Problems and Remarks:

Each session of the Symposium was concluded by a period devoted to remarks and open problems. These are given in this section, in the chronological order in which they were presented.

1. Remark. For every $n \in \mathbb{N}$ let k_n be an integer with $0 \leq k_n \leq n$. For an arbitrary real number $\lambda_1 \in [0, 1[$ define $\lambda_n := \frac{1}{n!}\left(\lambda_1 + \sum_{\nu=1}^{n-1} \nu! k_\nu\right)$ for all $n \in \mathbb{N}$. It is well known that then

$$f\left(\frac{m}{n!}\right) = e^{2\pi m \lambda_n i} \qquad (m \in \mathbb{Z}, n \in \mathbb{N})$$

defines a homomorphism f from $(Q, +)$ into the torus group $(T, \cdot)$ and that conversely every $f \in \mathrm{Hom}(Q, T)$ is obtained in this way.

Theorem. *The function f is continuous ifand only if*

i) $k_n = 0$ *for almost all* $n \in \mathbb{N}$, *or*

ii) $k_n = n$ *for almost all* $n \in \mathbb{N}$.

If it is continuous f has the form $f(x) = e^{2\pi c x i}$ $(x \in \mathbb{Q})$, where $c \leq 0$ in case i) and $c < 0$ in case ii).

References

[1] Hewitt, E. and Ross, K. A., *Abstract Harmonic Analysis I*, Springer, Berlin–Göttingen–Heidelberg, 1963, pp. 367–368 & 404–405.

[2] Maak, W., *Fastperiodische Funktionen*, Springer, Berlin–Göttingen–Heidelberg, 1950, pp. 89–90.

[3] Vietoris, L., *Zur Kennzeichnung des Sinus und verwandter Funkcionen durch Funktionalgleichungen*, J. Reine Angew. Math. **186** (1944), p. 4.

J. Rätz

2. Remark and problem. Using a recently developed method for solving certain types of inhomogeneous difference equations, we needed the following system of functional equations for $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

$$d(x + y, y) = d(x, y); \qquad d(x, y) = d(y, x). \tag{1}$$

L. Paganoni has proved that (1) has solutions different from identically constant functions, which we describe below.

Let H be a Hamel basis for the reals over the rationals $\mathbb{Q}$ and let H_0 be an arbitrary subset of H. Further, Iet $S_0 = V(H_0, \mathbb{Q}, +, \cdot)$ be the subspace of reals generated by H_0. We define the function $h : \mathbb{R} \to \mathbb{R}$ by:

$$h(x) = \begin{cases} 1 & \text{if } x \in S_0 \\ 0 & \text{if } x \notin S_0. \end{cases}$$

Then the function

$$d(x, y) = 1 - h(x)h(y)$$

fulfils conditions (1) and is obviously not constant.

Quite different ís the situation if we suppose continuity of d. Under this assumption all solutions of (1) are identically constant functions. This can be proved in a quite elementary way.

Problem. *Is it true that under the supposition of measurability the general solution of* (1) *is given by a.e. constant functions?*

I. Fenyő

3. Remark. In [1], Lorentz transformations in $\mathbb{R}^n$ (where $n \geq 3$) were characterized in a way for which there is no analogue in $\mathbb{R}^2$. For the indefinite metric

$$d((x_1, y_1), (x_2, y_2)) := (x_1 - x_2)^2 - (y_1 - y_2)^2$$

on $\mathbb{R}^2$, the bijective mappings $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(0,0) = (0,0)$ satisfying

$$d((x_1, y_1), (x_2, y_2)) = 0 \qquad \text{iff} \qquad d(T(x_1, y_1), T(x_2, y_2)) = 0$$

are precisely those for which there exist $\delta \in \{-1, +1\}$ and $\phi, \psi : \mathbb{R} \to \mathbb{R}$ bijective such that $\phi(0) = 0 = \psi(0)$ and

$$T(x, y) = \left(\phi\left(\frac{x+y}{2}\right) + \psi\left(\frac{x-y}{2}\right), \delta\phi\left(\frac{x+y}{2}\right) - \delta\psi\left(\frac{x-y}{2}\right) \right)$$

for all $x, y \in \mathbb{R}$ ([2]). For these mappings, the condition

$$T(x_1, y_1) - T(x_2, y_2) = T(x_1, y_2) - T(x_2, y_2) \tag{E}$$

(where $x_1, x_2, y_1, y_2 \in \mathbb{R}$) is necessary and sufficient for T to be additive, while the condition that there exists $\sigma \in \{-1, +1\}$ such that whenever $x_1, x_2, y_1, y_2 \in \mathbb{R}$

$$d((x_1, y_1), (x_2, y_2)) > 0 \qquad \text{implies} \qquad \sigma d(T(x_1, y_1), T(x_2, y_2)) > 0 \tag{M}$$

is necessary and sufficient for T to be continuous.

References

[1] Borchers, H. J. and Hegerfeldt, G. C., *The structure of space-time transformations*, Comm. Math. Phys. **29** (1972), 259–266.

[2] A part of this result is due to R. Stettler (oral cammunication).

J. Rätz

4. Remark and Problem. Linearizing coordinate transformations for graph papers.

Semi-log and log-log graph papers provide a means of plotting exponential and monomial functions, respectively, as straight lines. This fact yields a convenient method for determining if empirical data are associated with one of these two types of functions.

The author [1] has developed analogous kinds of graph papers for functions satisfying the logistics equation:

$$\dot{x} = x(a - bx)$$

and the Gompertz equation:

$$\dot{x} = x(a - b \ln x).$$

Appropriately normalized solutions of these equations plot as straight lines on the graph papers. (Normalization is necessary since the general solutions of these equations involve four arbitrary parameters, while straight line are determined by two.)

The form of all four kinds of graph paper was determined from the explicit form of the functions in question, rather than from the form of the corresponding functional or differential equation. (In the case of semi-log and log-log papers, of course, the "corresponding equations" are the appropriate multiplicative forms of Cauchy's equation.) This leads to the following open problem: how can the suitable coordinate spacing for the axes of the linearizing graph paper be obtained directly from the functional or differential equation without finding the explicit form of its solution?

To solve this problem we may require information about f^{-1} (whose functional equation is often obtainable from the functional equation for f, assuming f is invertible), and we may also require some means of numerically approximating the solution of the functional equation directly from the equation (see [2]).

References

[1] Snow, D. R., *Logistics and Gompertz graph papers*, Amer. Math. Soc. Abstracts **1** (1980), 468.

[2] Snow, D. R., *Remark: On numerical approximation methods for functional equations*, Aequationes Math. **15** (1977), 293–294.

D. R. Snow

5. Remark. This is a result by C. Wagner (lnstitute of Advanced Studies in the Behavioural Sciences, Stanford, CA. and the University of Tennessee, Knoxville), C. T. Ng, Pl. Kannappan, and myself. Let $f : [0,s]^n \to \mathbb{R}_+ \ (= \{x : x \geq 0\})$ be such that $f(0,0,\ldots,0) = 0$ and

$$\sum_{i=1}^{m} x_{ij} = s \qquad (j = 1,2,\ldots,n) \qquad \text{implies} \qquad \sum_{i=1}^{m} f(x_{i1}, x_{i2}, \ldots, x_{in}) = s$$

(where $m > 2, n, s$ fixed). Then there exist $w_j \geq 0\,(j = 1,2,\ldots,n)$, $\sum_{j=1}^{n} w_j = 1$ such that

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} w_j x_j \qquad \text{for all} \quad (x_1, x_2, \ldots, x_n) \in [0,s]^n.$$

One of the possible interpretations is the following. A (say, grant) amount s should be allocated to m applicants. The decision maker (committee chairman) asks n advisors (committee members). The j-th advisor recommends that the i-th applicant obtain the amount x_{ij}. The decision maker allocates $f(x_{i1}, x_{i2}, \ldots, x_{in})$ to the i-th applicant. The only conditions are that each advisor and also the decision maker allocate non-negative amounts to each applicant and the entire amount s is allocated by them to all appticants taken together, and the decision maker has to respect unanimous rejection (0 allocation) by all advisors. (Notice that the result compels the decision maker to respect also all other unanimous advice. The w_j in the result will be the "weight" of the j-th advisor and the final allocation will be a weighted arithmetic mean of the individual recommendations.) This is a characterization of the weighted arithmetic mean.

The cases $m \leq 2$ are also completely settled (then there are other solutions too).

The above results are stronger (the conditions weaker) than those reported at the 1979 meeting.

J. ACZÉL

6. Remark. Concerning Professor Fenyő's remark (Remark 2, these Proceedings) about non-constant and regular solutions $d : \mathbb{R}^2 \to \mathbb{R}^2$ of the system

$$d(x+y,y) = d(x,y), \qquad d(x,y) = d(y,x).$$

Consider Paganoni's solution $d := 1 - \chi_{V \times V}$ (with χ denoting characteristic functian), where V is an arbitrary subgroup of the additive group of all reals. If V is countable, then d is Borel measurable and locally integrable.

K. BARON

7. Remark. M. Laczkovich (University of Budapest) has solved Kemperman's problem (Aequationes Math. **4** (1970), 248–249) by proving that every solution of

$$2f(x) \leq f(x+h) + f(x+2h)$$

(for all real x and all positive h) is nondcreasing.

J. ACZÉL

8. Problem. In connection with the construction of a collective preference from any n given individual preferences, the following problem arises:

Let $n, m \in \mathbb{N}; x^1, x^2, \dots, x^n \in S \subseteq \mathbb{R}^m$. Find all (continuous or even differentiable) vector-valued solutions $f^n : S^n \to S$ of the system of functional equations:

(1) $f^n(x^{\pi(1)}, \dots, x^{\pi(n)}) = f^n(x^1, \dots, x^n)$, for all permutations π and for all $x^1, \dots, x^n \in S$,

(2) $f^n(x, x, \dots, x) = x$ for all $x \in S$.

(3) $f^n(f^k(x^1, \dots, x^k), \dots, f^k(x^1, \dots, x^k), x^{k+1}, \dots, x^n) = f^n(x^1, \dots, x^k, x^{k+1}, \dots, x^n)$ for all natural numbers $k \leq n$ and for all $x^1, \dots, x^n \in S$,

where additionally the i-th component of f^n (i.e. f_i^n) is a strictly monotonically increasing function of the i-th components of the vectors $x^1, \dots, x^n$ (i.e. of the variables $x_i^1, x_i^2, \dots, x_i^n$).

Remark. It is known that the functions f^n defined by

$$f_i^n(x^1, \dots, x^n) = g^{-1}\left(\frac{1}{n}\sum_{l=1}^{n} g(x_i^l)\right) \qquad (i = 1, 2, \dots, m)$$

with an arbitrary strictly monotonic (continuous or even differentiable) function g, defined on a proper subset $G \subset \mathbb{R}$, are solutions for any $n \in \mathbb{N}$.

F. STEHLING

9. Remark. Let us consider the following functional equation:

$$f(x+y)[f(x) + f(y) - 1] = f(x)f(y) \qquad x, y \in S \tag{1}$$

where S is a given subset of the reals. l. Fenyő and L. Paganoni have proved the following theorem (see C. R. Math. Rep. Acad. Sci. Canada **2** (1980), 113–117).

Theorem 1. *The most general solution $f : S \to \mathbb{R}(S \subset \mathbb{R})$ of equation (1) is the following:*

$$f(x) = \begin{cases} 0 & \text{if } x \in S_0 \\ 1 & \text{if } x \notin S_1 \\ \frac{1}{1-g(x)} & \text{if } x \in S_2 \end{cases} \tag{2}$$

where S_0, S_1, S_2 are disjoint half-groupoids (some of which may be empty), whose union is the set S and which have the following properties:

$$S \cap (S_0 + S_2) \subset S_0, \tag{3a}$$

$$S \cap (S_1 + S_2) \subset S_1, \tag{3b}$$

and g is an arbitrary solution of the Cauchy functional equation which does not take the values 0 and 1.

Corollary. *If the domain of f contains the origin, then the most general solution of (1) is the characteristic function of a half-groupoid contained in S.*

The following problem suggested by J. Aczél arises: given an arbitrary subset S of the set of nonzero real numbers, is it in any case possible to cut it into three disjoint nonempty halfgroupoids so that conditions (3a) and (3b) are fulfilled? A partial answer to this problem is contained in the following theorem.

Theorem 2. *Let S be a subset of the nonzero reals; and let $V(S)$ be the rational subspace of $\mathbb{R}$ generated by S. If $\dim V(S) > 2$, then it is possible to find three disjoint nonempty halfgroupoids S_i $(i = 0, 1, 2)$ for which the conditions (3a) and (3b) are fulfilled.*

In a more general way we can state that the answer to the question above is surely affirmative if a maximal hyperplane H exists with $S \cap H \neq \emptyset$ and which divides all other elements of S into two disjoint parts.

I. Fenyő

10. Remark (concerning the talk of Professor J. Baker). Recently P. Cholewa (Silesian University, Katowice) has proved a generalization of Professor Baker's first result on a problem of E. Lukacs concerning the stability of the functional equation

$$f(x + y) = f(x)f(y).$$

In particular, if a nonempty set S, a positive real number δ, and a metric space (X, ρ) are given, then any function $f : S \to X$ fulfilling the condition

$$\rho(f(G(x, y)), H(f(x), f(y))) < \delta, \qquad x, y \in S,$$

has to be either (metrically) bounded or to satisfy the funetional equation

$$f(G(x, y)) = H(f(x), f(y)), \qquad x, y \in S,$$

where $G : S \times S \to S$ and $H : X \times X \to X$ are given functions subjected to some rather natural and fairly general assumptions.

R. Ger

11. Problem. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function with the properties that $f(0) = \frac{\partial f}{\partial x_i}(0) = 0$ $(i = 1, 2, \dots, n)$, and that the rank of the matrix $\left| \frac{\partial^2 f}{\partial x_i \partial x_j} \right|$ is r at each point of $\mathbb{R}^n$.

Does there exist a linear coordinate transformation such that f can be expressed as a function of just r variables?

The answer is known to be affirmative in the case $r = 2$ and is negative on certain proper subsets of $\mathbb{R}^n$.

M. A. McKiernan

12. Remark. The functional equation

$$f(xy) + f(x+y) = f(xy+x) + f(y) \tag{1}$$

where $f : R \to G$, and R is a ring, G is a group, was introduced at the 17^{th} International Symposium on Functional Equations at Oberwolfach. At the present Symposium, R. Ger has announced some results on this equation, so it may be of interest to show (below) that if f satisfies (1), then the function taking x to $f(-x)$ satisfies Hosszú's functional equation:

$$f(xy) + f(x+y-xy) = f(x) + f(y). \tag{H}$$

So assume f satisfies (1). Let $y = -1$ in (1). Then we deduce

$$f(x-1) = f(0) + f(-1) - f(-x). \tag{2}$$

Again in (1), let $x = u+1$, $v = y-1$, and use (2) to show

$$-f(u-v-uv) + f(u+v) = f(uv+v) - f(-v). \tag{3}$$

A final use of (1), with $x = v, y = u$ allows one to replace $f(uv+v)$ in (3) by $f(uv) + f(u+v) - f(u)$; and so (3) becomes:

$$f(uv) + f(u-v-uv) = f(u) + f(-v). \tag{4}$$

Replacing u by $-u$ we deduce

$$f(-(u+v-uv)) + f(-uv) = f(-u) + f(-v). \tag{5}$$

Hence, if we let $g(x) := f(-x)$, then g satisfies Hosszú's functional equation (H).

If R is a division ring with at least 5 elements, then solutions of Hosszú's equation satisfy

$$f(x+v) + f(0) = f(x) + f(y). \tag{6}$$

For such division rings R, therefore, the solutions of (1) are precisely tlte solutions of (6).

T. Davison

13. Remark. The characterization of the inner product in $\mathbb{R}^3$ given by J. Aczél ([1], p. 310, Satz l; [2], pp. 27–28) may be generalized as follows:

If $(X : \langle\cdot,\cdot\rangle)$ is a reat inner product space, let $\mathrm{SO}(X,2)$ denote the set of all linear isometries $T : X \to X$ with a 2-dimensional invariant subspace M such that the restrietion $T_M : M \to M$ of T is an orientation-preserving rotation of M (i.e. $T_M \in \mathrm{SO}(M : \langle\cdot,\cdot\rangle)$) and $Tx = x$ for every x in the orthogonal complement of M. Suppose that the mapping $g : X \times X \to \mathbb{R}$ has the properties

1) $g(Tx, Ty) = g(x, y)$ for all $x, y \in X$ and every $T \in \mathrm{SO}(X, 2)$.

2) $g(x_1 + x_2, y) = g(x_1, y) + g(x_2, y)$ for all $x_1, x_2, y \in X$,

3) $g(x, \lambda y) = \lambda g(x, y) = g(\lambda x, y)$ for all $x, y \in X$ and all $\lambda \in \mathbb{R}$.

Then the following statements can be proved:

a) If $\dim X \neq 2$, then $\langle x, y \rangle = 0$ implies $g(x, y) = 0$.

b) If $e, e' \in X$, with $\|e\| = \|e'\| = 1$, then $g(e, e) = g(e', e')$.

c) g is additive in its second variable, i.e. g is bilinear.

d) If $\dim X \neq 2$, g is symmetric.

e) If $\dim X \neq 2$, there exists $\alpha \in \mathbb{R}$ such that $g(x, y) = \alpha \langle x, y \rangle$ for all $x, y \in X$.

f) For the case $\dim X = 2$, the conclusions in a), d), and e) do not hold.

References

[1] Aczél, J., *Bemerkungen über die Multiplikation von Vektoren und Quaternionen*, Acta. Math. Acad. Sci. Hungar. **3** (1952), 309–316.

[2] Aczél, J., *Lectures on Functional Equations and their Applications*, Academic Press, New York–San Francisco–London, 1966.

J. Rätz

14. Remark. Some results of D. Zupnik on congruences and endomorphisms.

Let S be a set and n a positive integer. An n-ary operation on S is a function G from S^n into S. An equivalence relation – on S is a congruence on S with respect to G if $x_i \sim y_i$ for $i = 1, 2, \dots, n$ implies $G(x_1, \dots, x_n) = G(y_1, \dots, y_n)$. At the I976 Symposium at Lecce and Castro Marina, congruences were characterized in terms of functional equations (see Aequationes Math. **15** (1977), p. 284). Recently, D. Zupnik has developed this characterization and used it to obtain related results. Among these are the ones which follow.

Definition 1. A function f is an n-congruence on an n-ary operation G on S if $\mathrm{Dom}\, f = S$, f is idempotent, and

$$f(G(x_1, \dots, x_n)) = f(G(f(x_1), \dots, f(x_n))) \tag{1}$$

for all $x_1, \dots, x_n$ in S.

Theorem 1. *An equivalence relation – on S is a congruence on S with respect to the n-ary operation G on S if and only if there exists an n-congruence f on G such that $x - y$ iff $f(x) = f(y)$.*

An n-congruence f on G is always an endomorphism of the n-ary operation $f \circ G$, but need not be an endomorphism of G itself.

Theorem 2. *Let f be an n-congruence on the n-ary operation G. Let G_0 be the restriction of G to $(\mathrm{Ran}\, f)^n$. Then f is an endomorphism of G if and only if G_0 is an n-ary operation on $\mathrm{Ran}\, f$, or equivalently, if and only if*

$$f(G(x_1, \dots, x_n)) = G(x_1, \dots, x_n) \tag{2}$$

for all $x_1, \dots, x_n$ in $\mathrm{Ran}\, f$.

Definition 2. An n-congruence f on an n-ary operation G admits an endomorphism of G if there exists an invertible function f_1 such that $\operatorname{Dom} f_1 = \operatorname{Ran} f$ and $f_1 \circ f$ is an endomorphism of G.

It is easily seen that if f is an endomorphism of G; then f admits an endomorphism of G. Furthermore, we have:

Theorem 3. *Let f be an n-congruence on an n-ary operation G. Then f admits an endomorphism of G if and only if there exists a subset S_1 of S such that*

a) $\operatorname{Card} S_1 = \operatorname{Card}(\operatorname{Ran} f)$,

b) *if G_1 denotes the restricrion of G to S_1^n, then G_1 is an n-ary operation on S_1,*

c) *the n-ary operation G_1 is isomorphic to the n-ary operation $f_2 \circ G_0$, where G_0 is as in the preceding theorem.*

A. Sklar

15. Remark. G. Fredricks (Texas Tech University) has proved the following result.

Let U be open in $\mathbb{R}^k$, A a smooth map of U into the gróup of symmetric $n \times n$ matrices, p and q nonnegative integers with $p + q \leq n$. Then there exists a smooth map $G : U \to GL(n)$ satisfying

$$G(\overline{x})A(\overline{x})G^T(\overline{x}) = \operatorname{diag}(1, \dots, 1, -1, \dots, -1, 0, \dots, 0)$$

for all $\overline{x} \in U$ (with p 1's and q -1's in the diagonal matrix on the right) if A has p positive and q negative eigenvalúes at each $\overline{x} \in U$ and U is smoothly contractible.

B. Ebanks

16. Remark. The solution of a problem of Alsina, and its generalization.

Let F and G be functions from the unit square onto the unit interval that are associative, continuous, and non-decreasing in each place, and having no interior idempotents.

ln problem **P193** (Aequationes Math. **20** (1980), p. 308), C. Alsina proposed the equation

$$F(x, y) \cdot G(x, y) = xy.$$

Its only solutions consist of the one-parameter family

$$F_\alpha(x, y) = (x^{-\alpha} + y^{-\alpha} - 1)^{-\frac{1}{\alpha}}, \quad G_\alpha(x, y) = (x^\alpha + y^\alpha - x^\alpha y^\alpha)^{\frac{1}{\alpha}}, \qquad 0 < \alpha < \infty.$$

(Note the limiting case $F_\infty = \min$. $G_\infty = \max$.)

The related equation

$$F(x, y) + G(x, y) = x + y$$

is solved in my paper (Aequationes Math. **19** (1979),194–226). Extensions of this result to functions defined on unbounded intervals yield the solutions of the rnore general equation

$$H(F(x, y), G(x, y)) = H(x, y)$$

for any H which can be written $H(x, y) = k(h(x) + h(y))$, with continuous and monotonic h and k. In particular. when $h(0) = -\infty$ and $h(1) = 0$, the functions

$f_\alpha(x) = 1 - \exp[-\alpha h(x)]$, $0 < \alpha < \infty$, generate the family of solutions F_α.

M. J. Frank

17. Problems Let

$$D = \{(x,y) : x, y \in [0,1[, x + y \leq 1\}$$

and let

$$D_0 = \{(x,y) : x, y, x + y \in]0,1[\}$$

be its interior.

(1) Determine the general real-valued solutions f of

$$f(x,u) + (1-x)f\left(\frac{y}{1-x}, \frac{v}{1-u}\right) = f(y,v) + (1-y)f\left(\frac{x}{1-y}, \frac{u}{1-v}\right) \quad (1)$$

on $D_0 \times D_0$.

(2) Determine the general (real-valued) solutions F, G, H, K (all four functions unknown) of

$$F(x) + (1-x)^\alpha G\left(\frac{y}{1-x}\right) = H(y) + (1-y)^\alpha K\left(\frac{x}{1-y}\right) \quad (2)$$

on D_0, (α a fixed constant).

The second problem may lead to the solution of the first, but there may be a simpler way. Equation (1) has been solved on $D \times D$ and on $D \times D_0$ (the solutions are essentially diffierent): equation (2) has been solved on D.

(3) Determine the general solutions of (2) on D_0 when t^α is replaced on both sides by $m(t)$, $m :]0,1[\to \mathbb{R}$ being an arbitrary multiplicative function ($m(tu) = m(t)m(u)$, $t, u \in]0,1[$). Again, similar equations (but not this one) have been solved by Kannappan and Ng.

The general solution, on $D_0 \times D_0$, of equations similar to (1), but with $(1-x)$ replaced by $(1-x)^\alpha(1-u)^\beta$ [and $(1-y)$ by $(1-y)^\alpha(1-v)^\beta$] (α, β arbitrary constants but $(\alpha, \beta) \neq (0,1), (1,0)$), and of similar n-dimensional equations, have been determined by Ng.

J. Aczél

18. Remark. A relationship of Catalan Numbers to Pascal's Triangle. We will call the identity

$$\binom{n+1}{r} = \sum_{k=0}^{r} \binom{n-r+k}{k}$$

the "stocking theorem" for Pascal's triangle, for the reason suggested by the figure below.

Figure 1:

(where in this case the overlay pattern illustrates the special.case $10 = 1 \cdot 6 + 1 \cdot 3 + 1 \cdot 1$ of the "stocking theorem").

The author has obtained generalizations of Pascal's triangle through the use of functional equations, and for each of these, there is a stocking theorem, analogous to the one above, which expresses each element of the generalized triangle as a certain linear combination of "higher" elements of the triangle. The coefficients in this linear combination are the first r elements of the stocking sequence associated with the triangle. (In the case of Pascal's triangle, the stocking sequence is simply $1, 1, 1, \ldots$.)

The generalized Pascal triangle T01 gives the number of ways of choosing n objects r at a time where, if an element is used at all, it must be used twice. The recurrence relation for this triangle is

$$C(n+1, r) = C(n, r) + C(n, r-2),$$

and the associated stocking sequence is

$$1, 0, 1, 0, 2, 0, 5, 0, 14, 0, 42, 0, 132, 0, \ldots .$$

which turns out to be the sequence of Catalan numbers

$$T_i = \frac{1}{i+1}\binom{2i}{i},$$

with zeros interspersed (see [1]).

For T01, it can be easily shown that $C(n, r) = 0$ for odd r. If we remove these zero columns from T01, we get Pascal's triangle T1, which means that the stocking theorem for T01 can be reinterpreted as the following statement relating the binomial coefftcients to the Catalan numbers T_i (defined above):

$$\binom{n+1}{r} = \sum_{i=0}^{r-1} T_i \binom{n-2i}{r-i-1},$$

where, for negative m, $\binom{m}{k}$ is the (unique) number determined by the Pascal recurrence relation

$$\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}$$

and by $\binom{m}{0} = 0$ for all integers m.

References

[1] Sloane, N. J. A., *Handbook of Integer Sequences*, Academic Press, New York, 1973.

D. R. Snow

19. Problem. Assume that

$$\sum_{i=1}^{k} \mu_i f(x + \phi_i(t)) = f(x) \tag{1}$$

for all $x \in \mathbb{R}^n, t \in \Delta \subset \mathbb{R}$: where $\sum_{i=i}^{k} \mu_i = 1, \mu_i > 0$ for $i = 1, \dots, k$, and there exists an $\alpha \in \Delta$ such that $\phi_i(\alpha) = 0$ for $i = 1, \dots, k$.

If the set of $\phi_i'(\alpha)$ (for $i = 1, \dots, k$) spans $\mathbb{R}^N$, then every locally integrable solution f of (1) is a C^∞ function (see [1]).

Question. *Are all the locally integrable solutions of (1) C^∞ functions if $\{(\phi_i'(\alpha) : i = 1, \dots, k\}$ does not span $\mathbb{R}^N$, but $\{(\phi_i''(\alpha) : i = 1, \dots, k\}$ does?*

References

[1] Światak, H., *Criteria for the regularity of continuous and locally integrable solutions of a class of linear functional equarions*, .Aequationes Math. **6** (1971), 170–187.

H. Światak

20. Problem. Find all functions $F :]0, \infty[\to \mathbb{R}$ satisfying:

$$F(xy) = F(x)F(y) \qquad \text{and} \qquad F(x+y) \le F(x) + F(y)$$

for all $x > 0$ and $v > 0$.

This problem arises in the calculation of entropy functions of degree $\alpha < 1$. Discontinuous solutions of the system are known to exist.

Gy. Maksa

21. Remark. It has been pointed out by V. I. Arnold and A. A. Kirilov that the function $\operatorname{Min}(x, y)$ admits no representation of the form

$$\operatorname{Min}(x, y) = f(g(x) + g(y)),$$

where f and g are continuous. A stronger result is easily established:

Theorem. *Let $A = [a, b]$ be a subinterval of the extended real line, and let $T : A \times A \to A$ define a semigroup on A such that for some $a < \overline{x} < b$,*

$$T(a, a) = a, \qquad T(\overline{x}, \overline{x}) = \overline{x}, \qquad T(b, b) = b.$$

Then there are no continuous functions f, g such that T can be represented in the form $T(x, y) = f(g(x) + g(y))$.

G. Krause

22. Remark. The following problem of Colin Rogers arises in gas dynamics in connection with the theory of Bäcklund transformations. Given real constants α, a, b, c, d, find smooth solutions $\phi :]0, \infty[\to \mathbb{R}$ such that

$$\phi(x) = \alpha(x + c)^2 \left[\phi\left(a + \frac{b}{x + c}\right) + d\right], \qquad x > 0. \tag{1}$$

We assume a, b, and c are such that $a + \frac{b}{x+c}$ is defined and positive whenever $x > 0$. In the homogeneous case ($d = 0$) the real analytic solutions of (1) can be found explicitly (they are rational functions in nontrivial cases) with the aid of the following theorem.

Theorem 1. *Let D be an open connected subset of $\mathbb{C}$ (the complex numbers) and let $g : D \to D$ be analytic and have a fixed point z_0 such thnt $0 < |g'(z_0)| < 1$ and $g^k(z) \longrightarrow z_0$ as $k \longrightarrow +\infty$ for every $z \in D$. Also let $f : D \to \mathbb{C}$ be anatytic with $f(z_0) = 1$, let $\lambda \in \mathbb{C}$ and suppose that $\phi : D \to \mathbb{C}$ is analytic and such that*

$$\lambda\phi(z) = f(z)\phi(g(z)), \qquad z \in D. \tag{2}$$

Then there exist analytic functions $F, G : D \to \mathbb{C}$ such that

(i) *if $\lambda \neq (g'(z_0))^k$ for all $k = 0, 1, 2, \ldots$, then $\phi \equiv 0$ and*

(ii) *if $\lambda = (g'(z_0))^k$ for some $k = 0, 1, 2, \ldots$ then there exists $\gamma \in \mathbb{C}$such that $\phi(z) = \gamma F(z)[G(z)(z - z_0)]^k$, $z \in D$.*

If we let $\Phi_k(z) = F(z)[G(z)(z - z_0)]^k$, for $z \in D, k = 0, 1, 2, \ldots$, then we can prove:

Theorem 2. *Given $h : D \to \mathbb{R}$ analytic, there exist $\delta > 0$ and a complex sequence $\{c_k\}_{k=0}^{+\infty}$ such that*

$$h(z) = \sum_{k=0}^{+\infty} c_k \Phi_k(z)$$

for $|z - z_0| < \delta$. Moreover the convergence is almost uniform on $\{z \in D : |z - z_0| < \delta\}$.

Using Theorem 2, one can determine the real analytic solutions of (1) in the nonhomogeneous case.

J. A. Baker

23. Remark. A function f, holomorphic in $D = \{z : |z| < 1\}$, is said to be annular in case there is a sequence $\{J_n\} \subset D$ of Jordan curves about 0 such that

$$\lim_{n\to\infty} \min\{|f(z)| : z \in J_n\} = \infty.$$

One can base a prooi for the annuiarim of

$$f(z) = \sum_{n=0}^{\infty} a^{cn} z^{a^n} \tag{1}$$

(where $c > 0, a = a(c)$, a sufiiciently large integer), on known methods and the fact that f satisfies the functional equation

$$f(z) - a^c f(az) = z.$$

Hardy and Littlewood in 1916 related (1) via a functinonal equation to

$$F(\zeta) = \sum_{n=1}^{\infty} n^{\delta-} e^{\beta n \log n} \zeta^n \tag{2}$$

($\delta > 0, \beta > 0$ certain constants), and thereby one can show that (2) is also annular. Fatou showed that for certain rational functions, for example

$$R(z) = \frac{z(z-s)}{1-sz}$$

c complex. $0 < |s| < 1$, the nontrivial analytic solutions of the Schröder equation

$$f(R(z)) = -sf(z)$$

are annular.

I would appreciate hearing of othter connections between functional equations and annular functions.

F. CARROLL

24. Problem. Let $(F, +, \cdot)$ be a system with the following properties:

I. $(F, +)$ is a toop (with identity 0).

II. $(F - \{0\}, \cdot)$ is a group.

III. $(a + b) \cdot c = a \cdot c + b \cdot c$ and $c \cdot 0 = 0$, for all $a, b, c \in F$.

IV. (Limited associativity) $(x + a) + b$ is equal to $x + (a + b)$ if $b + a = 0$, and is equal to $x(b + a)^{-1}(a + b) + (a + b)$ otherwíse.

Question. *Do the conditions I-IV imply that $(F, +)$ is an abelian group?*

The answer is known to be affirmative in case F has finite cardinality, or under some other additional assumptions, such as $a(1+1) = a+a$; or $1+1+1 = 0$.

W. LEISSNER

25. Remark. Solution of Problem 17 (2) (of these Proceedings).

In answer to a problem of J. Aczél, we have proved the following:

Theorem. *Ler $\alpha \in \mathbb{R}$ be fixed. $D_0 = \{(x, y) \in \mathbb{R}^2 : x, y, x + y \in]0, 1[\}$. The functions $F, G, H, K :]0, 1[\to \mathbb{R}$ satisfy*

$$F(x) + (1-x)^\alpha G\left(\frac{y}{1-x}\right) = H(y) + (1-y)^\alpha K\left(\frac{x}{1-y}\right)$$

for all $(x, y) \in D_0$ if and only if, for all $x \in]0, 1[$,

$$F(x) = \begin{cases} \phi(x) + \phi(1-x) + a_1 x + a_2(1-x) + a_3 & \textit{if } \alpha = 1 \\ l_1(1-x) + l_2(x) + a_1 & \textit{if } \alpha = 0 \\ d(x) + a_1 x^2 + a_2(1-x)^2 + a_3 & \textit{if } \alpha = 2 \\ a_1 x^\alpha + a_2(1-x)^\alpha + a_3 & \textit{otherwise} \end{cases}$$

$$G(x) = \begin{cases} \phi(x) + \phi(1-x) + a_1' x & \\ \quad +(a_1 - b_1 + a_3 - b_3 - b_1' + a_1' + b_2')(1-x) & \\ \quad +b_1 - a_2 - a_3 + b_3 - a_1' & \textit{if } \alpha = 1 \\ l_1(1-x) + l_3(x) - l_3(1-x) + b_1 - a_1 + b_1' & \textit{if } \alpha = 0 \\ \quad -d(x) + b_1 x^2 + a_2'(1-x)^2 - a_2 & \textit{if } \alpha = 2 \\ b_1 x^\alpha + a_2'(1-x)^\alpha - a_2 & \textit{otherwise} \end{cases}$$

$$H(x)=\begin{cases}\phi(x)+\phi(1-x)+b_1x+b_2(1-x)+b_3 & \text{if } \alpha=1\\ l_1(1-x)+l_2(1-x)+l_3(x)-l_3(1-x)+b_1-a_1+b_1' & \text{if } \alpha=0\\ \quad -d(x)+b_1x^2+b_2(1-x)^2+a_3 & \text{if } \alpha=2\\ b_1x^\alpha+b_2(1-x)^\alpha+a_3 & \text{otherwise}\end{cases}$$

$$K(x)=\begin{cases}\phi(x)+\phi(1-x)+b_1'x+b_2'(1-x) & \\ \quad +a_1+a_3-b_2-b_3-b_1' & \text{if } \alpha=1\\ l_1(1-x)+l_2(x)-l_3(1-x)+b_1' & \text{if } \alpha=0\\ d(x)+a_1x^2+a_2'(1-x)^2-b_2 & \text{if } \alpha=2\\ a_1x^\alpha+a_2'(1-x)^\alpha-b_2 & \text{otherwise}\end{cases}$$

where $\phi :]0,\infty[\to \mathbb{R}$ *satisfies*

$$\phi(xy)=x\phi(y)+y\phi(x),$$

for all $x,y\in]0,\infty[$, $l_j :]0,\infty[\to\mathbb{R}$ *satisfies*

$$l_i(xy)=l_i(x)+l_i(y)$$

for all $x,y\in]0,\infty[$ *and* $i=1,2,3$, *the function* d *is a real derivation and* a_i, b_i, a_k', b_k' $(i=1,2,3; k=1,2)$ *are arbitrary real constants.*

Gy. Maksa

26. Remark Solution of Problem 17 (1) (of these Proceedings).

In view of Gy. Maksa's solution (see Remark 25 above) to Problem 17 (2), the equation

$$f(x,u)+(1-x)f\left(\frac{y}{1-x},\frac{v}{1-u}\right)=f(y,v)+(1-y)f\left(\frac{x}{1-y},\frac{u}{1-v}\right) \quad (1)$$

for all $(x,y)\in D_0, (u,v)\in D_0$, where

$$D_0=\{(s,t): s,t,s+t\in]0,1[\}.$$

can be solved as follows.

Keeping u,v constant, (1) goes over into

$$F(x)+(1-x)^\alpha G\left(\frac{y}{1-x}\right)=H(y)+(1-y)^\alpha K\left(\frac{x}{1-y}\right)$$

for all $(x,y)\in D_0$.

From Maksa's solution of this equation ($\alpha=1$).

$$f(s,u)=F(s)=\phi(s)+\phi(1-s)+a_1s+b_1,$$
$$f(s,y)=H(s)=\phi(s)+\phi(1-s)+a_2s+b_2,$$

that is, letting u vary again,

$$f(x,u)=\phi(x)+\phi(1-x)+A(u)x+B(u). \quad (2)$$

Here

$$\phi(xy)=x\phi(y)+y\phi(x) \quad (3)$$

(for $x,y\in]0,1[$) and in consequence,

$$\phi\left(\frac{s}{t}\right)=\frac{t\phi(s)-s\phi(t)}{t^2}$$

(where $s, t, \frac{s}{t} \in]0,1[$).

By substituting (2) into (1), we get

$$\begin{aligned}
&\phi(x) + \phi(1-x) + A(u)x + B(u) + \phi(y) - \phi(1-x) + \phi(1-x-y) \\
&\quad + A\left(\frac{v}{1-u}\right) y + B\left(\frac{v}{1-u}\right)(1-x) \\
&= \phi(y) + \phi(1-y) + A(v)y + B(v) + \phi(x) - \phi(1-y) + \phi(1-x-y) \\
&\quad + A\left(\frac{u}{1-v}\right) x + B\left(\frac{u}{1-v}\right)(1-y).
\end{aligned}$$

After cancellations and comparing the coefficients of x and the terms independent of x and y on both sides we get

$$A(u) = A\left(\frac{u}{1-v}\right) + B\left(\frac{v}{1-u}\right)$$

and

$$B(u) - B\left(\frac{v}{1-u}\right) = B(v) + B\left(\frac{u}{1-v}\right)$$

for all $(u, v) \in D_0$. By adding these two equations and writing $C = A + B, p = \frac{u}{1-v}, q = \frac{v}{1-u}$, ($p, q \in]0,1[$, but otherwise arbitrary), we get

$$C(pq) = C(p) + B(1-q) \qquad (p, q \in]0,1[).$$

This is a Pexider type equation with the general solution (cf. [1]) $B(1-q) = l(q), C(u) = l(u) + c$. So

$$B(u) = l(1-u), \qquad A(u) = l(u) - l(1-u) + c$$

where l is an arbitrary solution of

$$l(uv) = l(u) + l(v) \qquad (u, v \in]0,1[), \tag{4}$$

(cf [2],[3]). Since the converse part is obvious, we have proved the following.

Theorem. *The general solution of* (1) *is given by*

$$f(x, u) = \phi(x) + \phi(1-x) + xl(u) + (1-x)l(1-u) + cx,$$

where c is an arbitrary constant and ϕ and l are arbitrary solutions of (3) *and* (4) *respectively.*

Note. *By interchanging (x, y) and (u, v), we can also use Maksa's $\alpha = 0$ result for the same purpose.*

References

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J. Aczél