On Polynomials Related to Derivatives of the Generating Function of Catalan Numbers

Wolfdieter L a n g [1]

Institut für Theoretische Physik
Universität Karlsruhe
Kaiserstrasse 12, D-76128 Karlsruhe, Germany

1 Introduction and summary

In [3] it has been shown that powers of the generating function $c(x)$ of *Catalan* numbers $\{C_n\}_{n\in\mathbf{N}_0} = \{1,1,2,5,14,42,...\}$ where $\mathbf{N}_0 := \{0,1,2,...\}$ (nr.1459 and $A000108$ of [8], and references of [3]) can be expressed in terms of a linear combination of 1 and $c(x)$ with coefficients replaced by certain scaled *Chebyshev* polynomials of the second kind. In this paper derivatives of $c(x)$ are studied in a similar manner. The starting point is the following expression for the first derivative.

$$\frac{d\ c(x)}{dx} \equiv c'(x) \ = \ \frac{1}{x(1-4x)}\Big(1+(-1+2x)\ c(x)\Big)\ . \tag{1}$$

This equation is equivalent to the simple recurrence relation valid for C_n:

$$(n+2)\ C_{n+1}\ -\ 2(2n+1)\ C_n\ =\ 0\ \ ,\ \ n=-1,0,1,...,\ \text{with}\ \ C_{-1}=-1/2\ \ . \tag{2}$$

Equation (1) can, of course, also be found from the explicit form $c(x) = (1-\sqrt{1-4x}\ )/(2x)$. The result for the n-th derivative is of the form

$$\frac{1}{n!}\frac{d^n c(x)}{dx^n}\ \ =\ \ \frac{1}{(x(1-4x))^n}\Big(a_{n-1}(x)\ +b_n(x)\ c(x)\Big), \tag{3}$$

with certain polynomials $a_{n-1}(x)$ of degree $n-1$ and $b_n(x)$ of degree n. These polynomials are found to be

$$b_n(x)\ \ =\ \ \sum_{m=0}^{n}(-1)^m B(n,m)\ x^{n-m}$$

with

$$B(n,m)\ :=\ \binom{2n}{n}\binom{n}{m}\Big/\binom{2m}{m}\ \ , \tag{4}$$

which defines a triangle of numbers for $n,m\in\mathbf{N}$, $n\geq m\geq 0$, where $\mathbf{N} := \{1,2,3,...\}$. The first terms are depicted in *TAB. 1* with $B(n,m)=0$ for $n<m$. Another representation for the polynomials $b_n(x)$ is also found, *viz.*

$$b_n(x)\ \ =\ \ -2\sum_{k=0}^{n} C_{k-1}\ x^k\ (4x-1)^{n-k}\ . \tag{5}$$

[1] E-mail: wolfdieter.lang@physik.uni-karlsruhe.de, http://www-itp.physik.uni-karlsruhe.de/˜wl

Equating both forms of $b_n(x)$ leads to a formula involving convolutions of Catalan numbers with powers of an arbitrary constant $\lambda := (4x-1)/x$. This formula is given in (31). Equation (5) reveals the generating function of the polynomials $b_n(x)$ because it is a convolution of two functional sequences. The result is

$$g_b(x;z) \;:=\; \sum_{n=0}^{\infty} b_n(x)\; z^n \;=\; \frac{\sqrt{1-4xz}}{1+(1-4x)\;z}\;. \tag{6}$$

The other family of polynomials is

$$a_n(x) \;=\; \sum_{k=0}^{n}(-1)^k\; A(n+1,k+1)\; x^{n-k}$$

with the triangular array $A(n,m)$ defined for $m=0$ by $A(n,0)=C_n$, and for $n,m\in\mathbf{N}$ with $n\geq m>0$ by the numbers

$$A(n,m) \;=\; \frac{1}{2}\binom{n}{m-1}\left[4^{n-m+1} \;-\; \binom{2n}{n}\Big/\binom{2(m-1)}{m-1}\right]\;. \tag{7}$$

The first terms of this triangular array of numbers are shown in *TAB. 2* with $A(n,m)=0$ for $n<m$. Both results, (4) and (7), are solutions to recurrence relations which hold for $b_n(x)$ and $a_n(x)$ and their respective coefficients $B(n,m)$ and $A(n,m)$.

Another representation for the polynomials $a_n(x)$ is found to be

$$a_n(x) \;=\; \sum_{k=0}^{n} C_k\; x^k\; (4x-1)^{n-k}\;, \tag{8}$$

which shows that the generating function of these polynomials is

$$g_a(x;z) \;:=\; \sum_{n=0}^{\infty} a_n(x)\; z^n \;=\; \frac{c(xz)}{1+(1-4x)\;z}\;. \tag{9}$$

Comparing (5) with (8) yields the following relation between these two types of polynomials

$$b_n(x) \;=\; (4x-1)^n \;-\; 2x\; a_{n-1}(x)\;\;,\quad n\in\mathbf{N}_0 \quad \text{with}\;\; a_{-1}(x)\equiv 0\;, \tag{10}$$

and between the coefficients

$$B(n,m) \;=\; \binom{n}{m}\; 4^{n-m} \;-\; 2\; A(n,m+1)\;. \tag{11}$$

The triangle of numbers $A(n,m)$ is related to a rectangular array of integers $\hat{A}(n,m)$, with $\hat{A}(0,m)\equiv 1$, $\hat{A}(n,0)=-C_n$ for $n\in\mathbf{N}$, and for $n\geq m\geq 1$ by

$$A(n,m) \;=\; -\hat{A}(n-m,m) \;+\; 2^{2(n-m)+1}\binom{n-1}{m-1}\;, \tag{12}$$

or with (7), for $m\in\mathbf{N}, n\in\mathbf{N}_0$, by

$$\hat{A}(n,m) \;=\; \frac{1}{2}\binom{n+m}{n+1}\left[\binom{2(n+m)}{n+m}\Big/\binom{2(m-1)}{m-1} \;-\; 4^{n+1}\;\frac{m-1}{n+m}\right]. \tag{13}$$

Part of the array $A(n,m)$ is shown in *TAB. 3*, where it is called $C4(n,m)$.

It turns out that the $m-$th column of the triangle of numbers $A(n,m)$ for $m = 0,1,...$ is determined by the generating function $c(x)(\frac{x}{1-4x})^m$. The $m-$th column of the triangle of numbers $B(n,m)$ for $m = 0,1,...$, is generated by $\frac{1}{\sqrt{1-4x}}\,(\frac{x}{1-4x})^m$. This fact identifies the infinite dimensional matrices $\mathbf{A}$ and $\mathbf{B}$ as examples of *Riordan* matrices in the terminology of [7]. The matrix $\hat{\mathbf{A}}$ associated with $\hat{A}(n,m)$ is an example of a *Riordan* array.

Because differentiation of $c(x) = \sum_{k=0}^{\infty}\ C_k\ x^k$ leads to

$$\frac{1}{n!}\,\frac{d^n c(x)}{dx^n} \;=\; \sum_{k=0}^{\infty} C(n,k)\ x^k\ ,\ \text{with}\ C(n,k)\ :=\ \frac{1}{n!}\prod_{j=1}^{n}(k+j)\ C_{n+k} \;=\; \frac{(2(n+k))!}{n!k!(n+k+1)!}\ , \tag{14}$$

where $C(0,k) = C_k$, one finds, together with (3), the following identities, for $n \in \mathbf{N}$, $p \in \{0,1,2,...,n-1\}$

$$(D1):\ \sum_{k=0}^{p}(-1)^k\ C_k\binom{n}{p-k}\Big/\binom{2(n-p+k)}{n-p+k} \;=\; \frac{1}{2}\binom{n}{p+1}\left\{2^{2(p+1)}\Big/\binom{2n}{n}\ -\ 1\Big/\binom{2(n-p-1)}{n-p-1}\right\}$$

$$=\ A(n,n-p)\Big/\binom{2n}{n}\ , \tag{15}$$

and, for $n \in \mathbf{N}, k \in \mathbf{N}_0$,

$$(D2):\ \sum_{j=0}^{n}(-1)^j\left(\binom{n}{j}\Big/\binom{2j}{j}\right)\sum_{l=0}^{k}4^l\binom{n+l-1}{n-1}C_{k+j-l} \;=\; C(n,k)\Big/\binom{2n}{n}\ . \tag{16}$$

The remainder of this paper provides proofs for the above statements.

2 Derivatives

The starting point is equation (1) which can either be verified from the explicit form of the generating function $c(x)$, or by converting the recursion relation (2) for *Catalan* numbers into an equation for their generating function. A computation of

$$\frac{1}{(n+1)!}\,\frac{d^{n+1}c(x)}{dx^{n+1}} \;=\; \frac{1}{n+1}\,\frac{d}{dx}\left(\frac{1}{n!}\,\frac{d^n c(x)}{dx^n}\right)$$

with (3) taken as granted and equation (1) produces the following mixed relations between the quantities $a_n(x)$ and $b_n(x)$ and their first derivatives, valid for $n \in \mathbf{N}_0$,

$$(n+1)\ a_n(x) \;=\; x(1-4x)\ a'_{n-1}(x)\ +\ b_n(x)\ +\ n(8x-1)\ a_{n-1}(x)\ , \tag{17}$$

$$(n+1)\ b_{n+1}(x) \;=\; x(1-4x)\ b'_n(x)\ +\ (-(n+1)+2(1+4n)x)\ b_n(x)\ , \tag{18}$$

with inputs $a_{-1}(x) \equiv 0$ and $b_0(x) \equiv 1$.

From (18) it is clear by induction that $b_n(x)$ is a polynomial of degree n. Again by induction, the same statement holds for $a_n(x)$ in (17). Therefore we write, for $n \in \mathbf{N}_0$,

$$a_n(x) \;=\; \sum_{k=0}^{n}(-1)^k\ a(n,k)\ x^{n-k}\ , \tag{19}$$

$$b_n(x) \;=\; \sum_{k=0} (-1)^k \; B(n,k) \; x^{n-k} \;, \tag{20}$$

with the triangular arrays of numbers $a(n,k)$ and $B(n,k)$ with row number n and column number $k \leq n$. The triangular array $a(n,k)$ will later be enlarged to another one which will then be called $A(n,k)$.

We first solve $b_n(x)$ in (18) by inserting (20) and deriving the recursion relation for the coefficients $B(n,m)$ after comparing coefficients of x^{n+1}, x^0, and x^{n-k} for $k = 0, 1, ..., n-1$.

$$x^{n+1} \;:\quad (n+1)\; B(n+1,0) \;=\; 2(2n+1)\; B(n,0) \;, \tag{21}$$

$$x^0 \;:\quad B(n+1,n+1) \;=\; B(n,n) \;, \tag{22}$$

$$x^{n-k} \;:\quad (n+1)\; B(n+1,k+1) \;=\; (k+1)\; B(n,k) \;+\; 2\Big(2(n+k)+3\Big)\; B(n,k+1) \;. \tag{23}$$

With the input $B(0,0) = 1$ one deduces from (21) for the leading coefficient of $b_n(x)$

$$B(n,0) \;=\; 2^n \frac{(2n-1)!!}{n!} \;=\; \frac{(2n)!}{n!\; n!} \;=\; \binom{2n}{n} \;, \tag{24}$$

and from (22)

$$B(n,n) \;\equiv 1 \;, \quad \text{i.e.,} \quad b_n(0) = (-1)^n \;. \tag{25}$$

In (24) the double factorial $(2n-1)!! := 1 \cdot 3 \cdot 5 \cdots (2n-1)$ appeared.

In order to solve (23) we conjecture from *TAB. 1* that for $n, m \in \mathbf{N}$

$$B(n,m) \;=\; 4\; B(n-1,m) \;+\; B(n-1,m-1) \;, \tag{26}$$

with input $B(n,0) = \binom{2n}{n}$ from (24).

If in (23) we use this conjecture, written with $n \to n-1$, $k \to m-1$, we are led to consider the simple recursion

$$B(n,m) \;=\; \frac{n+1-m}{2(2m-1)}\; B(n,m-1) \;. \tag{27}$$

The solution of this recursion is, for $n, m \in \mathbf{N}_0$,

$$B(n,m) \;=\; \frac{1}{2^m (2m-1)!!} \frac{n!}{(n-m)!} \binom{2n}{n} \;=\; \frac{m!\; n!}{(2m)!\; (n-m)!} \binom{2n}{n} \;=\; \binom{2n}{n} \binom{n}{m} \Big/ \binom{2m}{m} \;. \tag{28}$$

With the *Pochhammer* symbol $(a)_n := \Gamma(n+a)/\Gamma(a)$ this result can also be written as

$$B(n,m) \;=\; ((2m+1)/2)_{n-m} 4^{m-n}/(n-m)! \;.$$

This result satisfies (21), *i.e.*, (24), as well as (22), *i.e.*, (25). It is also the solution to (23) provided we prove the conjecture (26) using $B(n,m)$ in (28). This can be done by using the equality $B(n,m) = \frac{(2n)!\; m!}{(2m)!\; n!\; (n-m)!}$ in (26). Thus we have proved:

Proposition 1: *We have*

$$b_n(x) \;=\; \sum_{k=0}^{n} (-1)^k \, B(n,k) \, x^{n-k}$$

where $B(n,k) \;=\; \binom{2n}{n} \binom{n}{k} \Big/ \binom{2k}{k}$.

This triangle of numbers as shown in *TAB. 1* appears as A046521 in the database [8].

One can derive another explicit representation for the polynomials $b_n(x)$ by using (27) in (20):

$$(1-4x)\ b_n'(x)\ +\ 2(2n-1)\ b_n(x)\ +\ 2\binom{2n}{n}\ x^n\ =\ 0\ . \tag{29}$$

This leads, together with (18), to the following inhomogeneous recursion relation for $b_n(x)$.

$$b_{n+1}(x)\ =\ (4x-1)\ b_n(x)\ -\ 2C_n\ x^{n+1}\ \ ,\ \ b_0(x)\equiv 1\ . \tag{30}$$

Equation (29) can also be solved as a first order linear and inhomogeneous differential equation for $b_n(x)$.

Proposition 2: *We have*

$$b_n(x)\ =\ -2\sum_{k=0}^{n} C_{k-1}\ x^k\ (4x-1)^{n-k}\ ,$$

where the C_k'*s are the* Catalan *numbers for* $k\in\mathbf{N}_0$, *and* $C_{-1}=-1/2$.

Proof: Iteration of (30). □

Proposition 3: *The generating function* $g_b(x;z)\ :=\sum_{n=0}^{\infty}\ b_n(x)\ x^n$ *for* $\{b_n(x)\}$ *is given by* (14).

Proof: The alternative form of $b_n(x)$, given by (5), is a convolution of the functional sequences $\{-2C_{k-1}\ x^k\}_{n\in\mathbf{N}_0}$ and $\{(4x-1)^n\}_{n\in\mathbf{N}_0}$, with generating functions $1-2xz\ c(xz)\ =\ \sqrt{1-4xz}$ and $1/(1+(1-4x)z)$, respectively. Therefore, $g_b(x;z)$ is the product of these two generating functions. □

Comparing this alternative form (5) for $b_n(x)$ with the one given by (20), together with (28), proves the following identity in n and $\lambda:=(4x-1)/x$. The term $k=0$ in the sum (5) has been written separately.

Corollary 1 (Convolution of *Catalan* sequence and the sequence of powers of λ):

For $n\in\mathbf{N}$ *and* $\lambda\neq\infty$,

$$s_{n-1}(\lambda)\ :=\lambda^{n-1}\sum_{k=0}^{n-1}\frac{C_k}{\lambda^k}\ =\frac{1}{2}\left(\lambda^n\ -\ \binom{2n}{n}\sum_{k=0}^{n}(-1)^k\ (4-\lambda)^k\ \binom{n}{k}\Big/\binom{2k}{k}\right)\ . \tag{31}$$

Therefore, the generating function for the sequence $s_n(\lambda)$ is

$$g(\lambda;x)\ \ :=\ \sum_{n=0}^{\infty} s_n(\lambda)\ x^n\ =\ c(x)/(1-\lambda\ x)\ .$$

From the generating function the recurrence relation is found to be $s_n(\lambda)=\lambda\ s_{n-1}(\lambda)+C_n\ , s_{-1}(\lambda)\equiv 0$. The connection with the polynomial $b_n(x)$ is $s_n(\lambda)\ =\ \frac{1}{2}\left(\lambda^{n+1}\ -\ (4-\lambda)^{n+1}\ b_{n+1}(1/(4-\lambda))\right)$.

The case $\lambda=0$ ($x=1/4$) is also covered by this formula. It produces from $s_n(0)=C_n$ the following identity.

Example 1: Case $\lambda=0$ ($x=1/4$)

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\ 4^k\Big/\binom{2k}{k}\ =\ \frac{1}{2n-1}\ . \tag{32}$$

This identity occurs in one of the exercises 2.7, 2, p. 32 of [4].

We note that from (5) one has $-2b_{n+1}(1/4) = C_n/4^n$. The large n behaviour of this sequence is known to be $C_n/4^n \sim \frac{1}{\sqrt{\pi}}\,\frac{1}{n^{3/2}}$; *cf.* [2], Exercise 9.60.

If one puts in (5) $4x-1=x$, *i.e.* $x=1/3$, one can identify the partial sum $s_n(1)$ of *Catalan* numbers:

$$s_n(1) \;:=\; \sum_{k=0}^{n} C_k = \frac{1}{2}(1 \;-\; 3^{n+1}\; b_{n+1}(1/3)\;). \tag{33}$$

This sequence $\{1,2,4,9,23,65,197,626,2056,...\}$ appears as A014137 in the web encyclopedia [8]. If one puts $\lambda = 1$ in *Corollary 1* one also finds the following

Example 2:

$$2\; s_{n-1}(1) \;=\; 1 \;+\; \binom{2n}{n} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k}\, 3^k \Big/ \binom{2k}{k}\;. \tag{34}$$

Another interesting example is the case $\lambda = 4$ $(x=\infty)$. Here one finds a simple result for the convolution of *Catalan*'s sequence with powers of 4, *viz.*

Example 3: $\lambda = 4\;\;(x=\infty)$

$$2\; s_{n-1}(4) \;=\; 4^n \;-\; \binom{2n}{n}\;. \tag{35}$$

This sequence $\{1,5,22,93,386,1586,6476,...\}$ appears in the book [8] as Nr. 3920 and as A000346 in the web encyclopedia. It will show up again in this work as $A(n+1,1)$, the second column in the $A(n,m)$ triangle (*cf. TAB. 2*).

The sequence for $\lambda = -1$ $(x = 1/5)$ is also non-negative, as can be seen by writing $s_{2k}(-1) = C_2 + \sum_{l=2}^{k}(C_{2l}-C_{2l-1})$ for $k \in \mathbf{N}$ and $s_{2k+1}(-1) = \sum_{l=1}^{k}(C_{2l+1}-C_{2l})$, and using $\triangle C_n := C_n - C_{n-1} = 3\frac{n-1}{n+1}\,C_{n-1} \geq 0$. This is the sequence $\{1,0,2,3,11,31,101,328,1102,3760,...\}$ which appears now as *A*032357 in the web encyclopedia [8] .

Recursion (26) for $B(n,m)$ can be transformed into an equation for the generating function for the sequence appearing in the $m-$th column of the $B(n,m)$ triangle

$$G_B(m;x) := \sum_{n=m}^{\infty} B(n,m)\; x^n \quad , \tag{36}$$

with input $G_B(0;x) \;=\; \sum_{n=0}^{\infty}\binom{2n}{n}\,x^n \;=\; 1/\sqrt{1-4x}$, the generating function for the central binomial numbers. So (26) implies for $m \in \mathbf{N}_0$,

$$G_B(m;x) \;=\; \Big(\frac{x}{1-4x}\Big)^m \,\frac{1}{\sqrt{1-4x}}. \tag{37}$$

For $x\frac{d}{dx}G_B(m;x)$ see (53). Therefore, we have proved:

Proposition 4 (Column sequences of the $B(n,m)$ triangle)

The sequence $\{B(n,m)\}_{n=m}^{\infty}$, *defined, for fixed* $m \in \mathbf{N}_0$ *and* $n \in \mathbf{N}_0$ by (28) *is the convolution of the central binomial sequence* $\{\binom{2k}{k}\}_{k\in\mathbf{N}_0}$ *and the* $m-$*th convolution of the (shifted) power sequence*$\{0,1,4^1,4^2,...\}$.

Note 1: The infinite dimensional matrix $\mathbf{B}$ with elements $B(n,m)$ given for $n \geq m \geq 0$ by (28) and $B(n,m) \equiv 0$ for $n < m$ is an example of a *Riordan* matrix [7]. With the notation of this reference $\mathbf{B} = (\frac{1}{\sqrt{1-4x}}, \frac{x}{1-4x})$.

Note 2: *Sheffer*-type identities from *Riordan*-matrices

Triangular *Riordan*-matrices $\mathbf{M} = (M_{i,j})_{i\geq j\geq 0} = (g(x), f(x))$, $M_{i,j} = 0$ for $j > i$, in the notation of [7], lead to polynomials which satisfy *Sheffer*-type identities (see [5] and its references, and [1])

$$S_n(x+y) = \sum_{k=0}^{n} S_k(y)\, P_{n-k}(x) = \sum_{k=0}^{n} P_k(y)\, S_{n-k}(x) \ , \tag{38}$$

$$P_n(x+y) = \sum_{k=0}^{n} P_k(y)\, P_{n-k}(x) = \sum_{k=0}^{n} P_k(x)\, P_{n-k}(y) \ , \tag{39}$$

where the polynomials $S_n(x)$ and $P_n(x)$ are defined by

$$S_n(x) = \sum_{m=0}^{n} M_{n,m} \frac{x^m}{m!},\ n \in \mathbf{N}_0 \quad , \quad P_n(x) = \sum_{m=1}^{n} P_{n,m} \frac{x^m}{m!},\ n \in \mathbf{N},\ P_0(x) \equiv 1 \ , \tag{40}$$

with $P_{n,m} := [z^n]\,(f^m(z))$, $n \geq m \geq 1$. Here $g(x)$ defines the first column of $\mathbf{M}$: $M_{n,0} = [x^n]\, g(x)$.

If one uses $s_n(x) := n!\, S_n(x)$ and $p_n(x) := n!\, P_n(x)$ one obtains the *Sheffer*-identities (also called binomial identities) treated in [5]. Then $s_n(x)$ is *Sheffer* for $(1/g(\bar{f}(t)), \bar{f}(t))$, and $p_n(x)$ is associated to $\bar{f}(t)$ (or *Sheffer* for $(1, \bar{f}(t))$) in the terminology of [5]. Here $\bar{f}(t)$ stands for the inverse of $f(t)$ as a function.

Let us give the relation between $g_b(x;z)$ and $G_B(m;x)$.

Proposition 5: *We have*

$$g_b(x;z) = \sum_{m=0}^{\infty} (-1)^m\ G_B(m;xz)\ \left(\frac{1}{x}\right)^m \ . \tag{41}$$

Proof: One inserts the value of $b_n(x)$ given in (20) into the definition (6) of $g_b(x;z)$ and rewrites the *Cauchy*-sum as two infinite sums which are then interchanged. Finally, the definition of $G_B(m;x)$ in (36) is used. □

One can check (41) by using the explicit form of $G_B(m;xz)$ given in (36) and comparing with (6).

In a similar vein we can solve $a_n(x)$ in (17) with $b_n(x)$ given by (20) and (28). The coefficients $a(n,k)$, defined by (19), have to satisfy, after comparing coefficients of x^n, x^0, and x^{n-k} for $k = 1, 2, ..., n-1$ and $n \in \mathbf{N}_0$:

$$x^n : \quad a(n,0) = 4\ a(n-1,0) + C_n, \tag{42}$$

$$x^0 : \quad (n+1)\ a(n,n) = 1 + n\ a(n-1,n-1) \ , \tag{43}$$

$$x^{n-k} : \quad (n+1)\ a(n,k) = k\ a(n-1,k-1) + 4(n+1+k)\ a(n-1,k) + B(n,k) \ . \tag{44}$$

In (42) we have used (24), *i.e.*, $B(n,0) = (n+1)\ C_n$, and in 43 we have used (25), *i.e.*, $B(n,n) \equiv 1$. From (42) one finds with input $a(0,0) = 1$

$$a(n,0) = \sum_{k=0}^{n} C_k\ 4^{n-k} \ , \tag{45}$$

and from (43)

$$a(n,n) \equiv 1, \text{ or } a_n(0) = (-1)^n \,. \tag{46}$$

Note that $a(n,0) = s_n(4)$ of (31) with solution (35). It is convenient to define $a(n-1,-1) := C_n$, $n \in \mathbf{N}_0$. Then the sequence $\{a(n,0)\}_{-1}^{\infty}$ is, with $a(-1,0) := 0$, the convolution of the sequence $\{a(k,-1)\}_{-1}^{\infty}$ and the shifted power sequence $\{0,1,4^1,4^2,...\}$. Before solving (44), with $B(n,k)$ from (28) inserted, we add to the trianglular array of numbers $a(n,m)$ the $m=-1$ column and an extra row for $n=-1$, and define a new enlarged triangular array for $n,m \in \mathbf{N}_0$ as

$$A(n,m) \;:=\; a(n-1,m-1) \tag{47}$$

with $A(n,0) = a(n-1,-1) = C_n$ and $A(0,m) = a(-1,m-1) = \delta_{0,m}$. An inspection of the $A(n,m)$ triangular array, partly depicted in *TAB. 2*, leads to the conjecture

$$A(n,m) \;=\; 4\; A(n-1,m) \;+\; A(n-1,m-1) \;, \tag{48}$$

with $A(n,0) = C_n$ and $A(n,m) \equiv 0$ for $n < m$. This recursion relation can be used to extend the array $A(n,m)$ to negative integer values of m. This conjecture is correct for $A(n+1,1) = a(n,0)$ found in (45), as well as for $A(n+1,n+1) = a(n,n) \equiv 1$ known from (46). The generating function for the sequence appearing in the $m-$th column,

$$G_A(m;x) \;:=\; \sum_{n=m}^{\infty} A(n,m)\; x^n \;, \tag{49}$$

due to (48) satisfies $G_A(m;x) = \frac{x}{1-4x}\, G_A(m-1;x)$, remembering that $A(m-1,m) \equiv 0$, and that $G_A(0;x) = c(x)$. Therefore

$$G_A(m;x) \;=\; \Big(\frac{x}{1-4x}\Big)^m\; c(x) \;. \tag{50}$$

Note 3: The infinite dimensional matrix $\mathbf{A}$ with elements $A(n,m)$ given for $n \geq m \geq 0$ by (48) and $A(n,m) \equiv 0$ for $n < m$ is another example of a *Riordan* matrix, written in the notation of [7] as $(c(x), x/(1-4x))$.

Because of (37) and $\sqrt{1-4x}\; c(x) = 2 - c(x)$, these generating functions of the conjectured $A(n,m)$ column sequences obey

$$G_A(m;x) \;=\; (2-c(x))\; G_B(m;x) \;. \tag{51}$$

If we use the conjecture (48) in (44) which is written with (47) in the form

$$(n+1)\; A(n+1,m+1) \;=\; m\; A(n,m) \;+\; 4(n+m+1)\; A(n,m+1) \;+\; B(n,m) \;,$$

for $n \in \mathbf{N}_0,\; m \in \{1,2,..,n-1\}$, we have

$$m\; A(n+1,m+1) \;-\; (n+1)\; A(n,m) \;+\; B(n,m) \;=\; 0 \;. \tag{52}$$

This recursion relation can be written with the help of the generating functions (36) and (49) as

$$\left(x\frac{d}{dx}+1\right)\; G_A(m;x) \;-\; \frac{m}{x}\, G_A(m+1;x) \;=\; G_B(m;x) \;, \tag{53}$$

or with (50) (*i.e.* the conjecture) as

$$\Big(x\frac{d}{dx}+1 \;-\; \frac{m}{1-4x}\Big)\; G_A(m;x) \;=\; G_B(m;x) \;. \tag{54}$$

Together with (51) this means

$$x\frac{d}{dx}\Big((2-c(x))\ G_B(m;x)\Big) \ = \ \Big[\Big(\frac{m}{1-4x}-1\Big)(2-c(x))\ +1\Big]\ G_B(m;x)\ . \qquad (55)$$

If we can prove this equation with $G_B(x)$ given by (37) we have shown that (44) is equivalent to the conjecture (48). In order to prove (55) we first compute from (37), for $m \in \mathbf{N}_0$,

$$x\frac{d}{dx}G_B(m;x) \ = (2+\frac{m}{x})\ G_B(m+1;x) \ = \ \frac{2x+m}{1-4x}\ G_B(m;x)\ . \qquad (56)$$

With this result, (55) reduces to

$$\Big(-x\ c'(x)\ +\ (2-c(x))\ \frac{1-2x}{1-4x}\ -\ 1\Big)\ G_B(m;x)\ =\ 0\ , \qquad (57)$$

and with (1) the factor in front of $G_B(m;x)$ vanishes identically for $x \neq 1/4$. Therefore, we have proved the following two propositions concerning the column sequences of the $A(n,m)$ triangular array and the triangular $A(n,m)$ array respectively.

Proposition 6: *The triangular array of numbers* $A(n,m)$, *defined for* $n,m \in \mathbf{N}_0$ *by equation* (48), $A(n,0) = C_n$, $A(n,m) \equiv 0$ *for* $n < m$ *has as* $m-$*th column sequence* $\{A(n,m)\}_{n=m}^{\infty}$ *the convolution of the* Catalan *sequence and the* $m-$*th convolution of the shifted power sequence* $\{0,1,4^1,4^2,...\}$.

Proof: Use (50) with (49). □

Proposition 7: *The triangular array* $A(n,m)$ *of* Proposition 6 *coincides with the one defined by* (47) *and* (42), (43) *and* (44) *with* $B(n,m)$ *given by* (28).

Proof: On the one hand $a(n,0) = A(n+1,1)$ and $a(n,n) = A(n+1,n+1) \equiv 1$ of (42) and (43), *i.e.*, (45) and (46), respectively, satisfy (45). On the other hand (44) is rewritten with the aid of (47) as (52), and (52) has been proved by (53) to (57). □

Alternatively, one can use the now proven conjecture (48), together with (47), in (44) and derive for $n \in \mathbf{N}_0,\ m \in \mathbf{N}_0$,

$$4m\ a(n-1,m)\ =\ (n+1-m)\ a(n-1,m-1)\ -\ B(n,m)\ . \qquad (58)$$

This is written in terms of the polynomials $a_{n-1}(x)$ of (19) and $b_n(x)$ of (20) as

$$x(1-4x)\ a'_{n-1}(x)\ +\ (1-4x+4nx)\ a_{n-1}(x)\ -\ \binom{2n}{n}\ x^n\ +\ b_n(x)\ =\ 0\ . \qquad (59)$$

With this result (17) becomes an inhomogeneous recursion relation for $a_n(x)$, *viz.*

$$a_n(x)\ =\ (4x-1)\ a_{n-1}(x)\ +\ C_n\ x^n\ \ ,\ \ a_0(x) \equiv 1\ . \qquad (60)$$

Moreover, (59) can also be considered as an inhomogeneous linear differential equation for $a_{n-1}(x)$ with given $b_n(x)$. To find the solution this way is, however, a bit tedious. Let us give an alternative form for $a_n(x)$:

Proposition 8: *The solution of the recursion relation* (60) *is given by* (8).

Proof: Iteration of (60). ☐

Next we give a

Corollary 2: *The generating function* $g_a(x;z) \ := \ \sum_{n=0}^{\infty} \ a_n(x) \ z^n$ *is given by* (9).

Proof: Equation (8) shows that $a_n(x)$ is a convolution of the functional sequences $\{C_k \ x^k\}_{n\in\mathbf{N}_0}$ and $\{(4x-1)^k\}_{k\in\mathbf{N}_0}$ with generating functions $c(xz)$ and $1/(1+(1-4x)z)$. Therefore, $g_a(x;z)$ is the product of these generating functions. ☐

We now have a relation between $g_a(x;z)$ and $G_A(m;x)$:

Proposition 9:

$$g_a(x;z) \ = \ \frac{1}{1-4xz} \ \sum_{m=0}^{\infty} (-1)^m \ G_A(m;xz) \ \left(\frac{1}{x}\right)^m \ . \tag{61}$$

Proof: Analogous to the proof of *Proposition 5.* ☐

One can ckeck (61) by putting in the explicit form (50) of $G_A(m;x)$ and compare with (9). Let us state the relation between $b_n(x)$ and $a_{n-1}(x)$ as

Proposition 10: *For* $n \in \mathbf{N}_0$ *and* $a_{-1}(x) \equiv 0$, *the relation between* $b_n(x)$ *and* $a_{n-1}(x)$ *is given by* (10) .

Proof: The alternative expressions (5) and (8) for these two families of polynomials are used. One splits off the $k=0$ term in (5) with $C_{-1} = -1/2$ from the sum and shifts the summation variable. ☐

Corollary 3: The coefficients of the triangular arrays $A(n,m)$ and $B(n,m)$ are related as given by (11).

Proof: The relation (10) between the polynomials is, with the help of (19) and (20), written for the coefficients $a(n-1,m)$, or by (47) for $A(n,m+1)$, and $B(n,m)$. ☐

It remains to compute the explicit expression for the coefficients $a(n,k)$ of $a_n(x)$ defined by (19). Because of (47) it suffices to determine $A(n,m)$.

Corollary 4: *The triangular array numbers* $A(n,m)$ *are given explicitely by formula* (7).

Proof: The formula (4) written for $B(n,m-1)$ is used in relation (11). ☐

Note 4: This formula for $A(n,m)$ satisfies indeed the recursion relation (48) with the given input. The first term, $\frac{1}{2} \ 4^{n-m+1} \ \binom{n}{m-1}$, satisfies it because of the binomial identity $\binom{n}{m-1} \ = \ \binom{n-1}{m-1} \ + \ \binom{n-1}{m-2}$. For the second term of $A(n,m)$ in (7) one has to prove

$$\binom{n}{m-1} \binom{2n}{n} \ = \ 4 \binom{n-1}{m-1} \binom{2(n-1)}{n-1} + \binom{n-1}{m-2} \binom{2(n-1)}{n-1} \frac{2(2m-3)}{m-1} \ ,$$

or after division by $\binom{2(n-1)}{n-1}$,

$$\frac{2n-1}{n} \binom{n}{m-1} \ = \ 2 \binom{n-1}{m-1} + \binom{n-1}{m-2} \frac{2m-3}{m-1} \ ,$$

which reduces to the trivial identity $2n-1 = 2(n-m+1) + 2m-3$. Both terms together, *i.e.*, (7), satisfy the input $A(n,n) \equiv 1$.

Note 5: $A(n,m)$ was found originally after iteration in the form (with $n \geq m > 0$ and $(-1)!! := 1$)

$$A(n,m) \;=\; 2\cdot 4^{n-m}\binom{n}{m-1} \;-\; \frac{\prod_{k=1}^{m}(2(n-m)+2k-1)}{(2m-3)!!}\, C_{n-m}\;. \tag{62}$$

$A(n,0) = C_n$. It is easy to establish the equivalence with (7).

In the original derivation of the formula (7) for $A(n,m)$ it turned out to be convenient to introduce a rectangular array of integers $\hat{A}(n,m)$ for $n,m \in \mathbf{N}_0$ as follows: $\hat{A}(0,m) \equiv 1$, $\hat{A}(n,0) := -C_n$ for $n \in \mathbf{N}$, and for $m \in \mathbf{N}$ and $n \in \mathbf{N}_0$, $\hat{A}(n,m)$ is defined by (7), or equivalently, by (8). The $A(n,m)$ recursion (48) translates (with the help of the *Pascal*-triangle identity) into

$$\hat{A}(n,m) \;=\; 4\,\hat{A}(n-1,m) \;+\; \hat{A}(n,m-1)\;. \tag{63}$$

This leads, after iteration and use of $\hat{A}(0,m) \equiv 1$ from (12) with $A(n,n) \equiv 1$, to

$$\hat{A}(n,m) \;=\; 4^n\sum_{k=0}^{n}\hat{A}(k,m-1)/4^k\;. \tag{64}$$

Thus, the following proposition describes column sequences of the $\hat{A}(n,m) \equiv C4(n,m)$ array.

Proposition 11: *The $m-th$ column sequence of the $\hat{A}(n,m)$ array, $\{\hat{A}(n,m)\}_{n\in\mathbf{N}_0}$, is the convolution of the sequence $\{\hat{A}(n,0)\}_{n\in\mathbf{N}_0} = \{1,-1,-2,-5,...\}$, generated by $2-c(x)$, and the $m-th$ convolution of the power sequence $\{4^k\}_{k\in\mathbf{N}_0}$.*

Proof: Iteration of (64) with the $\hat{A}(n,0)$ input. □

Corollary 5: *The ordinary generating function of the $m-th$ column sequence of the $\hat{A}(n,m)$* array (13) *is for $m \in \mathbf{N}_0$ given by*

$$G_{\hat{A}}(m;x) \;:=\; \sum_{n=0}^{\infty}\hat{A}(n,m)\,x^n \;=\; (2-c(x))\left(\frac{1}{1-4x}\right)^m\;. \tag{65}$$

Proof: Use *Proposition 11* written for generating functions. □

Because of the convolution of the (negative) *Catalan* sequence with powers of 4 we shall call this $\hat{A}(n,m)$ array also $C4(n,m)$. A part of it is shown in *TAB. 3.* The second column sequence is given by $\hat{A}(n,1) \equiv C4(n,1) = \binom{2n+1}{n}$ and appears as nr.2848 in the book [8], or as $A001700$ in the web encyclopedia [8]. The sequence of the third column $\{\hat{A}(n,2) \equiv C4(n,2)\}_{n\in\mathbf{N}_0} = \{1,7,38,187,...\}$ is from (64) and (62) with (12) determined by $4^n\sum_{k=0}^{n}\binom{2k+1}{k}/4^k = (2n+3)(2n+1)\,C_n - 2^{2n+1}$, and is listed as $A000531$ in [8]. There the fourth column sequence is now listed as $A029887$.

Note 6: The infinite dimensional lower triangular matrix $\tilde{\mathbf{A}}$ related to the array $\hat{A}(n,m) \equiv C4(n,m)$ by $\tilde{A}(n,m) := \hat{A}(n-m,m+1)$ for $n \geq m \geq 0$ and $\tilde{A}(n,m) := 0$ for $n < m$ is again an example of a *Riordan* matrix [7]. In the notation of [7], $\tilde{\mathbf{A}} = (c(x)/\sqrt{1-4x}, x/\sqrt{1-4x})$.

Finally, we derive identities by using, for $n \in \mathbf{N}_0$, equation (14) for the left hand side of (3) and the results for $a_{n-1}(x)$ and $b_n(x)$ for the right hand side.

Because there are no negative powers of x on the left hand side of (3), such powers have to vanish on the right hand side. This leads to the first family of identities. Because $(1-4x)^{-n} = \sum_{k=0}^{\infty} \frac{(n)_k}{k!}\, 4^k\, x^k$, with *Pochhammer*'s symbol defined after (28), this means that $[x^p]\,(a_{n-1}(x) + b_n(x)\; c(x))$, the coefficient proportional to x^p, has to vanish for $p = 0, 1, ..., n-1$, $n \in \mathbf{N}$. This requirement reads

$$(-1)^{n-1-p}\; a(n-1, n-1-p) \;+\; \sum_{k=0}^{p} (-1)^{n-k}\; B(n, n-k)\; C_{p-k} \;\equiv 0\;. \tag{66}$$

The sum is restricted to $k \leq p\ (< n)$ because no number C_l with negative index is found in $c(x)$. Inserting the known coefficients, this produces (15).

Proposition 12: *For $n \in \mathbf{N}$ and $p \in \{0, 1, ..., n-1\}$ identity $(D1)$, given by* (15), *holds.*

Proof: With (47), (66) becomes

$$\sum_{k=0}^{p} (-1)^{p-k}\; C_{p-k}\; B(n, n-k) \;=\; A(n, n-p)\;, \tag{67}$$

which is $(D1)$ of (15) if the summation index k is changed into $p-k$, and the symmetry of the binomial coefficients is used. □.

Example 4: Take $p = n-1 \in \mathbf{N}_0$:

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{1}{2k+1} \;=\; 4^n \bigg/ \binom{2n}{n} \;-\; 1 \;=\; 2A(n,1) \bigg/ \binom{2n}{n}\quad . \tag{68}$$

With this identity we have found a sum representation for the convolution of the *Catalan* sequence and powers of 4:

$$s_{n-1}(4) \quad := \quad 4^{n-1} \sum_{k=0}^{n-1} C_k/4^k \;=\; \frac{1}{2}\binom{2n}{n} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{1}{2k+1}$$

(*cf.* (35) with (31)).

The second family of identities, $(D2)$ of (16), results from comparing powers x^k with $k \in \mathbf{N}_0$ on both sides of (3) after expansion of $(1-4x)^{-n}$ as given above in the text before (66). Only the second term $b_n(x)\; c(x)$ contributes because $a_{n-1}(x)/x^n$ has only negative powers of x. Thus, with definition (14), one finds for $k \in \mathbf{N}_0$ and $n \in \mathbf{N}$,

$$C(n,k) \;=\; \sum_{l=0}^{k} \frac{(n)_l\; 4^l}{l!} \sum_{j=0}^{n} (-1)^{n-j}\, B(n, n-j)\, C_{n-j+k-l} \tag{69}$$

which is, after interchange of the summations and insertion of $B(n, n-j)$ from (4) the desired identity $(D2)$ if also the summation index j is changed to $n-q$.

Thus we have shown:

Proposition 13: *For $k \in \mathbf{N}_0$ and $n \in \mathbf{N}$ identity $(D2)$ of* (16) *with $C(n,k)$ defined by* (14) *holds true.*

Example 5: Take $k = 0$, $n \in \mathbf{N}$. So we have

$$\sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} \;\equiv 1\;, \tag{70}$$

which is elementary.

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TAB. 1: B(n,m) Central Binomial Triangle

n \ m	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	2	1	0	0	0	0	0	0	0	0	0
2	6	6	1	0	0	0	0	0	0	0	0
3	20	30	10	1	0	0	0	0	0	0	0
4	70	140	70	14	1	0	0	0	0	0	0
5	252	630	420	126	18	1	0	0	0	0	0
6	924	2772	2310	924	198	22	1	0	0	0	0
7	3432	12012	12012	6006	1716	286	26	1	0	0	0
8	12870	51480	60060	36036	12870	2860	390	30	1	0	0
9	48620	218790	291720	204204	87516	24310	4420	510	34	1	0
10	184756	923780	1385670	1108536	554268	184756	41990	6460	646	38	1

TAB. 2: A(n,m) Catalan Triangle

n \ m	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0
2	2	5	1	0	0	0	0	0	0	0	0
3	5	22	9	1	0	0	0	0	0	0	0
4	14	93	58	13	1	0	0	0	0	0	0
5	42	386	325	110	17	1	0	0	0	0	0
6	132	1586	1686	765	178	21	1	0	0	0	0
7	429	6476	8330	4746	1477	262	25	1	0	0	0
8	1430	26333	39796	27314	10654	2525	362	29	1	0	0
9	4862	106762	185517	149052	69930	20754	3973	478	33	1	0
10	16796	431910	848830	781725	428772	152946	36646	5885	610	37	1

TAB. 3: C4(n,m) Catalan array

n \ m	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	-1	3	7	11	15	19	23
2	-2	10	38	82	142	218	310
3	-5	35	187	515	1083	1955	3195
4	-14	126	874	2934	7266	15086	27866
5	-42	462	3958	15694	44758	105102	216566
6	-132	1716	17548	80324	259356	679764	1546028
7	-429	6435	76627	397923	1435347	4154403	10338515
8	-1430	24310	330818	1922510	7663898	24281510	65635570
9	-4862	92378	1415650	9105690	39761282	136887322	399429602
10	-16796	352716	6015316	42438076	201483204	749032492	2346750900

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