A Probabilistic View of Certain Weighted Fibonacci Sums

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1 Introduction

In this paper we investigate sums of the form

$$a_n := \sum_{k\geq 1} \frac{k^n F_k}{2^{k+1}}. \tag{1}$$

For any given n, such a sum can be determined [3] by applying the $x\frac{d}{dx}$ operator n times to the generating function

$$G(x) := \sum_{k \geq 1} F_k x^k = \frac{x}{1 - x - x^2},$$

then evaluating the resulting expression at $x = 1/2$. This leads to $a_0 = 1$, $a_1 = 5$, $a_2 = 47$, and so on. These sums may be used to determine the expected value and higher moments of the number of flips needed of a fair coin until two consecutive heads appear [3]. In this article, we pursue the reverse strategy of using probability to derive a_n and develop an exponential generating function for a_n in Section 3. In Section 4, we present a method for finding an exact, non-recursive, formula for a_n.

2 Probabilistic Interpretation

Consider an infinitely long binary sequence of independent random variables $b_1, b_2, b_3, \ldots$ where $P(b_i = 0) = P(b_i = 1) = 1/2$. Let Y denote the random variable denoting the beginning of the first 00 substring. That is, $b_Y = b_{Y+1} = 0$ and no 00 occurs before then. Thus $P(Y = 1) = 1/4$. For $k \geq 2$, we have $P(Y = k)$ is equal to the probability that our sequence begins $b_1, b_2, \ldots, b_{k-2}, 1, 0, 0$, where no 00 occurs among the first $k - 2$ terms. Since the probability of occurence of each such string is $(1/2)^{k+1}$, and it is well known [1] that there are exactly F_k binary strings of length $k - 2$ with no consecutive 0's, we have for $k \geq 1$,

$$P(Y = k) = \frac{F_k}{2^{k+1}}.$$

Since Y is finite with probability 1, it follows that

$$\sum_{k\geq 1} \frac{F_k}{2^{k+1}} = \sum_{k\geq 1} P(Y=k) = 1.$$

For $n \geq 0$, the expected value of Y^n is

$$a_n := E(Y^n) = \sum_{k\geq 1} \frac{k^n F_k}{2^{k+1}}. \tag{2}$$

Thus $a_0 = 1$. For $n \geq 1$, we use conditional expectation to find a recursive formula for a_n. We illustrate our argument with $n = 1$ and $n = 2$ before proceeding with the general case.

For a random sequence $b_1, b_2, \ldots$, we compute $E(Y)$ by conditioning on b_1 and b_2. If $b_1 = b_2 = 0$, then $Y = 1$. If $b_1 = 1$, then we have wasted a flip, and we are back to the drawing board; let Y' denote the number of remaining flips needed. If $b_1 = 0$ and $b_2 = 1$, then we have wasted two flips, and we are back to the drawing board; let Y'' denote the number of remaining flips needed in this case. Now by conditional expectation we have

$$\begin{aligned} E(Y) &= \frac{1}{4}(1) + \frac{1}{2}E(1+Y') + \frac{1}{4}E(2+Y'') \\ &= \frac{1}{4} + \frac{1}{2} + \frac{1}{2}E(Y') + \frac{1}{2} + \frac{1}{4}E(Y'') \\ &= \frac{5}{4} + \frac{3}{4}E(Y) \end{aligned}$$

since $E(Y') = E(Y'') = E(Y)$. Solving for $E(Y)$ gives us $E(Y) = 5$. Hence,

$$a_1 = \sum_{k\geq 1} \frac{kF_k}{2^{k+1}} = 5.$$

Conditioning on the first two outcomes again allows us to compute

$$\begin{aligned}
E(Y^2) &= \frac{1}{4}(1^2) + \frac{1}{2}E\left[(1+Y')^2\right] + \frac{1}{4}E\left[(2+Y'')^2\right] \\
&= \frac{1}{4} + \frac{1}{2}E(1+2Y+Y^2) + \frac{1}{4}E(4+4Y+Y^2) \\
&= \frac{7}{4} + 2E(Y) + \frac{3}{4}E(Y^2).
\end{aligned}$$

Since $E(Y) = 5$, it follows that $E(Y^2) = 47$. Thus,

$$a_2 = \sum_{k\geq 1} \frac{k^2 F_k}{2^{k+1}} = 47.$$

Following the same logic for higher moments, we derive for $n \geq 1$,

$$\begin{aligned}
E(Y^n) &= \frac{1}{4}(1^n) + \frac{1}{2}E[(1+Y)^n] + \frac{1}{4}E[(2+Y)^n] \\
&= \frac{1}{4} + \frac{3}{4}E(Y^n) + \frac{1}{2}\sum_{k=0}^{n-1}\binom{n}{k}E(Y^k) + \frac{1}{4}\sum_{k=0}^{n-1}\binom{n}{k}2^{n-k}E(Y^k).
\end{aligned}$$

Consequently, we have the following recursive equation:

$$E(Y^n) = 1 + \sum_{k=0}^{n-1}\binom{n}{k}[2+2^{n-k}]E(Y^k)$$

Thus for all $n \geq 1$,

$$a_n = 1 + \sum_{k=0}^{n-1}\binom{n}{k}[2+2^{n-k}]a_k. \tag{3}$$

Using equation (3), one can easily derive $a_3 = 665$, $a_4 = 12,551$, and so on.

3 Generating Function and Asymptotics

For $n \geq 0$, define the exponential generating function

$$a(x) = \sum_{n\geq 0} \frac{a_n}{n!}x^n.$$

It follows from equation (3) that

$$\begin{aligned} a(x) &= 1 + \sum_{n\geq 1} \frac{\left(1 + \sum_{k=0}^{n-1} \binom{n}{k} [2 + 2^{n-k}] a_k\right)}{n!} x^n \\ &= e^x + 2a(x)(e^x - 1) + a(x)(e^{2x} - 1). \end{aligned}$$

Consequently,

$$a(x) = \frac{e^x}{4 - 2e^x - e^{2x}}. \tag{4}$$

For the asymptotic growth of a_n, one need only look at the leading term of the Laurent series expansion [4] of $a(x)$. This leads to

$$a_n \approx \frac{\sqrt{5} - 1}{10 - 2\sqrt{5}} \left(\frac{1}{\ln(\sqrt{5} - 1)} \right)^{n+1} n!. \tag{5}$$

4 Closed Form

While the recurrence (3), generating function (4), and asymptotic result (5) are satisfying, a closed form for a_n might also be desired. For the sake of completeness, we demonstrate such a closed form here.

To calculate

$$a_n = \sum_{k\geq 1} \frac{k^n F_k}{2^{k+1}},$$

we first recall the Binet formula for F_k [3]:

$$F_k = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^k - \left(\frac{1 - \sqrt{5}}{2} \right)^k \right) \tag{6}$$

Then (6) implies that (1) can be rewritten as

$$a_n = \frac{1}{2\sqrt{5}} \sum_{k\geq 1} k^n \left(\frac{1 + \sqrt{5}}{4} \right)^k - \frac{1}{2\sqrt{5}} \sum_{k\geq 1} k^n \left(\frac{1 - \sqrt{5}}{4} \right)^k. \tag{7}$$

Next, we remember the formula for the geometric series:

$$\sum_{k\geq 0} x^k = \frac{1}{1-x} \tag{8}$$

This holds for all real numbers x such that $|x| < 1$. We now apply the $x\frac{d}{dx}$ operator n times to (8). It is clear that the left-hand side of (8) will then become

$$\sum_{k\geq 1} k^n x^k.$$

The right-hand side of (8) is transformed into the rational function

$$\frac{1}{(1-x)^{n+1}} \times \sum_{j=1}^{n} e(n,j)x^j, \tag{9}$$

where the coefficients $e(n,j)$ are the Eulerian numbers [2, Sequence A008292], defined by

$$e(n,j) = j \cdot e(n-1,j) + (n-j+1) \cdot e(n-1,j-1) \quad \text{with} \quad e(1,1) = 1.$$

(The fact that these are indeed the coefficients of the polynomial in the numerator of (9) can be proven quickly by induction.) From the information found in [2, Sequence A008292], we know

$$e(n,j) = \sum_{\ell=0}^{j} (-1)^\ell (j-\ell)^n \binom{n+1}{\ell}.$$

Therefore,

$$\sum_{k\geq 1} k^n x^k = \frac{1}{(1-x)^{n+1}} \times \sum_{j=1}^{n} \left[\sum_{\ell=0}^{j} (-1)^\ell (j-\ell)^n \binom{n+1}{\ell} \right] x^j. \tag{10}$$

Thus the two sums

$$\sum_{k\geq 1} k^n \left(\frac{1+\sqrt{5}}{4}\right)^k \quad \text{and} \quad \sum_{k\geq 1} k^n \left(\frac{1-\sqrt{5}}{4}\right)^k$$

that appear in (7) can be determined explicitly using (10) since

$$\left|\frac{1+\sqrt{5}}{4}\right| < 1 \quad \text{and} \quad \left|\frac{1-\sqrt{5}}{4}\right| < 1.$$

Hence, an exact, non-recursive, formula for a_n can be developed.

References

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