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Riccati meets Fibonacci

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Abstract

The generating functions for generalized *Fibonacci* and *Lucas* numbers are shown to be the unique solutions to *Riccati* equations with certain initial conditions. This observation leads directly to recursion relations for k-fold convolutions of these numbers which have been partly found earlier in a different way [1]. A generalization of this method to so called $p-$*Fibonacci* numbers is also considered.

1 Introduction and Summary

The generating function for generalized *Fibonacci* numbers $\{F_n(a,b)\}_0^\infty$, defined by the three term recurrence relation

$$F_n(a,b) \; = \; a\,F_{n-1}(a,b) \; + \; b\,F_{n-2}(a,b), \quad F_0(a,b) \; = \; 0, \; F_1(a,b) \; = \; 1, \tag{1}$$

with given real $a \neq 0$ and $b \neq 0$, is well-known. For arbitrary a and b, $F_n(a,b)$ can be considered as a polynomial in two variables. If one considers the numbers, or polynomials, $U_n(a,b) \; := \; F_{n+1}(a,b)$ one has from the recursion with input $U_0(a,b) = 1$ and $U_1(a,b) = a$ (or $U_{-1}(a,b) = 0$)

$$U(a,b;x) \; := \; \sum_{n=0}^{\infty} U_n(a,b)\,x^n \; = \; \frac{1}{1 \, - \, a\,x \, - \, b\,x^2} \; . \tag{2}$$

Similarly, for the generalized *Lucas* numbers $\{L_n(a,b)\}_0^\infty$ which satisfy the same recursion eq. 1 but with inputs $L_0(a,b) = 2, L_1(a,b) = a$, one finds, with $V_n(a,b) \; := \; L_{n+1}(a,b)/a$, remembering that $a \neq 0$,

$$V(a,b;x) \; := \; \sum_{n=0}^{\infty} V_n(a,b)\,x^n \; = \; \frac{1 + 2\,b\,x/a}{1 \, - \, a\,x \, - \, b\,x^2} \; . \tag{3}$$

The input is now $V_0(a,b) = 1$ and $V_1(a,b) = (a^2 \, + \, 2\,b)/a$ (or $V_{-1}(a,b) = 2/a$).
These (ordinary) generating functions can also be written in terms of the characteristic roots corresponding to the recursion relation eq. 1

$$\lambda_\pm \; \equiv \; \lambda_\pm(a,b) \; := \; \frac{1}{2}\,(a \, \pm \, \sqrt{a^2 + 4\,b}) \tag{4}$$

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as follows.

$$U(a,b;x) \;=\; \frac{1}{x\,(\lambda_+ - \lambda_-)}\left(\frac{1}{1\,-\,\lambda_+\,x}\;-\;\frac{1}{1\,-\,\lambda_-\,x}\right), \tag{5}$$

$$V(a,b;x) \;=\; \frac{1}{\lambda_+ + \lambda_-}\left(\frac{\lambda_+}{1\,-\,\lambda_+\,x}\;+\;\frac{\lambda_-}{1\,-\,\lambda_-\,x}\right). \tag{6}$$

The corresponding *Binet* forms of the corresponding numbers are in the non-degenerate case $\lambda_+ \neq \lambda_-$, *i.e.* $D(a,b) := a^2 + 4\,b \neq 0$,

$$U_n(a,b) \;=\; \frac{\lambda_+^{n+1}\;-\;\lambda_-^{n+1}}{\lambda_+ - \lambda_-}, \tag{7}$$

$$V_n(a,b) \;=\; \frac{\lambda_+^{n+1}\;+\;\lambda_-^{n+1}}{\lambda_+ + \lambda_-}. \tag{8}$$

In the degenerate case one has

$$U_n(a) \;:=\; U_n(a,-\frac{a^2}{4}) \;=\; (n+1)\left(\frac{a}{2}\right)^n, \tag{9}$$

$$V_n(a) \;:=\; V_n(a,-\frac{a^2}{4}) \;=\; \left(\frac{a}{2}\right)^n. \tag{10}$$

The explicit form of these polynomials is

$$U_n(a,b) \;=\; \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-l}{l}\, a^{n-2l}\, b^l, \tag{11}$$

$$V_n(a,b) \;=\; \sum_{l=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{n+1}{n+1-l}\binom{n+1-l}{l}\, a^{n-2l}\, b^l. \tag{12}$$

This result for $U_n(a,b)$ follows from a combinatorial interpretation of the recurrence relation, and the one for $V_n(a,b)$ is due to the *Girard - Waring* formula in its simplest version (for this *cf.* [4], [3], also for original refs.).

The observation reported in this work is that the generating functions eq. 2 (or eq. 5) and eq. 3 (or eq. 6) are the unique solutions to the following special type of *Riccati* eqs. (simultaneously a special type of *Bernoulli* eq. For the history of such eqs. see [9], ch.I, 1.1)

$$(a\,+\,2\,b\,x)\,\frac{\partial}{\partial x}U(a,b;x)\;+\;4\,b\,U(a,b;x)\;-\;(a^2\,+\,4\,b)\,U^2(a,b;x)\;=\;0, \tag{13}$$

with the initial condition $U(a,b;0) = 1$. Similarly,

$$(1\,+\,2\,\frac{b}{a}\,x)^2\,\frac{\partial}{\partial x}V(a,b;x)\;+\;2\,\frac{b}{a}\,(1\,+\,2\,\frac{b}{a}\,x)\,V(a,b;x)\;-\;(a\,+\,4\,\frac{b}{a})\,V^2(a,b;x)\;=\;0, \tag{14}$$

with the initial condition $V(a,b;0) = 1$. Such *Riccati* eqs. can be transformed to inhomogeneous linear differential eqs. of first order. Our interest here is to use these eqs. in order to obtain information about U^2 and V^2. The degenerate case $D(a,b) \;:=\; a^2\,+\,4\,b = 0$, for which the above given differential eqs. become linear, will be considered separately.

These *Riccati* eqs. immediately allow one to express convolutions of *Fibonacci* or *Lucas* sequences in terms of these numbers. In the non-degenerate case, $D(a,b) \neq 0$, one rewrites the *Riccati* eqs. as

$$U^2(a,b;x) \;=\; \frac{1}{a^2\,+\,4\,b}\,\Big((a\,+\,2\,b\,x)\,\frac{\partial}{\partial x}\;+\;4\,b\Big)\,U(a,b;x), \tag{15}$$

and

$$V^2(a,b;x) \;=\; \frac{1}{a\,+\,4\,\frac{b}{a}}\,\Big((1\,+\,2\,\frac{b}{a}\,x)^2\,\frac{\partial}{\partial x}\;+\;2\,\frac{b}{a}(1\,+\,2\,\frac{b}{a}\,x)\Big)\,V(a,b;x). \tag{16}$$

The (first) convolution of these sequences, which we call

$$U_n^{(1)}(a,b) := \sum_{q=0}^{n} U_{n-q}(a,b)\,U_q(a,b) \quad \text{and} \quad V_n^{(1)}(a,b) := \sum_{q=0}^{n} V_{n-q}(a,b)\,V_q(a,b)\ , \tag{17}$$

and whose generating functions are $U^2(a,b;x)$ and $V^2(a,b;x)$, respectively, satisfy therefore

$$U_n^{(1)}(a,b) \;=\; \frac{1}{a^2\,+\,4\,b}\,\Big(a\,(n+1)\,U_{n+1}(a,b)\;+\;2\,b\,(n+2)\,U_n(a,b)\Big), \tag{18}$$

and

$$V_n^{(1)}(a,b) \;=\; \frac{1}{a\,+\,4\,\frac{b}{a}}\,\Big((n+1)\,V_{n+1}(a,b)\;+\;2\,\frac{b}{a}\,(2\,n+1)\,V_n(a,b)\;+\;4\,\frac{b^2}{a^2}\,n\,V_{n-1}(a,b)\Big)\ . \tag{19}$$

After use of the recursion relation eq. 1 the last eq. can be written in the form

$$V_n^{(1)}(a,b) \;=\; \frac{1}{a\,(a^2\,+\,4\,b)}\,\Big(\big[a^2\,(n+1)\;+\;4\,b\,n\big]\,V_{n+1}(a,b)\;+\;2\,b\,a\,V_n(a,b)\Big)\ . \tag{20}$$

For $a = b = 1$ one recovers well-known formulae for convolutions of ordinary *Fibonacci*, resp. *Lucas* numbers (*e.g.* [7], p.183, eqs. (98) and (99) (with corrected $L_{n-1} \to L_{n-i}$)). To see this, observe that $U_n^{(1)}(1,1) = F_{n+2}^{(1)}$ and $V_n^{(1)}(1,1) = L_{n+2}^{(1)} - 4\,L_{n+2}$.

$$F_n^{(1)} \;=\; U_{n-2}^{(1)}(1,1) \;=\; \frac{1}{5}\,\Big((n-1)\,F_n\;+\;2\,n\,F_{n-1}\Big) \;=\; \frac{1}{5}\,(n\,L_n\;-\;F_n) \tag{21}$$

$$L_n^{(1)} \;=\; V_{n-2}^{(1)}(1,1)\;+\;4\,L_n \;=\; \frac{1}{5}\,\Big((5\,n-9)\,L_n\;+\;2\,L_{n-1}\Big)\;+\;4\,L_n \;=\; (n+2)\,L_n\;+\;F_n\ . \tag{22}$$

For the $k-$th convolution $U_n^{(k)}(a,b)$ and $V_n^{(k)}(a,b)$, with $k = 1,2,...$, one employs in the case $D(a,b) \neq 0$ the following identities which follow from the *Riccati* eqs.

$$U^{k+1}(a,b;x) \;=\; \frac{1}{(a^2\,+\,4\,b)\,k}\,\Big((a\,+\,2\,b\,x)\,\frac{\partial}{\partial x}\;+\;4\,k\,b\Big)\,U^k(a,b;x)\ , \tag{23}$$

and

$$V^{k+1}(a,b;x) \;=\; \frac{a}{(a^2\,+\,4\,b)\,k}\,\Big((1\,+\,2\,\frac{b}{a}\,x)^2\,\frac{\partial}{\partial x}\;+\;2\,k\,\frac{b}{a}(1\,+\,2\,\frac{b}{a}\,x)\Big)\,V^k(a,b;x)\ . \tag{24}$$

These identities allow one to express the $k-$th convolution, generated by $U^{k+1}(a,b;x) =: \sum_{n=0}^{\infty} U_n^{(k)}(a,b)\,x^n$, in terms of the $k-1$-st one according to

$$U_n^{(k)}(a,b) \;=\; \frac{1}{k\,(a^2\,+\,4\,b)}\,\Big(a\,(n+1)\,U_{n+1}^{(k-1)}(a,b)\;+\;2\,b\,(n+2\,k)\,U_n^{(k-1)}(a,b)\Big), \tag{25}$$

with input $U_n^{(0)}(a,b) \,=\, U_n(a,b)$, and

$$V_n^{(k)}(a,b) \;=\; \frac{1}{k\,a\,(a^2\,+\,4\,b)}\Big((n+1)\,a^2\,V_{n+1}^{(k-1)}(a,b)\;+\;2\,a\,b\,(2\,n\,+\,k)\,V_n^{(k-1)}(a,b) + 4\,b^2\,(n\,+\,k\,-\,1)\,V_{n-1}^{(k-1)}(a,b)\Big)\;, \tag{26}$$

with input $V_n^{(0)}(a,b) \,=\, V_n(a,b)$. The formula given in eq. 25 has been found earlier in [1] (p. 202, III and p.213, eq. (30)) without using the defining *Riccati* eq. for $U(a,b;x)$. The notations have to be translated with the help of $F_n^{(k)} \,\hat{=}\, U_n^{(k-1)}$, $a_1 \,\hat{=}\, a$, and $a_2 \,\hat{=}\, b$.

Before discussing iteration of these recursion relations we state the results for the degenerate case $D(a,b) \,:=\, a^2\,+\,4\,b \,=\, 0$. The *Riccati* eqs. 13 and 14 collapse to linear differential eqs. for $U(a;x) \,:=\, U(a,-a^2/4;x)$ and $V(a;x) \,:=\, V(a,-a^2/4;x)$

$$(1\,-\,\frac{a}{2}\,x)\,\frac{\partial}{\partial x}U(a;x) \;=\; a\,U(a;x) \quad,\quad U(a;0)\,=\,1, \tag{27}$$

$$(1\,-\,\frac{a}{2}\,x)\,\frac{\partial}{\partial x}V(a;x) \;=\; \frac{a}{2}\,V(a;x) \quad,\quad V(a;0)\,=\,1\;. \tag{28}$$

For the last eq. $x \neq 2/a$ was assumed. Because the solutions to these eqs. imply

$$\frac{\partial^2}{\partial x^2}U(a;x) \;=\; \frac{3}{2}\,a^2\,U^2(a;x) \quad,\quad \frac{\partial}{\partial x}V(a;x) \;=\; \frac{a}{2}\,V^2(a;x)\;, \tag{29}$$

the corresponding first $(k=1)$ convolutions of these numbers $U_n(a) \,:=\, U_n(a,-a^2/4)$ and $V_n(a) \,:=\, V_n(a,-a^2/4)$ are given by

$$U_n^{(1)}(a) \;=\; \frac{2}{3\,a^2}\,(n\,+\,2)\,(n\,+\,1)\,U_{n+2}(a) \quad,\quad V_n^{(1)}(a) \;=\; \frac{2}{a}\,(n\,+\,1)\,V_{n+1}(a)\;, \tag{30}$$

with eqs. 9 and 10.

In order to derive the result for the $k-$th convolution one starts with identities which follow from the solutions of eqs. 27 and 28, namely

$$U^{k+1}(a;x) \;=\; \frac{2}{a^2\,k\,(2\,k+1)}\,\frac{\partial^2}{\partial x^2}\Big(U^k(a;x)\Big)\;, \tag{31}$$

$$V^{k+1}(a;x) \;=\; \frac{2}{a\,k}\,\frac{\partial}{\partial x}\Big(V^k(a;x)\Big)\;. \tag{32}$$

These identities imply for the $k-$th convolutions

$$U_n^{(k)}(a) \;=\; \frac{2}{a^2\,k\,(2\,k+1)}\,(n+2)\,(n+1)\,U_{n+2}^{(k-1)}(a)\;, \tag{33}$$

$$V_n^{(k)}(a) \;=\; \frac{2}{a\,k}\,(n+1)\,V_{n+1}^{(k-1)}(a)\;, \tag{34}$$

with inputs $U_n^{(0)}(a) \,=\, U_n(a) \,=\, (n+1)\,(a/2)^n$ and $V_n^{(0)}(a) \,=\, V_n(a) \,=\, (a/2)^n$. See eqs. 9 and 10. The iteration of these eqs. yields the final result, which for $k \in \mathbb{N}_0$, and in the degenerate case $b \,=\, -a^2/4$, is

$$U_n^{(k)}(a) \;=\; \binom{n+2\,k+1}{2\,k+1}\left(\frac{a}{2}\right)^n\;, \tag{35}$$

$$V_n^{(k)}(a) \;=\; \binom{n+k}{k}\left(\frac{a}{2}\right)^n\;. \tag{36}$$

Thus $V_n^{(2l+1)}(a) \;=\; U_n^{(l)}(a)$, and it suffices to treat $V_n^{(k)}(a)$. For even a these are non-negative integer sequences. For $n, k \in \mathbb{N}_0$, $V_{n+k}^{(k)}(2\,l)$ constitutes a convolution triangle of numbers based on the $k = 0$ column sequence $V_n^{(0)}(2\,l) \;=\; l^n$ (powers of l). See [5] for these triangles of numbers.

In the non-degenerate case the recursion relation eq. 24 can be iterated in order to express the $k-$th convolution of $U_n(a,b)$ as linear combination of these numbers according to

$$U_n^{(k)}(a,b) \;=\; \frac{1}{k!\,(a^2+4\,b)^k}\left(AU_{k-1}(a,b;n)\,(n+1)\,a\,U_{n+1}(a,b) \;+\; BU_{k-1}(a,b;n)\,(n+2)\,b\,U_n(a,b)\right)\,, \tag{37}$$

with certain polynomials $AU_{k-1}(a,b;n)$ and $BU_{k-1}(a,b;n)$ of degree $k-1$ in the variable n, for arbitrary, but fixed, $a \neq 0$, $b \neq 0$, and $b \neq -a^2/4$.
The (mixed) recursion relations for these polynomials are deduced from eq. 24, and for $k = 1, 2, ...$, they are

$$AU_k(a,b;n) \;=\; a^2\,(n+2)\,AU_{k-1}(a,b;n+1) \;+\; 2\,b\,(n+2(k+1))\,AU_{k-1}(a,b;n) \;+\; b\,(n+3)\,BU_{k-1}(a,b;n+1)\,, \tag{38}$$

$$BU_k(a,b;n) \;=\; a^2\,(n+1)\,AU_{k-1}(a,b;n+1) \;+\; 2\,b\,(n+2(k+1))\,BU_{k-1}(a,b;n)\,, \tag{39}$$

with inputs $AU_0(a,b;n) = 1$ and $BU_0(a,b;n) = 2$.
In eqs. (26), resp. (27), of [1] one can find explicit results for $U_n^{(k)}(a,b)$ for the instances $k = 2$, resp. $k = 3$ (in eq. (26) of this ref. one has to multiply the *lhs* with 2!, and in the second line of N of eq.(27) it should read $B(2, n+1)$).
For the case $a = 1 = b$ the triangles of the coefficients of these polynomials can be viewed under the nrs. A057995 and A057280 in [5]. For $a = 2$, $b = 1$ see A058402 and A058403.
Similarly, iteration of recursion eq. 25 results, with the help of the recursion relation eq. 1, in

$$V_n^{(k)}(a,b) \;=\; \frac{1}{k!\,a\,(a^2+4\,b)^k}\left(AV_k(a,b;n)\,V_{n+1}(a,b) \;+\; BV_k(a,b;n)\,V_n(a,b)\right)\,, \tag{40}$$

with certain polynomials $AV_k(a,b;n)$ and $BV_k(a,b;n)$ of generic degree k in the variable n, for fixed $a \neq 0$, $b \neq 0$, with $b \neq -a^2/4$.
The (mixed) recursion relations for these polynomials are found from eq. 26, and for $k = 1, 2, ...$, they are

$$AV_k(a,b;n) \;=\; a^2\,(n+1)\,AV_{k-1}(a,b;n+1) \;+\; 2\,b\,(2\,n+k)\,AV_{k-1}(a,b;n) \;+\; a\,(n+1)\,BV_{k-1}(a,b;n+1) \;+\; 4\,\frac{b}{a}\,(n+k-1)\,BV_{k-1}(a,b;n-1)\,, \tag{41}$$

$$BV_k(a,b;n) \;=\; 2\,b\,(2\,n+k)\,BV_{k-1}(a,b;n) \;-\; 4\;b\,(n+k-1)\,BV_{k-1}(a,b;n-1) \;+\; a\,b\,(n+1)\,AV_{k-1}(a,b;n+1) \;+\; 4\,\frac{b^2}{a}\,(n+k-1)\,AV_{k-1}(a,b;n-1)\,, \tag{42}$$

with inputs $AV_0(a,b;n) = 0$ and $BV_0(a,b;n) = a$.
For $a = 1 = b$ the triangles of coefficients of these polynomials in n can be found under the nrs. A061188 and A061189 in [5]. Observe that $BV_1(1,1;n)$ is accidentally of degree 0. For $a = 2, b = 1$ see nrs. A062133 and A062134. For $a = 2$, $b = 1$ see A062133 and A062134.

Motivated by a recent paper [6] we consider also the following generalized $p-$*Fibonacci* numbers $U_n(p;a,b)$ defined for $p \in \mathbb{N}_0$ by the generating function

$$U(p;a,b;x) \;:=\; \frac{1}{1\;-\;a\,x\;-\;b\,x^{p+1}} \;=\; \sum_{n=0}^{\infty} U_n(p;a,b)\,x^n\,. \tag{43}$$

Of course, we assume $b \neq 0$ and also take $a \neq 0$. For $p = 1$ these numbers reduce to the $U_n(a,b)$ treated above, and for $p = 0$ they become the powers $(a+b)^n$. $U(p;1,1;x)$ appears in eq. 71 of [2]. The recursion relations are

$$U_n(p;a,b) \;=\; a\,U_{n-1}(p;a,b) \;+\; b\,U_{n-(p+1)}(p;a,b)\;, \tag{44}$$

with inputs $U_j(p;a,b) \;=\; a^j$ for $j = 0,1,...,p$. In order to derive expressions for the k−th convolution of these $p-$*Fibonacci* numbers consider first the following *Riccati* eq. satisfied by $U(p;a,b;x)$ written for the non-degenerate case $D(p;a,b) \;:=\; (p+1)^{p+1}\,b \;+\; a\,(a\,p)^p \;\neq\; 0$ if $p \in \mathbb{N}$, and $a+b \neq 0$ if $p = 0$ (*i.e.* one puts $(a\,p)^p \;=\; 1$ if $p = 0$).

$$U^2(p;a,b;x) \;=\; \frac{1}{(p+1)^{p+1}\,b \;+\; a\,(a\,p)^p}\left\{A_p(a,b;x)\,\frac{\partial}{\partial x} \;+\; b\,(p+1)^2\,B_{p-1}(a;x)\right\}U(p;a,b;x)\;, \tag{45}$$

with

$$A_p(a,b;x) \;=\; (a\,p)^p \;+\; b\,(p+1)\,x\,B_{p-1}(a;x)\;, \tag{46}$$

$$B_{p-1}(a;x) \;=\; (p+1)^{p-1}\sum_{i=0}^{p-1}\left(\frac{a\,p}{p+1}\right)^i x^i \;=\; \frac{(p+1)^p \;-\; (a\,p\,x)^p}{p+1 \;-\; a\,p\,x}. \tag{47}$$

For $p \;=\; 0$ one has to use $A_0(a,b;x) \;=\; 1$ and $B_{-1}(a;x) \;=\; 0$. For given non-vanishing a and b these polynomials $A_p(a,b;x)$, resp. $B_{p-1}(a;x)$, in the variable x of degree p, resp. $p-1$, have therefore the following explicit form.

$$A_p(a,b;x) \;=\; \sum_{m=0}^{p} A(a,b;p,m)\;x^m \quad , \quad B_{p-1}(a;x) \;=\; \sum_{m=0}^{p-1} B(a;p-1,m)\;x^m\;, \tag{48}$$

with the coefficients

$$A(a,b;p,m) \;=\; \begin{cases} 0 & \text{if } m \;>\; p, \\ 1 & \text{if } m = 0 \text{ and } p = 0, \\ (a\,p)^p & \text{if } m = 0 \text{ and } p \geq 1, \\ b\,(p+1)^p\left(\frac{a\,p}{p+1}\right)^{m-1} & \text{if } m \geq 1\;. \end{cases} \tag{49}$$

$$B(a;p,m) \;=\; \begin{cases} 0 & \text{if } p \;<\; m \text{ or } p = -1\;, \\ (p+2)^p\left(\frac{a\,(p+1)}{p+2}\right)^m & \text{if } p \geq m \geq 0\;. \end{cases} \tag{50}$$

For $a \;=\; 1 \;=\; b$ these triangles of coefficients can be viewed under the numbers $A055858$ and $A055864$ in [5] where further details may be found.
Even though we cannot integrate the linear differential eq., which is equivalent to the *Riccati* eq. 45 for $p \neq 0,1$, analytically, the above given solution is its unique one due to the existence and uniqueness theorem for the linear first order differential eq. satisfied by $Z(p;a,b;x) \;:=\; 1/U(p;a,b;x)$ with initial value $Z(p;a,b;0) \;=\; 1$.
The result for the first ($k = 1$) convolution of the numbers $U_n(p;a,b)$ which flows from the *Riccati* eq. is

$$U_n^{(1)}(p;a,b) \;=\; \frac{1}{b\,(p+1)^{p+1} \;+\; a\,(a\,p)^p}\sum_{j=0}^{p} C_j(n;p;a,b)\,U_{n+1-j}(p;a,b)\;, \tag{51}$$

with

$$C_j(n;p;a,b) \;=\; \begin{cases} n+1 & \text{if } p=0=j, \\ (n+1)\,(a\,p)^p & \text{if } p\geq 1 \text{ and } j=0\,, \\ b\,(p+1)^p\,(n+p+2-j)\,\left(\frac{a\,p}{p+1}\right)^{j-1} & \text{if } p\geq 1 \text{ and } j=1,...,p. \end{cases} \tag{52}$$

The $U_n(p;a,b)$ recursion cannot be used to simplify the sum in eq. 51.
This result can now be compared, after putting $a = 1 = b$, with a different formula for the same convolution found in [6], eq.(14). For given $p \in \mathbb{N}_0$ and $k = 2, 3, ...$, the recursion for $F_p^{(2)}(k) \doteq U^{(1)}_{k-2}(p;1,1)$ in [6] involves all $k-1$ terms $F_p(n) \doteq U_{n-1}(p;1,1)$, for $n = 1, ..., k-1$, whereas our result needs only $p+1$ terms for all k. For example, $F_3^{(2)}(7) \doteq U_5^{(1)}(3;1,1)$ is reduced to six terms involving $F_3(1) \doteq U_0(3;1,1)$,..., $F_3(6) \doteq U_5(3;1,1)$ in [6], but only to four terms, involving $U_8(3;1,1) \doteq F_3(9)$, $U_7(3;1,1) \doteq F_3(8)$,..., $U_5(3;1,1) \doteq F_3(6)$ in eq. 51.
For the $k-$th convolution one uses the following identity which derives from the *Riccati* eq. 45. It is written for the non-degenerate case, and is valid for $k \in \mathbb{N}$.

$$U^{k+1}(p;a,b;x) \;=\; \frac{1}{k\left(b\,(p+1)^{p+1} \;+\; a\,(a\,p)^p\right)} \left\{A_p(a,b;x)\,\frac{\partial}{\partial x} \;+\; k\,b\,(p+1)^2\,B_{p-1}(a;x)\right\} U^k(p;a,b;x)\;. \tag{53}$$

For $p \in \mathbb{N}_0$ the corresponding recursion relation for the $k-$th convolution is (remember that one puts $(a\,p)^p \;=\; 1$ if $p=0$)

$$U_n^{(k)}(p;a,b) \;=\; \frac{1}{k\left(b\,(p+1)^{p+1} \;+\; a\,(a\,p)^p\right)} \sum_{j=0}^{p} C_j^{(k)}(n;p;a,b)\,U^{(k-1)}_{n+1-j}(p;a,b)\;. \tag{54}$$

with

$$C_j^{(k)}(n;p;a,b) \;=\; \begin{cases} n+1 & \text{if } p=0=j, \\ (n+1)\,(a\,p)^p & \text{if } p\geq 1 \text{ and } j=0\,, \\ b\,(p+1)^p\,(n+1+k\,(p+1)-j)\,\left(\frac{a\,p}{p+1}\right)^{j-1} & \text{if } p\geq 1 \text{ and } j=1,...,p. \end{cases} \tag{55}$$

Instead of showing the rather unwieldy formula for this iterated recursion relation after employing the fundamental recursion eq. 1 we prefer to state the result for the instance $p = 2, k = 2, a = 1 = b$, with the notation $U_n^{(1)}(2;1,1) \equiv U_n^{(1)}(2)$ and $U_n(2;1,1) \equiv U_n(2)$:

$$\begin{aligned} U_n^{(2)}(2) \;&=\; \frac{1}{2\cdot 31}\left(4\,(n+1)\,U^{(1)}_{n+1}(2) \;+\; 9\,(n+6)\,U^{(1)}_n(2) \;+\; 6\,(n+5)\,U^{(1)}_{n-1}(2)\right) \\ &=\; \frac{1}{2\cdot 31^2}\Big((217\,n^2 \;+\; 1425\,n \;+\; 1922)\,U_n(2) \;+\; 2\,(n+2)\,(62\,n \;+\; 305)\,U_{n-1}(2) \;+ \\ &\qquad 4\,(n+1)\,(31\,n+143)\,U_{n-2}(2)\Big)\;. \end{aligned} \tag{56}$$

The recursion relation eq. 1 has been used twice.

In the degenerate case $D(p;a,b) \;:=\; (p+1)^{p+1}\,b \;+\; a\,(a\,p)^p \;=\; 0$ (where one puts $(a\,p)^p \;\equiv\; 1$ if $p=0$) one finds for $U(p;a,b=b(p;a);x) \;=:\; U(p;a;x)$, where $b(p;a) \;:=\; -p^p\,(a/(p+1))^{p+1}$, the linear differential eq.

$$\left\{A_p(a,b(p;a);x)\,\frac{\partial}{\partial x} \;+\; b(p;a)\,(p+1)^2\,B_{p-1}(a;x)\right\} U(p;a;x) \;=\; 0 \tag{57}$$

with B_{p-1} and A_p taken in their explicit form known from eqs. 47 and 46. For general p and $D(p;a,b)=0$ we cannot say anything about convolution because we have no suitable expression for $U^2(p;a,b;x)$. The recurrence eq. 1 with depth $p+1$ can be replaced by one with only depth p. See eq. 73.

In the non-degenerate case one could also consider the other p linear independent (*Lucas*-type) sequences defined by the recurrence eq. and appropriate inputs but we will not do this here.

The remainder of this paper provides proofs for the above given statements.

2 Riccati equations for Fibonacci and Lucas generating functions

Lemma 1: $U(a,b;x)$ defined in eq. 2 satisfies eq. 13.

Proof: This follows from the identity

$$\frac{\partial}{\partial x}\frac{1}{1-a\,x-b\,x^2} \;=\; (a+2\,b\,x)\,\frac{1}{(1-a\,x-b\,x^2)^2} \tag{58}$$

after multiplication with $(a\,+\,2\,b\,x)$ and separation of the last term in eq. 13. $\square$.

Corollary 1: For the non-degenerate case $a^2+4\,b\neq 0$ this is a *Riccati* eq. (also a special *Bernoulli* eq.) which can be written as shown in eq. 15.
In the degenerate case $U(a;x)\;:=\;U(a,-a^2/4;x)$ satisfies the first order linear differential eq. 27 as well as the second order non-linear differential eq. given as the first of eqs. 29.

Proof: Elementary. $\square$

Note 1: The degenerate case is equivalent to $\lambda_+(a,b)\;=\;\lambda_-(a,b)$ with the definition of the characteristic roots of the recursion relation eq. 1 given in eq. 4. We may assume that not both, a and b, vanish and $x\neq 1/\lambda_\pm(a,b)\;=\;-\lambda_\mp/b$. In each case $U(a,b;0)\;=\;1$.

Lemma 2: $V(a,b;x)$ defined in eq. 3 for $a\neq 0$ satisfies eq. 14.

Proof: This follows from the identity

$$\frac{\partial}{\partial x}\,V(a,b;x) \;=\; \left((2\,\frac{b}{a}\,+\,a)\;+\;2\,b\,x\;+\;2\,\frac{b^2}{a}\,x^2\right)\frac{1}{(1-a\,x-b\,x^2)^2}\;, \tag{59}$$

after multiplication with $(a+2\,b\,x)^2$ and separation of the $V^2(a,b;x)$ term shown in eq. 14. $\square$

Corollary 2: For the non-degenerate case $a^2+4\,b\neq 0$ eq. 14 is a *Riccati* eq. which can be written as shown in eq. 16.
In the degenerate case $V(a;x)\;:=\;V(a,-a^2/4;x)$ satisfies the first order linear differential eq. 28 as well as the first order non-linear differential eq. given as the second of eqs. 29.
In each case $V(a,b,0)\;=\;1$.

Proof: Elementary. $\square$

Proposition 1 (Solution of eq. 13):

a) For $a^2+4\,b\neq 0$ the solution of the *Riccati* eq. 13 with $U(a,b;0)\;=\;1$ is given by eq. 2.

b) For $a^2+4\,b\;=\;0$ the solution of eqs. 27 is $U(a,-a^2/4;x)$ from eq. 2.

Proof: a) Rewrite this *Riccati* eq., which is also of the *Bernoulli* type, $y'\,+\,f(x)\,y\,+\,g(x)\,y^2\;=\;0$ with $f(x)\;=\;4\,b/(a\,+\,2\,b\,x)$ and $g(x)\;=\;-(a^2\,+\,4\,b)/(a\,+\,2\,b\,x)$, provided $x\;\neq\;-a/(2\,b)$, as a first order inhomogeneous linear differential eq. for $z\;=\;1/y$: $z'\;-\;f(x)\,z\;-\;g(x)\;=\;0$. Its standard solution is $y^{-1}\;=\;z\;=\;exp(F(x))\,\left[C\,+\,\int dx\,exp(F(x))\,g(x)\right]$ with $F(x)\;:=\;\int dx\,f(x)\;=\;ln((a\,+\,2\,b\,x)^2)$. The

integral becomes $(1 + a^2/(4b))/(a + 2bx)^2$ and C is fixed from the initial value $y^{-1}(0) = z(0) = 1$ to $-1/(4b)$. The solution $1/(1 - ax - bx^2)$ is also correct if $x = -a/(2b)$.
b) For $a^2 + 4b = 0$ and $x \neq 2/a$ the solution of the standard linear differential eq. 27 coincides with eq. 2. For $x = 2/a$ eq. 27 demands a singular $(ln\, U)'$; a requirement which is satisfied by this solution. □

Proposition 2 (Solution of eq. 14):

a) For $a^2 + 4b \neq 0$ the solution of the *Riccati* eq. 14 with $V(a, b; x)$ is given by eq. 3.

b) For $a^2 + 4b = 0$ the solution of eqs. 28 is $V(a, -a^2/4; 0) = 1$ from eq. 3.

Proof: a) Similar to the proof of *Proposition 1* as standard solution to the inhomogeneous linear differential eq. for $y^{-1} = z$ where now $f(x) = 2\frac{b}{a}/(1 + 2\frac{b}{a}x)$, $g(x) = -(a + 4\frac{b}{a})/(1 + 2\frac{b}{a}x)^2$ and $1 + 2\frac{b}{a}x \neq 0$. The initial value is $y^{-1}(0) = z(0) = 1$ which fixes the solution to be $y = (1 + 2\frac{b}{a}x)/(1 - ax - bx^2)$. This is also correct for $1 + 2\frac{b}{a}x = 0$.
b) For $a^2 + 4b = 0$ and $x \neq 2/a$ the solution of eq. 28 with $V(a; 0) = 1$ coincides with $V(a, -a^2/4; x) = 1/(1 - ax/2)$. For $x = 2/a$ eq. 28 needs a singular $(ln\, V)'$ because $a \neq 0$, and this requirement is indeed fulfilled. □

3 Convolutions of generalized Fibonacci and Lucas sequences

The *Riccati* eqs. 13, resp. 14, are replaced in the non-degenerate case by eqs. 15, resp. 16. Because the $k-$th power of the (ordinary) generating functions of a sequence generates $k-$fold convolutions of this sequence one obtains the expression eq. 18, resp. eq. 19. For the definition of the first convolutions $U^{(1)}(a, b)$ and $V^{(1)}(a, b)$ see eqs. 17. Eq. 19 simplifies to eq. 20 after use of the recurrence relation eq. 1. In this way the first convolutions can be determined in each case from linear combinations of the two independent original sequences. For $a = 1 = b$ these formulae are well-known (see the *Introduction* after eq. 20). The generalization to arbitrary $k-$fold convolutions is now straightforward.

Lemma 3 (Recurrencce for $k-$fold convolutions):

In the non-degenerate case $a^2 + 4b \neq 0$ the recurrence eq. 25, resp. eq. 26, holds for the $k-$fold convolution of generalized *Fibonacci*, resp. *Lucas*, sequences.

Proof: This is a direct consequence of the identities for the $k + 1$st power of $U(a, b; x)$, resp. $V(a, b; x)$ shown in eq. 23, resp. eq. 24. These identities, in turn, result after induction over k with the basis ($k = 1$) provided by eq. 15, resp. eq. 16. Because of $U^{k+1}(a, b; x) = \sum_{n=0}^{\infty} U_n^{(k)}(a, b)\, x^n$, and the same eq. with U replaced by V, one arrives at eq. 25, resp. eq. 26. □

Lemma 4 (Recurrence for $k-$fold convolution, degenerate case):

For $b = -\frac{a^2}{4} \neq 0$ the recurrence formulae for the $k-$fold convolution of the generalzed *Fibonacci*, resp. *Lucas*, sequences are those stated in eqs. 33, resp. 34.

Proof: This statement is equivalent to eq. 31, resp. eq. 32 for the powers of the corresponding generating functions. They are deduced from the the second, resp. first, order differential eq. given in eq. 29 which is identical with the $k = 1$ assertion. To verify the general k claim eq. 31, resp. eq. 32, one may use $U(a; x) = U(a, -\frac{a^2}{4}; x) = 1/(1 - ax/2)^2$ from eq. 2, resp. $V(a; x) = V(a, -\frac{a^2}{4}; x) = 1/(1 - ax/2)$ from eq. 3. □

Lemma 5: The explicit form for the $k-$fold convolution in the degenerate case is given by eq. 35, resp. 36, for the generalized *Fibonacci*, resp. *Lucas*, case.

Proof: Iteration of the recurrence eq. 33, resp. eq. 34, with input $U_n^{(0)}(a) = U_n(a) = (a/2)^n$, resp. $V_n^{(0)}(a) = V_n(a) = (a/2)^n$, which originates from the generating functions $U(a, -\frac{a^2}{4}; x)$, resp. $V(a, -\frac{a^2}{4}; x)$. □

Proposition 3 (Iteration of recurrence for $k-$fold convolutions; non-degenerate *Fibonacci* case):

For $a^2 + 4\,b \neq 0$ the $k-$fold convolution of the generalized *Fibonacci* sequence $\{U_n(a,b)\}$ is expressed as linear combinations of the two independent solutions of recurrence eq. 1 as given in eq. 37. The coefficient polynomials $AU_k(a,b;n)$ and $BU_k(a,b;n)$ satisfy the mixed recurrence relations eqs. 38 and 39.

Proof: If one considers eq. 37 as *ansatz* and puts it into the recurrence eq. 25 one finds, after elimination of $U_{n+2}(a,b)$ *via* its recursion relation and a comparison of the coefficients of the linear independent $U_n(a,b)$ and $U_{n-1}(a,b)$ sequences, the mixed recurrence relations for $AU_k(a,b;n)$ and $BU_k(a,b;n)$. The inputs $AU_0(a,b;n) = 1$ and $BU_0(a,b;n) = 2$ are necessary in order that for $k=1$ eq. 37 coincides with eq. 18. With these inputs and the mixed recurrence one proves, by induction over k, that $AU_k(a,b;n)$ and $BU_k(a,b;n)$ are polynomials in n of degree k, provided a and b are fixed with $b \neq -a^2/4$, $b \neq 0$ and $a \neq 0$. □

Note 2: For integers a and $b \neq -a^2/4$ the coefficients of the polynomials $AU_n(a,b;x)$ and $BU_n(a,b;x)$ furnish two lower triangular integer matrices. For the ordinary *Fibonacci* case ($a = 1 = b$) these positive integer triangles can be found in [5] under the nrs. A057995 and A057280. For the *Pell* case ($a = 2, b = 1$) see nrs. A058402 and A058403.

Proposition 4 (Iteration of recurrence for $k-$fold convolutions; non-degenerate *Lucas* case):

For $a^2 + 4\,b \neq 0$ the $k-$fold convolution of the generalized *Lucas* sequence $\{V_n(a,b)\}$ is expressed as linear combination of the two independent solutions of recurrence eq. 1 as given in eq. 40. The coefficient polynomials $AV_k(a,b;n)$ and $BV_k(a,b;n)$ satisfy the mixed recurrence relations eq. 41 and eq. 42.

Proof: Analogous to the one of *Proposition 3*. □

Note 3: For integers a and $b \neq -a^2/4$ the coefficients of the polynomials $AV_n(a,b;x)$ and $BV_n(a,b;x)$ furnish two lower triangular integer matrices. For the ordinary *Lucas* case ($a = 1 = b$) these positive integer triangles can be found in [5] under the nrs. A061188 and A061189. For the *Pell* case ($a = 2, b = 1$) see nrs. A062133 and A062134.

4 Convolutions of generalized $p-$Fibonacci sequences

Generalized $p-$*Fibonacci* numbers $U_n(p;a,b)$ are defined by eq. 1 for $p \in \mathbb{N}_0$, $b \neq 0$ and $a \neq 0$, together with the inputs $U_j(p;a,b) := a^j$ for $j = 0, ..., p$. For $p = 1$ one recovers the generalized *Fibonacci* numbers $U_n(a,b)$ treated above.

Lemma 6: The generating function $U(p;a,b;x)$ for the generalized $p-$*Fibonacci* numbers is given by eq. 43.

Proof: From the recurrence with inputs given in eq. 1. □

Lemma 7 (*Riccati* eq. for the generalized $p-$Fibonacci case):

If $D(p;a,b) := (p+1)^{p+1}\,b + a\,(a\,p)^p \neq 0$ (non-degenerate case) then $U(p;a,b;x)$ satisfies the *Riccati* eq. 45 with the polynomials $A_p(a,b;x)$ and $B_{p-1}(a,b;x)$ defined in eqs. 46 and 47.

Proof: Use the *ansatz*

$$U^2(p;a,b;x) = \Big(\sum_{i=0}^{p} A_i(p;a,b)\,x^i\Big)\,\frac{\partial}{\partial x}\,U(p;a,b;x) + \Big(\sum_{i=0}^{p-1} B_i(p;a,b)\,x^i\Big)\,U(p;a,b;x)\;, \tag{60}$$

together with

$$\frac{\partial}{\partial x} U(p;a,b;x) \;=\; (a \,+\, (p+1)\, b\, x^p)\, U^2(p;a,b;x) \tag{61}$$

from eq. 43. This implies

$$\Big(\sum_{i=0}^{p} A_i(a,b;x)\, x^i\Big)\, (a \,+\, (p+1)\, b\, x^p) \;+\; \Big(\sum_{i=0}^{p-1} B_i(a,b;x)\, x^i\Big)\, (1 \,-\, a\, x \,-\, b\, x^{p+1}) \;=\; 1\;. \tag{62}$$

Comparing first coefficients of the powers x^k, for $k = p+1, p+2, ..., 2\,p$, leads to the eqs. (remember that $b \neq 0$)

$$B_i(p;a,b) \;=\; (p+1)\, A_{i+1}(p;a,b)\;, \quad \text{for } i = 0,1,...,p-1. \tag{63}$$

With this result the coefficients of the powers x^k for $k = 0,1,...,p-1$ become, after iteration,

$$A_j(p;a,b) \;=\; \left(\frac{a\,p}{p+1}\right)^{j-1} \frac{1}{p+1}\, (1 \,-\, a\, A_0(p;a,b)) \text{ for } j = 1,2,...,p. \tag{64}$$

Therefore the eqs. 63 are now

$$B_i(p;a,b) \;=\; \left(\frac{a\,p}{p+1}\right)^{i} (1 \,-\, a\, A_0(p;a,b))\;, \text{ for } i = 0,1,...,p-1. \tag{65}$$

Finally, the coefficient of x^p has to satisfy $(p+1)\, b\, A_0 \,+\, a\,(A_p \,-\, B_{p-1}) \;=\; 0$ which implies, together with the above results,

$$A_0(p;a,b) \;=\; \left(\frac{a\,p}{p+1}\right)^{p} \frac{1}{(p+1)\, b \,+\, a\, \left(\frac{a\,p}{p+1}\right)^{p}}\;. \tag{66}$$

Inserting this into the above found expressions finally leads to

$$A_j(p;a,b) \;=\; \left(\frac{a\,p}{p+1}\right)^{j-1} \frac{b}{(p+1)\, b \,+\, a\, \left(\frac{a\,p}{p+1}\right)^{p}}\;, \quad \text{for } j = 1,2,...,p, \tag{67}$$

$$B_j(p;a,b) \;=\; \left(\frac{a\,p}{p+1}\right)^{j} \frac{b\,(p+1)}{(p+1)\, b \,+\, a\, \left(\frac{a\,p}{p+1}\right)^{p}}\;, \quad \text{for } j = 0,1,...,p-1, \tag{68}$$

$$A_0(p;a,b) \;=\; \left(\frac{a\,p}{p+1}\right)^{p} \frac{1}{(p+1)\, b \,+\, a\, \left(\frac{a\,p}{p+1}\right)^{p}}\;. \tag{69}$$

With

$$\mathcal{A}_p(a,b;x) \;:=\; ((p+1)^{p+1}\, b \,+\, a\, \left(\frac{a\,p}{p+1}\right)^{p}) \sum_{i=0}^{p} A_i(p;a,b)\, x^i\;, \tag{70}$$

$$b\,(p+1)^2\, \mathcal{B}_{p-1}(a;x) \;:=\; ((p+1)^{p+1}\, b \,+\, a\, \left(\frac{a\,p}{p+1}\right)^{p}) \sum_{i=0}^{p-1} B_i(p;a,b)\, x^i\;, \tag{71}$$

one thus finds eqs. 47 and 46. The assertion can now be proved directly with the found polynomials. □

Note 4: i) If $p = 0$, $U(0; a, b, ; x) = 1/(1 - (a + b)\, x)$ generates powers of $a + b$, and one has to put $A_0(a, b; x) \equiv 1$ and $B_{-1}(a; x) \equiv 0$. This means that one puts $(a\, p)^p = 1$ for $p = 0$.
ii) For given non-vanishing a and b $A_p(a, b; x)$ is a polynomial in x of degree p, and $B_{p-1}(a, b; x)$ is one of degree $p - 1$. The sum in $B_{p-1}(a, b; x)$ can be evaluated to yield the second of eqs. 47 provided $p \neq 0$.

Lemma 8 (Coefficient triangles of numbers for polynomials $A_p(a, b; x)$ and $B_p(a, b; x)$):

The coefficients of the polynomials defined in eqs. 48 are given by eqs. 49 and 50.

Proof: $B_p(a, b; x)$ from eq. 47 leads immediately to eq. 50, remembering that $B_{-1}(a; x) \equiv 0$. Then eq. 49 follows from eq. 46 and $A_0(a, b; x) \equiv 1$. □

Proposition 5 (Uniqueness of *Riccati* solution; non-degenerate case):

If $D(a, b) \neq 0$ then $y \equiv U(p; a, b; x) = 1/(1 - a\, x - b\, x^{p+1})$ is the unique solution of the *Riccati* eq. 45 with eqs. 46, 47 and initial value $U(p; a, b; 0) = 1$.

Proof: As a special *Bernoulli* eq. the *Riccati* eq. $y' + f(x)\, y + g(x)\, y^2 = 0$ is equivalent to the inhomogeneous linear differential eq. for $z = 1/y$: $z' \equiv F(x, z) = f(x)\, z + g(x)$. Because $F(x, z)$ is continuous in the strip $0 \leq x \leq A < \infty$, $|z| < \infty$ and is there ($k = k(p; a, b; A)$)-*Lipschitz*, the existence and uniqueness theorem for linear differential eqs. proves the assertion (see *e.g.*[8], § 6,I, p.62ff). In order to find k one uses the summed expression for B_{p-1} from eq. 47 and repeated applications of the triangle inequality. □

Lemma 9 (First convolution of $\{U_n(p; a, b)\}$; non-degenerate case):

The first convolution of the sequence $\{U_n(p; a, b)\}$, which is defined analogously to the first of eqs. 17, is given by eq. 51 with eq. 52.

Proof: One has to compute the coefficients of x^n of the *lhs* of eq. 45, taking into account the $x-$dependence of A_p and B_{p-1} with the coefficients from eqs. 49 and 50. The case $p = 0$ has to be considered separately. The recurrence eq. 1 cannot be used in order to simplify the sum in eq. 51. □

Proposition 6 (Recurrence for $k + 1$st power of the generating function $U(p; a, b; x)$; non-degenerate case):

For $D(p; a, b) := b\,(p + 1)^{p+1} + a\,(a\, p)^p \neq 0$ eq. 53 gives $U^{k+1}(p; a, b; x)$.

Proof:

$$\begin{aligned} & \frac{1}{D(p; a, b)}\, A_p(a, b; x)\, \frac{\partial}{\partial x}\, U^k(p; a, b; x) \;=\; k\, U^{k-1} \frac{A_p}{D} \frac{\partial}{\partial x}\, U \\ = \;& k\, U^k(p; a, b; x)\, (-b\,(p + 1)^2\, B_{p-1}(a; x) \;+\; U(p; a, b; x)) \;, \end{aligned} \qquad (72)$$

due to eq. 45.

Proposition 7 ($k-$fold convolution of $\{U_n(p; a, b)\}$; non-degenerate case):

For $D(p; a, b) \neq 0$ the entry $U_n^{(k)}(p; a, b)$ of the $k-$fold convolution of the sequence $\{U_n(p; a, b)\}$ is given by eq. 54.

Proof: This follows immediately from *Proposition 5* after comparing coefficients of x^n using the definition $U^{k+1}(p; a, b; x) =: \sum_{n=0}^{\infty} U_n^{(k)}(p; a, b)\, x^n$. □

Lemma 10 (Degenerate case $D(p; a, b) = 0$):

If $D(p; a, b) := (p + 1)^{p+1}\, b + a\,(a\, p)^p = 0$ then $U(p; a; x) = 1/(1 - a\, x - b(p; a)\, x^{p+1}) = 1/(1 - a\, x\, (p/(p + 1)^p\, (a\, x)^p)$ satisfies the first order linear differential eq. 57.

Proof: One proves $(a + (p+1)\, b\, x^p)\, A_p(a,b;x) \; + \; b\,(p+1)^2\,(1 - a\,x - b\,x^{p+1})\, B_{p-1}(a;x) \; = \; 0$ with eqs. 47 and 46 in the version where the sum has been evaluated (the case $p = 0$ is treated separately). If one factors out $b/(p+1-a\,p\,x)$ one finds that all terms cancel provided one replaces $a\,(a\,p)^p$ by $-b\,(p+1)^{p+1}$. □

Note 5: The solution $1/(1 - a\,x - b\,x^{p+1})$ of this linear differential eq. with input $U(p;a,b;0) = 1$ is unique. The proof is analogous to the one of *proposition 5*.

Note 6: If $U(p;a,b;x) \; = \; 1/(1 - a\,x + (((a\,p\,x/(p+1))^{p+1})/p)$ we do not have a formula for $U^2(p;a,b;x)$, valid for all p, like in the non-degenerate case. Therefore, we cannot derive results for convolutions along the line shown above.

Lemma 11 (Recurrence in the degenerate case):

If $D(p;a,b) \; := \; (p+1)^{p+1}\, b \; + \; a\,(a\,p)^p \; = \; 0$ (and $b \neq 0$) then one can replace the recurrence eq. 1 which has depth $p+1$, by the following one with depth $p \in \mathbb{N}$.

$$U_{n+1}(p;a) \;\; = \;\; \frac{a}{(p+1)\,(n+1)} \sum_{j=1}^{p} \left(\frac{a\,p}{p+1}\right)^{j-1} \;(n+p+2-j)\, U_{n+1-j}(p;a) \;\; , \tag{73}$$

where one uses the inputs $U_j(p;a) \; = \; a^j$ for $j = 0, 1, ..., p-1$.

Proof: This derives from the sum on the *rhs* of eq. 51 which now vanishes. If the coefficients C_j from eq. 52 are used with the replacement of $a\,(a\,p)^p$ by $-b\,(p+1)^{p+1}$ one arrives at the desired recurrence, after the common factor b has been dropped. The inputs are adopted from the original recurrence except that U_p can now be computed to be a^p. □

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