International Journal of Modern Physics B, Vol. 16, No. 9 (2002) 1269–1299

ANISOTROPIC STEP, SURFACE CONTACT, AND AREA WEIGHTED DIRECTED WALKS ON THE TRIANGULAR LATTICE

A. C. OPPENHEIM, R. BRAK and A. L. OWCZAREK*

Department of Mathematics and Statistics,
The University of Melbourne, Victoria 3010, Australia
**aleks@ms.unimelb.edu.au*

Received 2 January 2002

We present results for the generating functions of single fully-directed walks on the triangular lattice, enumerated according to each type of step and weighted proportional to the area between the walk and the surface of a half-plane (wall), and the number of contacts made with the wall. We also give explicit formulae for total area generating functions, that is when the area is summed over all configurations with a given perimeter, and the generating function of the moments of heights above the wall (the first of which is the total area). These results generalise and summarise nearly all known results on the square lattice: all the square lattice results can be obtaining by setting one of the step weights to zero. Our results also contain as special cases those that already exist for the triangular lattice. In deriving some of the new results we utilise the Enumerating Combinatorial Objects (ECO) and marked area methods of combinatorics for obtaining functional equations in the most general cases. In several cases we give our results both in terms of ratios of infinite q-series and as continued fractions.

PACS number(s): 05.50.+q, 02.10Ab, 61.41.+e

Keywords: Directed lattice walks; triangular lattice; Enumerating Combinatorial Objects method; marked area.

1. Introduction

The study of directed lattice walks have been of increasing interest for the past two decades, since the article of Fisher[1] demonstrating the many modelling uses for these lattice objects as simple polymer models[2] and as domain walls between phases in various systems. Because of their intrinsic interest as a basic type of lattice object, and their many relations to other types of combinatorial objects such as lattice trees and partitions of integers,[3,4] they have been studied in the combinatorics literature for more than a century. Recently the connections between combinatorics and physics have been strengthened through their appearance in a range of exactly solvable lattice models and in relation to various q-series identities that arise in these studies. The study of a single walk is usually the basis for studying arbitrary numbers of walks in that the solution of many walk generating functions can often be written in terms of one walk generating functions. Hence the single walk is the starting place for many studies of directed walk systems, and so it is important to have a compendium of one walk results from which to consult. Directed walks often appear in physics as weighted configurations and so it is of some importance to study single lattice walks with a variety of key properties distinguished by different weights.

Single walks on various lattices under several different boundary conditions were studied in now little-known papers about paths on a chessboard,[5–8] and some cases were treated as lattice permutations in Ref. 9. On the square lattice, denoted $\mathbb{S}$, many problems concerning one and more directed walks have been solved exactly, and so this task has essentially been accomplished.[10–15] However, a less complete set of problems have also been solved on the triangular lattice, denoted $\mathbb{T}$. The literature pertinent to $\mathbb{T}$ directed lattice walks is still fairly large;[16–26] more references are listed in Refs. 27 and 28. The triangular lattice is interesting in the combinatorial sense as the regular planar lattice other than the square where each site is topologically equivalent. Also, by considering weighted paths the square lattice can be treated as a special case. Triangular lattice results are of interest in statistical mechanics because they allow one to test the hypothesis of universality for various quantities, in particular in regard to the study of different corrections to scaling. Mathematically, the triangular lattice can force one to approach the solution of problems a little differently and so lead one to introduce new techniques. Here we take the opportunity to make contact with the combinatorial literature on the subject of directed walks and survey most of the physically useful results and related methods. Hence, while many sub-cases of the results appearing here have appeared previously in the combinatorics literature our most general results containing the contact weights as well as area and step weights have not appeared. This paper then provides a review of the most physically interesting triangular lattice results by providing a compendium of general formula for those results already known. Our work also generalises those results by the inclusion of two types of surface contact weight. Additionally, we introduce methods from the combinatorics literature that may prove useful in future work.[29]

As stated above we consider directed walks on the triangular lattice. Our triangular lattice is a tiling of isosceles right-angled triangles so that two adjacent triangles meeting along their hypotenuses form a square. Let us refer to those bonds as diagonal bonds as they form diagonals of squares. In this way the square lattice, rather than any arbitrary parallelogram lattice, can be obtained by the removal of a subset of bonds. This is simply an aesthetic consideration here because we consider walks with general step weights. We shall refer to this lattice as the "*squared*-triangular" lattice to distinguish it from the normal isotropic triangular lattice made from equilateral triangles.

We consider a "wall" parallel to some diagonal edge so that steps of the walk are allowed only on one side of the wall or on the wall. It is more convenient visually to display such single walk configurations by rotating the lattice through 45° (see

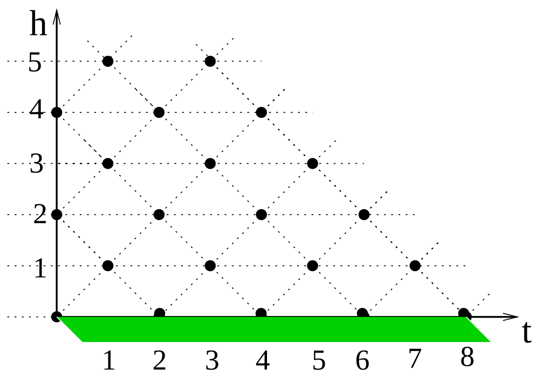

Fig. 1. The half-plane of the squared-triangular lattice considered here with the coordinate system displayed. Sites are coincident with each axis at only even integer values of the coordinates. The shaded region represents the wall with the t axis (*direction* axis) lying along the wall.

Fig. 1). We consider single walks in the half-plane on one side of (or on) the wall with one end of the walk on (touching) the wall. We use the site of attachment of the walk to the wall (i.e. left-most such touching) as the origin of a coordinate system. We use a coordinate system (t, h) where the t coordinate measures distance along the wall (along a *direction* axis) and the h coordinate height above the wall, both scaled so that the first site beyond the origin that is on an axis is at distance 2. By considering orienting the walks away from the origin in the first quadrant they are directed so that every step in the path has non-negative projection on the axis parallel to the wall (direction axis). For convenience we consider the direction axis to be on the wall.

In this paper, two classes of single walks most interesting to physics and fundamental in combinatorics are studied in particular. These classes are shown on the $\mathbb{T}$ lattice in Fig. 2. The classes of walk studied are types of single walks in the half-plane described above that have starting sites on the wall but can finish on or above the wall. In general these walks are called *ballot walks*. The two related classes of ballot walks considered here are

- A *return walk* is a ballot walk that, in addition, ends on the wall.
- An *elevated walk* is a return walk that touches the wall only at its starting and ending sites.

We will demonstrate later how to obtain results for general ballot walks from results concerning return walks.

The paper has the following structure. We begin in the next section by considering only step weights. This allows us to introduce the forms of the generating function solution and provide some fundamental formulae required in later sections. We then add contact weights in Sec. 3. In Sec. 4 we consider the added complication of counting area and introduce the Enumerating Combinatorial Objects (ECO)

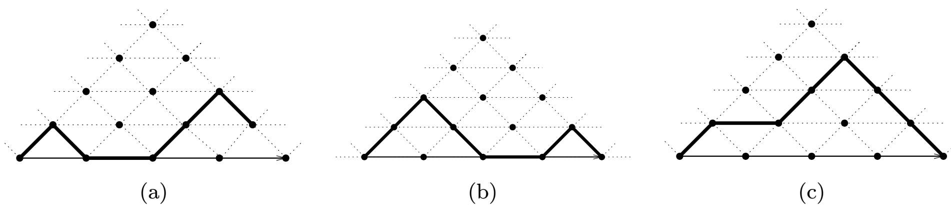

Fig. 2. Single walks with a wall on the $\mathbb{T}$ lattice: (a) a ballot walk (b) a return walk (c) an elevated walk.

method from combinatorics. First area-moment and height moments generating functions are then tackled in the following two sections, allowing us to introduce another combinatorial method known as marked area.

2. Return Walks with Step Weights

The walks that start and end at the wall in a one-wall system have been studied by various authors and go by many names: they have been called positive paths or walks,[10,30,31] zero paths,[32] return paths,[33] restricted walks[34] or under-diagonal walks,[35] amongst other names. On the $\mathbb{S}$ lattice, the walks are often referred to as Dyck paths, since when represented as words of x's and y's they are Dyck words.[a] Here, walks that start and end at the wall are always referred to as *return walks.* The empty or zero-step walk that is the site at the origin is classed as a return walk.

We introduce variables x, y and d associated with down, up and horizontal steps of a walk respectively.

Example 2.1. There are six return walks on the $\mathbb{T}$ lattice that consist of an up step, a down step and two horizontal steps. Each of the six walks contribute an xyd^2 term to an anisotropic length generating function for return walks. The walks are shown in Fig. 3; the walk labelled as (e) in Fig. 3 is an elevated walk.

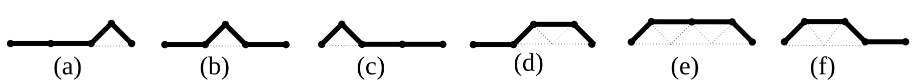

Fig. 3. Return walks of weight xyd^2.

Because a return walk finishes at the wall, it has the same number of down steps and up steps, and so the x and y variables have been coalesced in this case without loss of generality into a variable $v = xy$ that counts the number of up-down step pairs.

[a]Dyck words (and the Dyck language) are named after Walther Franz Anton (von) Dyck 1856–1934, a German mathematician.

Thus, with $\mathcal{W}$ denoting the set of return walks on the $\mathbb{T}$ lattice, let $L(v,d)$ be a generating function

$$L(v,d) = \sum_{w \in \mathcal{W}} v^{a(w)} d^{c(w)} \tag{2.1}$$

that enumerates return walks on the $\mathbb{T}$ lattice by types of steps, so that a walk w with $a(w)$ up (and so down) steps and $c(w)$ horizontal steps contributes a $v^{a(w)}d^{c(w)}$ term to $L(v,d)$.

Two equivalent representations for $L(v,d)$ are derived in this section. While $L(v,d)$ is relatively simple, it is a building block of the more complex generating functions and allows us to introduce some of the methods used to calculate such generating functions. The first is as an algebraic function derived as the solution to a quadratic equation, whilst the second is as an infinite continued fraction. Corresponding functions for elevated walks are then deduced.

Both derivations of $L(v,d)$ begin by partitioning the set of return walks in the following manner. The set $\mathcal{W}$ of return walks can be partitioned by splitting each walk in the set (other than the zero-step walk) into two parts at its first return to the wall. The section before this first return is either (i) a horizontal step, or (ii) an elevated walk. The section after the first return is a (possibly empty) return walk. If the zero-step walk is included as a separate category, the set of return walks has the partition shown schematically in Fig. 4, taken from, for example.[36,37]

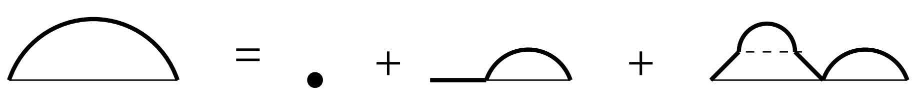

Fig. 4. Partition of return walks on the $\mathbb{T}$ lattice by first return to the wall.

From the decomposition of each walk leading to the partition of the set of walks one can deduce

$$L(v,d) = 1 + dL(v,d) + vL(v,d)^2\,. \tag{2.2}$$

2.1. *Deriving $L(v,d)$ as an algebraic function*

An algebraic function representation of $L(v,d)$ is found by solving the quadratic of Eq. (2.2). The equation has two roots,

$$L_{\pm} = \frac{1 - d \pm \sqrt{(1-d)^2 - 4v}}{2v}\,, \tag{2.3}$$

only one of which can be the solution unless the discriminant is zero. A generating function enumerating walks by number of steps via the variable z is $L(z^2, z)$. An examination of the first few terms in the Laurent expansions of $L_{\pm}$ show that L_- is the only allowed candidate. Hence,

$$L(v,d) = \frac{1 - d - \sqrt{(1-d)^2 - 4v}}{2v}\,, \tag{2.4}$$

where the generating function for the return walks in the variables x, y, d is given by $L(xy, d)$.

2.2. *Deriving $L(v, d)$ as a continued fraction*

A continued fraction representation for functions similar to $L(v, d)$ is often used in the literature, for example in Refs. 10, 32 and 38, and can be easily derived for $L(v, d)$. Indeed, Eq. (2.2) can be rearranged as

$$L(v,d) = \frac{1}{1 - d - vL(v,d)}\,, \tag{2.5}$$

which by iteration gives the infinite continued fraction representation

$$L(v,d) = \frac{1}{1 - d - \frac{v}{1-d-\frac{v}{1-d-\cdots}}}, \tag{2.6}$$

or, in more compact notation,

$$L(v,d) = \frac{1}{1-d-}\quad\frac{v}{1-d-}\quad\frac{v}{1-d-\cdots}\,. \tag{2.7}$$

This representation of $L(v, d)$ may be further specified by including the height coordinate of the site at the start of each step. Indeed, from the partition of the set of return walks in Fig. 4 and the relation in Eq. (2.5) for $L(v, d)$, the first denominator of the fraction in Eq. (2.7), i.e. $1 - d - v$, represents steps (or step pairs xy in the case of v) that start and end at height 0. Continuing, the second denominator represents steps (or step pairs) that start and end at height 1 and so on. If d_i denotes a horizontal step at height i and v_i an up-down step pair that starts and ends at height i (see Fig. 5 for examples), then from Ref. 10, a generating function that enumerates return walks by types of steps and in addition specifies the height of each step is the infinite continued fraction

$$L^{\#} = \frac{1}{1-d_0-}\quad\frac{v_0}{1-d_1-}\quad\frac{v_1}{1-d_2-}\quad\cdots\quad\frac{v_h}{1-d_{h+1}-\cdots}\,. \tag{2.8}$$

2.3. *Elevated walks*

Elevated walks, i.e. return walks that do not touch the wall between their starting and ending sites, have also been called prime paths,[36] lead paths,[39] elevated paths[40]

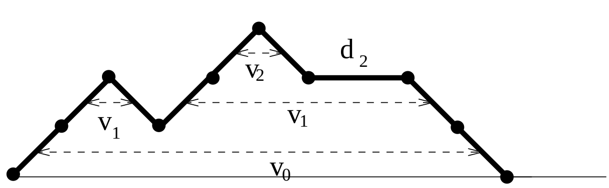

Fig. 5. A return walk with step heights specified.

and excursions (from the wall),[41] among other names. The zero-step walk does not leave the wall and so is not considered an elevated walk. Elevated walks are a subset of return walks, and their anisotropic length generating functions can be obtained from the corresponding functions for return walks. An elevated walk begins with an up step and ends with a down step, and in between these two steps is a return walk that starts and ends at height 1. Thus elevated walks can be constructed by "expanding" return walks away from the wall. A generating function, $\widehat{L}(v,d)$, enumerating elevated walks by its types of steps is then

$$\hat{L}(v,d) = vL(v,d) \tag{2.9}$$

2.4. *Length metrics*

Before we consider some special cases of the results above let us define a *length metric* on the $\mathbb{T}$ lattice to facilitate the discussions. The results with arbitrary weights x, y and d can be used to consider those cases where a *length* alone is assigned to each configuration. If we let the (α, β, γ) "metric" to be such that

$$x = z^{\alpha}, \qquad y = z^{\beta}, \qquad d = z^{\gamma}, \tag{2.10}$$

then α, β and γ assign *length* multiples to each type of step so that a length generating function for walks is

$$G(z^{\alpha}, z^{\beta}, z^{\gamma}) = \sum_{n\geq 0} l_n z^n, \tag{2.11}$$

where l_n is the number of walks of "length" $n = a(\alpha+\beta) + c\gamma$ with a being the number of up-down pairs of steps and c being the number of horizontal steps. It is given that α, β, γ are usually all non-negative integers. There is one exception to allow for cases where, say $d = 0$, we use the convention that a metric can have the "value" $-$, e.g. $\gamma = -$. Under metrics with $\gamma = -$ walks that have steps counted by d do not contribute to the generating functions, and so such cases consider the square lattice with $n = a(\alpha+\beta)$, where a is the number of up-down step pairs.

2.5. *Special cases*

Case 2.1. An early application of a one-wall lattice system for an enumerative problem appears in Ref. 5. The problem can be described as finding the number of 2-row Young tableaux[42] in which each row had n entries. These tableaux can be represented as return walks on the $\mathbb{S}$ lattice from the origin to the site at $(t,h) = (2n,0)$, i.e. Dyck paths. The $\mathbb{S}$ lattice scaled so that each step has unit length is the same as the $\mathbb{T}$ lattice under the $(1,1,-)$ metric. Thus the length generating function enumerating return walks by steps on the $\mathbb{S}$ lattice, or by number of configurations to $(2n,0)$, is

$$L(z^2, 0) = \frac{1-\sqrt{1-4z^2}}{2z^2} = C(z^2), \tag{2.12}$$

where $C(z) = \sum_{n\geq 0} c_n z^n$ is the generating function for Catalan numbers.[b] The coefficients of $L(z^2, 0)$ are often referred to as aerated Catalan numbers because of the zero term in between each successive Catalan number (see Refs. 40 and 46 for example).

Case 2.2. The $(1, 1, 2)$ metric is a natural metric of the lattice under the (t, h) coordinate system since it implies that whether a walk takes an up and then a down step to traverse across the diagonal of a square or simply takes a diagonal "step" both are counted as 2 steps. Hence using this metric is equivalent to considering walks enumerated by length on the *squared*-triangular lattice. The generating function for the return walk configurations ending at successive sites along a wall on the $\mathbb{T}$ lattice under a $(1, 1, 2)$ metric, given by evaluating $L(v, d)$ at $v = z^2, d = z^2$, is

$$L(z^2, z^2) = \frac{1 - z^2 - \sqrt{1 - 6z^2 + z^4}}{2z^2} = R(z^2)\,, \tag{2.13}$$

where $R(z^2) = \sum_{n\geq 0} r_n z^{2n}$ is the generating function for the (aerated) large Schröder numbers.[c] Here diagonal (horizontal) steps are treated as twice the length of up or down steps. This case is interesting since it is a case where all return walks of equal length end at the same lattice site. This condition fails when the isotropic lattice is considered.

As just mentioned, the two previous examples are exceptional in that all return walks of length n end at the one site, $(t, h) = (n, 0)$. The final two examples do not have this property; nonetheless a generating function enumerating return walks by length can be calculated for each.

Case 2.3. Walks on a lattice of unit step length equilateral triangles can be counted using the results above on the $\mathbb{T}$ lattice under the $(1, 1, 1)$ metric. Return walks under this metric have the length generating function

$$L(z^2, z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2} = M(z)\,, \tag{2.14}$$

where $M(z)$ is the generating function for Motzkin numbers.[d]

[b] The sequence of terms $\{\frac{1}{n+1}\binom{2n}{n}\}$ for $n \geq 0$, now commonly referred to as the sequence of Catalan numbers c_n, has a long and often misrepresented history starting in the middle of the eighteenth century;[4,43] see also Refs. 44 and 45.

[c] The Schröder number sequences are named after the author of a paper concerning bracketing problems[47] in which the small Schröder numbers s_n (related linearly to r_n via $s_0 = 1, s_n = \frac{1}{2}r_n$) were mentioned. The sequence $\{s_n\}$ also has a curious history, beginning apparently in ancient Greece [Refs. 48 (mark 732)], 49 [mark 1047], for which see Refs. 50 and 51. Schröder numbers also occur often in combinatorics; see Refs. 4, 22 and 52, for example.

[d] The Motzkin number sequence m_n was first introduced in another area of combinatorics in Ref. 53. Further details of the many classes of combinatorial objects counted by Motzkin numbers can be found in Refs. 4, 46 and 54–56. The "standard" walks enumerated by Motzkin numbers are walks not on a lattice of equilateral triangles, but on a graph that is a different generalisation of the $\mathbb{S}$ lattice.[10,30,40,57]

Case 2.4. Return walks under the $(1,2,2)$ metric have the length generating function

$$L(z^3, z^2) = \frac{1 - z^2 - \sqrt{1 - 2z^2 - 4z^3 + z^4}}{2z^3} . \tag{2.15}$$

This is essentially the generating function of Sequence A025250 in the Sloane encyclopaedia,[58] and so there is now a lattice derivation of the terms of that sequence.

3. Single Walks with Return Contact Weights

We now assign a weight κ to a site on the wall arrived at by a down step, and assign a weight μ to a site on the wall arrived at by a horizontal step along the wall (see Fig. 6). These weights describe the *return contacts* of the walk with the wall. Walks have been counted by their number of returns to the wall in combinatorics also; examples are found in Refs. 33, 37 and 59–61.

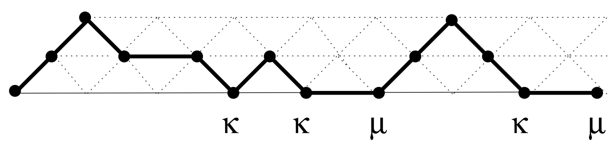

Fig. 6. Weights of (return) contacts representing the interaction between the walk and the wall. The weight κ is associated with sites that are on the wall having incident steps that are down steps, while the weight μ is associated with sites that are on the wall having incident steps that are horizontal steps.

Let $\mathbf{L}(v, d; \kappa, \mu)$ be a generating function

$$\mathbf{L}(v, d; \kappa, \mu) = \sum_{w \in \mathcal{W}} v^{a(w)} d^{c(w)} \kappa^{m(w)} \mu^{n(w)} \tag{3.16}$$

that enumerates return walks by types of steps and also by number of return contacts with the wall. A return walk w of $a(w)$ up (and so down) steps and $c(w)$ horizontal steps that has $m(w)$ down steps incident on the wall and $n(w)$ horizontal steps along the wall then contributes a unit $v^{a(w)} d^{c(w)} \kappa^{m(w)} \mu^{n(w)}$ term to $\mathbf{L}(v, d; \kappa, \mu)$.

The function $\mathbf{L}(v, d; \kappa, \mu)$ is simple to derive using the decomposition of return walks introduced in Sec. 2. Here, though, as a method which generalises to walk pairs it is derived by splitting a return walk into components by each return to the wall. The length generating function of return walks on the $\mathbb{S}$ lattice was derived using different terminology in Ref. 9 (p. 128) by separating a return walk into its elevated components, i.e.

$$L(v, 0) = \frac{1}{1 - \hat{L}(v, 0)} . \tag{3.17}$$

In general, on the $\mathbb{T}$ lattice, where in addition to elevated walk components, a return walk may also have single horizontal step components along the wall,

$$L(v,d) = \frac{1}{1-d-\hat{L}(v,d)}\,, \tag{3.18}$$

which, since $L(v,d) = v\hat{L}(v,d)$, is then the same as Eq. (2.5), thus the concatenation of elevated components provides a direct combinatorial interpretation of Eq. (2.5).

An elevated walk has a single return contact with the wall from the down step to its ending site (that is weighted by κ). A horizontal step along the wall has a single *return* contact at its ending site (that is weighted by μ). By incorporating these two weights of return contact to Eq. (3.18), one obtains

$$\mathbf{L}(v,d;\kappa,\mu) = \frac{1}{1-\mu d-\kappa\hat{L}(v,d)} = \frac{2}{2-\kappa+d(\kappa-2\mu)+\kappa\sqrt{(1-d)^2-4v}}\,. \tag{3.19}$$

Cases of $\mathbf{L}(v,d;\kappa,\mu)$ for various values of the contact weights are given below.

3.1. *Special cases*

Case 3.1 (Vanishing wall). If $(\kappa,\mu) = (2,1)$, then

$$\mathbf{L}(v,d;2,1) = \frac{1}{\sqrt{(1-d)^2-4v}}\,, \tag{3.20}$$

which is a generating function for bilateral walks on $\mathbb{T}$, i.e. for walks not in the half-plane but in the infinite plane that end at height 0 ($h=0$). In other words, setting κ to 2 is the same as removing the wall and counting all walks that traverse the plane (in directed manner) freely so long as they end at height 0. This can be thought of as having two choices for each elevated walk component returning to the wall, one above the wall, and the other its reflection below the wall. This behaviour for $\kappa = 2$ was first noted for the $\mathbb{S}$ lattice in Ref. 2, where it represents the behaviour of a system at its critical point.

Case 3.2 ($\boldsymbol{\mu} = 0$ or bouncy walks). The return walks that have weights (κ,μ) set to $(1,0)$ can touch the wall at sites other than their starting and ending sites but do not include steps along the wall. Thus such walks are concatenations of elevated walks. The single site (zero-step walk) is from these definitions included in this set.

A generating function enumerating walks that "bounce" off the wall (*bouncy* walks) is then

$$\check{L}(v,d) = \mathbf{L}(v,d;1,0) = \frac{1}{1-vL(v,d)} \tag{3.21}$$

$$= \frac{1}{1-\hat{L}(v,d)} = \frac{2}{1+d+\sqrt{(1-d)^2-4v}}\,. \tag{3.22}$$

On any parallelogram lattice, i.e. the $\mathbb{T}$ lattice under an $(\alpha,\beta,-)$ metric, $\check{L}(v,d) = L(v,d)$ since no horizontal steps (represented by the variable d) are allowed along the wall or elsewhere in a walk. On triangular lattices, however, $\check{L}(v,d) \neq L(v,d)$. Two specific subcases for which the numbers of bouncy walks of length n are well-known sequences are given below.

Subcase 3.2.1. Under the $(1,1,2)$ metric, the isotropic length generating function for bouncy walks is

$$\check{L}(z^2,z^2) = \frac{1+z^2-\sqrt{1-6z^2+z^4}}{4z^2} = S(z^2)\,, \tag{3.23}$$

where $S(z^2)$ is the generating function for (aerated) small Schröder numbers s_n. The first lattice derivation of a generating function for the sequence $\{s_n\}$ apparently is in Ref. 22, but this was in another context. From Eq. (3.21), the small and large Schröder numbers can be related using

$$S(z^2) = \frac{1}{1-z^2R(z^2)}\,. \tag{3.24}$$

Subcase 3.2.2. Under the $(1,1,1)$ metric, i.e. on the isotropic triangular lattice, the perimeter generating function for bouncy walks is

$$\check{L}(z^2,z) = \frac{1}{2z}\left(1-\sqrt{\frac{1-3z}{1+z}}\right)\,. \tag{3.25}$$

The sequence of coefficients $\{[z^n]\check{L}(z^2,z)\}$ was first considered in the context of rooted trees in Ref. 39, where it was referred to as $\{\gamma_n\}$. The numbers γ_n count the number of bouncy walks of length n on the $\mathbb{T}$ lattice under the $(1,1,1)$ metric.[e] One example of their relationship to Motzkin numbers (that counted return walks under the same metric), is, from Eq. (3.21),

$$\check{L}(z^2,z) = \frac{1}{1-z^2M(z)}\,. \tag{3.26}$$

This relation was shown algebraically in Ref. 54 (p. 277).

3.2. *Ballot walks*

A ballot walk, in the literature often called a ballot path or a left factor, is a walk that starts on the wall remains in the half-plane on or above the wall, and finishes at an arbitrary (integer) height above the wall. On the $\mathbb{S}$ lattice, such a walk is a representation of a ballot between two candidates A and B, where at any point in the counting, A always has at least as many votes as B. Ballot problems have been generalised to include various kinds of winning margins, further candidates and in other ways. One summary of their early history is found in Ref. 62.

A ballot walk can be decomposed into sections, called "terraces", by considering the last site at which the walk is at each height value less than the final height.[61]

[e]More recently, the coefficients have been called the ring numbers, and also Riordan numbers.[56] Further properties of the sequence γ_n are found in Refs. 46 and 54 and in particular in Ref. 56.

The section of a ballot walk that is after the last step that leaves a height $h = i$, and before the last step that leaves height $h = i+1$, is a (sub)-walk that starts and ends at height $h+1$, and does not drop below height $h+1$. This is shown in Fig. 7. Thus ballot walks can be described in terms of return walks.

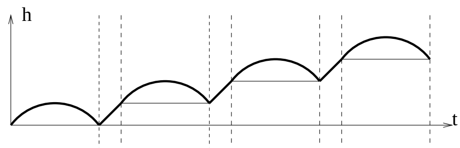

Fig. 7. A ballot walk can be described in terms of return walks.

Anisotropic length generating functions on the $\mathbb{T}$ lattice can be used to construct length generating functions for ballot walks ending at height k under various metrics. Indeed, if $\mathbf{B}_k(x, y, d; \kappa, \mu)$ is a generating function enumerating ballot walks that end at height k above the wall by types of steps and also by return contacts, then

$$\mathbf{B}_k(x, y, d; \kappa, \mu) = \mathbf{L}(v, d; \kappa, \mu) y^k L(xy, d)^k\,, \tag{3.27}$$

and a function enumerating ballot walks by types of steps, return contacts and height is then

$$\mathbf{B}(x, y, d; \kappa, \mu; w) = \sum_{k\geq 0} \mathbf{B}_k(x, y, d; \kappa, \mu) w^k = \frac{\mathbf{L}(v, d; \kappa, \mu)}{1 - yL(xy, d)w}\,. \tag{3.28}$$

4. Return Walks by Length and Area

4.1. *Continued fraction generating functions*

With $\mathcal{W}$ denoting the set of return walks on the $\mathbb{T}$ lattice, let $A(v, d; q)$ be the generating function

$$A(v, d; q) = \sum_{w\in\mathcal{W}} v^{a(w)} d^{c(w)} q^{i(w)}$$

that enumerates return walks by types of steps and standard area, where the standard area of a return walk on the $\mathbb{T}$ lattice is the number of triangular cells enclosed between it and the wall. A walk w with $a(w)$ up (and so down) steps and $c(w)$ horizontal steps that encloses $i(w)$ units of area between it and the wall then contributes a $v^{a(w)} d^{c(w)} q^{i(w)}$ term to $A(v, d; q)$.

Example 4.1. There are six return walks on the $\mathbb{T}$ lattice that consist of an up step, a down step and two horizontal steps. Of these, the walks of (a), (b) and (c) in Fig. 8 each enclose one (triangular) cell, (d) and (f) each enclose three cells

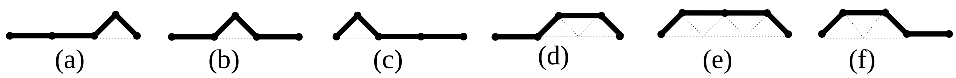

Fig. 8. Cells enclosed by return walks that each have anisotropic length term of vd^2.

and the elevated walk (e) encloses five cells. Thus these six walks contribute a $vd^2(3q+q^3+q^5)$ term to $A(v,d;q)$.

A continued fraction representation of $A(v,d;q)$ is found here by decomposing return walks at their first return to the wall, as was done to find $L(v,d)$. By considering the area enclosed before the first return for each category of walk in the decomposition of Fig. 4 the equation

$$A(v,d;q) = 1 + dA(v,d;q) + vqA(vq^2,dq^2;q)A(v,d;q) \tag{4.29}$$

is obtained. The similar relation (2.2) for $L(v,d)$ had an algebraic solution, but Eq. (4.29) does not, due to the $A(vq^2,dq^2;q)$ term. By collecting the $A(v,d;q)$ terms, however, a continued fraction representation of $A(v,d;q)$ is found as

$$\begin{aligned} A(v,d;q) &= \frac{1}{1-d-vqA(vq^2,dq^2;q)} \\ &= \frac{1}{1-d-}\frac{vq}{1-dq^2-}\frac{vq^3}{1-dq^4-}\cdots\frac{vq^{2h+1}}{1-dq^{2h+2}-\cdots}. \end{aligned} \tag{4.30}$$

A corresponding function for elevated walks is then

$$\hat{A}(v,d;q) = vqA(vq^2,dq^2;q)\frac{vq}{1-dq^2-}\frac{vq^3}{1-dq^4-}\cdots\frac{vq^{2h+1}}{1-dq^{2h+2}-\cdots}. \tag{4.31}$$

An alternative derivation of these continued fractions, presented in Ref. 10 and also noted in Ref. 52, is to include in the function $L^{\#}$ for example, i.e. in Eq. (2.8), the area under steps at each height level via substitutions such as $v_i \to q^{2i+1}v^i$.

4.2. *Infinite sum generating functions*

In previous sections the set of return walks was assumed to be already constructed, and relations satisfied by generating functions were derived by categorising the walks in the set by their component sections between contacts with the wall. If, instead, the set of return walks is built up by starting with walks of minimal length and applying operators to construct progressively longer walks, then relations satisfied by generating functions can also be derived from the construction process.

In particular, a "local expansion" of the *last fall* (of down steps to the ending site) of return walks that was introduced in Ref. 63 can be used to construct walks recursively from other walks. An equation satisfied by a generating function enumerating return walks by their length, area and number of steps in their last fall can be derived from the expansion. This method of enumerating combinatorial objects by constructing objects of given size (here length) from objects of smaller size has been applied to find generating functions enumerating return walks by length and standard area, and also length and the so-called non-decreasing-point-area (which we will denote $\triangle$-area: see later for definition) in Refs. 31 and 63. It is known as the *Enumerating Combinatorial Objects method* (or ECO method) and can be used for the enumeration of more general combinatorial objects also. More details and examples can be found in Refs. 64 and 65 and the references therein.

In this section, the ECO method is used to find an equation satisfied by a generating function that enumerates *elevated* walks on the $\mathbb{T}$ lattice by types of steps, standard area and steps in their last fall. The function $\hat{A}(v, d; q)$ is obtained from this equation by an iterative technique. We then consider contact weights.

4.3. *The Enumerating Combinatorial Objects method*

The essence of the Enumerating Combinatorial Objects (ECO) method is the following proposition that is found in Ref. 63, amongst others:

Proposition 4.1. *Let $\mathcal{S}$ be a class of combinatorial objects and $\mathcal{S}_n$ the subsets of objects having a fixed size n. Define the operator Θ on $\mathcal{S}_n$ as a function from $\mathcal{S}_n$ to the power set of the elements of $\mathcal{S}_{n+1}$. Suppose that Θ is an operator on $\mathcal{S}$ (and so on $\mathcal{S}_n$ for all n).*

If for all $Y \in \mathcal{S}_{n+1}$, there exists an $X \in \mathcal{S}_n$ such that $Y \in \Theta(X)$, and, if for all $X_1, X_2 \in \mathcal{S}_n$ with $X_1 \neq X_2$, $\Theta(X_1) \cap \Theta(X_2) = \emptyset$, then the set family $\mathcal{O} = \{\Theta(X) | X \in \mathcal{S}_n\}$ is a partition of $\mathcal{S}_{n+1}$.

That is, if for a given class $\mathcal{S}$ such an operator Θ can be found, each element of $\mathcal{S}_{n+1}$ can be constructed via Θ from one and only one element of $\mathcal{S}_n$. The elements of $\mathcal{S}_n$ would then have a recursive description. Often, as is the case here, a functional equation for a generating function enumerating elements of $\mathcal{S}$ can be derived from this description.

Here, the ECO method is used to enumerate elevated walks; from this an expression for the area-perimeter function $\hat{A}(v, d; q)$ is obtained. The method requires a set of objects and an operator on that set in order to generate the objects recursively. Let the set of elevated walks on the $\mathbb{T}$ lattice be denoted by $\mathcal{E}$. An appropriate class of elevated walks on which to apply the ECO method is the set $\mathcal{E}_{2n}$ defined here as those walks that have ending site at $(t, h) = (2n, 0)$. A satisfactory choice of operator is one that from $\mathcal{E}_{2n}$ constructs all elements of the set of elevated walks $\mathcal{E}_{2n+2}$ by inserting an up-and-down peak (represented by $xy = v$) or a horizontal step (d) into the last fall of w. If this operator is denoted as $\Theta_{\mathcal{E}}$, then

$$\begin{aligned} &\Theta_{\mathcal{E}} : \mathcal{E}_{2n} \to \mathcal{E}_{2n+2}, \\ &\Theta_{\mathcal{E}}(w) = \{u \in \mathcal{E}_{2n+2}, u = w'Cw'', \text{ with } w = w'w'', \\ &\qquad\qquad w'' \in \{y, yy, yyy, \ldots\}, C \in \{v, d\}\}. \end{aligned} \tag{4.32}$$

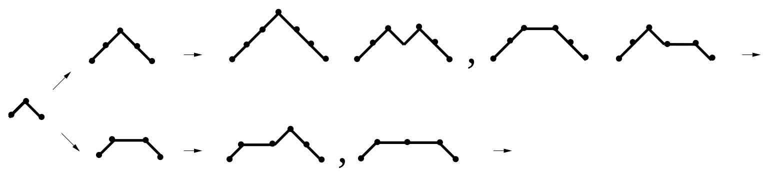

Fig. 9. Construction of the set of elevated walks from the up-down step pair walk via the operator $\Theta_{\mathcal{E}}$.

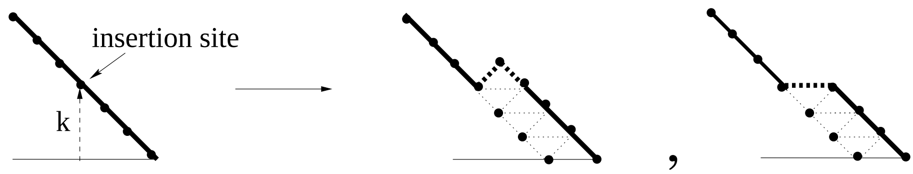

Fig. 10. Elevated walks obtained by inserting a peak or horizontal step in the last fall of an elevated walk, with extra cells enclosed shown.

Proof that this operator $\Theta_{\mathcal{E}}$ on $\mathcal{E}$ satisfies Proposition 4.1 follows the proofs of similar results in Refs. 31 and 63 and is not detailed here. The construction of the sets $\mathcal{E}_{2n}$ from the (shortest) elevated walk, the up-down step pair, is shown for small n in Fig. 9.

The insertion of an extra step or step pair to the last fall adds both steps and units of area to an elevated walk. Let the variable s count the number of steps in the last fall of the walk and let $\hat{W}(s; v, d; q)$ (or $\hat{W}(s)$ for short) be the generating function

$$\hat{W}(s; v, d; q) = \sum_{w \in \mathcal{E}} s^{\mathrm{fall}(w)} v^{a(w)} d^{c(w)} q^{i(w)} \tag{4.33}$$

that enumerates elevated walks by steps in the last fall, types of steps and standard area. The desired function, the area-perimeter function $\hat{A}(v, d; q)$, is then simply $\hat{W}(1; v, d; q)$.

The set of elevated walks $\mathcal{E}$ can be reclassified into sets according to the number of steps fall(w) of the last fall. For any elevated walk w, suppose the insertion in the last fall is made at height k, where $1 \leq k \leq \mathrm{fall}(w)$ since the result of applying $\Theta_{\mathcal{E}}$ must still be an elevated walk. If the insertion is a peak, this constructs a new walk with fall$(w) = k + 1$ and area $2k + 1$ units larger than that of w. Similarly, inserting a horizontal step constructs a new walk with fall$(w) = k$ and area increased over that of w by $2k$ units. These insertions are shown in Fig. 10.

A functional equation for $\hat{W}(s)$ is found by collecting together the results of applying $\Theta_{\mathcal{E}}$ on all elevated walks, and adding to this the shortest elevated walk, the up-down step pair, that cannot be generated by $\Theta_{\mathcal{E}}$.

Thus

$$\begin{aligned}\hat{W}(s) &= svq + \sum_{w\in\mathcal{E}} \sum_{k=1}^{\mathrm{fall}(w)} \{s^{k+1} v^{a(w)+1} d^{c(w)} q^{i(w)+2k+1} + s^k v^{a(w)} d^{c(w)+1} q^{i(w)+2k}\} \\ &= svq + (d + svq)\frac{sq^2}{1 - sq^2} \sum_{w\in\mathcal{E}} v^{a(w)} d^{c(w)} q^{i(w)} (1 - (sq^2)^{\mathrm{fall}(w)}) \\ &= svq + (d + svq)\frac{sq^2}{1 - sq^2}(\hat{W}(1) - \hat{W}(sq^2))\,. \end{aligned} \tag{4.34}$$

4.4. *Iterative solutions of functional equations*

The functional equation (4.34) contains the desired $\hat{W}(1)$ term but also a $\hat{W}(s)$ and a $\hat{W}(sq^2)$ term. It cannot be solved immediately for $\hat{W}(1)$ if information about the area enclosed by the walk, given by the exponent of q, is to be retained, since some contribution to this exponent comes from the $\hat{W}(sq^2)$ term. Commonly a solution for $\hat{W}(1)$ is found by iterating the functional equation to remove the $\hat{W}(sq^2)$ term. The telescoping technique used below is well-known; the presentation of it here is based upon that in Ref. 66.

If in Eq. (4.34), s is replaced with sq^2, then the equation

$$\hat{W}(sq^2) = svq^3 + (d + svq^3)\frac{sq^4}{1 - sq^4}(\hat{W}(1) - \hat{W}(sq^4)) \tag{4.35}$$

is obtained, so that the $\hat{W}(sq^2)$ term can be removed in Eq. (4.34), leaving

$$\begin{aligned}\hat{W}(s) = svq &- (d + svq)\frac{sq^2}{1 - sq^2} svq^3 \\ &+ (d + svq)\frac{sq^2}{1 - sq^2}\left(1 - (d + svq^3)\frac{sq^4}{1 - sq^4}\right)\hat{W}(1) \\ &+ (d + svq)\frac{sq^2}{1 - sq^2}(d + svq^3)\frac{sq^4}{1 - sq^4}\hat{W}(sq^4)\,. \end{aligned} \tag{4.36}$$

The substitution $s \to sq^2$ can then be used in Eq. (4.35) to obtain a relation between $\hat{W}(sq^4)$, $\hat{W}(1)$ and $\hat{W}(sq^6)$, and then again in this latter relation to obtain a further relation between $\hat{W}(sq^6), \hat{W}(1)$ and $\hat{W}(sq^8)$, and so on. After N iterations of the substitution,

$$\begin{aligned}\hat{W}(s) = svq &\left(1 + \sum_{n\geq 1}^{N} (-1)^n \prod_{j=1}^{n} sq^2 \frac{q^{2j}(d + svq^{2j-1})}{1 - sq^{2j}}\right) \\ &+ \left(\sum_{n\geq 0}^{N} (-1)^n \prod_{j=0}^{n} sq^2 \frac{q^{2j}(d + svq^{2j+1})}{1 - sq^{2j+2}}\right)\hat{W}(1) \\ &- \left((-1)^N \prod_{j=0}^{N} sq^2 \frac{q^{2j}(d + svq^{2j+1})}{1 - sq^{2j+2}}\right)\hat{W}(sq^{2N+2})\,. \end{aligned} \tag{4.37}$$

The coefficient that is the left part of the final term of Eq. (4.37) is a product, not a sum of products. If the length variables d and v in the equation are isotropised to z^γ and $z^{\alpha+\beta}$, then the minimum exponent of z in the entire final term will be at least either $(N+1)\gamma$ or $(N+1)(\alpha+\beta)$, both of which increase with N. Thus in the limit $N \to \infty$, then, Eq. (4.37) is

$$\hat{W}(s) = svq\left(1 + \sum_{n\geq 1}(-1)^n \prod_{j=1}^{n} sq^2 \frac{q^{2j}(d+svq^{2j-1})}{1-sq^{2j}}\right) + \left(\sum_{n\geq 0}(-1)^n \prod_{j=0}^{n} sq^2 \frac{q^{2j}(d+svq^{2j+1})}{1-sq^{2j+2}}\right)\hat{W}(1)\,. \tag{4.38}$$

The functional equation (4.34) has now been reduced to an equation with just $\hat{W}(1)$ and $\hat{W}(s)$ terms. The variable s counting the last fall is not needed in $\hat{A}(v,d;q) = \hat{W}(1)$, so setting s to 1 in Eq. (4.38) and rearranging gives the desired $\hat{A}(v,d;q)$ as

$$\hat{A}(v,d;q) = \frac{vq\sum_{n\geq 0}(-1)^n q^{n(n+3)}(q^2;q^2)_n^{-1}\prod_{j=1}^{n}(d+vq^{2j-1})}{\sum_{n\geq 0}(-1)^n q^{n(n+1)}(q^2;q^2)_n^{-1}\prod_{j=1}^{n}(d+vq^{2j-1})}, \tag{4.39}$$

where the notation

$$(a;q)_j = \prod_{j=0}^{n-1}(1-aq^j), \qquad (a;q)_0 = 1 \tag{4.40}$$

has been used.

Hence the generating function for return walks is found from Eqs. (4.30), (4.31) and (4.39) as

$$A(v,d;q) = \frac{\sum_{n\geq 0}(-1)^n q^{n(n+1)}(q^2;q^2)_n^{-1}\prod_{j=1}^{n}(d+vq^{2j-1})}{\sum_{n\geq 0}(-1)^n q^{n(n-1)}(q^2;q^2)_n^{-1}\prod_{j=1}^{n}(d+vq^{2j-1})}\,. \tag{4.41}$$

Case 4.1 (Return walks on the isotropic $\mathbb{S}$ lattice). The $\mathbb{S}$ lattice is the $\mathbb{T}$ lattice under the $(1,1,-)$ metric, so from Eq. (4.41) we have

$$\begin{aligned} A(z^2,0;q) &= \frac{1}{1-}\,\frac{z^2q}{1-}\,\frac{z^2q^3}{1-}\cdots\frac{z^2q^{2h+1}}{1-\cdots} \\ &= \frac{\sum_{n\geq 0}(-z^2)^n q^{2n^2+n}(q^2;q^2)_n^{-1}}{\sum_{n\geq 0}(-z^2)^n q^{2n^2-n}(q^2;q^2)_n^{-1}}\,. \end{aligned} \tag{4.42}$$

A lattice derivation of the first line of this equation can be found in Ref. 10, whilst the second line is a special case of a more general result [Ref. 14, Thm.12] which was derived using a different method. The equation is also an identity related to the so-called Rogers–Ramanujan continued fraction.[67]

Both representations of $A(z^2,0;q)$ in Eq. (4.42) are q-analogues of $L(z^2,0)$. Similarly, the representation of $\hat{A}(v,d;q)$ in Eq. (4.39) is a q-analogue of $\hat{L}(v,d)$ just as was the representation in Eq. (4.31). Unlike continued fractions, however, with ratios of infinite sums it is not possible to set q to 1 and obtain the anisotropic length generating function, because the expressions are singular at $q = 1$. Nonetheless, length generating functions can still be obtained from intermediate results in the derivation of the infinite sums. For example, the function $\hat{L}(v, d)$ can be obtained by setting $q = 1$ in Eq. (4.34) and collecting terms. Such a derivation is an example of what has become known as the *kernel method*, for which see Ref. 68 (with references therein) and also Ref. 69

4.5. *Return walks with contact weights and area*

In a similar manner to the perimeter-only generating functions in Sec. 3, contact weights can be included in the discussion by using the decomposition of return walks into pieces including elevated walks. The same can be done for length-area generating functions. Indeed, a return walk can be split by returns to the wall into single horizontal steps along the wall and elevated walks, and we note that horizontal steps along the wall do not enclose any area. Hence, the generating function $\mathbf{A}(v, d; q; \kappa, \mu)$ enumerating return walks according to their steps, the area under the walks, and the types of return contact (as described in Sec. 3) weighted by κ and μ satisfies the functional equation

$$\mathbf{A}(v, d; q; \kappa, \mu) = \frac{1}{1 - \mu d - \kappa \hat{A}(v, d; q)}. \tag{4.43}$$

Substitution of Eq. (4.39) and some rearrangement leads to

$$\mathbf{A}(v, d; q; \kappa, \mu) = \frac{\sum_{n\geq 0}(-1)^n q^{n(n+1)}(q^2; q^2)_n^{-1} \prod_{j=1}^{n}(d + vq^{2j-1})}{\sum_{n\geq 0} \mathbf{G}_n(d; q; \kappa, \mu)(-1)^n q^{n(n-1)}(q^2; q^2)_n^{-1} \prod_{j=1}^{n}(d + vq^{2j-1})}, \tag{4.44}$$

where

$$\mathbf{G}_n(d; q; \kappa, \mu) = \kappa(1 - q^{2n}) + q^{2n} + (\kappa - \mu)dq^{2n}. \tag{4.45}$$

4.6. *Walks by length and △-area*

The △-area of a return walk on the $\mathbb{T}$ lattice is the number of *up* triangular cells enclosed between the walk and the wall. An *up* triangle, denoted △, is one which has an apex pointing upwards. Functions that are derived using △-area are given a up-pointing triangular ($_\triangle$) subscript. The △-area is also known as the non-decreasing-point-area.

Example 4.2. Of the six return walks on the $\mathbb{T}$ lattice represented by the anisotropic length term vd^2, the walks (a)–(c) in Fig. 11 enclose one unit of △-area, (d) and (f) each enclose two units and (e) encloses three units.

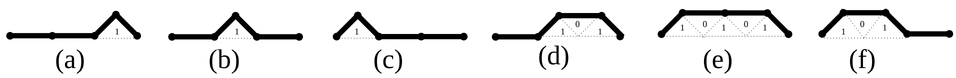

Fig. 11. The $\triangle$-area of walks that are represented by the anisotropic length term vd^2, as the sum of the number of up triangles.

Generating functions enumerating elevated or return walks by $\triangle$-area as well as other characteristics can be found by the same processes that led to the corresponding standard area functions above. Such functions, however, are linked by the observation that the standard area for single walks can be found in terms of the $\triangle$-area:

$$A_{\triangle}(v, d; q) = A(\sqrt{q}v, d; \sqrt{q}) \tag{4.46}$$

and

$$\hat{A}_{\triangle}(v, d; q) = \hat{A}(\sqrt{q}v, d; \sqrt{q})\,. \tag{4.47}$$

It appears that $\triangle$-area was first used as a definition of the area of a walk in Ref. 63 for Motzkin paths; these biject to walks under the $(1, 1, 1)$ metric, so giving $A_{\triangle}(z^2, z; q)$. The $\triangle$-area has since been used to find $A_{\triangle}(z^2, dz^2; q)$, i.e. to consider return walks under the $(1, 1, 2)$ metric also enumerated by the number of horizontal steps; these walks were then related to permutations with forbidden sequences enumerated by number of inversions in Ref. 31. Both of these previous uses considered walks on one lattice; here they have been generalised by the enumeration of walks by types of steps on the $\mathbb{T}$ lattice, and by the inclusion of contact weights.

4.7. *Fountains of coins*

A *fountain* of coins is an arrangement of identical circles in rows such that any circle not in the bottom row is supported by, i.e. touches, exactly two circles in the row below. An (n, k)-fountain is then an arrangement of n coins into rows such that there are k coins in the bottom row.[70] A $(16, 8)$-fountain is shown in Fig. 12(a). Fountains of coins have been studied in both statistical mechanics and combinatorics over the past fifty years. An early study of fountains of coins is Ref. 71, in which fountains were used to enumerate partitions of a set into smaller sets under certain restrictions. The fountains used had a single contiguous block of coins in each row (since called *block fountains*[72]). Another partition enumeration problem, solved in Ref. 73, gave a combinatorial interpretation of the continued fraction of $A_{\triangle}(z^2, 0; q)$, in that the number of partitions of $n + \binom{k}{2}$ for which the largest part is k and for all $i, 1 \leq i \leq k$, the ith part is at least i, is $[z^{2k}q^n]A_{\triangle}(z^2, 0; q)$. This second partition problem can also be represented by fountains (see Ref. 70).

An expression for $A_{\triangle}(z^2, 0; q)$ as a generating function enumerating fountains of coins is found in Ref. 70. There, the function was found by decomposing fountains in a manner similar to decomposing walks by their first return to the wall. These

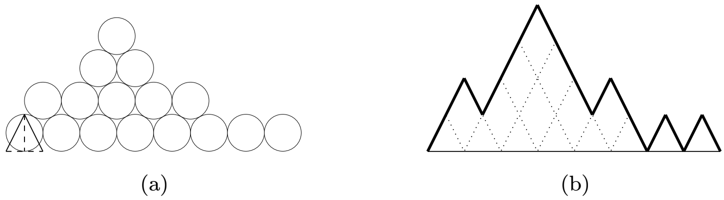

Fig. 12. (a) A $(16, 8)$-fountain; (b) the return walk on the parallelogram lattice corresponding to the fountain.

general (i.e. not just block) fountains had previously been mentioned in Ref. 74. Here, in a different method, a construction linking fountains and triangles of $\triangle$-area of return walks on the $\mathbb{S}$ lattice is used to show that $A_\triangle(z, 0; q)$ is a generating function enumerating fountains by coins in the bottom row (counted by z) and coins in total (counted by q).

A section of a parallelogram lattice can be constructed from a fountain as is clear from Fig. 12. Each coin corresponds to one unit of $\triangle$-area, as can be seen by inserting a horizontal line across the middle of each parallelogram cell of the lattice and marking the up-pointing triangles. This parallelogram lattice is equivalent under a planar isomorphism to the $\mathbb{S}$ lattice. Thus an (n, k)-fountain bijects to a return walk on the $\mathbb{S}$ lattice of length $2k$ and $\triangle$-area of n units. A similar bijection was given in Ref. 75.

A generating function for (n, k)-fountains according to total number of coins (counted by q) and number of coins in the bottom row (counted by z) is then

$$A_\triangle(z, 0; q) = \frac{1}{1-}\frac{zq}{1-}\frac{zq^2}{1-}\cdots\frac{zq^{h+1}}{1-\cdots} = \frac{\sum_{n\geq 0}(-z)^n q^{n(n+1)}(q;q)_n^{-1}}{\sum_{n\geq 0}(-z)^n q^{n^2}(q;q)_n^{-1}}\,. \tag{4.48}$$

It is well-known (for example, see Refs. 4 and 70) that the number of fountains with k coins in the bottom row (regardless of n) is the kth Catalan number c_k; this can be deduced by treating fountains as return walks.

Another question asked about fountains, however, seeks to find the number of fountains of n coins in total, regardless of the number in the bottom row. This question is then equivalent to one asking for the number of return walks of given $\triangle$-area on the $\mathbb{S}$ lattice, regardless of length, and is answered by setting $z = 1$ and considering only the area variable q in Eq. (4.48).

5. First Area-Moment of Walks

Let $\mathbf{TA}(v, d; \kappa, \mu)$ be the generating function

$$\mathbf{TA}(v, d; \kappa, \mu) = \sum_{w\in\mathcal{W}} i(w) v^{a(w)} d^{c(w)} \kappa^{m(w)} \mu^{n(w)} \tag{5.49}$$

that enumerates return walks by types of steps and by return contacts, and in which each walk is given a coefficient weighting equal to its standard area $i(w)$.

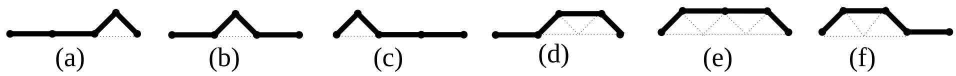

Fig. 13. Return walks that have anisotropic length term of vd^2.

The coefficient of $v^a d^c$ in $\mathbf{TA}(v, d; \kappa, \mu)$ is then the contact-weighted total of the areas of all return walks with that set of steps. The functions $\mathbf{TA}_\triangle(v, d; \kappa, \mu)$, for which $\triangle$-area is counted, and $\widehat{TA}(v, d)$ and $\widehat{TA}_\triangle(v, d)$, which enumerate the first area-moment enclosed by elevated walks in units of standard area and $\triangle$-area respectively, are defined similarly. Here generating functions enumerating the first area-moment of walks by their length are called *first area-moment* functions.

Example 5.1. Of the six return walks on the $\mathbb{T}$ lattice that have anisotropic length term of vd^2, three enclose one unit of standard area each, two enclose three units each and one encloses five units. Thus the total of the standard areas of those return walks is 14 units. Since the $\triangle$-area of a return walk is the number of up triangles enclosed between the walk and the wall, the total of the $\triangle$-areas of return walks that have anisotropic length term of xyd^3 is 10 units.

In theory, all area-moment functions could be calculated as q-derivatives of the corresponding length-area functions found in Sec. 4. For example,

$$\mathbf{TA}(v, d; \kappa, \mu) = \left[q \frac{\partial}{\partial q} \mathbf{A}(v, d; q; \kappa, \mu) \right]_{q=1} . \tag{5.50}$$

It is possible to find such derivatives from the functional equations, such as Eq. (4.29) for first area-moment functions. We however take a different route, more combinatorial in nature that can prove useful in more complex situations. Here, first area-moment functions are found considering the two-dimensional area as a sum of one-dimensional heights. The standard area of a walk on the $\mathbb{T}$ lattice is also the sum of the heights of the walk at all integer values of the t coordinate since the area in one column of the walk is simply the height of that column. If the height of the walk, w, at $t = i$ is denoted by $h_i(w)$, then

$$\mathbf{TA}(v, d; \kappa, \mu) = \sum_{w \in \mathcal{W}} \left\{ \sum_{i \geq 0} h_i(w) \right\} v^{a(w)} d^{c(w)} \kappa^{m(w)} \mu^{n(w)} . \tag{5.51}$$

The first area-moment of the set of elevated walks of a given length on the $\mathbb{T}$ lattice under the $(1, 1, 2)$ metric has been studied in Ref. 76 and the generating function $\widehat{TA}(z^2, z^2)$ in Ref. 52. In Refs. 77 and 78, both this function and the corresponding $\mathbb{S}$ lattice function $\widehat{TA}(z^2, 0)$ were considered, and in Ref. 40, the first area-moments of "generalised Motzkin paths", including both of these previous cases, were studied. The first area-moment of return walks on the $\mathbb{S}$ lattice, i.e. $TA(z^2, 0)$, was considered in Refs. 59 and 79, and the function $TA(z^2, z^2)$ was studied in Ref. 52. The generalisation to return walks on graphs that are not necessarily

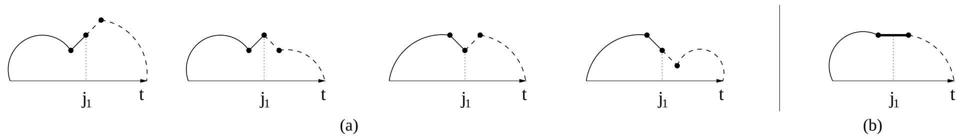

Fig. 14. Elevated walks that pass through (j_1, k): (a) at the end of a step; (b) in the middle of a horizontal step.

lattices formed from tilings of the plane was made in Ref. 80, and more recently in Ref. 14.

The total $\triangle$-area of a set of walks, however, does not appear to have been studied. The alternate (and original) definition of the $\triangle$-area of a walk is as the sum of the heights of the walk at the endpoint of each up step or horizontal step. Thus, if $\mathcal{I}_\triangle(w)$ is used to denote the subset of the integer t coordinates at which up or horizontal steps of a walk w end,

$$\mathbf{TA}_\triangle(v, d; \kappa, \mu) = \sum_{w \in \mathcal{W}} \left\{ \sum_{i \in \mathcal{I}_\triangle(w)} h_i \right\} v^{a(w)} d^{c(w)} \kappa^{m(w)} \mu^{n(w)} . \tag{5.52}$$

In this section, general expressions are found for $\widehat{TA}(v, d)$ and $\mathbf{TA}(v, d; \kappa, \mu)$, and for the corresponding $\triangle$-area functions also. Length metrics are then used to give examples of first area-moment functions on various lattice systems.

5.1. *First area-moment of elevated walks*

To find a generating function enumerating the first area-moment of elevated walks by their length, it is convenient to take the collection of height values of all the walks and regroup them by the value of the height. Here, this is done for the standard area case; the $\triangle$-area case can be derived in the same manner.

If $\widehat{K}_k(v, d)$ is a generating function in which the coefficient of $v^{a(w)} d^{c(w)}$ is the number of points at integer coordinates of t that are at height k in all elevated walks with that set of steps, then

$$\widehat{TA}(v, d) = \sum_{k \geq 0} k \hat{K}_k(v, d) . \tag{5.53}$$

The function $\widehat{TA}(v, d)$ is found here by first deriving an expression for $\hat{K}_k(v, d)$ via a convolution previously used in Refs. 40 and 79 which in turn is based upon the one in Ref. 81.

Suppose an elevated walk ends at $(t, h) = (j_2, 0)$. If the walk passes through the coordinate (j_1, k), then at that point, the walk is either at the end of a step or in the middle of a horizontal step, as is shown in Fig. 14.

In the first case, the number of walk configurations from $(0, 0)$ to (j_1, k) is equal to the number of ballot walks from the origin to (j_1, k) that do not return to the wall. The remainder of the elevated walk, from (j_1, k) to the endpoint, is then the *reverse* of a similarly restricted ballot walk from the origin to $(j_2 - j_1, k)$. In the second case, since the walk must include a horizontal step from $(j_1 - 1, k)$ to $(j_1 + 1, k)$, the relevant ballot walks are from the origin to $(j_1 - 1, k)$ and $(j_2 - j_1 - 1, k)$. The anisotropic length generating function of ballot walks from the origin to a site at height k that do not return to the wall is, from Eq. (3.27), equal to $y^k L(xy, d)^k$ and so the corresponding function for reversed ballot walks from height k that end at the wall but do not drop down to it before then is $x^k L(xy, d)^k$. Thus, summing ballot walks over j_1 and j_2 gives, for $k \geq 1$, and where as previously $v = xy$,

$$\widehat{K}_k(v, d) = v^k L(v, d)^{2k} (1 + d)\,, \tag{5.54}$$

from which

$$\begin{aligned}\widehat{TA}(v, d) = (1 + d) \sum_{k \geq 0} k v^k L(v, d)^{2k} &= (1 + d) \frac{vL(v, d)^2}{\left(1 - vL(v, d)^2\right)^2} \\ &= (1 + d) \frac{v}{\left(1 - d - 2vL(v, d)\right)^2} \\ &= \frac{v(1 + d)}{(1 - d)^2 - 4v}\,,\end{aligned} \tag{5.55}$$

where the second-last equality follows from Eq. (2.2) and the final equality from Eq. (2.4).

The corresponding function for the total $\triangle$-area of elevated walks of a given set of steps, $\widehat{TA}_{\triangle}(v, d)$, can be found by a similar convolution of ballot walks as

$$\widehat{TA}_{\triangle}(v, d) = \frac{v}{(1 - d)^2 - 4v} \frac{1 + dL(v, d)}{L(v, d)} = \frac{v}{(1 - d)^2 - 4v} \frac{1 + d + \sqrt{(1 - d)^2 - 4v}}{2}\,. \tag{5.56}$$

5.2. *First area-moment of return walks*

Generating functions enumerating the first area-moment of return walks by types of steps and return contacts can be derived from the corresponding functions for elevated walks. The method used in this section, as was used for similar derivations in the first two sections, is that of decomposing a return walk at its returns to the wall.

The following lemma relates the value of a function summed across all integer coordinates passed through by an elevated walk to the value of the same function summed across a return walk, and is styled on a similar lemma for walks on the $\mathbb{S}$ lattice in Ref. 59.

Lemma 1. *Let ϕ be a real-valued function on the non-negative integers and suppose that $\phi(0) = 0$. For $\mathcal{W}$ (sim. $\mathcal{E}$) the set of return (elevated) walks on the $\mathbb{T}$*

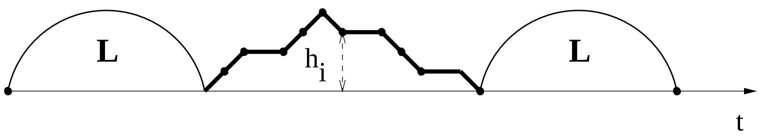

Fig. 15. Isolation of an elevated component of a return walk.

lattice, $\mathcal{I}(w)$ the set of integer values of the t coordinate passed through by a walk w and h_i the h coordinate of a walk at $t = i$, the generating functions

$$\mathbf{F}_\phi(v, d; \kappa, \mu) = \sum_{w \in \mathcal{W}} \left\{ \sum_{i \in \mathcal{I}(w)} \phi(h_i) \right\} v^{a(w)} d^{c(w)} \kappa^{m(w)} \mu^{n(w)} \tag{5.57}$$

and

$$\widehat{F}_\phi(v, d) = \sum_{w \in \mathcal{E}} \left\{ \sum_{i \in \mathcal{I}(w)} \phi(h_i) \right\} v^{a(w)} d^{c(w)} \tag{5.58}$$

satisfy

$$\mathbf{F}_\phi(v, d; \kappa, \mu) = \kappa \hat{F}_\phi(v, d) \mathbf{L}(v, d; \kappa, \mu)^2 \,, \tag{5.59}$$

where $\mathbf{L}(v, d; \kappa, \mu)$ is a generating function enumerating return walks by types of steps and return contacts.

Proof. The function ϕ is summed across its values at the heights of the walk at each integer t coordinate. Since ϕ takes the value zero at any coordinate at which the walk is at height 0, the coefficient of the term in $\mathbf{F}_\phi(v, d; \kappa, \mu)$ contributed by a return walk is therefore the sum of the areas of each of its elevated walk components. Any return walk w that does not have an elevated walk component is a concatenation of zero or more horizontal steps along the wall. The sum of the values of $\phi(h_i)$ across such a walk is then zero, so walks without elevated components can be discarded from consideration.

A new set of walks can be constructed by making as many copies of every walk $w \in \mathcal{W}$ as there are elevated components in w, and in each copy marking one of the components. This new set of *marked area walks* can be reordered by grouping together all walks that have the same marked component. Each element in a group can be characterised as being an elevated walk wedged between two (possibly empty) return walks, as is shown in Fig. 15. These two return walks are of arbitrary length, so if $w_1 \in \mathcal{E}$ is the marked elevated component, then the group of walks with that marked component contributes a

$$\mathbf{L}(v, d; \kappa, \mu) \left(\sum_{i \in \mathcal{I}(w_1)} \phi(h_i) v^{a(w_1)} d^{c(w_1)} \kappa \right) \mathbf{L}(v, d; \kappa, \mu) \tag{5.60}$$

term to $\mathbf{F}_\phi(v, d; \kappa, \mu)$. The result follows from summing over all possible elevated walk components. □

Corollary 1. *With $\phi(h_i) = h_i$,*

$$\mathbf{TA}(v, d; \kappa, \mu) = \kappa\widehat{TA}(v, d)\mathbf{L}(v, d; \kappa, \mu)^2 \,. \tag{5.61}$$

Lemma 5.1 can be modified to relate the generating functions that enumerate the total $\triangle$-area of elevated and return walks by their types of steps. Indeed, if the set $\mathcal{I}$ is changed to $\mathcal{I}_\triangle$, i.e. the integer values of only the t coordinates at the endpoint of up and horizontal steps of a walk, then for $\phi(h_i) = h_i$

$$\mathbf{TA}_\triangle(v, d; \kappa, \mu) = \kappa\widehat{TA}_\triangle(v, d)\ \mathbf{L}(v, d; \kappa, \mu)^2 \,. \tag{5.62}$$

Lemma 5.1 could also be generalised to replace the ϕ functions by, for example, the "possibility functions" described in Ref. 10, or by further changing the restrictions on the coordinate set $\mathcal{I}$, but these extensions are not studied here.

5.3. *Special cases*

Five special cases of first area-moment generating functions for elevated walks now follow. The first three cases use the standard area of a walk, whilst the final two use the $\triangle$-area. The first area-moment generating function for return walks is easily found in each case from Eq. (5.61) or (5.62), and is not always mentioned in the examples.

Case 5.1. Under the $(1, 1, -)$ metric, i.e. on the $\mathbb{S}$ lattice,

$$\widehat{TA}(z^2, 0) = \frac{z^2}{1 - 4z^2} \,, \tag{5.63}$$

so the total standard area of all elevated walks of length $2n + 2$ is 4^n triangles or $4^n/2$ squares. This is well-known. The result for return walks that have unit contact weights on the square lattice, $\mathbf{TA}(z^2, 0; 1, 1)$, i.e. $TA(z^2, 0)$, is

$$TA(z^2, 0) = \widehat{TA}(z^2, 0)L(z^2, 0)^2 = \frac{(1 - \sqrt{1 - 4z^2})^2}{4z^2(1 - 4z^2)} \,, \tag{5.64}$$

which was derived by other means in Ref. 80 and also in Refs. 59 and 79. Asymptotics for this and other first area-moment generating functions are found in Ref. 80.

Case 5.2. Under the $(1, 1, 2)$-metric,

$$\widehat{TA}(z^2, z^2) = \frac{z^2(1 + z^2)}{1 - 6z^2 + z^4}. \tag{5.65}$$

The coefficient of z^{2n} in $\widehat{TA}(z^2, z^2)$ is usually represented as f_n. In Ref. 76, it was shown that $f_n = \sum_{k\geq 0} 2^k \binom{2n-1}{2k}$, and also that

$$f_{n+1} = 6f_n - f_{n-1}, \qquad n \geq 2 \,, \tag{5.66}$$

where $f_0 = 1$ and $f_1 = 7$. This recurrence was mentioned again in Ref. 52, and combinatorial proofs of the recurrence which involve lattice walks are found in Refs. 77 and 78. Other articles that discuss this recurrence or the sequence obtained from it are Refs. 85–82. The asymptotics of the corresponding return walk function, $TA(z^2, z^2)$, are considered in Ref. 52.

Case 5.3. Under the $(1,1,1)$ metric,

$$\widehat{TA}(z^2, z) = \frac{z^2}{1-3z}\,, \tag{5.67}$$

which may have a combinatorial interpretation similar to that given in Ref. 77 for walks under the $(1,1,2)$ metric.

Case 5.4. Under the $(1,1,-)$ metric, i.e. the $\mathbb{S}$ lattice, the generating function enumerating the total $\triangle$-area of elevated walks by types of steps is

$$\widehat{TA}_{\triangle}(z^2, 0) = \frac{z^2}{1-4z^2}\frac{2z^2}{1-\sqrt{1-4z^2}} = \frac{z^2}{1-4z^2}\frac{1}{C(z^2)}\,, \tag{5.68}$$

where, again, $C(z)$ is the generating function of the Catalan numbers. The result for return walks is

$$TA_{\triangle}(z^2, 0) = \widehat{TA}_{\triangle}(z^2, 0)L(z^2, 0)^2 = \frac{1-\sqrt{1-4z^2}}{2(1-4z^2)} = \frac{z^2C(z^2)}{1-4z^2}\,. \tag{5.69}$$

Case 5.5. Under the $(1,1,1)$-metric,

$$\widehat{TA}_{\triangle}(z^2, z) = \frac{z^2}{1-2z-3z^2}\frac{1+z+\sqrt{1-2z-3z^2}}{2} \tag{5.70}$$

and

$$TA_{\triangle}(z^2, z) = \widehat{TA}_{\triangle}(z^2, z)L(z^2, z)^2 = \frac{z^2}{1-2z-3z^2}\frac{1-z-2z^2-\sqrt{1-2z-3z^2}}{2z^3}\,. \tag{5.71}$$

Although apparently not studied as sequences derived from lattice walks, each of the sequences of coefficients from these four $\triangle$-area functions is listed in Ref. 58; some already have other combinatorial interpretations.

6. Height Moments of Walks

For $r \geq 1$, the *rth total height moment* of a return walk w is here defined to be $\sum_{i\geq 0}(h_i)^r$, i.e. the sum of the rth powers of the height of the walk at each integer t coordinate. For example, the walk in Fig. 16 has sequence of heights $0, 1, 2, 1, 2, 3, 2, 2, 2, 1, 0$ as t ranges from 0 to 10. The first total height moment of the walk is then the sum of the heights (i.e. 16), the second moment the sum of the squares of the heights (32), the third moment the sum of the cubes of the heights (70) and so on. The first total height moment is also the first area moment but higher moments differ.

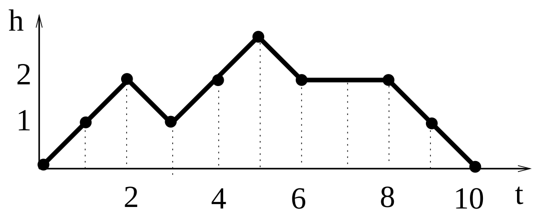

Fig. 16. An elevated walk with first total height moment of 16, second total height moment of 32 and third total height moment of 70.

For $r \geq 1$, let $\widehat{M}_r(v, d)$ be the generating function

$$\widehat{M}_r(v, d) = \sum_{w \in \mathcal{E}} \left\{ \sum_{i \geq 0} (h_i)^r \right\} v^{a(w)} d^{c(w)} \tag{6.72}$$

that enumerates the rth total height moments of elevated walks (the set of which is denoted $\mathcal{E}$) by types of steps. Here, with some abuse of notation, this function is called the *rth moment of elevated walks*. Similar functions have previously been considered in Refs. 40 and 59.

The corresponding moments for return walks are found from Lemma 5.1, with $\phi(h) = h^r$, as

$$\mathbf{M}_r(v, d; \kappa, \mu) = \kappa \widehat{M}_r(v, d) \mathbf{L}(v, d; \kappa, \mu)^2 \,. \tag{6.73}$$

We now derive a formula for $\widehat{M}_r(v, d)$ using a technique similar to that used to solve a $\mathbb{S}$ lattice moment problem in Ref. 86. From Eqs. (6.72), (5.53) and (5.54), an expression for the rth moment of elevated walks is

$$\widehat{M}_r(v, d) = \sum_{k \geq 0} k^r \widehat{K}_k(v, d) = \sum_{k \geq 0} k^r \; (1 + d) v^k L(v, d)^{2k} \,. \tag{6.74}$$

An exponential generating function for the moments is

$$\widehat{K}(z) = \sum_{k \geq 1} \widehat{K}_k(v, d) e^{kz} = (1 + d) \frac{v L(v, d)^2 e^z}{1 - v L(v, d)^2 e^z} \,, \tag{6.75}$$

in that

$$\left[\frac{z^r}{r!} \right] \widehat{K}(z) = \sum_{k \geq 1} k^r \widehat{K}_k(v, d) = \widehat{M}_r(v, d) \,, \tag{6.76}$$

so

$$\widehat{M}_r(v, d) = \left[\frac{\partial^r}{\partial z^r} \left(\frac{v(1 + d) L(v, d)^2 e^z}{1 - v L(v, d)^2 e^z} \right) \right]_{z=0} \,. \tag{6.77}$$

In Ref. 86, it was mentioned that for a given function D, and for $A(r, j)$ the Eulerian numbers (for which see Ref. 21),

$$\frac{\partial^r}{\partial z^r} \frac{D e^z}{1 - D e^z} = \frac{\sum_{j=1}^{r} A(r, j) D^j e^{jz}}{(1 - D e^z)^{r+1}} \,, \tag{6.78}$$

and also it was mentioned that that the coefficients $m(r,s)$ for which

$$\sum_{j=1}^{r} A(r,j)x^j = \sum_{s=1}^{\lceil r/2 \rceil} m(r,s)x^s(1+x)^{r+1-2s} \tag{6.79}$$

satisfy the recurrence

$$m(r,s) = sm(r-1,s) + 2(r-2s+2)m(r-1,s-1), \qquad r,s>1 \tag{6.80}$$

with $m(1,1)=1, m(1,p)=0$ for $p \neq 1$. A table of the values of $m(r,s)$ for small r and s is given in Table 1.

Table 1. Table of values of $m(r,s)$ for small r and s.

$s\backslash r$	1	2	3	4
1	1			
2	1			
3	1	2		
4	1	8		
5	1	22	16	
6	1	52	136	
7	1	114	720	272

If Eqs. (6.78)–(6.80) are applied in turn to Eq. (6.77), then

$$\begin{aligned}\widehat{M}_r(v,d) &= \left[\frac{(1+d)\sum_{j=1}^{r} A(r,j)v^j L(v,d)^{2j}e^{jz}}{(1-vL(v,d)^2e^z)^{r+1}}\right]_{z=0} \\ &= \frac{(1+d)\sum_{s=1}^{\lceil r/2\rceil} m(r,s)v^s L(v,d)^{2s}(1+vL(v,d)^2)^{r+1-2s}}{(1-vL(v,d)^2)^{r+1}}\,,\end{aligned} \tag{6.81}$$

which, after two uses of the functional Eq. (2.2) for $L(v,d)$, becomes

$$\widehat{M}_r(v,d) = \frac{1+d}{\sqrt{((1-d)^2-4v)^{r+1}}} \sum_{s=1}^{\lceil r/2\rceil} m(r,s)v^s(1-d)^{r+1-2s}\,. \tag{6.82}$$

This expression for the rth moment of elevated walks relies only on a linear combination of polynomials in the step variables v and d. From this final expression it also is clear that odd order moments are rational, whilst even order moments are algebraic.

The moment problem considered in Ref. 86 was that of finding the moments of the distance between two non-intersecting paths on the $\mathbb{S}$ lattice. This was as an instance of moments of Shapiro's "Catalan triangle".[87] The array, $\{w_{n,k}\}$ say, of the numbers of points at height k in the set of elevated walks on the $\mathbb{S}$ lattice that end at $(t,h)=(2n,0)$ is then another instance of the Catalan triangle, and thus the derivation of $\widehat{M}_r(v,d)$ given here is an extension of the moment problem in Ref. 86 to a system beyond the $\mathbb{S}$ lattice.

Acknowledgments

Financial support from the Australian Research Council is gratefully acknowledged by RB and ALO. One of the authors, ACO thanks the Graduate School of The University of Melbourne for an Australian Postgraduate Award.

References

1. M. E. Fisher, *J. Stat. Phys.* **34**, 667 (1984).
2. R. Brak, J. Essam and A. L. Owczarek, *J. Stat. Phys.* **93**, 155 (1998).
3. R. Simion, *Discrete Math.* **217**, 367 (2000).
4. R. P. Stanley, *Enumerative Combinatorics* (Cambridge University Press, 1999). In two volumes, Vol. 1 a corrected reprint of the original.
5. H. A. Delannoy, Emploi de l'échiquier pour la solution de problèmes arithmétiques, in *Comptes Rendus: Association française pour l'avancement des Sciences, Congr̀es de Nancy*, volume 2, pages 183–188, 1886.
6. H. A. Delannoy, Emploi de l'échiquier pour la résolution de divers problèmes de probabilité, in *Comptes Rendus: Association française pour l'avancement des Sciences, Congr̀es de Paris*, volume 2, pages 43–52, 1889.
7. H. A. Delannoy, Emploi de l'échiquier pour la résolution de certaines problèmes de probabilités, in *Comptes Rendus: Association française pour l'avancement des Sciences, Congr̀es de Bordeaux*, volume 2, pages 70–90, 1895.
8. E. Lucas, *Theorie des Nombres* (Blanchard, Paris, 1891), p. 1961.
9. P. A. MacMahon, *Combinatory Analysis* (Chelsea, New York, 1960).
10. P. Flajolet, *Discrete Math.* **32**, 125 (1980).
11. R. Brak, J. Essam and A. L. Owczarek, *J. Stat. Phys.* **93**, 155 (1998).
12. R. Brak, J. Essam and A. L. Owczarek, *J. Phys.* **A32**, 2921 (1999).
13. R. Brak, J. Essam and A. L. Owczarek, *J. Stat. Phys.* **102**, 997 (2001).
14. P. Duchon, *Discrete Math.* **225**, 121 (2000).
15. E. Deutsch, *Discrete Math.* **204**, 167 (1999).
16. D. F. Lawden, *Math. Gaz.* **36**, 193 (1952).
17. L. Moser, *Math. Gaz.* **39**, 54 (1955).
18. L. Moser and W. Zayachkowski, *Scripta Math.* **26**, 223 (1963).
19. R. G. Stanton and D. D. Cowan, *SIAM Review* **12**, 277 (1970).
20. R. D. Fray and D. P. Roselle, *Pacific J. Math.* **37**, 85 (1971).
21. L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions* (Reidel, Dordrecht, 1974).
22. D. G. Rogers, A Schröder Triangle: Three Combinatorial Problems, in *Combinatorial Mathematics V, Melbourne 1976. Proceedings*, ed. C. H. C. Little, volume 622 of *Lecture Notes in Math.*, pages 175–195, 1977.
23. H. T. Laquer, *Stud. Appl. Math.* **64**, 271 (1981).
24. P. Erdős, A. Hildebrand, A. Odlyzko, P. Pudaite and B. Reznick, *Pacific J. Math.* **126**, 227 (1987).
25. M. Q. Zhang and T. G. Marr, *J. Theor. Biol.* **174**, 119 (1995).
26. M. D. Hirschhorn, *Aust. Math. Gaz.* **27**, 104 (2000).
27. A. C. Oppenheim, Directed walkers on the square and triangular planar lattices, Honours project report, Department of Mathematics and Statistics, University of Melbourne, 1998.
28. A. C. Oppenheim, Some enumerative results for one and two directed walks on site-equivalent planar lattices, Master's thesis, Department of Mathematics and Statistics,

University of Melbourne, 2001.
29. A. C. Oppenheim, R. Brak and A. L. Owczarek, Anisotropic step, mutual contact and area weighted festoons and parellelogram polyominoes on the triangular lattice, Unpublished, 2002.
30. R. Sprugnoli, *Discrete Math.* **132**, 267 (1994).
31. E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, *Ann. Comb.* **3**, 171 (1999).
32. I. P. Goulden and D. M. Jackson, *Combinatorial enumeration* (Wiley, New York, 1983).
33. J. R. Goldman and T. Sundquist, *Adv. in Appl. Math.* **13**, 216 (1992).
34. D. G. Rogers and L. Shapiro, Some correspondences involving the Schröder numbers and relations, in *Proceedings of the international conference on combinatorial theory, Canberra, August 16-27, 1977*, eds. D. A. Holton and J. Seberry, volume 686 of *Lecture Notes in Math.*, pages 267–274, 1978.
35. D. Merlini, R. Sprugnoli and M. C. Verri, *Lecture Notes in Comput. Sci.* **787**, 218 (1994).
36. J.-M. Fédou, E. Roblet and X. G. Viennot, *Ann. Sci. Math. Québec* **21**, 67 (1997).
37. S. Getu and L. Shapiro, *Congr. Numer.* **135**, 65 (1998).
38. M. Jani and R. G. Rieper, *Electron. J. Combin.* **7**, R45 (2000).
39. J. Riordan and J. Combin. *Theory Ser.* **A19**, 214 (1975).
40. R. A. Sulanke, *J. Integer Seq.* **3** (2000).
41. E. J. J. van Rensberg, *The statistical mechanics of interacting walks, polygons, animals and vesicles* (Oxford University, 2000).
42. W. Fulton, *Young tableaux: with applications to representation theory and geometry*, Cambridge University Press, 1997.
43. P. Larcombe and P. D. C. Wilson, *Math. Today (Southend-on-Sea)* **34**, 114 (1998).
44. L. Euler, 154: Euler an Goldbach, in *Leonhard Euler und Christian Goldbach, Briefwechsel 1729–1764*, eds. A. P. Juškevič and E. Winter (Akademie–Verlag, Berlin, 1965) pp. 339–340.
45. Mingantu, Ge Yuan Mi and Lü Jie Fa, in *Zhongguo kexue jishu dianji tonghui, shu xue juan*, volume 4, pages 858–943, Henan Jiaoyu chubanshe, Zhengzhou, (1839) 1993.
46. R. Donaghey and L. W. Shapiro, *J. Combin. Theory Ser.* **A23**, 291 (1977).
47. E. Schröder, *Z. für M. Phys.* **15**, 361 (1870), Corrig. pp. 179-180 of a later issue.
48. Plutarch, *Moralia vol. IX: Table-Talk* (Heinemann Harvard, 1961), Loeb Library, E. L. Minar, Jr (trans.).
49. Plutarch, *Moralia vol. XIII: On Stoic Self-Contradictions*, Harvard and Heinemann (Cambridge, 1976), Loeb Library, H. Cherniss (trans.).
50. R. P. Stanley, *Amer. Math. Monthly* **104**, 344 (1997).
51. L. Habsieger, M. Kazarian and S. Lando, *Amer. Math. Monthly* **105**, 446 (1998).
52. J. Bonin, L. Shapiro and R. Simion, *J. Stat. Plann. Inference* **34**, 35 (1993).
53. T. Motzkin, *Bull. Amer. Math. Soc.* **54**, 352 (1948).
54. E. Barcucci, R. Pinzani and R. Sprugnoli, *Pure Math. Appl. Ser.* **A2**, 249 (1991).
55. M. Aigner, *European J. Combin.* **19**, 663 (1998).
56. F. R. Bernhart, *Discrete Math.* **204**, 73 (1999).
57. J. Labelle and Y.-N. Yeh, *Discrete Math.* **82**, 1 (1990).
58. N. J. A. Sloane, An on-line version of the encyclopedia of integer sequences, http://www.research.att.com/~njas/sequences/.
59. R. Chapman, *Discrete Math.* **204**, 113 (1999).
60. D. Callan, *Math. Mag.* **72**, 295 (1999).
61. R. Brak and J. W. Essam, Return polynomials for non-intersecting paths above a surface on the directed square lattice, To appear in *J. Phys. A*.

62. L. Takács, On the ballot theorems, in *Advances in combinatorial methods and applications to probability and statistics*, ed. N. Balakrishnan, pages 97–114, Boston, 1997, Birkhuser Boston.
63. E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, A construction for enumerating k-coloured Motzkin paths, in *Computing and combinatorics: First annual international conference, COCOON '95, Xi'an, China, August 24–26 1995, Proceedings*, eds. D.-Z. Du and M. Li, volume 959 of *Lecture Notes in Comput. Sci.*, pages 254–263, 1995.
64. E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, Towards a methodology for tree enumeration, in *Proc. 7th conference on formal power series and algebraic combinatorics, Marne-la-Vallee*, eds. B. Leclerc and J. Y. Thibon, pages 53–65, 1995.
65. E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, *Discrete Math.* **180**, 45 (1998), A more recently published article by the same authors is ECO: a methodology for the enumeration of combinatorial objects, *J. Differ. Equations Appl.* 5:435–490, 1999.
66. A. L. Owczarek and T. Prellberg, *J. Stat. Phys.* **70**, 1175 (1993).
67. B. C. Berndt, *Ramanujan's notebooks, part III* (Springer-Verlag, New York, 1991).
68. C. Banderier *et al.*, On generating functions of generating trees, Research report 3661, INRIA, 1999, Appeared in *Proc. 11th conference on formal power series and algebraic combinatorics (FPSAC'99), Barcelona, June 1999.*
69. A. D. Rechnitzer, *Some problems in the counting of lattice animals, polyominoes, polygons and walks*, PhD thesis, University of Melbourne, 2001.
70. A. M. Odlyzko and H. S. Wilf, *Amer. Math. Monthly* **95**, 840 (1988).
71. F. C. Auluck, *Proc. Cambridge Philos. Soc.* **47**, 679 (1951).
72. H. S. Wilf, *Generatingfunctionology*, Academic Press, San Diego etc., second edition, 1994.
73. G. Szekeres, *Canad. Math. Bull.* **11**, 405 (1968).
74. R. K. Guy, *Amer. Math. Monthly* **95**, 697 (1988), General fountains are mentioned as being the idea of J. Propp.
75. H. S. Snevily and D. B. West, *Amer. Math. Monthly* **105**, 131 (1998).
76. G. Kreweras, *Cahiers du B.U.R.O* **24**, 9 (1976).
77. E. Pergola and R. Pinzani, *Electron. J. Combin.* **6**, R40 (1999).
78. R. A. Sulanke, *Electron. J. Combin.* **5**, R47 (1998).
79. W.-J. Woan, *Discrete Math.* **226**, 439 (2001).
80. D. Merlini, R. Sprugnoli and M. C. Verri, The area determined by underdiagonal lattice paths, in *Trees in Algebra and Programming: CAAP '96, 21st international colloquium, Linköping, Sweden, April 1996*, ed. H. Kirchner, volume 1059 of *Lecture Notes in Comput. Sci.*, pages 59–71, 1996.
81. W.-J. Woan, L. Shapiro and D. G. Rogers, *Amer. Math. Monthly* **104**, 926 (1997).
82. M. A. Gruber *et al.*, *Amer. Math. Monthly* **4**, 24 (1897). Proposed as Problem 45 [in Diophantine Analysis] vol. 3, p. 153.
83. M. Newman, D. Shanks and H. C. Williams, *Acta Arith.* **38**, 129 (1980/81).
84. E. Barcucci, S. Brunetti, A. Del Lungo and F. Del Ristoro, *Discrete Math.* **190**, 235 (1998).
85. A. S. Fraenkel, *Discrete Math.* **224**, 273 (2000).
86. L. W. Shapiro, W.-J. Woan and S. Getu, *SIAM J. Algebraic Disc. Methods.* **4**, 459 (1983).
87. L. W. Shapiro, *Discrete Math.* **14**, 83 (1976).