On Polynomials Related to Derivatives of the Generating Function of Catalan's Numbers

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1 Introduction and Summary

In an accompanying paper [3] it has been shown that powers of the generating function $c(x)$ of *Catalan*'s numbers $\{C_n\}_0^\infty = \{1,1,2,5,14,42,...\}$ (nr.1459 and $A000108$ of [8], and refs. quoted in [3]) can be expressed in terms of a linear combination of 1 and $c(x)$ with coefficients replaced by certain scaled *Chebyshev* polynomials of the second kind. In this paper derivatives of $c(x)$ are studied in a similar manner. The starting point is the following expression for the first derivative.

$$\frac{d\ c(x)}{dx} \equiv c'(x) \ = \ \frac{1}{x(1-4x)}\Big(1+(-1+2x)\ c(x)\Big)\ . \tag{1}$$

This eq. is equivalent to the simple recurrence relation valid for C_n:

$$(n+2)\ C_{n+1}\ -\ 2(2n+1)\ C_n\ =\ 0\quad ,\quad n=-1,0,1,...,\ \text{with}\ \ C_{-1}=-1/2\ \ . \tag{2}$$

Eq.(1) can, of course, also be found from the explicit form $c(x) \ = \ (1-\sqrt{1-4x}\ )/(2x)$. The result for the n-th derivative is of the form

$$\frac{1}{n!}\frac{d^n c(x)}{dx^n} \ = \ \frac{1}{(x(1-4x))^n}\Big(a_{n-1}(x)\ +b_n(x)\ c(x)\Big), \tag{3}$$

with certain polynomials a_{n-1} of degree $n-1$ and b_n of degree n. These polynomials are found to be $b_n(x) \ = \ \sum_{m=0}^{n}(-1)^m B(n,m)\ x^{n-m}$ with

$$B(n,m)\ :=\ \binom{2n}{n}\ \binom{n}{m}/\binom{2m}{m}\ , \tag{4}$$

which defines a triangle of numbers for $n,m\in\mathbf{N}$, $n\geq m\geq 0$. Its head is depicted in *TAB. 1* with $B(n,m)=0$ for $n<m$. Another representation for the b_n polynomials is also found, *viz*

$$b_n(x)\ \ =\ \ -2\sum_{k=0}^{n} C_{k-1}\ x^k\ (4x-1)^{n-k}\ . \tag{5}$$

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Equating both forms of $b_n(x)$ leads to a formula involving convolutions of Catalan numbers with powers of an arbitrary constant $\lambda := (4x-1)/x$. This formula is given in *section 2* as eq.(31). Eq.(5) reveals the generating function of the $\{b_n(x)\}$ polynomials because it is a convolution of two functional sequences. The result is

$$g_b(x;z) \;:=\; \sum_{n=0}^{\infty} b_n(x)\, z^n \;=\; \frac{\sqrt{1-4xz}}{1+(1-4x)\, z}\ . \tag{6}$$

The other family of polynomials is $a_n(x) = \sum_{k=0}^{n} (-1)^k\, A(n+1,k+1)\, x^{n-k}$ with the triangular array $A(n,m)$ defined for $m=0$ by $A(n,0) = C_n$, and for $n \in \mathbf{N}, m \in \mathbf{N}$ with $n \geq m > 0$ by the numbers

$$A(n,m) \;=\; \frac{1}{2}\binom{n}{m-1}\Big[4^{n-m+1} \;-\; \binom{2n}{n}\Big/\binom{2(m-1)}{m-1}\Big]\ . \tag{7}$$

The head of this triangular array of numbers is shown in *TAB.2* with $A(n,m) = 0$ for $n < m$. Both results, (4) and (7), are solutions to recurrence relations which hold for $b_n(x)$ and $a_n(x)$ and their respective coefficients $B(n,m)$ and $A(n,m)$.

Another representation for the a_n polynomials is found to be

$$a_n(x) \;=\; \sum_{k=0}^{n} C_k\, x^k\, (4x-1)^{n-k}\ , \tag{8}$$

which shows that the generating function of these polynomials is

$$g_a(x;z) \;:=\; \sum_{n=0}^{\infty} a_n(x)\, z^n \;=\; \frac{c(xz)}{1+(1-4x)\, z}\ . \tag{9}$$

Comparing (5) with (8) yields the following relation between these two types of polynomials

$$b_n(x) \;=\; (4x-1)^n \;-\; 2x\, a_{n-1}(x)\ ,\quad n \in \mathbf{N}_0 \quad \text{with}\ \ a_{-1}(x) \equiv 0\ , \tag{10}$$

or for the coefficients

$$B(n,m) \;=\; \binom{n}{m}\, 4^{n-m} \;-\; 2\, A(n,m+1)\ . \tag{11}$$

The triangle of numbers $A(n,m)$ is related to a rectangular array of integers $\hat{A}(n,m)$, with $\hat{A}(0,m) \equiv 1$, $\hat{A}(n,0) = -C_n$ for $n \in \mathbf{N}$, and for $n \geq m \geq n \geq 1$ by

$$A(n,m) \;=\; -\hat{A}(n-m,m) \;+\; 2^{2(n-m)+1}\binom{n-1}{m-1}\ , \tag{12}$$

or with (7), for $m \in \mathbf{N}, n \in \mathbf{N}_0$, by

$$\hat{A}(n,m) \;=\; \frac{1}{2}\binom{n+m}{n+1}\Big[\binom{2(n+m)}{n+m}\Big/\binom{2(m-1)}{m-1} \;-\; 4^{n+1}\,\frac{m-1}{n+m}\Big]. \tag{13}$$

Part of the array $\hat{A}(n,m)$ is shown in *TAB. 3*, where it is called $C4(n,m)$.

It turns out that the mth column of the number triangle $A(n,m)$ is for $m = 0,1,...$ determined by the generating function $c(x)(\frac{x}{1-4x})^m$. The mth column of the number triangle $B(n,m)$ is, for $m = 0,1,...$, generated by $\frac{1}{\sqrt{1-4x}}\,(\frac{x}{1-4x})^m$. This fact identifies the infinite dimensional matrices $\mathbf{A}$ and $\mathbf{B}$ as examples of *Riordan* matrices in the terminology of [7]. The matrix $\hat{\mathbf{A}}$ associated with $\hat{A}(n,m)$ is an example of a *Riordan* array.

Because differentiation of $c(x) = \sum_{k=0}^{\infty} C_k\, x^k$ leads to

$$\frac{1}{n!}\,\frac{d^n c(x)}{dx^n} \;=\; \sum_{k=0}^{\infty} C(n,k)\; x^k \;,\; \text{with } C(n,k) \;:=\; \frac{1}{n!}\,\prod_{j=1}^{n}(k+j)\; C_{n+k} \;=\; \frac{(2(n+k))!}{n!k!(n+k+1)!}\;, \tag{14}$$

with $C(0,k) = C_k$, one finds, together with (3), the following identities, for $n \in \mathbf{N}$, $p \in \{0,1,2,...,n-1\}$

$$(D1):\;\; \sum_{k=0}^{p}(-1)^k\; C_k\, \binom{n}{p-k}\Big/\binom{2(n-p+k)}{n-p+k} \;=\; \frac{1}{2}\binom{n}{p+1}\Big\{2^{2(p+1)}\Big/\binom{2n}{n} \;-\; 1\Big/\binom{2(n-p-1)}{n-p-1}\Big\}$$

$$=\; A(n,n-p)\Big/\binom{2n}{n}\;, \tag{15}$$

and, for $n \in \mathbf{N}, k \in \mathbf{N}_0$,

$$(D2):\;\; \sum_{j=0}^{n}(-1)^j\;\Big(\binom{n}{j}\Big/\binom{2j}{j}\Big)\;\sum_{l=0}^{k} 4^l\;\binom{n+l-1}{n-1}\; C_{k+j-l} \;=\; C(n,k)\Big/\binom{2n}{n}\;. \tag{16}$$

The remainder of this paper provides proofs for the above given statements.

2 Derivatives

The starting point is eq.(1) which can either be verified from the explicit form of the generating function $c(x)$, or by converting the recursion relation (2) for *Catalan*'s numbers into an eq. for their generating function. A computation of $\frac{1}{(n+1)!}\,\frac{d^{n+1}c(x)}{dx^{n+1}} \;=\; \frac{1}{n+1}\,\frac{d}{dx}\big(\frac{1}{n!}\,\frac{d^n c(x)}{dx^n}\big)$ with (3) taken as ansatz and eq. (1) produces the following mixed relations between the quantities $a_n(x)$ and $b_n(x)$ and their first derivatives, valid for $n \in \mathbf{N}_0$,

$$(n+1)\; a_n(x) \;=\; x(1-4x)\; a'_{n-1}(x) \;+\; b_n(x) \;+\; n(8x-1)\; a_{n-1}(x)\;, \tag{17}$$

$$(n+1)\; b_{n+1}(x) \;=\; x(1-4x)\; b'_n(x) \;+\; (-(n+1)+2(1+4n)x)\; b_n(x)\;, \tag{18}$$

with inputs $a_{-1}(x) \equiv 0$ and $b_0(x) \equiv 1$.

From (18) and the input it is clear by induction that $b_n(x)$ is a polynomial in x of degree n. With this information (17) and the input show, again by induction, that the same statement holds for $a_n(x)$. Therefore we write, for $n \in \mathbf{N}_0$,

$$a_n(x) \;=\; \sum_{k=0}^{n}(-1)^k\; a(n,k)\; x^{n-k}\;, \tag{19}$$

$$b_n(x) \;=\; \sum_{k=0}^{n}(-1)^k\; B(n,k)\; x^{n-k}\;, \tag{20}$$

with the triangular arrays of numbers $a(n,k)$ and $B(n,k)$ with row number n and column number $k \leq n$. The triangular array $a(n,k)$ will later be enlarged to another one which will then be called $A(n,k)$.

We first solve the $b_n(x)$ eq.(18) by inserting (20) and deriving the recursion relation for the coefficients $B(n,m)$ after comparing coefficients of x^{n+1}, x^0, and x^{n-k} for $k = 0, 1, ..., n-1$.

$$x^{n+1} : \quad (n+1)\ B(n+1,0) \ = \ 2(2n+1)\ B(n,0)\ , \tag{21}$$

$$x^0 : \quad B(n+1,n+1) \ = \ B(n,n)\ , \tag{22}$$

$$x^{n-k} : \quad (n+1)\ B(n+1,k+1) \ = \ (k+1)\ B(n,k) \ + \ 2(2(n+k)+3)\ B(n,k+1)\ . \tag{23}$$

With the input $B(0,0) = 1$ one deduces from (21) for the leading coefficient of $b_n(x)$

$$B(n,0) \ = \ 2^n \frac{(2n-1)!!}{n!} \ = \ \frac{(2n)!}{n!\ n!} \ = \ \binom{2n}{n}\ , \tag{24}$$

and from (22)

$$B(n,n) \ \equiv 1 \ , \ \text{ i.e. } \ b_n(0) = (-1)^n\ . \tag{25}$$

In order to solve (23) we inspect the $B(n,m)$ triangle of numbers *TAB.1*, and conjecture that for $n, m \in \mathbf{N}$

$$B(n,m) \ = \ 4\ B(n-1,m) \ + \ B(n-1,m-1)\ , \tag{26}$$

with input $B(n,0) = \binom{2n}{n}$ from (24).

If we use this conjecture in (23), written with $n \to n-1,\ k \to m-1$ we are led to consider the simple recursion

$$B(n,m) \ = \ \frac{n+1-m}{2(2m-1)}\ B(n,m-1)\ , \tag{27}$$

with input $B(n,0) = \binom{2n}{n}$ from (24).

The solution of this recursion is, for $n, m \in \mathbf{N}_0$,

$$B(n,m) \ = \ \frac{1}{2^m (2m-1)!!} \frac{n!}{(n-m)!} \binom{2n}{n} \ = \ \frac{m!\ n!}{(2m)!\ (n-m)!} \binom{2n}{n} \ = \ \binom{2n}{n} \binom{n}{m} / \binom{2m}{m}\ . \tag{28}$$

With the *Pochhammer* symbol $(a)_n := \Gamma(n+a)/\Gamma(a)$ this result can also be written as $B(n,m) = ((2m+1)/2)_{n-m} 4^{m-n}/(n-m)!$. This result satisfies (21), *i.e.* (24), as well as (22), *i.e.* (25). It is also the solution to (23) provided we prove the conjecture (26) for $B(n,m)$ of (28). This can be done by using the form $B(n,m) = \frac{(2n)!\ m!}{(2m)!\ n!\ (n-m)!}$ and extracting this expression on the *rhs.* of (26). Then one is left to prove $1 \ = \ \frac{4}{2}\ \frac{n-m-1}{2n-1} \ + \ \frac{2m-1}{2n-1}$, which is trivial. Thus we have proved:

Proposition 1: Explicit form of $b_n(x)$

$B(n,m)$ given by eq. (28) is the solution to eqs.(21), (22), and (23). Hence $b_n(x)$, defined by (20) with $B(n,m)$ from (28), solves eq. (18) with $b_0(x) \equiv 1$.

This triangle of numbers whose head is shown in *TAB.1* appears now as nr. A046521 in the database [8].

One can derive another explicit representation for the $b_n(x)$ polynomials by converting the simple recurrence relation (27) into the following eq. for $b_n(x)$ defined by (20).

$$(1-4x)\ b_n'(x) \ + \ 2(2n-1)\ b_n(x) \ + \ 2 \binom{2n}{n}\ x^n \ = \ 0\ . \tag{29}$$

This leads, together with (18), to the following inhomogeneous recursion relation for $b_n(x)$.

$$b_{n+1}(x) \;=\; (4x-1)\; b_n(x) \;-\; 2C_n\; x^{n+1} \quad , \quad b_0(x) \equiv 1\ . \tag{30}$$

Eq.(29) can also be solved as first order linear and inhomogeneous differential eq. for $b_n(x)$.

Proposition 2: Alternative form for $b_n(x)$

The solution to eq.(30) is given by eq.(5), with $C_{-1} = -1/2$ and the *Catalan* numbers C_k for $k \in \mathbf{N}_0$.

Proof: Iteration of (30). □

Proposition 3: Generating function for $\{b_n(x)\}$

The generating function $g_b(x;z) \;:= \sum_{n=0}^{\infty}\; b_n(x)\; x^n$ is given by eq.(14).

Proof: The alternative form of $b_n(x)$, given by (5), is a convolution of the functional sequences $\{-2C_{k-1}\,x^k\}_0^\infty$ and $\{4x-1\}_0^\infty$, with generating functions $1-2xz\,c(xz) = \sqrt{1-4xz}$ and $1/(1+(1-4x)z)$, respectively. Therefore, $g_b(x;z)$ is the product of these two generating functions. □

Comparing this alternative form (5) for $b_n(x)$ with the one given by (20), together with (28), proves the following identity in n and $\lambda := (4x-1)/x$. The term $k=0$ in the sum (5) has been written separately.

Corollary 1: Convolution of *Catalan* sequence and powers of λ

$$s_{n-1}(\lambda) \;:=\; \lambda^{n-1} \sum_{k=0}^{n-1} \frac{C_k}{\lambda^k} \;=\frac{1}{2}\Big(\lambda^n \;-\; \binom{2n}{n} \sum_{k=0}^{n}\, (-1)^k\; (4-\lambda)^k\; \binom{n}{k}\Big/\binom{2k}{k}\Big)\ , \tag{31}$$

for $n \in \mathbf{N}$ and $\lambda \neq \infty$. Observe that $s_n(\lambda)$ is the convolution of the *Catalan* sequence with the sequence of powers of λ. Therefore, the generating function for the sequence $s_n(\lambda)$ is $g(\lambda;x) := \sum_{n=0}^{\infty}\; s_n(\lambda)\; x^n \;=\; c(x)/(1-\lambda\; x)$.

From the generating function the recurrence relation is found to be $s_n(\lambda) = \lambda\; s_{n-1}(\lambda) + C_n$, $s_{-1}(\lambda) \equiv 0$. The connection to the $b_n(x)$ polynomial is $s_n(\lambda) \;=\; \frac{1}{2}\Big(\lambda^{n+1} \;-\; (4-\lambda)^{n+1}\; b_{n+1}(1/(4-\lambda))\Big)$.

The case $\lambda = 0$ $(x = 1/4)$ is also covered by this formula. It produces from $s_n(0) = C_n$ the following identity.

Example 1: Case $\lambda = 0$ $\;(x = 1/4)$

$$\sum_{k=0}^{n}\; (-1)^{k+1}\; \binom{n}{k}\; 4^k\Big/\binom{2k}{k} \;=\; \frac{1}{2n-1}\ . \tag{32}$$

This identity occurs in one of the exercises 2.7, 2, p.32, in [4].

We note that from (5) one has $-2b_{n+1}(1/4) = C_n/4^n$. The large n behaviour of this sequence is known to be $C_n/4^n \sim \frac{1}{\sqrt{\pi}}\;\frac{1}{n^{3/2}}$, *cf.* [2], Exercise 9.60.

If one puts in (5) $4x - 1 = x$, *i.e.* $x = 1/3$, one can identify the partial sum of *Catalan* numbers, $s_n(1)$ as follows.

$$s_n(1) \;= \sum_{k=0}^{n}\; C_k \;=\; \frac{1}{2}(1 \;-\; 3^{n+1}\; b_{n+1}(1/3)\ ). \tag{33}$$

This sequence $\{1,2,4,9,23,65,197,626,2056,...\}$ appears as A014137 in the on-line encyclopedia [8].

If one puts $\lambda = 1$ in *Corollary 1* one also finds

Example 2:

$$2\; s_{n-1}(1) \;=\; 1 \;+\; \binom{2n}{n} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k}\; 3^k/\binom{2k}{k}\;. \tag{34}$$

Another interesting example is the case $\lambda = 4$ ($x = \infty$). Here one finds a simple result for the convolution of *Catalan*'s sequence with powers of 4, *viz*

Example 3: $\lambda = 4 \quad (x = \infty)$

$$2\; s_{n-1}(4) \;=\; 4^n \;-\; \binom{2n}{n}\;. \tag{35}$$

This sequence $\{1,5,22,93,386,1586,6476,...\}$ appears in the book [8] as Nr. 3920 and as A000346 in the on-line encyclopedia. It will show up again in this work as $A(n+1,1)$, the second column in the $A(n,m)$ triangle (*cf. TAB.2*).

The sequence for $\lambda = -1$ ($x = 1/5$) is also non-negative, as can be seen by writing $s_{2k}(-1) = C_2 + \sum_{l=2}^{k}(C_{2l} - C_{2l-1})$ for $k \in \mathbf{N}$ and $s_{2k+1}(-1) = \sum_{l=1}^{k}(C_{2l+1} - C_{2l})$, and using $\triangle C_n := C_n - C_{n-1} = 3\frac{n-1}{n+1}\, C_{n-1} \geq 0$. This is the sequence $\{1,0,2,3,11,31,101,328,1102,3760,...\}$ which appears now as *A*032357 in the on-line encyclopedia [8] .

Recursion (26) for $B(n,m)$ can be transformed into an eq. for the generating function for the sequence appearing in the mth column of the $B(n,m)$ triangle

$$G_B(m;x) := \sum_{n=m}^{\infty} B(n,m)\; x^n \quad , \tag{36}$$

with input $G_B(0;x) = \sum_{n=0}^{\infty}\binom{2n}{n}\, x^n = 1/\sqrt{1-4x}$, the generating function for the central binomial numbers. (26) implies for $m \in \mathbf{N}_0$

$$G_B(m;x) \;=\; \Big(\frac{x}{1-4x}\Big)^m \frac{1}{\sqrt{1-4x}}. \tag{37}$$

For $x\frac{d}{dx}G_B(m;x)$ see (53).

Therefore, we have proved:

Proposition 4: Column sequences of the $B(n,m)$ triangle

The sequence $\{B(n,m)\}_{n=m}^{\infty}$, defined, for fixed $m \in \mathbf{N}_0$ and $n \in \mathbf{N}_0$ by (28) is the convolution of the central binomial sequence $\{\binom{2k}{k}\}_0^{\infty}$ and the mth convolution of the (shifted) power sequence $\{0,1,4^1,4^2,...\}$.

Note 1: The infinite dimensional matrix $\mathbf{B}$ with elements $B(n,m)$ given for $n \geq m \geq 0$ by (28) and $B(n,m) \equiv 0$ for $n < m$ is an example of a *Riordan* matrix [7]. With the notation of this ref. $\mathbf{B} = (\frac{1}{\sqrt{1-4x}}, \frac{x}{1-4x})$.

Note 2: *Sheffer*-type identities from *Riordan*-matrices

Triangular *Riordan*-matrices $\mathbf{M} = (M_{i,j})_{i\geq j\geq 0} = (g(x), f(x))$, $M_{i,j} = 0$ for $j > i$, in the notation of ref.[7] lead to polynomials which satisfy *Sheffer*-type identities (see ref.[5], also for original refs., and [1])

$$S_n(x+y) = \sum_{k=0}^{n} S_k(y)\, P_{n-k}(x) = \sum_{k=0}^{n} P_k(y)\, S_{n-k}(x)\ , \tag{38}$$

$$P_n(x+y) = \sum_{k=0}^{n} P_k(y)\, P_{n-k}(x) = \sum_{k=0}^{n} P_k(x)\, P_{n-k}(y)\ , \tag{39}$$

for the polynomials

$$S_n(x) = \sum_{m=0}^{n} M_{n,m}\, \frac{x^m}{m!},\ n \in \mathbf{N}_0\quad ,\quad P_n(x) = \sum_{m=1}^{n} P_{n,m}\, \frac{x^m}{m!},\ n\in \mathbf{N},\ P_0(x) \equiv 1\ , \tag{40}$$

with $P_{n,m} := [z^n]\,(f^m(z))$, $n \geq m \geq 1$. $g(x)$ defines the first column of $\mathbf{M}$: $M_{n,0} = [x^n]\, g(x)$.

If one uses $s_n(x) := n!\, S_n(x)$ and $p_n(x) := n!\, P_n(x)$ one obtains the *Sheffer*-identities (also called binomial identities) treated in ref. [5]. $s_n(x)$ is then *Sheffer* for $(1/g(\bar{f}(t)), \bar{f}(t))$, and $p_n(x)$ is associated to $\bar{f}(t)$ (or *Sheffer* for $(1, \bar{f}(t))$) in the language of ref.[5]. Here $\bar{f}(t)$ stands for the compositional inverse of $f(t)$.

Proposition 5: Relation between $g_b(x;z)$ and $G_B(m;x)$

$$g_b(x;z) = \sum_{m=0}^{\infty} (-1)^m\, G_B(m;xz)\, \left(\frac{1}{x}\right)^m\quad . \tag{41}$$

Proof: One inserts $b_n(x)$ of (20) into the definition (6) of $g_b(x;z)$ and rewrites the *Cauchy*-sum as two infinite sums which are then interchanged. Finally, the definition of $G_B(m;x)$ from (36) is used. □

One can check (41) by putting in the explicit form (36) for $G_B(m;xz)$ and compare with (6).

In a similar vein we solve the $a_n(x)$ eq.(17) with $b_n(x)$ given by (20) and (28). The coefficients $a(n,k)$, defined by (19), have to satisfy, after comparing coefficients of x^n, x^0, and x^{n-k} for $k = 1, 2, ..., n-1$ and $n \in \mathbf{N}_0$:

$$x^n\ :\quad a(n,0) = 4\ a(n-1,0)\ +\ C_n, \tag{42}$$

$$x^0\ :\quad (n+1)\ a(n,n) = 1\ +\ n\ a(n-1,n-1)\ , \tag{43}$$

$$x^{n-k}\ :\quad (n+1)\ a(n,k) = k\ a(n-1,k-1)\ +\ 4(n+1+k)\ a(n-1,k)\ +\ B(n,k)\ . \tag{44}$$

We have used (24), *i.e.* $B(n,0) = (n+1)\ C_n$ in (42), as well as (25), *i.e.* $B(n,n) \equiv 1$, in (43). From (42) one finds with input $a(0,0) = 1$

$$a(n,0) = \sum_{k=0}^{n} C_k\ 4^{n-k}\ , \tag{45}$$

and from (43)

$$a(n,n) \equiv 1,\ \text{or}\ \ a_n(0) = (-1)^n\ . \tag{46}$$

Note that $a(n,0) = s_n(4)$ of (31) with solution (33). It is convenient to define $a(n-1,-1) := C_n$, $n \in \mathbf{N}_0$. Then the sequence $\{a(n,0)\}_{-1}^{\infty}$ is, with $a(-1,0) := 0$, the convolution of the sequence $\{a(k,-1)\}_{-1}^{\infty}$ and the shifted power sequence $\{0, 1, 4^1, 4^2, ...\}$. Before solving (44), with $B(n,k)$ from (28) inserted we therefore add to the trianglular array of numbers $a(n,m)$ the $m = -1$ column and an extra row for $n = -1$, and define a new enlarged triangular array for $n, m \in \mathbf{N}_0$ as

$$A(n,m) \ := \ a(n-1, m-1) \tag{47}$$

with $A(n,0) = a(n-1,-1) = C_n$ and $A(0,m) = a(-1, m-1) = \delta_{0,m}$. An inspection of the $A(n,m)$ triangular array, partly depicted in *TAB. 2*, leads to the conjecture

$$A(n,m) \ = \ 4 \ A(n-1,m) \ + \ A(n-1,m-1) \ , \tag{48}$$

with $A(n,0) = C_n$ and $A(n,m) \equiv 0$ for $n < m$. This recursion relation can be employed to extend the array $A(n,m)$ to negative integer m values. This conjecture is correct for $A(n+1,1) = a(n,0)$ found in (45), as well as for $A(n+1,n+1) = a(n,n) \equiv 1$ known from (46). The generating function for the sequence appearing in the mth column,

$$G_A(m;x) \ := \ \sum_{n=m}^{\infty} \ A(n,m) \ x^n \ , \tag{49}$$

satisfies due to (48) $G_A(m;x) = \frac{x}{1-4x} \ G_A(m-1;x)$, remembering that $A(m-1,m) \equiv 0$, or because of $G_A(0;x) \ = \ c(x)$

$$G_A(m;x) \ = \ \left(\frac{x}{1-4x}\right)^m \ c(x) \ . \tag{50}$$

Note 3: The infinite dimensional matrix $\mathbf{A}$ with elements $A(n,m)$ given for $n \geq m \geq 0$ by (48) and $A(n,m) \equiv 0$ for $n < m$ is another example of a *Riordan* matrix, written in the notation of [7] as $(c(x), x/(1-4x))$.

Because of (37) and $\sqrt{1-4x} \ c(x) \ = \ 2 - c(x)$ these generating functions of the conjectured $A(n,m)$ column sequences obey

$$G_A(m;x) \ = \ (2 - c(x)) \ G_B(m;x) \ . \tag{51}$$

If we use the conjecture (48) in (44) which is written with (47) in the form $(n+1) \ A(n+1, m+1) \ = \ m \ A(n,m) \ + \ 4(n+m+1) \ A(n,m+1) \ + \ B(n,m)$, for $n \in \mathbf{N}_0$, $m \in \{1, 2, .., n-1\}$, we have

$$m \ A(n+1,m+1) \ - \ (n+1) \ A(n,m) \ + \ B(n,m) \ = \ 0 \ . \tag{52}$$

This recursion relation can be written with the help of the generating functions (36) and (49) as

$$(x\frac{d}{dx} + 1) \ G_A(m;x) \ - \ \frac{m}{x} \ G_A(m+1;x) \ = \ G_B(m;x) \ , \tag{53}$$

or with (50) (*i.e.* the conjecture) as

$$\left(x\frac{d}{dx} + 1 \ - \ \frac{m}{1-4x}\right) \ G_A(m;x) \ = \ G_B(m;x) \ . \tag{54}$$

Together with (51) this means

$$x\frac{d}{dx}\Big((2-c(x)) \ G_B(m;x)\Big) \ = \ \Big[(\frac{m}{1-4x} - 1)(2 - c(x)) \ + 1\Big] \ G_B(m;x) \ . \tag{55}$$

If we can prove this eq. with $G_B(x)$ given by (37) we have shown that (44) is equivalent to the conjecture (48). In order to prove (55) we first compute from (37), for $m \in \mathbf{N}_0$,

$$x\frac{d}{dx}G_B(m;x) \ = (2 + \frac{m}{x}) \ G_B(m+1;x) \ = \ \frac{2x+m}{1-4x} \ G_B(m;x) \ . \tag{56}$$

With this result (55) reduces to

$$\left(-x\ c'(x)\ +\ (2-c(x))\ \frac{1-2x}{1-4x}\ -\ 1\right)\ G_B(m;x)\ =\ 0\ , \tag{57}$$

and with (1) the factor in front of $G_B(m;x)$ vanishes identically for $x \neq 1/4$. Therefore, we have proved the following two propositions.

Proposition 6: Column sequences of the $A(n,m)$ triangular array

The triangular array of numbers $A(n,m)$, defined for $n, m \in \mathbf{N}_0$ by eq.(48), $A(n,0) = C_n$, $A(n,m) \equiv 0$ for $n < m$ has as mth column sequence $\{A(n,m)\}_{n=m}^{\infty}$ the convolution of *Catalan*'s sequence and the mth convolution of the shifted power sequence $\{0, 1, 4^1, 4^2, ...\}$.

Proof: (50) with (49). □

Proposition 7: Triangular $A(n,m)$ array

The triangular array $A(n,m)$ of *proposition 6* coincides with the one defined by (47) and (42), (43) and (44) with $B(n,m)$ given by (28).

Proof: $a(n,0) = A(n+1,1)$ and $a(n,n) = A(n+1,n+1) \equiv 1$ of (42) and (43), *i.e.* (45) and (46), respectively, satisfy (45). (44) is rewritten with the aid of (47) as (52), and (52) has been proved by (53) to (57). □

Alternatively, one can use the now proven conjecture (48), together with (47), in (44) and derive for $n \in \mathbf{N}_0,\ m \in \mathbf{N}_0$

$$4m\ a(n-1,m)\ =\ (n+1-m)\ a(n-1,m-1)\ -\ B(n,m)\ . \tag{58}$$

This is written in terms of the polynomials $a_{n-1}(x)$ of (19) and $b_n(x)$ of (20) as

$$x(1-4x)\ a'_{n-1}(x)\ +\ (1-4x+4nx)\ a_{n-1}(x)\ -\ \binom{2n}{n}\ x^n\ +\ b_n(x)\ =\ 0\ . \tag{59}$$

With this result (17) becomes an inhomogeneous recursion relation for $a_n(x)$, *viz*

$$a_n(x)\ =\ (4x-1)\ a_{n-1}(x)\ +\ C_n\ x^n\quad ,\quad a_0(x) \equiv 1\ . \tag{60}$$

(59) can also be considered as inhomogeneous linear differential eq. for $a_{n-1}(x)$ with given $b_n(x)$. To find the solution this way is, however, a bit tedious.

Proposition 8: Alternative form for $a_n(x)$

The solution of the recursion relation (60) is given by (8).

Proof: Iteration of (60). □

Corollary 2: Generating function for $\{a_n(x)\}$

The generating function $g_a(x;z)\ :=\ \sum_{n=0}^{\infty}\ a_n(x)\ z^n$ is given by eq.(9).

Proof: (8) shows that $a_n(x)$ is a convolution of the functional sequences $\{C_k\, x^k\}_0^\infty$ and $\{(4x-1)^k\}_0^\infty$ with generating functions $c(xz)$ and $1/(1+(1-4x)z)$. Therefore, $g_a(x;z)$ is the product of these generating functions. □

Proposition 9 Relation between $g_a(x;z)$ and $G_A(m;x)$

$$g_a(x;z) \;=\; \frac{1}{1-4xz}\,\sum_{m=0}^{\infty}(-1)^m\; G_A(m;xz)\;\left(\frac{1}{x}\right)^m\,. \tag{61}$$

Proof: Analogous to the proof of *proposition 5.* □

One can ckeck (61) by putting in the explicit form (50) of $G_A(m;x)$ and compare with (9).

Proposition 10: Relation between $b_n(x)$ and $a_{n-1}(x)$

For $n \in \mathbf{N}_0$ and $a_{-1}(x) \equiv 0$ the relation between $b_n(x)$ and $a_{n-1}(x)$ is given by eq.(10) .

Proof: The alternative expressions (5) and (8) for these two families of polynomials are used. One splits off the $k=0$ term in (5) with $C_{-1}=-1/2$ from the sum and shifts the summation variable. □

Corollary 3: Relation between $A(n,m)$ and $B(n,m)$

The coefficients of the triangular arrays $A(n,m)$ and $B(n,m)$ are related as given by eq.(11) .

Proof: The relation (10) between the polynomials is, with the help of (19) and (20), written for the coefficients $a(n-1,m)$, or by (47) for $A(n,m+1)$, and $B(n,m)$. □

It remains to compute the explicit expression for the $a_n(x)$ coefficients $a(n,k)$ defined by (19). Because of (47) it suffices to determine $A(n,m)$.

Corollary 4: Explicit form of $A(n,m)$

The triangular array numbers $A(n,m)$ are given explicitely by formula (7).

Proof: The formula (4) written for $B(n,m-1)$ is used in relation (11). □

Note 4: This formula for $A(n,m)$ satisfies indeed recursion relation (48) with the given input. The first term, $\frac{1}{2}\,4^{n-m+1}\,\binom{n}{m-1}$, satisfies it because of the binomial identity $\binom{n}{m-1} \;=\; \binom{n-1}{m-1} + \binom{n-1}{m-2}$ (*Pascal*'s triangle). For the second term of $A(n,m)$ in (7) one has to prove $\binom{n}{m-1}\binom{2n}{n} \;=\; 4\,\binom{n-1}{m-1}\binom{2(n-1)}{n-1} + \binom{n-1}{m-2}\binom{2(n-1)}{n-1}\frac{2(2m-3)}{m-1}$, or after division by $\binom{2(n-1)}{n-1}$, $\frac{2n-1}{n}\,\binom{n}{m-1} \;=\; 2\,\binom{n-1}{m-1} + \binom{n-1}{m-2}\frac{2m-3}{m-1}$, which reduces to the trivial identity $2n-1 = 2\,(n-m+1)+2m-3$. Both terms together, *i.e.* (7), satisfy the input $A(n,n)\equiv 1$.

Note 5: $A(n,m)$ was found originally after iteration in the form (with $n \geq m > 0$ and $(-1)!! := 1$)

$$A(n,m) \;=\; 2\cdot 4^{n-m}\begin{pmatrix} n \\ m-1 \end{pmatrix} \;-\; \frac{\prod_{k=1}^{m}(2(n-m)+2k-1)}{(2m-3)!!}\,C_{n-m}\,. \tag{62}$$

$A(n,0) = C_n$. It is easy to establish equivalence with (7).

In the original derivation of the $A(n,m)$ formula (7) it turned out to be convenient to introduce a rectangular array of integers $\hat{A}(n,m)$ for $n,m \in \mathbf{N}_0$ as follows. $\hat{A}(0,m)\equiv 1$, $\hat{A}(n,0) := -C_n$ for $n \in \mathbf{N}$, and for $m \in \mathbf{N}$ and $n \in \mathbf{N}_0$ $\hat{A}(n,m)$ is defined by (7), or equivalently, by (8). The $A(n,m)$ recursion (48) translates (with the help of the *Pascal*-triangle identity) into

$$\hat{A}(n,m) \;=\; 4\,\hat{A}(n-1,m) \;+\; \hat{A}(n,m-1)\;. \tag{63}$$

This leads, after iteration and use of $\hat{A}(0,m) \equiv 1$ from (12) with $A(n,n) \equiv 1$, to

$$\hat{A}(n,m) \;=\; 4^n \sum_{k=0}^{n} \hat{A}(k,m-1)/4^k\;. \tag{64}$$

Thus, the following proposition holds.

Proposition 11: Column sequences of the $\hat{A}(n,m) \equiv C4(n,m)$ array

The mth column sequence of the $\hat{A}(n,m)$ array, $\{\hat{A}(n,m)\}_{n=0}^{\infty}$, is the convolution of the sequence $\{\hat{A}(n,0)\}_0^{\infty} = \{1,-1,-2,-5,...\}$, generated by $2-c(x)$, and the mth convolution of the power sequence $\{4^k\}_0^{\infty}$.

Proof: Iteration of (64) with the $\hat{A}(n,0)$ input. □

Corollary 5: Generating functions for columns of the $\hat{A}(n,m) \equiv C4(n,m)$ array

The ordinary generating function of the mth column sequence of the $\hat{A}(n,m)$ array (13) is for $m \in \mathbf{N}_0$ given by

$$G_{\hat{A}}(m;x) \;:=\; \sum_{n=0}^{\infty} \hat{A}(n,m)\;x^n \;=\; (2-c(x))\,\left(\frac{1}{1-4x}\right)^m\;. \tag{65}$$

Proof: Proposition 11 written for generating functions. □

Because of the convolution of the (negative) *Catalan* sequence with powers of 4 we shall call this $\hat{A}(n,m)$ array also $C4(n,m)$. A part of it is shown in *TAB.3*. The second column sequence is given by $\hat{A}(n,1) \equiv C4(n,1) = \binom{2n+1}{n}$ and appears as nr.2848 in the book [8], or as $A001700$ in the on-line encyclopedia [8]. The sequence of the third column $\{\hat{A}(n,2) \equiv C4(n,2)\}_0^{\infty} \;=\; \{1,7,38,187,...\}$ is from (64) and (62) with (12) determined by $4^n\;\sum_{k=0}^{n}\binom{2k+1}{k}/4^k \;=\; (2n+3)\;(2n+1)\;C_n \;-\; 2^{2n+1}$, and is listed as $A000531$ in the mentioned on-line encyclopedia. There the fourth column sequence is now listed as $A029887$.

Note 6: The infinite dimensional lower triangular matrix $\tilde{\mathbf{A}}$ related to the array $\hat{A}(n,m) \equiv C4(n,m)$ by $\tilde{A}(n,m) := \hat{A}(n-m,m+1)$ for $n \geq m \geq 0$ and $\tilde{A}(n,m) := 0$ for $n < m$ is again an example of a *Riordan* matrix [7]. In the notation of [7] $\tilde{\mathbf{A}} \;=\; (c(x)/\sqrt{1-4x},x/\sqrt{1-4x})$.

Finally, we derive identities by using, for $n \in \mathbf{N}_0$, eq.(14) for the *lhs.* of (3) and the results for $a_{n-1}(x)$ and $b_n(x)$ for the *rhs.*

Because there are no negative powers of x on the *lhs.* of (3), such powers have to vanish on the *rhs.* This leads to the first family of identities. Because $(1-4x)^{-n} \;=\; \sum_{k=0}^{\infty}\frac{(n)_k}{k!}\;4^k\;x^k$, with *Pochhammer*'s symbol defined after eq. (28), this means that $[x^p]\;(a_{n-1}(x)+b_n(x)\;c(x))$, the coefficient proportional to x^p, has to vanish for $p = 0,1,...,n-1$, $n \in \mathbf{N}$. This requirement reads

$$(-1)^{n-1-p}\;a(n-1,n-1-p) \;+\; \sum_{k=0}^{p}(-1)^{n-k}\;B(n,n-k)\;C_{p-k}\;\equiv 0\;. \tag{66}$$

The sum is restricted to $k \leq p(< n)$ because no C_l number with negative index is found in $c(x)$. Inserting the known coefficients this produces identity $(D1)$ of (15).

Proposition 12: Identity $(D1)$ of (15)

For $n \in \mathbf{N}$ and $p \in \{0, 1, ..., n-1\}$ identity $(D1)$, given by (15), holds.

Proof: With (47) (66) becomes

$$\sum_{k=0}^{p} (-1)^{p-k} \, C_{p-k} \, B(n, n-k) \; = \; A(n, n-p) \; , \tag{67}$$

which is $(D1)$ of (15) if the summation index k is changed into $p-k$, and symmetry of the binomial coefficients is used. $\square$.

Example 4: $(D1)$ identity for $p = n-1 \in \mathbf{N}_0$

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{1}{2k+1} \; = \; 4^n / \binom{2n}{n} \; - \; 1 \; = \; 2A(n,1) / \binom{2n}{n} \; . \tag{68}$$

With this identity we have found a sum representation for the convolution of the *Catalan* sequence and powers of 4: $s_{n-1}(4) := 4^{n-1} \sum_{k=0}^{n-1} C_k / 4^k \; = \; \frac{1}{2} \binom{2n}{n} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{1}{2k+1}$ (*cf.* (35) with (31)). The second family of identities, $(D2)$ of (16), results from comparing powers x^k with $k \in \mathbf{N}_0$ on both sides of eq.(3) after expansion of $(1-4x)^{-n}$ as given above in the text before eq. (66). Only the second term $b_n(x) \; c(x)$ contributes because $a_{n-1}(x)/x^n$ has only negative powers of x. Thus, with definition (14) one finds for $k \in \mathbf{N}_0$ and $n \in \mathbf{N}$,

$$C(n,k) \; = \; \sum_{l=0}^{k} \frac{(n)_l \; 4^l}{l!} \sum_{j=0}^{n} (-1)^{n-j} \; B(n, n-j) \; C_{n-j+k-l} \tag{69}$$

which is, after interchange of the summations and insertion of $B(n, n-j)$ from (4) the desired identity $(D2)$ if also the summation index j is changed to $n-q$.

Thus we have shown:

Proposition 13: Identity $(D2)$ of (16)

For $k \in \mathbf{N}_0$ and $n \in \mathbf{N}$ identity $(D2)$ of (16) with $C(n,k)$ defined by (14) holds.

Example 5: Identity $(D2)$ for $k = 0$, $n \in \mathbf{N}$

$$\sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} \; \equiv 1 \; , \tag{70}$$

which is elementary.

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TAB. 1 : $B(n,m)$ *Central Binomial Triangle*

n\m	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	2	1	0	0	0	0	0	0	0	0	0
2	6	6	1	0	0	0	0	0	0	0	0
3	20	30	10	1	0	0	0	0	0	0	0
4	70	140	70	14	1	0	0	0	0	0	0
5	252	630	420	126	18	1	0	0	0	0	0
6	924	2772	2310	924	198	22	1	0	0	0	0
7	3432	12012	12012	6006	1716	286	26	1	0	0	0
8	12870	51480	60060	36036	12870	2860	390	30	1	0	0
9	48620	218790	291720	204204	87516	24310	4420	510	34	1	0
10	184756	923780	1385670	1108536	554268	184756	41990	6460	646	38	1

TAB. 2 : $A(n,m)$ *Catalan triangle*

n\m	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0
2	2	5	1	0	0	0	0	0	0	0	0
3	5	22	9	1	0	0	0	0	0	0	0
4	14	93	58	13	1	0	0	0	0	0	0
5	42	386	325	110	17	1	0	0	0	0	0
6	132	1586	1686	765	178	21	1	0	0	0	0
7	429	6476	8330	4746	1477	262	25	1	0	0	0
8	1430	26333	39796	27314	10654	2525	362	29	1	0	0
9	4862	106762	185517	149052	69930	20754	3973	478	33	1	0
10	16796	431910	848830	781725	428772	152946	36646	5885	610	37	1

TAB. 3 : $C4(n,m)$ *Catalan array*

n\m	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	-1	3	7	11	15	19	23
2	-2	10	38	82	142	218	310
3	-5	35	187	515	1083	1955	3195
4	-14	126	874	2934	7266	15086	27866
5	-42	462	3958	15694	44758	105102	216566
6	-132	1716	17548	80324	259356	679764	1546028
7	-429	6435	76627	397923	1435347	4154403	10338515
8	-1430	24310	330818	1922510	7663898	24281510	65635570
9	-4862	92378	1415650	9105690	39761282	136887322	399429602
10	-16796	352716	6015316	42438076	201483204	749032492	2346750900

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