = 0$. □

Proposition 5.8. *$D_{\ell+1}^{\#}(n) = 2(n - 2^\ell)$ if $2^\ell \leq n < J_{\ell+1}$.*

Proof. If $2^\ell \leq n < J_{\ell+1}$ then, in the notation of Theorem 3.5, $q = 0$ and $r = n$. Since $n \geq 2^\ell$, we are in the second case of Theorem 3.5 so

$$\begin{aligned}
D_{\ell+1}^{\#}(n) &= \frac{n - 2^{\ell+1} + n}{2^{\ell+2}}(0 + 2^{\ell+2}) \\
&= 2n - 2^{\ell+1} \\
&= 2(n - 2^\ell).
\end{aligned}$$

□

Proposition 5.9. *For n and ℓ as given above, $2(n-2^\ell) < n - J_\ell$.*

Proof. We begin by showing that $J_{\ell+1} - 2^\ell = 2^\ell - J_\ell$. We have,

$$\begin{aligned} J_{\ell+1} - 2^\ell &= \frac{2^{\ell+2} + (-1)^{\ell+1}}{3} - 2^\ell \\ &= \frac{4 \cdot 2^\ell - 3 \cdot 2^\ell - (-1)^\ell}{3} \\ &= \frac{3 \cdot 2^\ell - 2 \cdot 2^\ell - (-1)^\ell}{3} \\ &= 2^\ell - \frac{2^{\ell+1} + (-1)^\ell}{3} \\ &= 2^\ell - J_\ell, \end{aligned}$$

and hence,

$$\begin{aligned} 2(n-2^\ell) &= n - 2^\ell + n - 2^\ell \\ &< n - 2^\ell + J_{\ell+1} - 2^\ell \\ &= n - 2^\ell + 2^\ell - J_\ell \quad \text{from the above argument} \\ &= n - J_\ell \end{aligned}$$

so we have our result. □

We are now ready to prove our Theorem.

Theorem 5.10. *$A(n)$ is even if n is not a Jacobsthal number.*

Proof. Our goal is to show that there is some k such that $N_k^\#(n)$ is strictly greater than $D_k^\#(n)$ since, by Remark 5.5, we have shown that $N_k^\#(n) \geq D_k^\#(n)$ for all positive integers k and n.

Given n, not a Jacobsthal number, there exists a positive integer ℓ such that $J_\ell < n < J_{\ell+1}$. Then $N_{\ell+1}^\#(n) = n - J_\ell$ by Proposition 5.6, and since $n > J_\ell$, $N_{\ell+1}^\#(n) > 0$. On the other hand, by Proposition 5.7, if $n < 2^\ell$, then $D_{\ell+1}^\#(n) = 0$. If $2^\ell \leq n < J_{\ell+1}$, then by Proposition 5.8, $D_{\ell+1}^\#(n) = 2(n-2^\ell)$ which is strictly less than $n - J_\ell = N_{\ell+1}^\#(n)$ by Proposition 5.9. Hence, in every case, $N_{\ell+1}^\#(n)$ is strictly greater than $D_{\ell+1}^\#(n)$ so there is at least one factor of two in $A(n)$ and we have our result. □

6 A Closing Thought

We close by noting that we can prove a result stronger than Theorem 5.10. If $J_\ell < n < J_{\ell+1}$, then

$$N_{\ell+1}^\#(n) - D_{\ell+1}^\#(n) = \begin{cases} n - J_\ell & \text{if} \quad J_\ell < n \leq 2^\ell \\ J_{\ell+1} - n & \text{if} \quad 2^\ell \leq n < J_{\ell+1} \end{cases}$$

by Propositions 5.6, 5.7, 5.8 and Lemma 3.2.

Let $ord_2(n)$ be defined as the highest power of 2 that divides n. By Remark 5.5, $N_k^\#(n) - D_k^\#(n) \geq 0$ for all n and for all k, so that

$$ord_2(A(n)) \geq \begin{cases} n - J_\ell & \text{if} \quad J_\ell < n \leq 2^\ell \\ J_{\ell+1} - n & \text{if} \quad 2^\ell \leq n < J_{\ell+1} \end{cases},$$

which strengthens Theorem 5.10.

Finally, we see that $ord_2(A(2^\ell)) = J_{\ell-1}$ since, for all $k < \ell+1, N_k^\#(2^\ell) = N_k^{\#(1)}(2^\ell) = D_k^{\#(1)}(2^\ell) = D_k^\#(2^\ell)$, and $2^\ell - J_\ell = J_{\ell+1} - 2^\ell = J_{\ell-1}$. So, for example, we know that $A(2^{10})$ is divisible by 2^{J_9}, which equals 2^{341}, and that $A(2^{10})$ is not divisible by 2^{342}.

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