On Polynomials Related to Powers of the Generating Function of Catalan's Numbers

Wolfdieter L a n g [1]

Institut für Theoretische Physik
Universität Karlsruhe
Kaiserstrasse 12, D-76128 Karlsruhe, Germany

1 Introduction and Summary

Catalan's sequence of numbers $\{C_n\}_0^\infty = \{1, 1, 2, 5, 14, 42, ...\}$ (nr.1459 and A000108 of [14]) emerges in the solution of many combinatorial problems (see [2],[4],[5],[16] (also for further references)). The moments $\mu_{2\,k}$ of the normalized weight function of *Chebyshev*'s polynomials of the second kind are given by $C_k/2^k$ (see e.g. [3] Lemma 4.3, p. 160 for $l = 0$, [17], p.II-3). This sequence also shows up in the asymptotic moments of zeros of scaled *Laguerre* and *Hermite* polynomials [9] eqs.(3.34) and (3.35). The generating function $c(x) = \sum_{n=0}^\infty C_n\, x^n$ is the solution of the quadratic equation $x\, c^2(x) - c(x) + 1 = 0$ with $c(0) = 1$. Therefore, every positive integer power of $c(x)$ can be written as

$$c^n(x) \;=\; p_{n-1}(x)1 \;+\; q_{n-1}(x)\; c(x) \quad , \tag{1}$$

with certain polynomials p_{n-1} and q_{n-1}, both of degree $(n-1)$, in $1/x$. In *section* 2 they are shown to be related to *Chebyshev*'s polynomials of the second kind:

$$p_{n-1}(x) \;=\; -(\frac{1}{\sqrt{x}})^n\; S_{n-2}(\frac{1}{\sqrt{x}}) \quad , \quad q_{n-1}(x) \;=\; (\frac{1}{\sqrt{x}})^{n-1}\; S_{n-1}(\frac{1}{\sqrt{x}}) \;=\; -x\; p_n(x)\;, \tag{2}$$

with $S_n(y) = U_n(y/2)$. It is therefore possible to extend the range of the power n to negative integers (or to real or complex numbers). Tables for the $U_n(x)$ polynomials can be found, *e.g.*, in [1]. Because powers of a generating function correspond to convolutions of the generated number sequence the given decomposition of $c^n(x)$ will determine convolutions of the *Catalan* sequence. In passing, an explicit expression for general convolutions in the form of nested sums will also be given. Contact with the works of refs. [6],[12],[18], [5] will be made.

Together with the known (*e.g.* [4],[11]) result (valid for real n)

$$c^n(x) \;=\; \sum_{k=0}^{\infty} C_k(n)\; x^k \quad ,\text{with } C_k(n) \;=\; \frac{n}{n+2k}\binom{n+2k}{k} \;=\; \frac{n}{k+n}\binom{n-1+2k}{k}\;, \tag{3}$$

[1]E-mail: wolfdieter.lang@physik.uni-karlsruhe.de, http://www-itp.physik.uni-karlsruhe.de/~wl

one finds from the alternative expression (1) for positive n two sets of identities:

$$(P1) \qquad \sum_{l=0}^{p} (-1)^l \binom{n+1-p+l}{p-l} C_l = \binom{n-p}{p} , \tag{4}$$

for $n \in \mathbf{N}_0,\ p \in \{0,1,2,...\lfloor \frac{n}{2} \rfloor\}$, and

$$(P2) \qquad \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^l \binom{n-1-l}{l} C_{k+n-1-l} = C_k(n) , \tag{5}$$

for $n \in \mathbf{N}$, $k \in \mathbf{N}_0$.

For negative powers in (1) two other sets of identities result:

$$(P3) \qquad \sum_{l=0}^{min(\lfloor \frac{n-1}{2} \rfloor, k-1)} (-1)^l \binom{n-1-l}{l} C_{k-1-l} = (-1)^{k+1} \binom{n-k-1}{k-1} , \tag{6}$$

for $n \in \mathbf{N},\ k \in \{0,1,2,...\lfloor \frac{n}{2} \rfloor\}$, (for $k = 0$ both sides are by definition zero) and

$$(P4) \qquad \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^l \binom{n-1-l}{l} C_{k-1-l} = -C_k(-n) = \frac{n}{k} \binom{2k-n-1}{k-1}, \tag{7}$$

for $n \in \mathbf{N}$, $k \in \mathbf{N}$ with $k \geq \lfloor \frac{n}{2} \rfloor + 1$. These identities can be continued for appropriate values of real n.

Another expression for the coefficients of negative powers of $c(x)$ is

$$C_k(-n) = \sum_{l=1}^{min(n,k)} (-1)^l \binom{n}{l} C_{k-l}(n) , \tag{8}$$

for $n, k \in \mathbf{N}$, and $C_0(-n) = 1$, $C_n(0) = \delta_{n,0}$. Also, from (3) $C_k(-n) = -C_{k-n}(n)$ for $n, k \in \mathbf{N}$ with $k \geq n$.

The remainder of this paper provides proofs for the above given statements. *Section 2* deals with integer (and real) powers of the generating function $c(x)$. Convolutions of general sequences are expressed there in terms of nested sums. In *Section 3* some families of integer sequences related to the polynomials $q_n(x)$ (2) evaluated for $x = 1/m$ for $m = 4, 5, ...$ and $(-1)^n\ q_n(x)$ evaluated at $x = -1/m$, $m \in \mathbf{N}$ are considered.

2 Powers

The equation $x\ c^2(x) - c(x) + 1 = 0$ whose solution defines the generating function of *Catalan*'s numbers if $c(0) = 1$ can be considered as characteristic equation for the recursion relation

$$x\ r_{n+1} - r_n + r_{n-1} = 0 \quad , \quad n = 0, 1, ... \quad , \tag{9}$$

with arbitrary inputs $r_{-1}(x)$ and $r_0(x)$. A basis of two linearly independent solutions is given by the *Lucas*-type polynomials $\{\mathcal{U}_n\}$ and $\{\mathcal{V}_n\}$, with standard inputs $\mathcal{U}_{-1} = 0$, $\mathcal{U}_0 = 1$, $(\mathcal{U}_{-2} = -x)$, and $\mathcal{V}_{-1} = 1$, $\mathcal{V}_0 = 2$, $(\mathcal{V}_1 = 1/x)$, in the *Binet* form

$$\mathcal{U}_{n-1}(x) = \frac{c_+^n(x) - c_-^n(x)}{c_+(x) - c_-(x)} , \tag{10}$$

$$\mathcal{V}_n(x) = c_+^n(x) + c_-^n(x) = \frac{1}{x}(\mathcal{U}_{n-1}(x) - 2\,\mathcal{U}_{n-2}(x)) , \tag{11}$$

with the two solutions of the characteristic equation, *viz* $c_\pm(x) := (1 \pm \sqrt{1-4x}\ )/(2x)$. $c(x) := c_-(x)$ satisfies $c(0) = 1$, and $c_+(x) = 1/(xc(x))$, as well as $c_+(x) + c(x) = 1/x$. From the recurrence (9) it is clear that for positive $n \neq 0$ $\mathcal{U}_n$ is a polynomial in $1/x$ of degree $n-1$. If $c_+(x) - c_-(x) = 0$, *i.e.* $x = 1/4$, eq.(10) is replaced by $\mathcal{U}_n(1/4) = 2^n(n+1)$. The second eq. in (11) holds because both sides of the eq. satisfy recurrence (9) and the inputs for $\mathcal{V}_0$ and $\mathcal{V}_1$ match. One may associate with the recurrence relation (9) a transfer matrix

$$\mathbf{T}(x) = \begin{pmatrix} 1/x & -1/x \\ 1 & 0 \end{pmatrix} , \quad Det\ \mathbf{T}(x) = 1/x . \tag{12}$$

With this matrix one can rewrite (9) as

$$\begin{pmatrix} r_n \\ r_{n-1} \end{pmatrix} = \mathbf{T}(x) \begin{pmatrix} r_{n-1} \\ r_{n-2} \end{pmatrix} = \mathbf{T}^n(x) \begin{pmatrix} r_0(x) \\ r_{-1}(x) \end{pmatrix} \tag{13}$$

Because $\mathbf{T}^n = \mathbf{T}\ \mathbf{T}^{n-1}$ with input $\mathbf{T}^1 = \mathbf{T}(\mathbf{x})$ given by (12), one finds from the recurrence relation (9) with $r_n = \mathcal{U}_n$

$$\mathbf{T}^n(x) = \begin{pmatrix} \mathcal{U}_n(x) & -\frac{1}{x}\,\mathcal{U}_{n-1}(x) \\ \mathcal{U}_{n-1}(x) & -\frac{1}{x}\,\mathcal{U}_{n-2}(x) \end{pmatrix} . \tag{14}$$

Note that for $x = 1$ one has $c_\pm(1) = (1 \pm i\sqrt{3})/2$, which are 6th roots of unity, and the related period 6 sequences are $\{\mathcal{U}_n(1)\}_{-1}^{\infty} = \{\overline{0,1,1,0,-1,-1,}\}$, as well as $\{\mathcal{V}_n(1)\}_0^{\infty} = \{\overline{2,1,-1,-2,-1,1,}\}$. This follows from eqs. (10) and (11). It is convenient to map the recursion relation (9) to the familiar one for *Chebyshev*'s $S_n(x) = U_n(x/2)$ polynomials of the second kind, *viz*

$$S_n(x) = x\ S_{n-1}(x) - S_{n-2}(x) \ , \quad S_{-1} = 0,\ S_0 = 1 , \tag{15}$$

with characteristic equation $\lambda^2 - x\lambda + 1 = 0$ and solutions $\lambda_\pm(x) = \frac{x}{2}(1 \pm \sqrt{1-(2/x)^2}\ )$, satisfying $\lambda_+(x)\ \lambda_-(x) = 1$ and $\lambda_+(x) + \lambda_-(x) = x$. The relation to $c_\pm(x)$ is

$$\sqrt{x}\ c_\pm(x) = \lambda_\pm(1/\sqrt{x}) . \tag{16}$$

The *Binet* form of the corresponding two independent polynomial systems is

$$S_{n-1}(x) = \frac{\lambda_+^n(x) - \lambda_-^n(x)}{\lambda_+(x) - \lambda_-(x)} , \tag{17}$$

$$2\ T_n(x/2) = \lambda_+^n(x) + \lambda_-^n(x) , \tag{18}$$

and $T_n(x/2) = (S_n(x) - S_{n-2}(x))/2$ are *Chebyshev*'s polynomials of the first kind. Tables of *Chebyshev*'s polynomials can be found in [1]. The coefficient triangles of the $S_n(x)$, $U_n(x)$ and $T_n(x)$ polynomials can also be viewed under the numbers A049310, A053117 and A053120, respectively, in the on-line data-base [14].

The extension to negative integer indices runs as follows

$$\mathcal{U}_{-n}(x) = -x^{n-1}\,\mathcal{U}_{n-2}(x)\ , \tag{19}$$

$$S_{-(n+2)}(x) = -S_n(x)\ . \tag{20}$$

This follows from (10) and (17). Note that from (9) $\mathcal{U}_n$ is for positive n a monic polynomial in $1/x$ of degree n, and for negative n in general a non-monic polynomial in x of degree $\lfloor -\frac{n}{2}\rfloor$. It is possible to extend the range of n to complex numbers using the *Binet* forms.

A connection between both systems of polynomials is made, after using (10), (16) and (17),by

$$\mathcal{U}_n(x) = \left(\frac{1}{\sqrt{x}}\right)^n\ S_n(1/\sqrt{x})\ . \tag{21}$$

This holds for $n \in \mathbf{Z}$, in accordance with (19) and (20).

After these preliminaries we are ready to state:

Proposition 1: The nth power of $c(x)$, the generating function of *Catalan*'s numbers, can, for $n \in \mathbf{Z}$, be written as

$$c^n(x) = -\frac{1}{x}\,\mathcal{U}_{n-2}(x) + \mathcal{U}_{n-1}(x)\ c(x)\ , \tag{22}$$

$$= -(\frac{1}{\sqrt{x}})^n\ S_{n-2}(1/\sqrt{x}) + (\frac{1}{\sqrt{x}})^{n-1}\ S_{n-1}(1/\sqrt{x})\ c(x). \tag{23}$$

Proof: Due to $c^2(x) = (c(x)-1)/x$ and $c^{-1}(x) = 1 - x\ c(x)$ one can, for $n \in \mathbf{Z}$, write $c^n(x) = p_{n-1}(x) + q_{n-1}(x)\ c(x)$. From $c^n(x) = c(x)\ c^{n-1}(x)$ one is led to $q_{n-1} = p_{n-2} + \frac{1}{x}\ q_{n-2}$ and $p_{n-1} = -\frac{1}{x}\ q_{n-2}$, or $q_{n-1} = (q_{n-2} - q_{n-3})/x$ with input $q_{-1} = 0,\ q_0 = 1$. Therefore, $q_{n-1}(x) = \mathcal{U}_{n-1}(x)$ and $p_{n-1}(x) = -\ \mathcal{U}_{n-2}(x)/x$. (23) then follows from (21). □

Note 1: Because $S_n(y) = \sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\binom{n-j}{j}y^{n-2j}$ the explicit form of these polynomials (2) is $p_{n-1}(x) = \sum_{j=0}^{\lfloor n/2\rfloor-1}(-1)^{j+1}\binom{n-2-j}{j}x^{-(n-1-j)}$, $p_{-1} = 1$, $p_0 = 0$, and $q_{n-1}(x) = \sum_{j=0}^{\lfloor(n-1)/2\rfloor}(-1)^j\binom{n-1-j}{j}x^{-(n-1-j)}$, $q_{-1} = 0$. For negative index one has, due to (20), $p_{-(n+1)}(x) = (\sqrt{x})^n\ S_n(1/\sqrt{x}) = \sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\ \binom{n-j}{j}\ x^j$, and $q_{-(n+1)}(x) = -(\sqrt{x})^{n+1}\ S_{n-1}(1/\sqrt{x}) = -x\ \sum_{j=0}^{\lfloor(n-1)/2\rfloor}(-1)^j\ \binom{n-1-j}{j}\ x^j$.

In the *Table* one can find the coefficient triangle for the polynomials $\{p_n(x)\}_{-1}^{12}$ with column m corresponding to $(\frac{1}{x})^m$, $m \geq 0$.

Note 2: An alternative proof of *proposition 1* can be given starting with eqs.(17) and (18) which show, together with $\lambda_+(x) - \lambda_-(x) = \sqrt{x^2-4}$, that

$$\lambda_\pm^n(x) = T_n(x/2) \pm \sqrt{(x/2)^2-1}\ S_{n-1}(x)\ , \tag{24}$$

or, from $\pm\sqrt{(x/2)^2-1} = \lambda_\pm(x) - x/2$ and the S_n recurrence relation (15)

$$\lambda_\pm^n(x) = T_n(x/2) - \tfrac{1}{2}\,(S_n(x) + S_{n-2}(x)) + S_{n-1}(x)\ \lambda_\pm(x) \tag{25}$$

$$= -S_{n-2}(x) + S_{n-1}(x)\ \lambda_\pm(x)\ . \tag{26}$$

Now (23) follows from (16). This also proves that one may replace in *proposition 1* $c(x)$ by $c_+(x) = 1/(xc(x))$ from which one recovers the c^{-n} formula for $n \in \mathbf{N}$ in accordance with (19) and (20).

Note 3: For the transfer matrix $\mathbf{T}(\mathbf{x})$, defined in (12), one can prove for $n \in \mathbf{N}$ in an analogous manner

$$\mathbf{T}^n \;=\; -(\frac{1}{\sqrt{x}})^n\; S_{n-2}(1/\sqrt{x})\; \mathbf{1} \;+\; (\frac{1}{\sqrt{x}})^{n-1}\; S_{n-1}(1/\sqrt{x})\; \mathbf{T}(x), \tag{27}$$

by employing the *Cayley-Hamilton* theorem for the 2×2 matrix $\mathbf{T}$ with $tr\ \mathbf{T} \;=\; \frac{1}{x} \;=\; det\ \mathbf{T}$ which states that $\mathbf{T}$ satisfies the characteristic equation $\mathbf{T}^2 \;-\; \frac{1}{x}\,\mathbf{T} \;+\; \frac{1}{x}\,\mathbf{1} \;=\; 0$.

Powers of a function which generates a sequence generate convolutions of this sequence. Therefore, *proposition 1* implies that convolutions of the *Catalan* sequence can be expressed in terms of *Catalan* numbers and binomial coefficients. Before giving this result we shall present an explicit formula for the nth convolution of a general sequence $\{C_n\}$ generated by $c(x) \;=\; \sum_{l=0}^{\infty}\; C_l\; x^l$. Usually the convolution coefficients $C_l(n)$, defined by $c^n(x) = \sum_{l=0}^{\infty}\; C_l(n)\; x^l$, are written as

$$C_l(n) \;=\; \sum_{\sum_{j=1}^{n} i_j \;=\; l} C_{i_1}\; C_{i_2}\cdots C_{i_n} \quad,\quad \text{with } i_j \in \mathbf{N}_0\ . \tag{28}$$

An explicit formula with $(l-1)$ nested sums is the content of the next lemma.

Lemma 1: General convolutions

For $l = 2, 3, ...$

$$C_l(n) \;=\; C_0^{n-l}\; C_1^l \Big(\prod_{k=2}^{l}\sum_{i_k=a_k}^{\lfloor b_k\rfloor}\Big) \;< n, l, \{i_j\}_2^l >\; \prod_{j=2}^{l}\Big((\frac{C_j\; C_0}{C_1^j})^{i_j}\;\frac{1}{i_j!}\Big)\ , \tag{29}$$

with

$$b_2 = l/2 \quad,\quad b_k \;=\; (l - \sum_{j=2}^{k-1}\; j\; i_j)/k\ , \tag{30}$$

$$a_k = 0 \quad,\quad \text{for } k = 2, 3, ..., l-1;\quad a_l \;=\; max\Big(0, \lceil\frac{l - n - \sum_{j=2}^{l-1}(j-1)\; i_j)}{l-1}\rceil\Big) \tag{31}$$

$$< n, l, \{i_j\}_2^l > \;=\; \frac{n!}{(n - l + \sum_{j=2}^{l}(j-1)\; i_j)!(l - \sum_{j=2}^{l} j\; i_j)!}. \tag{32}$$

The first product in (29) is understood to be ordered such that the sums have indices $i_2, i_3, ...i_l$ when written from the left to the right. In addition: $C_0(n) = C_0^n$ and $C_1(n) = n\; C_0^{n-1}\; C_1$.

Proof: $C_l(n)$ of (28) is rewritten first as

$$C_l(n) \;=\; \sum\; (n, l, \{i_j\}_0^l)\; C_0^{i_0}\; C_1^{i_1}\cdots C_l^{i_l} \quad,\quad i_j \in \mathbf{N}_0\quad, \tag{33}$$

where the sum is restricted by

$$(i):\qquad \sum_{j=0}^{l}\; j\; i_j \;=\; l \qquad\text{and}\qquad (ii):\qquad \sum_{j=0}^{l}\; i_j \;=\; n\ . \tag{34}$$

$(n, l, \{i_j\}_0^l)$ is a combinatorial factor to be determined later on. (*E.g.* for $n \;=\; 3, l \;=\; 5$ one has 4 terms in the sum: $i_5 \;=\; 1, i_0 \;=\; 2$; $i_4 \;=\; 1, i_1 \;=\; 1, i_0 \;=\; 1$; $i_3 \;=\; 1, i_2 \;=\; 1, i_0 \;=\; 1$; $i_3 \;=\; 1, i_2 \;=\; 2$, with other indices vanishing, and the combinatorial factors are 3, 6, 6, 3, respectively.) (ii) restricts the sum to terms with n factors, and (i) produces the correct weight l. These restrictions are solved by (i') : $i_1 = l - \sum_{j=2}^{l} j\ i_j$ and (ii') : $i_0 = n - i_1 - \sum_{j=2}^{l} i_j = n - l + \sum_{j=2}^{l} (j-1)\ i_j$. From $i_1 \geq 0$,*i.e.* $l - \sum_{j=2}^{l} j\ i_j \geq 0$, one infers $i_2 \leq \lfloor \frac{l}{2} \rfloor$, thus $i_2 \in [0, \lfloor \frac{l}{2} \rfloor\ ]$. For given i_2 in this range $i_3 \leq \lfloor \frac{l-2i_2}{3} \rfloor$, *etc.*, in general $0 \leq i_k \leq \lfloor (l - \sum_{j=2}^{k-1} j\ i_j)/k \rfloor$ for $k = 2, 3, ..., l$ with the sum replaced by zero for $k = 2$. This accounts for the upper boundaries $\lfloor b_k \rfloor$ in (30). Now, because $i_0 \geq 0$ (ii') implies a lower bound for i_l, the index of the last sum, *viz* $i_l \geq \lceil (l - n - \sum_{j=2}^{l-1} (j-1)\ i_j)/(l-1) \rceil$ with the ceiling function $\lceil \cdot \rceil$. In any case $i_l \geq 0$, therefore, the lower boundary for the i_l-sum is a_l as given in (31). All restrictions have then be solved and the lower boundaries of the other sums are given by $a_k = 0$, for $k = i_2, ..., i_{l-1}$. As to the combinatorial factor, it now depends only on $n, l, \{i_j\}_2^l$ and is written as $< n, l, \{i_j\}_2^l >$. It counts the number of possibilities for the occurence of the considered term of the sum which is given by $\binom{n}{i_0} \binom{n-i_0}{i_1} \cdots \binom{n-\sum_{j=2}^{l-1} i_j}{i_l} = n!/(\prod_{j=0}^{l} i_j!\ (n - \sum_{j=0}^{l} i_j)!$. Inserting i_0 and i_1 from (ii') and (i'), respectively, remembering (ii), produces $< n, l, \{i_j\}_2^l >$ as given in (32). Finally, $\sum < n, l, \{i_j\}_2^l >\ C_0^{i_0}\ C_1^{i_1} \cdots C_l^{i_l}$ is transformed into $(l-1)$ nested sums with boundaries a_k and $\lfloor b_k \rfloor$ after replacement of i_1 and i_0. This completes the proof of (29) for the non-trivial $l \geq 2$ cases. □

Corollary 1: *Catalan* convolutions

For *Catalan*'s sequence $\{C_n\}_0^\infty$ the n-th convolution sequence is for $n \in \mathbf{N}$ given by $C_0(n) = 1$, $C_1(n) = n$ and, for $l = 2, 3, ..$, by

$$C_l(n) = \Big(\prod_{k=2}^{l} \sum_{i_k=a_k}^{\lfloor b_k \rfloor} \Big) < n, l, \{i_j\}_2^l > \prod_{j=2}^{l} \Big(\frac{C_j^{i_j}}{i_j!} \Big) , \qquad (35)$$

with (30), (31) and (32).

Proof: This is *lemma 1* with $C_0 = 1 = C_1$. □

Example 1: $C_4(3) = 3C_4 + 6C_3 + 3C_2^2 + 3C_2 = 90$.

Corollary 2: With the *Catalan* generating function $c(x)$ and the definition

$c^{-n}(x) =: \sum_{l=0}^{\infty} C_l(-n)\ x^l$, for $n \in \mathbf{N}$, one has for $l = 2, 3, ...$

$$C_l(-n) = (-1)^l \Big(\prod_{k=2}^{l} \sum_{i_k=a_k}^{\lfloor b_k \rfloor} \frac{(-1)^{(k-1)i_k}}{i_k!} \Big) < n, l, \{i_j\}_2^l > \prod_{j=2}^{l-1} C_j^{i_{j+1}} , \qquad (36)$$

with (30), (31), (32) and *Catalan*'s numbers C_k. In addition: $C_0(-n) = 1$, $C_1(-n) = -n$.

Proof: *Lemma 1* is used for powers of $c(x)$ replaced by those of $c^{-1}(x) = 1 - x\ c(x)$, with the *Catalan* generating function $c(x)$. Hence $c^{-1}(x) = \sum_{k=0}^{\infty} C_k(-1)\ x^k$ with

$C_k(-1) = \begin{cases} 1 & \text{for } k = 0 \\ -C_{k-1} & \text{for } k = 1, 2, ... \end{cases}$. Then in *lemma 1* C_k is replaced by $C_k(-1)$. □

Example 2: $C_4(-3) = -3C_3 + 6C_2 - 3 + 3 = -3$.

Convolutions of *Catalan*'s sequence have been encountered in various contexts, for example, in the enumeration of non-intersecting path pairs on a square lattice [12], [18], [5], and in the problem of inverting triangular matrices with *Pascal* triangle entries [6] (and earlier works cited there). They also appear in [15], p.148.

Note 4: *Shapiro*'s *Catalan* triangle has entries $B_{n,k} = \frac{k}{n}\binom{2n}{n-k}$ for $n \geq k \geq 1$, and $B_{n,k} = [x^n](x^k c^{2k}(x))$, with $[x^n]f(x)$ denoting the coefficient of x^n in the expansion of $f(x)$ around $x = 0$. Here $\hat{c}(x) = (c(x)-1)/x = c^2(x)$. (See [12], propositions (2.1) and (3.3) with $i_j \in \mathbf{N}$, *not* $\mathbf{N}_0$.) In [18] this triangle of numbers from [12] reappears as $b(n,k)$ and it is shown there that $B_{n,k} \equiv b(n,k) = [x^n](x\ c^2(x))^k$, in accordance with the identity $\hat{c}(x) = c^2(x)$. Therefore, only even powers of $c(x)$ appear in *Shapiro*'s *Catalan* triangle. In [5] $C_l(n)$ appears as special case ${}_2d_{2-n,l+1}$. In [6] all powers of $c(x)$ show up as convolutions for the special case of the S_1 sequence there. The entries of the S_1-array, p. 397, are $[x^n]c^{k+1}(x)$ for $n,k \in \mathbf{N}_0$.

The referee of this paper noticed that the inverse of the lower triangular matrix $S_{n,k} = [x^k]\,S_n(x)$, for $n,k \in \mathbf{N}_0$, with *Chebyshev*'s $S_n(x) = U_n(x/2)$ polynomials is the lower triangular convolution matrix obtained from its first (k=0) column sequence generated by $c(x^2)$ (*Catalan* numbers alternating with zeros). This follows from the fact that the $\mathbf{S}-$matrix is also a lower triangular convolution matrix with generating function $1/(1+x^2)$ of its first column. See [13] for such type of matrices $\mathbf{M}$ and the relation between the generating functions of the first columns of $\mathbf{M}$ and $\mathbf{M}^{-1}$. The head of this *Catalan* triangle can be viewed under number A053121 in the on-line data-base [14]. See also [6] for inverses of *Pascal*-type arrays.

Lemma 2: Explicit form of *Catalan* convolutions [12],[18],[6],[4],[11],[5]

For $n \in \mathbf{R}$, $l \in \mathbf{N}_0$:

$$C_l(n) = \frac{n}{l}\binom{2l+n-1}{l-1} = \frac{n}{n+2l}\binom{n+2l}{l} = \frac{n}{l+n}\binom{2l+n-1}{l}\,. \qquad (37)$$

Proof: Three equivalent expressions have been given for convenience. See [4], p. 201, eq.(5.60), with $\mathcal{B}_2(z) = c(z)$, $t \to 2, k \to l, r \to n$. The proof of this eq.(5.60) appears as (7.69) on p.349, with $m = 2, n = l \in \mathbf{R}$.

The same formula occurs as exercise nr. 213 in Vol.1 of [11] for $\beta = 2$ as a special instance of exercises nrs. 211, 212. Put $\alpha = n$ and $n = l$ in the solution of exercise nr. 213 on p. 301.

In order to prove this lemma from [12] or [18] one can use $C_l(n) = \sum_{j=0}^{\min(l,n)} \binom{n}{j}\ \hat{C}_l(j)$ obtained from $c(x) =: 1 + \hat{c}(x)$ with $\hat{c}^n(x) =: \sum_{k=n}^{\infty} \hat{C}_k(n)\ x^{k-n}$. The result in [12] and [18] is, with this notation, $\hat{C}_l(j) = B_{l,j} = b(l,j) = \frac{j}{l}\binom{2l}{l-j}$. Inserting this in the given sum, making use of the identity $j\,\binom{n}{j} = n\binom{n-1}{j-1}$ and the *Vandermonde* convolution identity, leads to *lemma 2* at least for positive integer n but one can continue this formula to real (or complex) n.

In [6] one finds this result as eq.(3.1), p.402, for $i = 1$: $s_1(l,n) = C_l(n)$.

In [5] ${}_2d_{2-n,l+1} = C_l(n)$ with the result given in theorem 2.3, eq. (2.6), p.71. □

Note 5: As a side remark we mention that from (37) $E_l(x) := l!\ C_l(x)$ (with real $n = x$) is a polynomial of degree l, *viz* $\prod_{j=0}^{l-1}(x+l+1+j)$. These polynomials which are not the subject of this work are known (see [8] and references given there) as exponential convolution polynomials satisfying $E_l(x+y) = \sum_{k=0}^{l}\binom{l}{k}\ E_k(x)\ E_{l-k}(y)$.

We now compute the coefficients $C_l(n) = [x^l]c^n(x)$ (see *Note 4* for this notation) from our formula given in *proposition 1*. This can be done for $n \in \mathbf{Z}$.

First consider $n \in \mathbf{N}_0$. For $n=0$ and $n=1$ there is nothing new due to the inputs $S_{-2}=-1$, $S_{-1}=0$ and $S_0=1$. $C_l(n)=0$ for negative integer l. Therefore, terms proportional to $1/x^l$ with $l \in \mathbf{N}$ have to cancel in (23), or (1). For $n=2,3,...$ terms of the type $1/x^{n-j}$ occur for $j \in \{1,2,...,\lfloor n/2 \rfloor\}$. The coefficient of $1/x^{n-j}$ in $p_{n-1}(x)$ is $(-1)^j \binom{n-1-j}{j-1}$ (see Note 3 for the explicit form of p_{n-1}). For the $1/x^{n-j}$ coefficient in $q_{n-1}(x)\, c(x)$ one finds the convolution $\sum_{l=0}^{j-1}(-1)^{j-l-1}\binom{n-(j-l)}{j-l-1}\, C_l$. Compensation of both coefficients leads to identity $(P1)$ given in (4), after $(j-1)$ has been traded for p. Thus, after a shift $n \to n+2$:

Proposition 2: Identity $(P1)$

For $n \in \mathbf{N}_0$ and $p = 0,1,...,\lfloor \frac{n}{2} \rfloor$ identity $(P1)$, given in eq.(4) holds.

Example 3: $n = 2(k-1),\ p = k-1$, and $n = 2k-1,\ p = k-1$ for $k \in \mathbf{N}$

$\sum_{l=0}^{k-1}(-1)^l \binom{k+l}{2l+1}\, C_l = 1$, $\sum_{l=0}^{k-1}(-1)^l \binom{k+l+1}{2(l+1)}\, C_l = k$.

e.g. $k=3$: $3\,C_0 - 4\,C_1 + 1\,C_2 = 1$, $6\,C_0 - 5\,C_1 + 1\,C_2 = 3$.

For $n=2,3,...$ terms in (1), or (23), proportional to x^k with $k \in \mathbf{N}_0$ arise only from $q_{n-1}(x)\, c(x)$, and they are given by the convolution (cf. Note 4) $\sum_{l=0}^{\lfloor (n-1)/2 \rfloor}(-1)^l \binom{n-1-l}{l}\, C_{k+n-1-l}$. For $n=1$ this is C_k. The *lhs.* of (1) contributes $C_k(n)$, and $C_k(1)=C_k$. Therefore:

Proposition 3: Identity $(P3)$

For $n \in \mathbf{N}$, $k \in \mathbf{N}_0$ identity $P(2)$, given in eq.(5) with (3) holds.

Example 4: $k=0$, $(n-1) \to n$: $\sum_{l=1}^{\lfloor n/2 \rfloor}(-1)^{l+1}\binom{n-l}{l}\, C_{n-l} = C_n - 1$,

e.g. $n=3$: $2\,C_2 = C_3 - 1$, $n=4$: $3\,C_3 - 1\,C_2 = C_4 - 1$.

Now consider negative powers in (1), *i.e.* $c^{-n}(x)$, $n \in \mathbf{N}$. No negative powers of x appear (*cf.* footnote 4 for the explicit form of $p_{-(n+1)}(x)$ and $q_{-(n+1)}(x)$). The coefficient of x^k, $k \in \mathbf{N}_0$, of the *rhs.* of (1) is $(-1)^k\binom{n-k}{k} - \sum_{l=0}^{\lfloor (n-1)/2 \rfloor}(-1)^l\binom{n-1-l}{l}\, C_{k-1-l}$, where the first term, arising from $p_{-(n+1)}(x)$, contributes only for $k \in \{0,1,...,\lfloor n/2 \rfloor\}$. In the summation one also needs $l \le k-1$ because no *Catalan* numbers with negative index occur in (1). The *lhs.* of (1) has $[x^k]c^{-n}(x) = C_k(-n)$. From the last eq. in (37) one finds $C_k(-n) = \frac{n}{n-k}\binom{2k-n-1}{k} = (-1)^k \frac{n}{n-k}\binom{n-k}{k}$. In the last eq. the upper index in the binomial has been negated (*cf.* [4], (5.14)). Two sets of identities follow, depending on the range of k:

Proposition 4: Identity $(P3)$

For $n \in \mathbf{N}$, $k \in \{0,1,...,\lfloor \frac{n}{2} \rfloor\}$ identity $(P3)$, given in eq.(6) holds.

Example 5: $k=3,\ n \ge 6$: $C_2 - (n-2)\,C_1 + \binom{n+3}{2}\,C_0 = \binom{n-4}{2}$.

Proposition 5: Identity $(P4)$

For $n \in \mathbf{N}$, $k \in \mathbf{N}$ with $k \ge \lfloor \frac{n}{2} \rfloor + 1$ identity $(P4)$, given in eq.(7) holds.

In $(P4)$ only the $q_{-(n+1)}(x)\, c(x)$ part of (1) contributed and we used the first expression for $C_k(-n)$ in (37). In $(P3)$, where also $p_{-(n+1)}(x)$ contributed, we used the negated binomial coefficient for $C_l(-n)$ and absorption in the resulting one.

Note that (37) implies $C_k(-n) = -C_{k-n}(n)$ for $k, n \in \mathbf{N}$, and $k \geq n$. $C_k(0) = \delta_{k,0}$.

Example 6: $n = 5,\ k \geq 3:\ C_{k-1} - 3\,C_{k-2} + C_{k-3} = \frac{5}{k}\binom{2k-6}{k-1}$, *e.g.* $k = 7:\ C_6 - 3\,C_5 + C_4 = 20$.

If one uses the binomial formula for $c^{-n}(x) = (1 - x\ c(x))^n$ and $c^n(x) = \sum_{k=0}^{\infty} C_k(n)\ x^k$ one arrives at eq.(8).

3 Some families of integer sequences

In this section we present some sequences of positive integers which are defined with the help of the $\mathcal{U}_n$ polynomials (10).

$$u_n(m) := \mathcal{U}_n(1/m) = (\sqrt{m}\,)^n\ S_n(\sqrt{m}\,)\ . \tag{38}$$

The last eq. is due to (21). It will be shown that $u_n(m)$ is for each $m = 4, 5, ...$ and $n = -1, 0, ...$ a non-negative integer. Also negative integers $-m,\ m \in \mathbf{N}$ are of interest. In this case we add a sign factor.

$$v_n(m) := (-1)^n\ \mathcal{U}_n(-1/m) = (-i\sqrt{m}\,)^n\ S_n(i\sqrt{m}\,)\ . \tag{39}$$

From the S_n recursion relation (15) one infers those for the $u_n(m)$ and $v_n(m)$ sequences.

$$u_n(m) = m\ (u_{n-1}(m) - u_{n-2}(m))\ ,\quad u_{-1}(m) \equiv 0\ ,\quad u_0(m) \equiv 1\ , \tag{40}$$

$$v_n(m) = m\ (v_{n-1}(m) + v_{n-2}(m))\ ,\quad v_{-1}(m) \equiv 0\ ,\quad v_0(m) \equiv 1\ . \tag{41}$$

This shows that $v_n(m)$ constitutes a non-negative integer sequences for positive integer m. It describes certain generalized *Fibonacci* sequences (see *e.g.* [7] with $v_n(m) = W_{n+1}(0, 1; m, m)$). $v_n(m)$ counts, for example, the length of the binary word $W(m; n)$ obtained at step n from the substitution rule $1 \to 1^m 0$, $0 \to 1^m$, starting at step $n = 0$ with 0. The number of 1's, resp. 0's in $W(m; n)$ is $2v_{n-1}(m)$, resp. $2v_{n-2}(m)$. *E.g.* $W(2; 3) = (110)^2 1^2 (110)^2 1^2$ and $v_3(2) = 16$, $2v_2(2) = 6$ and $2v_1(2) = 4$. For $m = 1$ this substitution rule produces the well-known Fibonacci-tree. Of course, one can define in a similar manner generalized *Lucas* sequences using the polynomials $\{\mathcal{V}_n\}$ given in (11). Each $u_n(m)$ sequence (which is identified with $W_{n+1}(0, 1; m, -m)$ of [7]) turns out to be composed of two simpler sequences, *viz* $u_{2k}(m) =: m^k\ \alpha_k(m)$ and $u_{2k-1}(m) =: m^k\ \beta_k(m)$, $k \in \mathbf{N}_0$. These new sequences, which are, due to (38), given by $\alpha_k = S_{2k}(\sqrt{m}\,)$ and $\beta_k(m) = S_{2k-1}(\sqrt{m}\,)/\sqrt{m}$, satisfy therefore the following relations.

$$\beta_{k+1}(m) = (m-2)\ \beta_k(m) - \beta_{k-1}(m)\ ,\quad \beta_0(m) \equiv 0\ ,\quad \beta_1(m) \equiv 1\ , \tag{42}$$

and

$$\alpha_{k-1}(m) = \beta_k(m) + \beta_{k-1}(m)\ . \tag{43}$$

From (42) it is now clear that $\beta_n(m)$ is a non-negative integer sequence for $m = 4, 5, ...$ (In [7] $\beta_n(m) = W_n(0, 1; m-2, -1)$.) This property is then inherited by the $\alpha_n(m)$ sequences due to (43), and then by the composed sequence $u_n(m)$.

The ordinary generating functions are

$$g_\beta(m; x) := \sum_{n=0}^{\infty} \beta_n(m)\ x^n = \frac{1}{x^2 - (m-2)x + 1}\ ,\quad g_\alpha(m; x) := \sum_{n=0}^{\infty} \alpha_n(m)\ x^n = \frac{1+x}{x^2 - (m-2)x + 1}\ , \tag{44}$$

$$g_u(m;x) := \sum_{n=0} u_n(m)\, x^n = \frac{1}{1 - m\, x + m\, x^2} \quad , \quad g_v(m;x) := \sum_{n=0} v_n(m)\, x^n = \frac{1}{1 - m\, x - m\, x^2} \,. \qquad (45)$$

Note 6: The $\{\beta_n(m)\}$ sequences for $m = 4,5,6,7,8,10$ appear in the book [14]. The case $m = 4$ produces the sequence of non-negative integers, $m = 5$ are the even indexed *Fibonacci* numbers. The $m = 9$ sequence appears in *Sloane's* On-Line-Encyclopedia [14] as $A004187$. The $\{\alpha_n(m)\}$ sequences for $m = 4,5,6$ and 8 appear in the book [14]. $m = 4$ yields the positive odd integer sequence; $m = 5$ is the odd indexed *Lucas* number sequence. The $m = 7$ sequence appears now as $A030221$ in the database [14]. The composed sequences $\{u_n(m)\}$ are not in the book but some of them are found in the database [14]. $m = 4$ is the sequence $(n+1)\ 2^n$, $A001787$, and $m = 5,6,7$ appear now as $A030191, A030192$, $A030240$, respectively. As mentioned above $\{v_{n+1}(1)\}$ is the *Fibonacci* sequence. The instances $m = 2$ and 3 appear as $A002605$ and $A030195$, respectively, in the database [14].

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TABLE : $p(n,m) \;=\; [1/x^m]\, p\_\{n\}(x)$ *coefficient matrix*

$n = -1..12,\; m = 0..12$

n\m	0	1	2	3	4	5	6	7	8	9	10	11	12
-1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	-1	0	0	0	0	0	0	0	0	0	0	0
2	0	0	-1	0	0	0	0	0	0	0	0	0	0
3	0	0	1	-1	0	0	0	0	0	0	0	0	0
4	0	0	0	2	-1	0	0	0	0	0	0	0	0
5	0	0	0	-1	3	-1	0	0	0	0	0	0	0
6	0	0	0	0	-3	4	-1	0	0	0	0	0	0
7	0	0	0	0	1	-6	5	-1	0	0	0	0	0
8	0	0	0	0	0	4	-10	6	-1	0	0	0	0
9	0	0	0	0	0	-1	10	-15	7	-1	0	0	0
10	0	0	0	0	0	0	-5	20	-21	8	-1	0	0
11	0	0	0	0	0	0	1	-15	35	-28	9	-1	0
12	0	0	0	0	0	0	0	6	-35	56	-36	10	-1