Riccati meets Fibonacci

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1 Introduction

Consider the *Riccati* differential equation

$$f(x)\, G'(x) \;=\; f_0(x)\, G^2(x) \;+\; f_1(x)\, G(x) \;+\; f_2(x)\ . \tag{1}$$

For $f_2(x) \equiv 0$ this reduces to a special *Bernoulli* equation (with exponent 2) which will be treated separately. For the history of such eqs. see [9], ch.I, 1.1. If $f_0(x)$ does not vanish we speak of the non-degenerate case, and

$$G^2(x) \quad = \; \alpha(x)\, G'(x) \;-\; \beta(x)\, G(x) \;-\; \gamma(x)\ , \tag{2}$$

with $\alpha(x) \,=\, f(x)/f_0(x)$, $\beta(x) \,=\, f_1(x)/f_0(x)$, and $\gamma(x) \,=\, f_2(x)/f_0(x)$.

Let $G(x)$ generate the number sequence $\{G_n\}_0^\infty$, *i.e.* $G(x) = \sum_{n=0}^\infty G_n\, x^n$. Because $G^2(x)$ is the generating function for the convolution of the sequence $\{G_n\}_0^\infty$ with itself, *i.e.* of $G_n^{(1)} := \sum_{k=0}^n G_k\, G_{n-k}$, one can use eq. 2 in order to express the convolution numbers $G_n^{(1)}$ in terms of $\{G_k\}_0^{n+1}$ and the numbers $\{\alpha_k\}_0^n$, $\{\beta_k\}_0^n$, and γ_n, which are generated by the functions $\alpha(x)$, $\beta(x)$, and $\gamma(x)$, respectively, as follows.

$$\begin{aligned} G_n^{(1)} &= \sum_{q=0}^n \left((n+1-q)\, G_{n+1-q}\, \alpha_q \;-\; G_{n-q}\beta_q\right) \;-\; \gamma_n \;, \\ &= \sum_{q=0}^n \left((q+1)\, G_{q+1}\, \alpha_{n-q} \;-\; G_q\beta_{n-q}\right) \;-\; \gamma_n \;. \end{aligned} \tag{3}$$

The $k-$th order convolution sequence $\{G_n^{(k)}\}_{n=0}^\infty$ is generated by $G^{k+1}(x)$, and can be obtained recursively if one first writes $G^{k+1}(x) = G^{k-1}(x)\, G^2(x)$ and then employs *Riccati* eq. 2:

$$G^{k+1}(x) = \left(\alpha(x)\, \frac{1}{k}\, \frac{d}{dx} \;-\; \beta(x)\right) G^k(x) \;-\; \gamma(x)\, G^{k-1}(x) \tag{4}$$

for $k \in \mathbb{N}$. This yields in terms of the expansion coefficients, from $G^{k+1}(x) =: \sum_{n=0}^\infty G_n^{(k)}\, x^n$,

$$G_n^{(k)} = \sum_{q=0}^n \left(\frac{1}{k}\,(q+1)\, G_{q+1}^{(k-1)}\, \alpha_{n-q} \;-\; G_q^{(k-1)}\beta_{n-q} \;-\; G_q^{(k-2)}\, \gamma_{n-q}\right) \;. \tag{5}$$

for $k \in \mathbb{N}$, with $G_q^{(-1)} := \delta_{q,0}$ (*Kronecker* symbol) and $G_q^{(0)} = G_q$.

As is well-known, *Riccati* eq. 2 can be transformed into a homogeneous second order differential equation of the type (we use $\alpha(x) \neq 0$)

$$\alpha(x)\, H''(x) \;+\; (\alpha'(x) \;-\; \beta(x))\, H'(x) \;+\; (\gamma(x)/\alpha(x))\, H(x) \;=\; 0 \;. \tag{6}$$

This transformation is accomplished by

$$G(x) \;=\; -\alpha(x)(ln\, H(x))' \quad \text{or} \quad H(x) \;=\; exp\left(-\int \frac{G(x)}{\alpha(x)}\, dx\right) \;, \tag{7}$$

Therefore, if a function $H(x)$ satisfies the differential eq. of type 6 with certain initial conditions for $H(0)$ and $H'(0)$ we can use recursion eq. 5 for the $k-$th convolution of the sequence $\{G_n\}_0^\infty$ generated by $G(x) = -\alpha(x)(ln\, H(x))'$, and $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ generate the coefficients in eq. 5

In the special *Bernoulli* case, when $\gamma(x) \equiv 0$, and $f(x) \neq 0$ and $\alpha(x) \neq 0$, the eq.

$$G'(x) = \frac{f_1(x)}{f(x)} G(x) + \frac{f_0(x)}{f(x)} G^2(x) = (\beta(x)\, G(x) + G^2(x))/\alpha(x) , \tag{8}$$

can be transformed into an inhomogeneous first order linear differential eq. for the inverse of $G(x)$; *i.e.* $H(x) := 1/G(x)$ satisfies

$$\alpha(x)\, H'(x) + \beta(x)\, H(x) = -1 , \tag{9}$$

with the solution

$$H(x) = \frac{1}{G(x)} = e^{F(x)} \left[C - \int \frac{e^{-F(x)}}{\alpha(x)}\, dx \right] , \tag{10}$$

where C is an integration constant, and $F(x) := -\int (\beta(x)/\alpha(x))\, dx$.

Therefore, if $H(x)$ satisfies a differential equation of type 9 with a certain initial condition for $H(0)$ we can use recursion eq. 5, with $\gamma_{n-q} \equiv 0$, for the $k-$th convolution of the sequence $\{G_n\}_0^\infty$ generated by $G(x) = 1/H(x)$. $\alpha(x)$ and $\beta(x)$ generate the remaining coefficients in eq. 5.

From this set-up we do not gain direct information about convolutions of the sequence of numbers generated by the functions $H(x)$ in both cases. This method becomes particularly useful if the coefficient functions $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are simple, for example if they are polynomials.

In this paper we concentrate on examples of *Riccati* equation 8 of the special *Bernoulli* type. It is shown that the generalized *Fibonacci* and corresponding *Lucas* numbers are generated by functions which satisfy such a *Riccati* equation. We discuss the resulting expressions for the $k-$th convolution of these number sequences. At the end we extend this method to the so-called generalized $p-$*Fibonacci* numbers which appeared in a recent paper [6].

2 Summary

The generating function for the generalized *Fibonacci* numbers $\{F_n(a,b)\}_0^\infty$, defined by the three term recurrence relation

$$F_n(a,b) \;=\; a\,F_{n-1}(a,b) \;+\; b\,F_{n-2}(a,b), \quad F_0(a,b) \;=\; 0,\; F_1(a,b) \;=\; 1, \tag{11}$$

with given real $a \neq 0$ and $b \neq 0$, is well-known. For arbitrary a and b, $F_n(a,b)$ can be considered as a polynomial in two variables. If we introduce the numbers, or polynomials, $U_n(a,b) \;:=\; F_{n+1}(a,b)$ we have from the recursion with input $U_0(a,b) = 1$ and $U_1(a,b) = a$ (or $U_{-1}(a,b) = 0$)

$$U(a,b;x) \;:=\; \sum_{n=0}^{\infty} U_n(a,b)\,x^n \;=\; \frac{1}{1\,-\,a\,x\,-\,b\,x^2}\;. \tag{12}$$

Similarly, for the generalized *Lucas* numbers $\{L_n(a,b)\}_0^\infty$ which satisfy the same recursion eq. 11 but with inputs $L_0(a,b) = 2, L_1(a,b) = a$, we find, with $V_n(a,b) \;:=\; L_{n+1}(a,b)/a$, remembering that $a \neq 0$,

$$V(a,b;x) \;:=\; \sum_{n=0}^{\infty} V_n(a,b)\,x^n \;=\; \frac{1 + 2\,b\,x/a}{1\,-\,a\,x\,-\,b\,x^2}\;. \tag{13}$$

The input is now $V_0(a,b) = 1$ and $V_1(a,b) = (a^2\,+\,2\,b)/a$ (or $V_{-1}(a,b) = 2/a$).

These (ordinary) generating functions can also be written in terms of the characteristic roots corresponding to recursion relation eq. 11

$$\lambda_\pm \;\equiv\; \lambda_\pm(a,b) \;:=\; \frac{1}{2}\,(a\,\pm\,\sqrt{a^2 + 4\,b}) \tag{14}$$

as follows.

$$U(a,b;x) \;=\; \frac{1}{x\,(\lambda_+ - \lambda_-)}\left(\frac{1}{1\,-\,\lambda_+\,x} \;-\; \frac{1}{1\,-\,\lambda_-\,x}\right)\,, \tag{15}$$

$$V(a,b;x) \;=\; \frac{1}{\lambda_+ + \lambda_-}\left(\frac{\lambda_+}{1\,-\,\lambda_+\,x} \;+\; \frac{\lambda_-}{1\,-\,\lambda_-\,x}\right)\,. \tag{16}$$

The corresponding *Binet* forms of the generated numbers are, in the non-degenerate case $\lambda_+ \neq \lambda_-$, *i.e.* $D(a,b) := a^2 + 4\,b \neq 0$,

$$U_n(a,b) \;=\; \frac{\lambda_+^{n+1} \;-\; \lambda_-^{n+1}}{\lambda_+ - \lambda_-}\,, \tag{17}$$

$$V_n(a,b) \;=\; \frac{\lambda_+^{n+1} \;+\; \lambda_-^{n+1}}{\lambda_+ + \lambda_-}\,. \tag{18}$$

In the degenerate case we have

$$U_n(a) \;:=\; U_n(a,-\frac{a^2}{4}) \;=\; (n+1)\left(\frac{a}{2}\right)^n\,, \tag{19}$$

$$V_n(a) \;:=\; V_n(a,-\frac{a^2}{4}) \;=\; \left(\frac{a}{2}\right)^n\,. \tag{20}$$

A sum representation of these polynomials is obtained by expanding the generating functions.

$$U_n(a,b) \;=\; \sum_{l=0}^{\lfloor\frac{n}{2}\rfloor} \binom{n-l}{l}\, a^{n-2l}\, b^l, \tag{21}$$

$$V_n(a,b) \;=\; \sum_{l=0}^{\lfloor\frac{n+1}{2}\rfloor} \frac{n+1}{n+1-l} \binom{n+1-l}{l}\, a^{n-2l}\, b^l. \tag{22}$$

This result for $U_n(a,b)$ follows also from a combinatorial interpretation of the recurrence relation, and the one for $V_n(a,b)$ is also due to the *Girard -Waring* formula in its simplest version (for this *cf.* [4], [3], also for original refs.).

The generating functions eq. 12 (or eq. 15) and eq. 13 (or eq. 16) are found to be the unique solutions of *Riccati* eqs. (simultaneously a special type of *Bernoulli* eq.) of the type shown in eq. 8. To be precise we have, identically in a and b,

$$(a \,+\, 2\,b\,x)\,\frac{\partial}{\partial x}U(a,b;x) \;+\; 4\,b\,U(a,b;x) \;-\; (a^2 \,+\, 4\,b)\,U^2(a,b;x) \;=\; 0, \tag{23}$$

with the initial condition $U(a,b;0) = 1$. Similarly,

$$(1 \,+\, 2\,\frac{b}{a}\,x)^2\,\frac{\partial}{\partial x}V(a,b;x) \;+\; 2\,\frac{b}{a}\,(1 \,+\, 2\,\frac{b}{a}\,x)\,V(a,b;x) \;-\; (a \,+\, 4\,\frac{b}{a})\,V^2(a,b;x) \;=\; 0, \tag{24}$$

with the initial condition $V(a,b;0) = 1$

Hence the coefficient functions from eq. 9 are at most first degree polynomials, namely $\alpha(x) \equiv \alpha(a,b;x) = (a + 2\,b\,x)/(a^2 + 4\,b)$ and $\beta(x) \equiv \beta(a,b;x) = -4\,b/(a^2 + 4\,b)$ in the *Fibonacci* case, and $\alpha(x) \equiv \alpha(a,b;x) = (1 + 2\,b\,x/a)^2/(a + 4\,b/a)$ and $\beta(x) \equiv \beta(a,b;x) = -2\,(b/a)\,(1 + 2\,b\,x/a)/(a + 4\,b/a)$ in the *Lucas* case, provided $a \neq 0$ and $a^2 + 4\,b \neq 0$.

The degenerate case $D(a,b) := a^2 + 4\,b = 0$, for which the above given differential eqs. become linear, will be considered separately. This case corresponds to vanishing $f_0(x)$ in *section 1.*

From the general results given in *section 1* the generating functions for the $k-$th convolution of these sequences satisfy

$$U^{k+1}(a,b;x) \;=\; \frac{1}{(a^2 + 4\,b)\,k}\Big((a + 2\,b\,x)\,\frac{\partial}{\partial x} \,+\, 4\,k\,b\Big)\,U^k(a,b;x)\ , \tag{25}$$

and

$$V^{k+1}(a,b;x) \;=\; \frac{a}{(a^2 + 4\,b)\,k}\Big((1 + 2\,\frac{b}{a}\,x)^2\,\frac{\partial}{\partial x} \,+\, 2\,k\,\frac{b}{a}(1 + 2\,\frac{b}{a}\,x)\Big)\,V^k(a,b;x)\ . \tag{26}$$

This implies, from eq. 5, that the $k-$th convolution $U_n^{(k)}$, defined by $U^{k+1}(a,b;x) =: \sum_{n=0}^{\infty} U_n^{(k)}(a,b)\,x^n$ can be expressed in terms of the $k-1$-st one according to

$$U_n^{(k)}(a,b) \;=\; \frac{1}{k\,(a^2 + 4\,b)}\Big(a\,(n+1)\,U_{n+1}^{(k-1)}(a,b) \;+\; 2\,b\,(n + 2\,k)\,U_n^{(k-1)}(a,b)\Big), \tag{27}$$

with input $U_n^{(0)}(a,b) = U_n(a,b)$, and similarly

$$\begin{aligned} V_n^{(k)}(a,b) \;=\; & \frac{1}{k\,a\,(a^2 + 4\,b)}\Big((n+1)\,a^2\,V_{n+1}^{(k-1)}(a,b) \;+\; 2\,a\,b\,(2\,n + k)\,V_n^{(k-1)}(a,b) \\ & + 4\,b^2\,(n + k - 1)\,V_{n-1}^{(k-1)}(a,b)\Big)\ , \end{aligned} \tag{28}$$

with input $V_n^{(0)}(a,b) = V_n(a,b)$. The formula given in eq. 27 has been found earlier in [1] (p. 202, III and p.213, eq. (30)) without using the defining *Riccati* eq. for $U(a,b;x)$. The notations have to be translated with the help of $F_n^{(k)} \hat{=} U_n^{(k-1)}$, $a_1 \hat{=} a$, and $a_2 \hat{=} b$.

For example, the convolution of $\{V_n(a,b)\}_0^\infty$ with itself ($k=1$) becomes, after use of recursion eq. 11

$$V_n^{(1)}(a,b) \;=\; \frac{1}{a\,(a^2\,+\,4\,b)}\Big(\big[a^2\,(n+1)\;+\;4\,b\,n\big]\,V_{n+1}(a,b)\;+\;2\,b\,a\,V_n(a,b)\Big)\;. \tag{29}$$

For $a=b=1$ one recovers well-known formulae for the first convolutions of ordinary *Fibonacci*, *resp.* *Lucas* numbers (*e.g.* [7], p.183, eqs. (98) and (99) (with corrected $L_{n-1}\to L_{n-i}$)). To see this, observe that $U_n^{(1)}(1,1)\;=\;F_{n+2}^{(1)}$ and $V_n^{(1)}(1,1)\;=\;L_{n+2}^{(1)}-4\,L_{n+2}$.

$$F_n^{(1)} \;=\; U_{n-2}^{(1)}(1,1) \;=\; \frac{1}{5}\Big((n-1)\,F_n\;+\;2\,n\,F_{n-1}\Big) \;=\; \frac{1}{5}\,(n\,L_n\;-\;F_n) \tag{30}$$

$$L_n^{(1)} \;=\; V_{n-2}^{(1)}(1,1)\;+\;4\,L_n \;=\; \frac{1}{5}\Big((5\,n-9)\,L_n\;+\;2\,L_{n-1}\Big)\;+\;4\,L_n \;=\; (n+2)\,L_n\;+\;F_n\;. \tag{31}$$

We note, in passing, a sum representation of these convolutions obtained from the expansion of the generating functions which is valid for $k\in\mathbb{N}_0$.

$$U_n^{(k)}(a,b) \;=\; \sum_{l=0}^{\lfloor\frac{n}{2}\rfloor}\binom{k+n-l}{k}\binom{n-l}{l}\,a^{n-2l}\,b^l, \tag{32}$$

$$V_n^{(k)}(a,b) \;=\; \sum_{p=0}^{min(n,k+1)} 2^p\binom{k+1}{p}\sum_{l=p}^{n}\binom{n-l+k}{k}\binom{n-l}{l-p}\,a^{n-2l}\,b^l. \tag{33}$$

Before discussing iteration of recursion relations 27 and 28 we state results for the degenerate case $D(a,b)\;:=\;a^2\;+\;4\,b\;=\;0$. *Riccati* eqs. 23 and 24 collapse to linear differential eqs. for $U(a;x)\;:=\;U(a,-a^2/4;x)$ and $V(a;x)\;:=\;V(a,-a^2/4;x)$

$$(1\;-\;\frac{a}{2}\,x)\,\frac{\partial}{\partial x}U(a;x) \;=\; a\,U(a;x)\quad,\quad U(a;0)\;=\;1, \tag{34}$$

$$(1\;-\;\frac{a}{2}\,x)\,\frac{\partial}{\partial x}V(a;x) \;=\; \frac{a}{2}\,V(a;x)\quad,\quad V(a;0)\;=\;1\;. \tag{35}$$

For the last eq. $x\neq 2/a$ was assumed. Because the solutions to these eqs. imply

$$\frac{\partial^2}{\partial x^2}U(a;x)\;=\;\frac{3}{2}\,a^2\,U^2(a;x)\quad,\quad\frac{\partial}{\partial x}V(a;x)\;=\;\frac{a}{2}\,V^2(a;x)\;, \tag{36}$$

the corresponding first ($k=1$) convolutions of these numbers $U_n(a)\;:=\;U_n(a,-a^2/4)$ and

$V_n(a) := V_n(a, -a^2/4)$ are given by

$$U_n^{(1)}(a) = \frac{2}{3\,a^2}\,(n+2)\,(n+1)\,U_{n+2}(a) \quad , \quad V_n^{(1)}(a) = \frac{2}{a}\,(n+1)\,V_{n+1}(a) \ , \tag{37}$$

with eqs. 21 and 22.

In order to derive the result for the $k-$th convolution we start with identities which follow from the solutions of eqs. 34 and 35, namely

$$U^{k+1}(a;x) = \frac{2}{a^2\,k\,(2\,k+1)}\,\frac{\partial^2}{\partial x^2}\Big(U^k(a;x)\Big) \ , \tag{38}$$

$$V^{k+1}(a;x) = \frac{2}{a\,k}\,\frac{\partial}{\partial x}\Big(V^k(a;x)\Big) \ . \tag{39}$$

These identities imply for the $k-$th convolutions

$$U_n^{(k)}(a) = \frac{2}{a^2\,k\,(2\,k+1)}\,(n+2)\,(n+1)\,U_{n+2}^{(k-1)}(a) \ , \tag{40}$$

$$V_n^{(k)}(a) = \frac{2}{a\,k}\,(n+1)\,V_{n+1}^{(k-1)}(a) \ , \tag{41}$$

with inputs $U_n^{(0)}(a) = U_n(a) = (n+1)\,(a/2)^n$ and $V_n^{(0)}(a) = V_n(a) = (a/2)^n$. See eqs. 19 and 20. The iteration of these eqs. yields the final result, which for $k \in \mathbb{N}_0$, and in the degenerate case $b = -a^2/4$, is

$$U_n^{(k)}(a) = \binom{n+2\,k+1}{2\,k+1}\left(\frac{a}{2}\right)^n \ , \tag{42}$$

$$V_n^{(k)}(a) = \binom{n+k}{k}\left(\frac{a}{2}\right)^n \ . \tag{43}$$

Thus $V_n^{(2l+1)}(a) = U_n^{(l)}(a)$, and it suffices to treat $V_n^{(k)}(a)$. For even a these are non-negative integer sequences. For $n,k \in \mathbb{N}_0$, $V_{n+k}^{(k)}(2\,l)$ constitutes a convolution triangle of numbers based on the $k=0$ column sequence $V_n^{(0)}(2\,l) = l^n$ (powers of l). See [5] for these triangles of numbers.

In the non-degenerate case recursion eq. 27 can be iterated in order to express the $k-$th convolution of $U_n(a,b)$ as linear combination of these numbers according to

$$U_n^{(k)}(a,b) \;=\; \frac{1}{k!\,(a^2+4\,b)^k}\,\Big(AU_{k-1}(a,b;n)\,(n+1)\,a\,U_{n+1}(a,b) \;+\; BU_{k-1}(a,b;n)\,(n+2)\,b\,U_n(a,b)\Big)\;, \tag{44}$$

with certain polynomials $AU_{k-1}(a,b;n)$ and $BU_{k-1}(a,b;n)$ of degree $k-1$ in the variable n, for arbitrary, but fixed, $a \neq 0$, $b \neq 0$, and $b \neq -a^2/4$.

The (mixed) recursion relations for these polynomials are deduced from eq. 27, and for $k = 1, 2, ...$, they are

$$\begin{aligned} AU_k(a,b;n) \;&=\; a^2\,(n+2)\,AU_{k-1}(a,b;n+1) \;+\; 2\,b\,(n+2(k+1))\,AU_{k-1}(a,b;n) \;+ \\ &\quad\; b\,(n+3)\,BU_{k-1}(a,b;n+1)\;, \end{aligned} \tag{45}$$

$$BU_k(a,b;n) \;=\; a^2\,(n+1)\,AU_{k-1}(a,b;n+1) \;+\; 2\,b\,(n+2(k+1))\,BU_{k-1}(a,b;n)\;, \tag{46}$$

with inputs $AU_0(a,b;n) = 1$ and $BU_0(a,b;n) = 2$.

In eqs. (26), *resp.* (27), of [1] one can find explicit results for $U_n^{(k)}(a,b)$ for the instances $k = 2$, *resp.* $k = 3$ (in eq. (26) of this ref. one has to multiply the *lhs* with 2!, and in the second line of N of eq.(27) it should read $B(2,n+1)$).

For the case $a = 1 = b$ the triangles of the coefficients of these polynomials can be viewed under the nrs. A057995 and A057280 in [5]. For $a = 2$, $b = 1$ see A058402 and A058403, and for $a = 1$, $b = 2$ A073401 and A073402.

Similarly, iteration of recursion eq. 28 results, with the help of recursion eq. 11, in

$$V_n^{(k)}(a,b) \;=\; \frac{1}{k!\,a\,(a^2+4\,b)^k}\,\Big(AV_k(a,b;n)\,V_{n+1}(a,b) \;+\; BV_k(a,b;n)\,V_n(a,b)\Big)\;, \tag{47}$$

with certain polynomials $AV_k(a,b;n)$ and $BV_k(a,b;n)$ of generic degree k in the variable n, for fixed $a \neq 0$, $b \neq 0$, with $b \neq -a^2/4$.

The (mixed) recursion relations for these polynomials are found from eq. 28, and for $k = 1, 2, ...,$ they are

$$\begin{aligned} AV_k(a,b;n) &= a^2\,(n+1)\,AV_{k-1}(a,b;n+1) \;+\; 2\,b\,(2\,n+k)\,AV_{k-1}(a,b;n) \;+ \\ &\quad a\,(n+1)\,BV_{k-1}(a,b;n+1) \;+\; 4\,\frac{b}{a}\,(n+k-1)\,BV_{k-1}(a,b;n-1)\;, \qquad (48) \\ BV_k(a,b;n) &= 2\,b\,(2\,n+k)\,BV_{k-1}(a,b;n) \;-\; 4\,b\,(n+k-1)\,BV_{k-1}(a,b;n-1) \;+ \\ &\quad a\,b\,(n+1)\,AV_{k-1}(a,b;n+1) \;+\; 4\,\frac{b^2}{a}\,(n+k-1)\,AV_{k-1}(a,b;n-1)\;, \qquad (49) \end{aligned}$$

with inputs $AV_0(a,b;n) = 0$ and $BV_0(a,b;n) = a$.

For $a = 1 = b$ the triangles of coefficients of these polynomials in n can be found under the nrs. A061188 and A061189 in [5]. Observe that $BV_1(1,1;n)$ is accidentally of degree 0. For $a = 2, b = 1$ see nrs. A062133 and A062134.

Motivated by a recent paper [6] we consider also the following generalized $p-$*Fibonacci* numbers $U_n(p;a,b)$ defined for $p \in \mathbb{N}_0$ by the generating function

$$U(p;a,b;x) \;:=\; \frac{1}{1\;-\;a\,x\;-\;b\,x^{p+1}} \;=\; \sum_{n=0}^{\infty} U_n(p;a,b)\,x^n\;. \qquad (50)$$

Of course, we assume $b \neq 0$ and also take $a \neq 0$. For $p = 1$ these numbers reduce to the $U_n(a,b)$ treated above, and for $p = 0$ they become the powers $(a+b)^n$. $U(p;1,1;x)$ appears in eq. 71 of [2]. The recursion relations are

$$U_n(p;a,b) \;=\; a\,U_{n-1}(p;a,b) \;+\; b\,U_{n-(p+1)}(p;a,b)\;, \qquad (51)$$

with inputs $U_j(p;a,b) \;=\; a^j$ for $j = 0, 1, ..., p$. In order to derive expressions for the $k-$th convolution of these $p-$*Fibonacci* numbers consider first the following *Riccati* eq. of type 8 satisfied by $U(p;a,b;x)$ written for the non-degenerate case $D(p;a,b) \;:=\; (p+1)^{p+1}\,b \;+\; a\,(a\,p)^p \;\neq\; 0$ if $p \in \mathbb{N}$, and $a + b \neq 0$ if $p = 0$ (*i.e.* one puts $(a\,p)^p \;=\; 1$ if $p = 0$).

$$U^2(p;a,b;x) \;=\; \frac{1}{(p+1)^{p+1}\,b \;+\; a\,(a\,p)^p}\left\{A_p(a,b;x)\,\frac{\partial}{\partial x} \;+\; b\,(p+1)^2\,B_{p-1}(a;x)\right\}U(p;a,b;x)\;, \qquad (52)$$

with

$$A_p(a,b;x) \;=\; (a\,p)^p \;+\; b\,(p+1)\,x\,B_{p-1}(a;x)\;, \tag{53}$$

$$B_{p-1}(a;x) \;=\; (p+1)^{p-1}\sum_{i=0}^{p-1}\left(\frac{a\,p}{p+1}\right)^i x^i \;=\; \frac{(p+1)^p \;-\; (a\,p\,x)^p}{p+1\;-\;a\,p\,x}. \tag{54}$$

Hence, the coefficient functions $\alpha(x)$, *resp.* $\beta(x)$ from the general set-up in *section 1* are polynomials of degree p *resp.* $p-1$, namely $\alpha(x) \equiv \alpha_p(a,b;x) = A_p(a,b;x)/D(p;a,b)$ *resp.* $\beta(x) \equiv \beta_p(a,b;x) = -b\,(p+1)^2\,B_{p-1}(a;x)/D(p;a,b)$, and $\gamma(x) \equiv 0$. For $p=0$ one has to use $A_0(a,b;x) = 1$ and $B_{-1}(a;x) = 0$. For given non-vanishing a and b these polynomials $A_p(a,b;x)$, *resp.* $B_{p-1}(a;x)$, in the variable x of degree p, *resp.* $p-1$, have therefore the following explicit form.

$$A_p(a,b;x) \;=\; \sum_{m=0}^{p} A(a,b;p,m)\;x^m \quad , \quad B_{p-1}(a;x) \;=\; \sum_{m=0}^{p-1} B(a;p-1,m)\;x^m\;, \tag{55}$$

with the coefficients

$$A(a,b;p,m) \;=\; \begin{cases} 0 & \text{if } m \;>\; p, \\ 1 & \text{if } m=0 \text{ and } p=0, \\ (a\,p)^p & \text{if } m=0 \text{ and } p\geq 1, \\ b\,(p+1)^p\left(\frac{a\,p}{p+1}\right)^{m-1} & \text{if } m\geq 1\;. \end{cases} \tag{56}$$

$$B(a;p,m) \;=\; \begin{cases} 0 & \text{if } p\;<\;m \text{ or } p=-1\;, \\ (p+2)^p\left(\frac{a\,(p+1)}{p+2}\right)^m & \text{if } p\geq m\geq 0\;. \end{cases} \tag{57}$$

For $a = 1 = b$ these triangles of coefficients can be viewed under the numbers $A055858$ and $A055864$ in [5] where further details may be found.

Even though we cannot compute the integral in the solution eq. 10 of the linear differential eq. 9, which is equivalent to *Riccati* eq. 52 for $p \neq 0, 1$, $H(x) = 1 - a\,x - b\,x^{p+1}$ is the unique solution due to the existence and uniqueness theorem for the linear first order differential eq. 9 with initial value $H(p;a,b;0) = 1$.

The result for the first ($k = 1$) convolution of the numbers $U_n(p;a,b)$ which flows from *Riccati* eq. 52 is

$$U_n^{(1)}(p;a,b) \;=\; \frac{1}{b\,(p+1)^{p+1} \;+\; a\,(a\,p)^p}\sum_{j=0}^{p} C_j(n;p;a,b)\,U_{n+1-j}(p;a,b)\;, \tag{58}$$

with

$$C_j(n;p;a,b) \;=\; \begin{cases} n+1 & \text{if } p=0=j,\\ (n+1)\,(a\,p)^p & \text{if } p\geq 1 \text{ and } j=0\,,\\ b\,(p+1)^p\,(n+p+2-j)\left(\frac{a\,p}{p+1}\right)^{j-1} & \text{if } p\geq 1 \text{ and } j=1,...,p. \end{cases} \tag{59}$$

The $U_n(p;a,b)$ recursion cannot be used to simplify the sum in eq. 58.

This result can now be compared, after putting $a = 1 = b$, with a different formula for the same convolution found in [6], eq.(14). For given $p \in \mathbb{N}_0$ and $k = 2, 3, ...$, the recursion for $F_p^{(2)}(k) \,\hat{=}\, U_{k-2}^{(1)}(p;1,1)$ in [6] involves all $k-1$ terms $F_p(n) \,\hat{=}\, U_{n-1}(p;1,1)$, for $n = 1, ..., k-1$, whereas our result needs only $p+1$ terms for all k. For example, $F_3^{(2)}(7) \,\hat{=}\, U_5^{(1)}(3;1,1)$ is reduced to six terms involving $F_3(1) \,\hat{=}\, U_0(3;1,1)$,..., $F_3(6) \,\hat{=}\, U_5(3;1,1)$ in [6], but only to four terms, involving $U_8(3;1,1) \,\hat{=}\, F_3(9)$, $U_7(3;1,1) \,\hat{=}\, F_3(8)$,..., $U_5(3;1,1) \,\hat{=}\, F_3(6)$ in eq. 58.

For the $k-$th convolution we use eq. 4 with $\gamma(x) \equiv 0$ and the above given functions $\alpha_p(a,b;x)$ and $\beta_p(a,b;x)$. For the non-degenerate case, and for $k \in \mathbb{N}$, we have

$$U^{k+1}(p;a,b;x) \;=\; \frac{1}{k\left(b\,(p+1)^{p+1} \;+\; a\,(a\,p)^p\right)}\left\{A_p(a,b;x)\,\frac{\partial}{\partial x} \;+\; k\,b\,(p+1)^2\,B_{p-1}(a;x)\right\}U^k(p;a,b;x)\;. \tag{60}$$

For $p \in \mathbb{N}_0$ the corresponding recursion relation for the $k-$th convolution is (remember that we put $(a\,p)^p \;=\; 1$ if $p = 0$)

$$U_n^{(k)}(p;a,b) \;=\; \frac{1}{k\left(b\,(p+1)^{p+1} \;+\; a\,(a\,p)^p\right)}\sum_{j=0}^{p} C_j^{(k)}(n;p;a,b)\,U_{n+1-j}^{(k-1)}(p;a,b)\;. \tag{61}$$

with

$$C_j^{(k)}(n;p;a,b) \;=\; \begin{cases} n+1 & \text{if } p = 0 = j, \\ (n+1)\,(a\,p)^p & \text{if } p \geq 1 \text{ and } j = 0\,, \\ b\,(p+1)^p\,(n+1+k\,(p+1)-j)\left(\frac{a\,p}{p+1}\right)^{j-1} & \text{if } p \geq 1 \text{ and } j = 1,...,p. \end{cases} \qquad (62)$$

Instead of showing the rather unwieldy formula for the iteration of this recursion relation, after employing the fundamental recursion eq. 51, we prefer to state the result for the instance $p = 2, k = 2, a = 1 = b$, with the notation $U_n^{(1)}(2;1,1) \equiv U_n^{(1)}(2)$ and $U_n(2;1,1) \equiv U_n(2)$:

$$\begin{aligned} U_n^{(2)}(2) &= \frac{1}{2\cdot 31}\Big(4\,(n+1)\,U_{n+1}^{(1)}(2) \;+\; 9\,(n+6)\,U_n^{(1)}(2) \;+\; 6\,(n+5)\,U_{n-1}^{(1)}(2)\Big) \\ &= \frac{1}{2\cdot 31^2}\Big(\,(217\,n^2 \;+\; 1425\,n \;+\; 1922)\,U_n(2) \;+\; 2\,(n+2)\,(62\,n \;+\; 305)\,U_{n-1}(2) \;+ \\ &\qquad 4\,(n+1)\,(31\,n+143)\,U_{n-2}(2)\Big)\,. \end{aligned} \qquad (63)$$

Recursion relation eq. 51 has been used twice.

In the degenerate case $D(p;a,b) \;:=\; (p+1)^{p+1}\,b \;+\; a\,(a\,p)^p \;=\; 0$ (where we put $(a\,p)^p \;\equiv\; 1$ if $p = 0$) we find for $U(p;a,b = b(p;a);x) \;=:\; U(p;a;x)$, where $b(p;a) \;:=\; -p^p\,(a/(p+1))^{p+1}$, the linear differential eq.

$$\Big\{A_p(a,b(p;a);x)\,\frac{\partial}{\partial x} \;+\; b(p;a)\,(p+1)^2\,B_{p-1}(a;x)\Big\}\,U(p;a;x) \;=\; 0 \qquad (64)$$

with B_{p-1} and A_p taken in their explicit form known from eqs. 55 with 57 and 56. For general p and $D(p;a,b) \;=\; 0$ we cannot say anything about convolutions because we have no suitable expression for $U^2(p;a,b;x)$. Recurrence eq. 51 with depth $p+1$ can be replaced by one with only depth p. See eq. 65.

In the non-degenerate case we could also consider the other p linear independent (*Lucas*-type) sequences defined by recurrence eq. 51 with appropriate inputs, but we will not do this here.

The remainder of this paper provides proofs for the above given statements.

3 Riccati equations for Fibonacci and Lucas generating functions

Proposition 1: $U(a,b;x)$ defined in eq. 12 is for $a^2 + 4\,b \neq 0$ equivalent to *Riccati* eq. 23 with initial condition $U(a,b;0) = 1$.

Proof: a) $H(a,b;x) = 1/U(a,b;x) = 1 - a\,x - b\,x^2$ satisfies eq. 9 with $\alpha(x) \equiv \alpha(a,b;x) = (a + 2\,b\,x)/(a^2 + 4\,b)$ and $\beta(x) \equiv \beta(a,b;x) = -4\,b/(a^2 + 4\,b)$. Therefore, $U(a,b;x)$ obeys eq. 8 which coincides with eq. 23.

b) With $\alpha(a,b;x)$ and $\beta(a,b;x)$ from eq. 23, as given in part a) we can compute the integral in eq. 10 and determine the constant C from the initial condition. This produces $1/U(a,b;x)$. $\square$

Lemma 1: In the degenerate case $U(a;x) := U(a,-a^2/4;x)$ yields the first order linear differential eq. 34 as well as the second order non-linear differential eq. given as the first of eqs. 36.

Proof: Elementary. $\square$

Note 1: The degenerate case is equivalent to $\lambda_+(a,b) = \lambda_-(a,b)$ with the definition of the characteristic roots of the recursion relation eq. 11 given in eq. 14. We may assume that not both, a and b, vanish and $x \neq 1/\lambda_\pm(a,b) = -\lambda_\mp/b$. In each case $U(a,b;0) = 1$.

Proposition 2: $V(a,b;x)$, defined in eq. 13 for $a \neq 0$, is for $a^2 + 4\,b \neq 0$ equivalent to *Riccati* eq. 24 with initial condition $V(a,b;0) = 1$.

Proof: Analogous to the proof of *Proposition 1.* $\square$

Lemma 2: In the degenerate case $V(a;x) := V(a,-a^2/4;x)$ satisfies the first order linear differential eq. 35 as well as the first order non-linear differential eq. given as the second of eqs. 36.

In each case $V(a,b,0) = 1$.

Proof: Elementary. $\square$

4 Convolutions of generalized Fibonacci and Lucas sequences

Because the $k+1$st power of the (ordinary) generating functions of a sequence generates $k-$fold convolutions of this sequence we obtain in the non-degenerate case, $a^2 + 4\,b \neq 0$, according to the general set-up of *section 1*, for the generalized *Fibonacci resp. Lucas* case, expression eq. 27, *resp.* eq. 28. For the definition of the $k-$th convolutions $U_n^{(k)}(a,b)$ (similarly of $V_n^{(k)}(a,b)$) see the line after eq. 26. The first convolutions ($k=1$) can be determined in each case from linear combinations of the two independent original sequences. See eq. 29 for the *Lucas* case. For $a = 1 = b$ these formulae are well-known (see *section 2* after eq. 29).

Lemma 3 (Recurrence for $k-$fold convolution, degenerate case):

For $b = -\frac{a^2}{4} \neq 0$ the recurrence formulae for the $k-$fold convolution of the generalzed *Fibonacci*,*resp. Lucas*, sequences are those stated in eqs. 40, *resp.* 41.

Proof: This statement is equivalent to eq. 25, *resp.* eq. 26 for the powers of the corresponding generating functions. They are deduced from the the second, *resp.* first, order differential eq., given in eqs. 36, which coincides with the $k=1$ assertion. To verify the general k case, eq. 38 *resp.* eq. 39, one may use $U(a;x) = U(a,-\frac{a^2}{4};x) = 1/(1-a\,x/2)^2$ from eq. 12, *resp.* $V(a;x) = V(a,-\frac{a^2}{4};x) = 1/(1-a\,x/2)$ from eq. 13. □

Lemma 4: The explicit form for the $k-$fold convolution in the degenerate case is given by eq. 42, *resp.* 43, for the generalized *Fibonacci*, *resp. Lucas*, case.

Proof: Iteration of recurrence eq. 40, *resp.* eq. 41, with input $U_n^{(0)}(a) = U_n(a) = (a/2)^n$, *resp.* $V_n^{(0)}(a) = V_n(a) = (a/2)^n$, which originates from the generating functions $U(a,-\frac{a^2}{4};x)$, *resp.* $V(a,-\frac{a^2}{4};x)$. □

Proposition 3 (Iteration of recurrence for $k-$fold convolutions; non-degenerate *Fibonacci* case):

For $a^2 + 4\,b \neq 0$ the $k-$fold convolution of the generalized *Fibonacci* sequence $\{U_n(a,b)\}$ is expressed as linear combinations of the two independent solutions of recurrence eq. 11 as given in eq. 44. The coefficient polynomials $AU_k(a, b; n)$ and $BU_k(a, b; n)$ satisfy the mixed recurrence relations eqs. 45 and 46.

Proof: If one considers eq. 44 as *ansatz* and puts it into recurrence eq. 27 we find, after elimination of $U_{n+2}(a, b)$ *via* its recursion relation and a comparison of the coefficients of the linear independent $U_n(a, b)$ and $U_{n-1}(a, b)$ sequences, the mixed recurrence relations for $AU_k(a, b; n)$ and $BU_k(a, b; n)$. The inputs $AU_0(a, b; n) = 1$ and $BU_0(a, b; n) = 2$ are necessary in order that for $k = 1$ eq. 44 coincides with eq. 27. With these inputs and the mixed recurrence one proves, by induction over k, that $AU_k(a, b; n)$ and $BU_k(a, b; n)$ are polynomials in n of degree k, provided a and b are fixed with $b \neq -a^2/4$, $b \neq 0$ and $a \neq 0$. □

Note 2: For fixed integers a and $b \neq -a^2/4$ the coefficients of the polynomials $AU_n(a, b; x)$ and $BU_n(a, b; x)$ furnish two lower triangular (infinite) integer matrices. For the ordinary *Fibonacci* case $a = 1 = b$ these positive integer triangles can be found in [5] under the nrs. A057995 and A057280. For the *Pell* case $a = 2, b = 1$ see nrs. A058402 and A058403, and for the case $a = 1, b = 2$ see nrs. A073401 and A073402.

Proposition 4 (Iteration of recurrence for $k-$fold convolutions; non-degenerate *Lucas* case):

For $a^2 + 4\,b \neq 0$ the $k-$fold convolution of the generalized *Lucas* sequence $\{V_n(a, b)\}$ is expressed as linear combination of the two independent solutions of recurrence eq. 11 as given in eq. 47. The coefficient polynomials $AV_k(a, b; n)$ and $BV_k(a, b; n)$ satisfy the mixed recurrence relations eq. 48 and eq. 49.

Proof: Analogous to the proof of *Proposition 3.* □

Note 3: For fixed integers a and $b \neq -a^2/4$ the coefficients of the polynomials $AV_n(a, b; x)$ and $BV_n(a, b; x)$ furnish two lower triangular (infinite) integer matrices. For the ordinary *Lucas* case $a = 1 = b$ these positive integer triangles can be found in [5] under the nrs. A061188 and A061189. For the *Pell* case $a = 2, b = 1$ see nrs. A062133 and A062134.

5 Convolutions of generalized $p-$Fibonacci sequences

Generalized $p-$*Fibonacci* numbers $U_n(p; a, b)$ are defined by eq. 51 for $p \in \mathbb{N}_0$, $b \neq 0$ and $a \neq 0$, together with the inputs $U_j(p; a, b) := a^j$ for $j = 0, ..., p$. For $p = 1$ we recover the generalized *Fibonacci* numbers $U_n(a, b)$ treated above.

Lemma 5: The generating function $U(p; a, b; x)$ for the generalized $p-$*Fibonacci* numbers is given by eq. 50.

Proof: From the recurrence with inputs given in eq. 51. □

Lemma 6 (*Riccati* eq. for the generalized $p-$Fibonacci case):

If $D(p; a, b) := (p+1)^{p+1}\, b + a\,(a\,p)^p \neq 0$ (non-degenerate case) then $U(p; a, b; x)$ satisfies *Riccati* eq. 52 with the polynomials $A_p(a, b; x)$ and $B_{p-1}(a, b; x)$ defined in eqs. 53 and 54.

Proof: $H(p; a, b; x) = 1/U(p; a, b; x) = 1 - a\,x - b\,x^{p+1}$ satisfies eq. 9 with $\alpha(x) \equiv \alpha_p(a, b; x) = A_p(a, b; x)/D(p; a, b)$ and $\beta(x) \equiv \beta_p(a, b; x) = -b\,(p+1)^2\, B_{p-1}(a; x)/D(p; a, b)$ with $A_p(a, b; x)$ and $B_{p-1}(a; x)$ given by eq. 53 and 54. This is shown by comparing coefficients of powers x^i for $i = 0, 1, ..., 2\,p$. According to *section 1 Riccati* eq. 8 ensues which becomes eq. 52. □

Note 4: i) If $p = 0$, $U(0; a, b; x) = 1/(1 - (a + b)\,x)$ generates powers of $a + b$, and one has to put $A_0(a, b; x) \equiv 1$ and $B_{-1}(a; x) \equiv 0$. This means that one puts $(a\,p)^p = 1$ for $p = 0$.

ii) For given non-vanishing a and b $A_p(a, b; x)$ is a polynomial in x of degree p, and $B_{p-1}(a; x)$ is one of degree $p - 1$. The sum in $B_{p-1}(a; x)$ can be evaluated to yield the second of eqs. 54 provided $p \neq 0$.

Lemma 7 (Coefficient triangles of numbers for polynomials $A_p(a, b; x)$ and $B_p(a; x)$):

The coefficients of the polynomials defined in eqs. 55 are given by eqs. 56 and 57.

Proof: $B_p(a;x)$ from eq. 54 leads immediately to eq. 57, remembering that $B_{-1}(a;x) \equiv 0$. Then eq. 56 follows from eq. 53 and $A_0(a,b;x) \equiv 1$. $\square$

Proposition 5 (Uniqueness of *Riccati* solution; non-degenerate case):

If $D(a,b) \neq 0$ then $y \equiv U(p;a,b;x) = 1/(1 - a\,x - b\,x^{p+1})$ is the unique solution of *Riccati* eq. 52 with eqs. 53, 54 and initial value $U(p;a,b;0) = 1$.

Proof: From *section 1* we know that the *Riccati* eq. is equivalent to the inhomogeneous linear differential eq. for the inverse $H = 1/U$: $H' \equiv F(x,H) = (-\beta(x)/\alpha(x))\,H - 1/\alpha(x)$. Because $F(x,H)$ is continuous in the strip $0 \leq x \leq A < \infty$, $|H| < \infty$ and is there $(K = K(p;a,b;A))$-*Lipschitz*, the existence and uniqueness theorem for linear differential eqs. proves the assertion (see *e.g.*[8], § 6,I, p.62ff). In order to find K we use the summed expression for B_{p-1} from eq. 54 and apply the triangle inequality repeatedly. $\square$

Proposition 6 (Recursion for $k-$th convolution of $\{U_n(p;a,b)\}$; non-degenerate case):

The $k-$th convolution of the sequence $\{U_n(p;a,b)\}$ is given in the non-degenerate case $D(p;a,b) := b\,(p+1)^{p+1} + a\,(a\,p)^p \neq 0$ recursively by eq. 61 with eq. 62.

Proof: This follows from the general set-up of *section 1*, eq. 5 with $\gamma_{n-q} \equiv 0$ and the appropriate coefficient functions $\alpha(x) = \alpha_p(a,b;x)$ and $\beta(x) = \beta_p(a,b;x)$ given after eq. 54. See the corresponding eq. 60 for the $k+1$-st power of the generating function. $\square$

Lemma 8 (Degenerate case $D(p;a,b) = 0$):

If $D(p;a,b) := (p+1)^{p+1}\,b + a\,(a\,p)^p = 0$ then $U(p;a;x) = 1/(1 - a\,x - b(p;a)\,x^{p+1}) = 1/(1 - a\,x\,(p/(p+1)^p\,(a\,x)^p)$ satisfies the first order linear differential eq. 64.

Proof: We prove $(a + (p+1)\,b\,x^p)\,A_p(a,b;x) + b\,(p+1)^2\,(1 - a\,x - b\,x^{p+1})\,B_{p-1}(a;x) = 0$ with eqs. 54 and 53 in the version where the sum has been evaluated (the case $p = 0$ is treated separately). If we factor out $b/(p+1-a\,p\,x)$ we see that all terms cancel provided we replace $a\,(a\,p)^p$ by $-b\,(p+1)^{p+1}$.$\square$

Note 5: The solution $1/(1 - a\,x - b\,x^{p+1})$ of this linear differential eq. 64 with input $U(p;a,b;0) \;=\; 1$ is unique. The proof is analogous to the one of *Proposition 5*.

Note 6: If $U(p;a,b;x) \;=\; 1/(1 - a\,x + (((a\,p\,x/(p{+}1))^{p+1})/p)$ we do not have a formula for $U^2(p;a,b;x)$, valid for all p, like in the non-degenerate case. Therefore, we cannot derive results for convolutions along the line shown above.

Lemma 9 (Recurrence in the degenerate case):

If $D(p;a,b) \;:=\; (p+1)^{p+1}\,b \,+\, a\,(a\,p)^p \;=\; 0$ (and $b \neq 0$) then one can replace recurrence eq. 51 which has depth $p+1$, by the following one with depth $p \in \mathbb{N}$.

$$U_{n+1}(p;a) \;=\; \frac{a}{(p+1)\,(n+1)} \sum_{j=1}^{p} \left(\frac{a\,p}{p+1}\right)^{j-1} (n+p+2-j)\,U_{n+1-j}(p;a)\;, \tag{65}$$

where one uses the inputs $U_j(p;a) \;=\; a^j$ for $j = 0, 1, ..., p-1$.

Proof: This derives from the sum on the *rhs* of eq. 58 which now vanishes. If the coefficients C_j from eq. 59 are used with the replacement of $a\,(a\,p)^p$ by $-b\,(p+1)^{p+1}$ one arrives at the desired recurrence, after the common factor b has been dropped. The inputs are adopted from the original recurrence except that U_p can now be computed to be a^p. □

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