An algebraic characterization of the set of succession rules

Luca Ferrari [*] Elisa Pergola [†] Renzo Pinzani [†]
Simone Rinaldi [†]

"Qui dedit beneficium taceat; narret qui accepit" (Seneca)
Merci Maurice

Abstract

In this paper we will give a formal description of succession rules in terms of linear operators satisfying certain conditions. This representation allows us to introduce a system of *well-defined operations* into the set of *succession rules* and then to tackle problems of combinatorial enumeration simply by using operators instead of generating functions. Finally we will suggest several open problems whose solution should lead to an algebraic characterization of the set of succession rules.

1 Introduction

A *succession rule* Ω is a system consisting of an *axiom* (b), $b \in \mathbb{N}^+$, and a set of *productions*:

$$\{(k_t) \rightsquigarrow (e_1(k_t))(e_2(k_t))\dots(e_{k_t}(k_t)) : \ t \in \mathbb{N}\} ,$$

where $e_i : \mathbb{N}^+ \to \mathbb{N}^+$, which explains how to derive the *successors* $(e_1(k))$, $(e_2(k)), \dots (e_k(k))$ of any given label (k), $k \in \mathbb{N}^+$. In general for a succession rule Ω, we use the more compact notation

[*]Dipartimento di Matematica, Viale Morgagni, 67/A, Firenze. ferrari@math.unifi.it
[†]Dipartimento di Sistemi e Informatica, Via Lombroso 6/17, Firenze. {elisa, pinzani, rinaldi}@dsi.unifi.it

$$\begin{cases} (b) \\ (k) \rightsquigarrow (e_1(k))(e_2(k))\dots(e_k(k)), \end{cases} \tag{1}$$

to mean that there can be infinitely many productions in the system, but at most one for each integer $k \in \mathbb{N}^+$.

The positive integers (b), (k), $(e_i(k))$, are called *labels* of Ω. The rule Ω can be represented by means of a *generating tree*, that is a rooted tree whose vertices are the labels of Ω; (b) is the label of the root and each node labeled (k) has k sons labeled by $e_1(k), \dots, e_k(k)$ respectively, according to the production of (k) in (1). A succession rule Ω defines a sequence of positive integers $\{f_n\}_{n\geq 0}$, f_n being the number of the nodes at level n in the generating tree defined by Ω. By convention the root is at level 0, so $f_0 = 1$. The function $f_\Omega(x) = \sum_{n\geq 0} f_n x^n$ is the *generating function* determined by Ω.

One of the most common succession rules is that defining Schröder numbers [4], $1, 2, 6, 22, 90, 394$, M2898 in [12]:

$$\begin{cases} (2) \\ (2) \rightsquigarrow (3)(3) \\ (k) \rightsquigarrow (3)\dots(k+1), \qquad k \geq 3. \end{cases} \tag{2}$$

In Fig. 1 the first levels of the generating tree of (2) are shown. We refer to [3] for further details and examples.

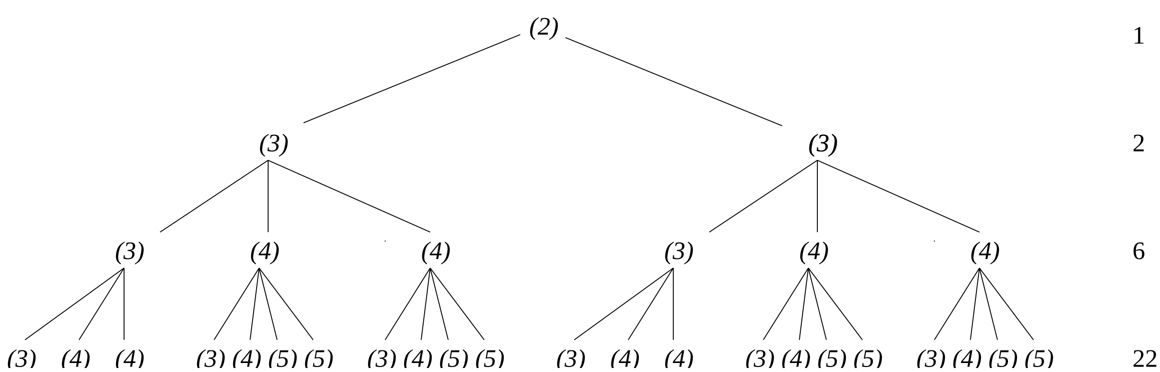

Figure 1: The first levels of the generating tree of (2), and its number sequence.

The concept of succession rule was first introduced in [6] by Chung et al. to study reduced Baxter permutations, and was later applied to the enumeration of permutations with forbidden subsequences [8, 13]. Moreover, they represent an excellent tool for ECO method [3], which is a general method for the enumeration of combinatorial objects. The basic idea of this method is the following: given a class $\mathcal{O}$ of combinatorial objects and a parameter p of $\mathcal{O}$, let us consider the set $\mathcal{O}_n = \{x \in \mathcal{O} : p(x) = n\}$. If we are able to define an operator ϑ which satisfies the following conditions:

1. for each $Q \in \mathcal{O}_{n+1}$ there exists $P \in \mathcal{O}_n$ such that $Q \in \vartheta(P)$,
2. for each $P_1, P_2 \in \mathcal{O}_n$ such that $P_1 \neq P_2$, then $\vartheta(P_1) \cap \vartheta(P_2) = \emptyset$,

then $\mathcal{F}_{n+1} = \{\vartheta(P) : \forall\ P \in \mathcal{O}_n\}$ is a partition of $\mathcal{O}_{n+1}$. Therefore, we have a recursive construction of the elements of $\mathcal{O}$. A generating tree is then associated to the operator ϑ, in such a way that the number of nodes appearing in the tree at level n gives the number of n-sized objects in the class, and the sons of each object are the objects it produces through ϑ. Such a generating tree can be formally represented by means of a succession rule of the form (1), meaning that the root object has b sons, and the k objects $O'_1, \ldots, O'_k$, produced by an object O through ϑ are such that $|\vartheta(O'_i)| = e_i(k)$, $1 \leq i \leq k$.

A succession rule is called rational, algebraic or trascendental if its generating function is rational, algebraic or trascendental, respectively. The relationship between the structural properties of the rules and their rationality, algebraicity or trascendence is studied in [1].
However, the complete analytic characterization of the set of algebraic succession rules and of the set of algebraic generating functions remains an open problem.
In literature, succession rules can have several different forms. However, this paper will focus only on the rules having the form (1), where each label (k) produces exactly k sons, also named *ECO-systems*.

Two rules Ω_1 and Ω_2 are said to be *equivalent*, $\Omega_1 \cong \Omega_2$, if they define the same number sequence, that is $f_{\Omega_1}(x) = f_{\Omega_2}(x)$. For example, the following rules are equivalent to (2), and define the Schröder numbers [4, 5]:

$$\begin{cases} (2) \\ (2k) \rightsquigarrow (2)(4)^2 \dots (2k)^2(2k+2) \end{cases}$$

$$\begin{cases} (2) \\ (2) \rightsquigarrow (3)(3) \\ (2k-1) \rightsquigarrow (3)^2(5)^2 \dots (2k-1)^2(2k+1) \end{cases}$$

$$\begin{cases} (2) \\ (2^k) \rightsquigarrow (2)^{2^{k-1}}(4)^{2^{k-2}}(8)^{2^{k-3}} \dots (2^{k-1})^2(2^k)(2^{k+1}) \end{cases}$$

where the power notation is used to express repetitions, that is $(h)^i$ stands for $\underbrace{(h)\dots(h)}_{i \text{ times}}$.

Next we slightly extend the definition of succession rule given at the beginning, and introduce *colored rules* as follows: a rule Ω is colored when there are at least two labels (k) and $(\bar{k})$ having the same value but different productions. For example, it is easily proved, that the sequence $1, 2, 3, 5, 9, 17, 33, \dots, 2^{n-1}+1$, having

$$\frac{1-x-x^2}{1-3x+2x^2}$$

as generating function, can only be described by means of colored rules, such as:

$$\begin{cases} (2) \\ (1) \rightsquigarrow (\bar{2}) \\ (2) \rightsquigarrow (1)(2) \\ (\bar{2}) \rightsquigarrow (\bar{2})(\bar{2}). \end{cases} \tag{3}$$

In this paper we first solve two open problems on the set of finite succession rules. In Section 3, we introduce the concept of *rule operator* associated with a succession rule, that is, the algebraic counterpart of the combinatorial concept of succession rule: it is a linear operator on $\mathbb{R}[x]$, considered as an $\mathbb{R}$-vector space, and it gives us a formal tool to deal with ECO-systems from an algebraic view-point. Indeed it allows us to define some operations in the set of rule operators, reflecting some well-known operations on the number sequences associated with them.

2 Finite succession rules

A succession rule Ω is *finite* if it has a finite number of different labels. For example, for any positive integer, the number sequences $\{a_{n,k}\}_n$, defined by the recurrences:

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}a_{n-j,k}=0 \qquad k\in\mathbb{N},$$

having $\frac{1}{(1-x)^k}$ as generating function, have finite succession rules:

$$\Omega(k):\left\{\begin{array}{l}(k)\\(1)\rightsquigarrow(1)\\(2)\rightsquigarrow(1)(2)\\(3)\rightsquigarrow(1)(2)(3)\\\ldots\qquad\ldots\\(k)\rightsquigarrow(1)(2)(3)\ldots(k-1)(k).\end{array}\right.$$

Moreover, let $\{a_n\}_n$ be the sequence of integers satisfying the recurrence:

$$a_n=ka_{n-1}+ha_{n-2}, \qquad k\in\mathbb{N}^+, h\in\mathbb{Z},$$

subject to the initial conditions $a_0=1$, $a_1=b\in\mathbb{N}^+$; thus every term of the sequence is a positive number if $k+h>0$. In this case, the sequence $\{a_n\}_n$ is defined by the finite succession rule:

$$\Omega_{\mathcal{F}^b_{k,h}}:\left\{\begin{array}{l}(b)\\(b)\rightsquigarrow(k)^{b-1}(k+h)\\(k)\rightsquigarrow(k)^{k-1}(k+h)\\(k+h)\rightsquigarrow(k)^{k+h-1}(k+h).\end{array}\right. \tag{4}$$

Finite succession rules play an important role in enumerative combinatorics, because of their strong relations with rational functions and regular languages; in particular they allow the enumeration of some restricted classes of combinatorial objects [9]. Let us first recall some basics about *PD0L systems* [11]. A PD0L system is a triple:

$$G=(\Sigma,h,w_0),$$

where Σ is an alphabet, h is an endomorphism defined on Σ^+ and w_0, named the *axiom*, is an element of Σ^+. The *language* of G is defined by:

$$L(G)=\{h^i(w_0):i\geq 0\}.$$

The function $f_G(n) = |h^n(w_0)|$, $n \geq 0$ is the *growth function* of G, and the sequence $|h^n(w_0)|$, $n \geq 0$ is termed *growth sequence.*

It is important to point out that we can regard any finite succession rule Ω as a particular PD0L system using the set of labels of Ω as the alphabet Σ, where h is defined by productions of Ω, and $w_0 \in \Sigma$. These remarks together with Theorem III.8.1 [11] lead us to the solution of the *equivalence problem* for finite succession rules.

Equivalence. *Let Ω_1 and Ω_2 be two finite succession rules having h_1 and h_2 labels respectively, then $\Omega_1 \cong \Omega_2$, if and only if the first $h_1 + h_2$ terms of the two sequences defined by Ω_1 and Ω_2 coincide.*

For example, let us consider the number sequences defined by (3) and by (4) with $b = 2, k = 1, h = 1$ (which is the rule for Fibonacci numbers). The sequences determined by (3) and (4) coincide for the first four terms, but not for the fifth.

Let $\mathcal{N}$ be the set of rational generating functions of positive sequences, $\mathcal{R}$ the set of generating functions of regular languages and $\mathcal{S}$ the set of generating functions of finite succession rules. The set of $\mathbb{N}$-rational functions $f(x)$, for which $f(0)$ equals 0 or 1, coincides with $\mathcal{R}$ [11]. Moreover, the analytic characterization of $\mathbb{N}$-rational functions is also given in [11]. With reference to [2], or by the methods of [11, 10], given a rational function $f(x)$, it is possible to establish whether $f(x) \in \mathcal{R}$. Furthermore, there are some examples of rational generating functions of positive sequences, which are not the generating functions of any regular language (see Section 5, [2]). Below, we state a result obtained through Theorem III.4.11 in [10], which gives an analytic characterization of the set of generating functions of PDOL growth sequences:

Generating functions. *The function $f(x)$ is the generating function of a finite succession rule if and only if:*

1. $f(x) = \frac{P(x)}{Q(x)}$, with $P(x), Q(x) \in \mathbb{Z}[x]$, and $Q(0) = P(0) = 1$;
2. $\frac{1}{x}(f(x) - 1) - f(x)$ is $\mathbb{N}$-rational.

This proves that each generating function of a finite succession rule is the generating function of a regular language, whereas the converse does not hold. For example, let $g(x) = \frac{1}{1-10x}$ and $h(x) = \frac{1-3x+36x^2}{(1-9x)(1+2x+81x^2)}$; $h(x)$ is a rational function having all positive coefficients (see [2] for the proof) but it is not $\mathbb{N}$-rational, since the poles of minimal modulus are complex numbers. Let

$$f(x) = g(x^2) + x[g(x^2) + h(x^2)] = k_1(x^2) + xk_2(x^2); \qquad (5)$$

$f(x)$ is $\mathbb{N}$-rational, since it is the merge in the sense of [10] of the two functions $k_1(x)$ and $k_2(x)$, each of them having a real positive dominating root, $x = 10$. This proves the existence of a regular language having $f(x)$ as its generating function. Moreover, it is clear that $f(x)$ defines a strictly increasing sequence of positive numbers. Neverthless $\frac{1}{x}(f(x) - 1) - f(x)$ is not $\mathbb{N}$-rational, since it is a merge of $g(x)$ and $h(x)$, and $h(x)$ is not $\mathbb{N}$-rational. Thus there are no finite succession rules having $f(x)$ as its generating function. We conclude that

$$\mathcal{S} \subset \mathcal{R} \subset \mathcal{N}.$$

The equivalence and the generating functions problems remain still open in the case of not finite succession rules.

3 Rule operators

In this section we introduce the concept of *rule operator*, which represents a simple algebraic tool to handle succession rules. This notion is not completely new in combinatorics, indeed it has been widely applied without a suitable algebraic formalization, especially when computing generating functions of succession rules [1, 3, 4].

Let us consider a succession rule having the form (1). We define the rule operator L_Ω associated with Ω as follows:

$$L_\Omega : \mathbb{R}[x] \to \mathbb{R}[x]$$

$$L_\Omega(1) = x^b;$$

$$L_\Omega(x^k) = x^{e_1(k)} + \ldots + x^{e_k(k)};$$

$$L_\Omega(k) = kx^k, \qquad \text{if the label } (k) \text{ is not in the generating tree of } \Omega\ ,$$

and then extending by linearity on $\mathbb{R}[x]$ (considered as a $\mathbb{R}$-vector space). In general, we use the power notation to express the iterated application of L_Ω: $L_\Omega^{n+1}(1) = L_\Omega(L_\Omega^n(1))$. In the sequel we will always write L in place of L_Ω, if not required by the context.

The following proposition characterizes the set of rule operators associated to ECO-systems:

Proposition 3.1 Let L be a linear operator on $\mathbb{R}[x]$. It is the rule operator associated with a ECO-system if and only if:

1) $L(x^k) \in \mathbb{N}[x]$, for all $k \in \mathbb{N}$;

2) $L(1) = x^b$, for some $b \in \mathbb{N}^+$;

3) $[L(x^k)]_{x=0} = 0$, $k \in \mathbb{N}$;

4) $[L(x^k)]_{x=1} = k$, $k \in \mathbb{N}$.

The linear operator L clearly retains the properties of the succession rule Ω; in particular, the sequence of positive integers $\{f_n\}$ defined by Ω can be easily obtained from L. We have the following proposition, which can be easily proved by induction on $n \in \mathbb{N}$:

Proposition 3.2 For any $n \in \mathbb{N}$ we have:

1) $f_n = [L^{n+1}(1)]_{x=1}$;

2) $f_n = [DL^n(1)]_{x=1}$;

where D is the derivative operator in the variable x.

We remark that condition 4) of Proposition 3.1 implies $[L^{n+1}(1)]_{x=1} = [DL^n(1)]_{x=1}$, as stated in Proposition 3.2.

Example 3.1 We present a small catalogue of ECO-systems and the corresponding rule operators associated with sequences of combinatorial interest. The identification numbers refer to [12].

Number sequence	ECO-system	rule operator
Fibonacci (**M0692**)	$\begin{cases} (2) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (1)(2); \end{cases}$	$L(1) = x^2, L(x) = x^2,$ $L(x^2) = x + x^2$
Factorial (**M1675**)	$\begin{cases} (2) \\ (k) \rightsquigarrow (k+1)^k \end{cases}$	$L(1) = x^2,$ $L(x^k) = kx^{k+1} = x^2 D(x^k)$;
Arrangements (**M1497**)	$\begin{cases} (2) \\ (k) \rightsquigarrow (k)(k+1)^{k-1} \end{cases}$	$L(1) = x^2,$ $L(x^k) = x^k + (k-1)x^{k+1}$;
Involutions (**M1221**)	$\begin{cases} (2) \\ (k) \rightsquigarrow (k-1)^{k-1}(k+1) \end{cases}$	$L(1) = x^2,$ $L(x^k) = (k-1)x^{k-1} + x^{k+1}$;
Bell (**M1484**)	$\begin{cases} (2) \\ (k) \rightsquigarrow (k)^{k-1}(k+1) \end{cases}$	$L(1) = x^2,$ $L(x^k) = (k-1)x^k + x^{k+1}$;
Catalan (**M1459**)	$\begin{cases} (2) \\ (k) \rightsquigarrow (2)(3)\ldots(k)(k+1) \end{cases}$	$L(1) = x^2,$ $L(x^k) = x^2 + \ldots + x^{k+1}$;
Motzkin (**M1184**)	$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (k) \rightsquigarrow (1)(2)\ldots(k-1)(k+1) \end{cases}$	$L(1) = x, L(x) = x^2,$ $L(x^k) = x + \ldots + x^{k-1} + x^{k+1}$.

Now we aim at extending the concept of rule operator also to the set of colored succession rules. Consider a 2-colored succession rule Ω written as follows:

$$\left\{\begin{array}{l}(a)\\(h)\rightsquigarrow(e_1(h))(e_2(h))\dots(e_\alpha(h))(\overline{e_{\alpha+1}(h)})\dots(\overline{e_h(h)})\\(\overline{k})\rightsquigarrow(c_1(k))(c_2(k))\dots(c_\beta(k))(\overline{c_{\beta+1}(k)})\dots(\overline{c_k(k)}).\end{array}\right. \quad (6)$$

The 2-colored operator L_Ω associated with (6) is then:

$$L_\Omega : \mathbb{R}[x] \oplus y\mathbb{R}[y] \to \mathbb{R}[x] \oplus y\mathbb{R}[y]$$

$$L_\Omega(1) = x^a;$$

$$L_\Omega(x^h) = x^{e_1(h)} + \ldots + x^{e_\alpha(h)} + y^{e_{\alpha+1}(h)} + \ldots + y^{e_h(h)};$$

$$L_\Omega(y^k) = x^{c_1(k)} + \ldots + x^{c_\beta(k)} + y^{c_{\beta+1}(k)} + \ldots + y^{c_k(k)}.$$

extended by linearity on the vector space $\mathbb{R}[x] \oplus y\mathbb{R}[y]$. Of course, this definition generalizes to n-colored rules. Operators for 2-colored rules possess analogous properties to those already stated for rule operators in the first part of this section.

Proposition 3.3 The linear operator L on $\mathbb{R}[x] \oplus y\mathbb{R}[y]$ is the rule operator of a 2-colored ECO-system if and only if the following conditions are satisfied:

1) $L(x^k), L(y^k) \in \mathbb{N}[x]$, for all $k \in \mathbb{N}$;

2) $[L(x^k)]_{x=y=0} = [L(y^k)]_{x=y=0} = 0$ for all $k \in \mathbb{N}$;

3) $[L(x^k)]_{x=y=1} = [L(y^k)]_{x=y=1} = k$ for all $k \in \mathbb{N}$.

Proposition 3.4 Let Ω be a 2-colored ECO-system, L the associated 2-colored rule operator, and $\{f_n\}$ the sequence defined by Ω. We have:

$$f_n = \left[L^{n+1}(1)\right]_{x=y=1} = \left[(D_x + D_y)L^n(1)\right]_{x=y=1},$$

for $n \in \mathbb{N}$, where D_x and D_y denote the partial derivative operators with respect to x and y, respectively.

4 Operations on succession rules

Now we aim at defining some operations, to be carried out on the set of rule operators, which reflect some well-known operations on the related number sequences. Let L_Ω and $L_{\Omega'}$ be two rule operators, associated to the succession rules Ω and Ω', defining the sequences $\{f_n\}_n$ and $\{g_n\}_n$, and having $f(x)$ and $g(x)$ as generating functions, respectively. Below we will deal with L_Ω and L'_Ω having the following general forms:

$$\begin{cases} L_\Omega(1) = x^a \\ \\ L_\Omega(x^h) = x^{e_1(h)} + x^{e_2(h)} + \ldots + x^{e_h(h)}, \end{cases}$$

$$\begin{cases} L_{\Omega'}(1) = x^b \\ \\ L_{\Omega'}(x^k) = x^{c_1(k)} + x^{c_2(k)} + \ldots + x^{c_k(k)}. \end{cases}$$

4.1 Sum of rule operators

Given two rule operators L_Ω and $L_{\Omega'}$, their *sum*, $L_\Omega \oplus L_{\Omega'}$, is the rule operator defining the sequence $\{h_n\}_n$ such that $h_0 = 1$ and $h_n = f_n + g_n$, when $n > 0$, and having $f(x) + g(x) - 1$ as generating function. We define:

$$L_\Omega \oplus L_{\Omega'} : \mathbb{R}[x] \oplus y\mathbb{R}[y] \oplus z\mathbb{R}[z] \to \mathbb{R}[x] \oplus y\mathbb{R}[y] \oplus z\mathbb{R}[z]$$

$$L_\Omega \oplus L_{\Omega'}(1) = z^{a+b},$$

$$L_\Omega \oplus L_{\Omega'}(z^{a+b}) = L_\Omega(x^a) + L_{\Omega'}(y^b),$$

$$L_\Omega \oplus L_{\Omega'}(x^h) = L_\Omega(x^h),$$

$$L_\Omega \oplus L_{\Omega'}(y^k) = L_{\Omega'}(y^k).$$

If we define $L_\Omega \oplus L_{\Omega'}$ as the identity on the remaining powers of x, y, z, and then we extend it by linearity, we obtain the desired rule operator which defines the sequence $\{h_n\}_n$.

4.2 Product of succession rules

Given two rule operators L_Ω and $L_{\Omega'}$, their *product* , $L_\Omega \otimes L_{\Omega'}$, is the rule operator defining the sequence $\left\{\sum_{k \leq n} f_{n-k} g_k\right\}_n$, and having $f(x) \cdot g(x)$ as generating function. We define:

$$L_\Omega \otimes L_{\Omega'} : \mathbb{R}[x] \oplus y\mathbb{R}[y] \to \mathbb{R}[x] \oplus y\mathbb{R}[y]$$

$$L_\Omega \otimes L_{\Omega'}(1) = x^{a+b},$$

$$L_\Omega \otimes L_{\Omega'}(x^{h+b}) = x^b L_\Omega(x^h) + L_{\Omega'}(y^b),$$

$$L_\Omega \times L_{\Omega'}(y^k) = L_{\Omega'}(y^k).$$

We will prove that:

$$\left[(L_\Omega \otimes L_{\Omega'})^{n+1}(1)\right]_{x=y=1} = \sum_{k\le n} f_{n-k} g_k.$$

Since $(L_\Omega \otimes L_{\Omega'})(x^b p(x)) = (L_\Omega \otimes L_{\Omega'})(\sum_k p_{n,k} x^{k+b}) =$
$\sum_k p_{n,k}(x^b L_\Omega(x^k) + L_{\Omega'}(y^b)) = x^b L_\Omega(p(x)) + p(1) L_{\Omega'}(y^b)$,
Lemma 4.1 follows:

Lemma 4.1 For each polynomial $p(x) = \sum_{k=1}^m p_{n,k} x^k$, we have:

$$(L_\Omega \otimes L_{\Omega'})(x^b p(x)) = x^b L_\Omega(p(x)) + p(1) L_{\Omega'}(y^b).$$

Proposition 4.1 For each $n \in \mathbb{N}$, we have

$$(L_\Omega \otimes L_{\Omega'})^n(1) = x^b L_\Omega^n(1) + \sum_{k=1}^{n-1} \left[L_\Omega^k(1)\right]_{x=1} \cdot L_{\Omega'}^{n+1-k}(1).$$

Proof. We work by induction on $n \in \mathbb{N}$. It is easy to show that the statement holds for $n = 1, 2, 3$. Supposing it holds for a fixed n, then we have:

$(L_\Omega \otimes L_{\Omega'})^{n+1}(1) = (L_\Omega \otimes L_{\Omega'})(L_\Omega \otimes L_{\Omega'})^n(1) = (L_\Omega \otimes L_{\Omega'})(x^b L_\Omega^n(1) +$

$\sum_{k=1}^{n-1}[L_\Omega^k(1)]_{x=1} \cdot L_{\Omega'}^{n+1-k}(1)) = x^b L_\Omega^{n+1}(1) + [L_\Omega^n(1)]_{x=1} L_{\Omega'}^2(1) +$

$\sum_{k=1}^{n-1}[L_\Omega^k(1)]_{x=1} \cdot L_{\Omega'}^{n+2-k}(1)) = x^b L_\Omega^{n+1}(1) + \sum_{k=1}^{n}[L_\Omega^k(1)]_{x=1} \cdot L_{\Omega'}^{n+2-k}(1).\square$

Corollary 4.1 For each $n \in \mathbb{N}$, we have

$$\left[(L_\Omega \otimes L_{\Omega'})^{n+1}(1)\right]_{x=y=1} = \sum_{k\le n} f_{n-k} g_k.$$

In a completely similar way it can also be proved that

$$[(D_x + D_y)(L_\Omega \otimes L_{\Omega'})^n(1)]_{x=y=1} = \sum_{k\leq n} f_{n-k} g_k.$$

Example 4.1 i) *Product of Catalan and Fibonacci numbers.* The rule operator obtained by applying the previously defined operation $\otimes$ to the rule operators for Catalan and Fibonacci numbers (see Example 3.1) is:

$$L_\mathcal{C} \otimes L_\mathcal{F}(1) = x^4$$

$$L_\mathcal{C} \otimes L_\mathcal{F}(x^{k+2}) = x + x^2 + x^4 + x^5 + \ldots + x^k + x^{k+1}$$

$$L_\mathcal{C} \otimes L_\mathcal{F}(x) = x^2$$

$$L_\mathcal{C} \otimes L_\mathcal{F}(x^2) = x + x^2.$$

and it defines the number sequence $1, 4, 12, 35, 103, 312....$ The reader can check that in this case the product can be expressed with no need of other variables.

ii) *The rule operator for the n-th power Catalan numbers.* We want to prove that the rule operator $L^n_\mathcal{C}$ for the sequence defined by $C(x)^n$ is the following:

$$\begin{aligned} &L^n_\mathcal{C}(1) = x^n \\ &L^n_\mathcal{C}(x^k) = L_\mathcal{C}(x^k) = x^2 + x^3 + x^4 + \ldots + x^k + x^{k+1}. \end{aligned} \tag{7}$$

We can prove this statement inductively, supposing it holds for $n \in \mathbb{N}$, and therefore verifying it for $n+1$. Since $L^{n+1}_\mathcal{C} = L_\mathcal{C} \otimes L^n_\mathcal{C}$, we have $L^{n+1}_\mathcal{C}(1) = x^{n+1}$. Moreover we have:
$L^{n+1}_\mathcal{C}\left(x^{k+1}\right) = L_\mathcal{C} \otimes L^n_\mathcal{C}\left(x^{k+1}\right) = xL^n_\mathcal{C}(x^k) + L_\mathcal{C}(x) = x^2 + x^3 + x^4 + \ldots$
$+ x^h + x^{h+1} + x^{h+2} = L^{n+1}_\mathcal{C}(x^{k+1})$.

4.3 The Star of a rule operator

The *star* of the rule operator L_Ω is denoted as L^*_Ω, briefly L^*, and it is the operator defining the number sequence having

$$g(x) = \frac{1}{1-f_0(x)} = 1 + f_0(x) + f_0^2(x) + \ldots + f^n(x) + \ldots = \sum_{n\geq 0} f_0^n(x)$$

as its generating function, where $f_0(x) = f(x) - 1$. Set $L(1) = x^a$, the operator L^* is defined as:

$$\begin{aligned} L^*(1) &= x^a = L(1) \\ L^*(x^a) &= x^a L(x^a) = L(1)L^2(1) \qquad (8) \\ L^*(x^{a+h}) &= x^a(L(x^a) + L(x^h)). \end{aligned}$$

We then prove that, for every $n \in \mathbb{N}$:

$$\left[(L^*)^{n+1}(1)\right]_{x=1} = [x^n]g(x),$$

where $[x^n]g(x)$ indicates, as usual, the coefficient of x^n in $g(x)$.

Lemma 4.2 For every polynomial $p(x) \in \mathbb{R}[x]$ such that $\deg p(x) \geq 1$, we have:

$$L^*(x^a p(x)) = x^a \left(L^2(1)p(1) + L(p(x))\right).$$

Proof. Let $p(x) = \sum_{k=1}^{n} p_{nk} x^k$. Therefore we have:

$$L^*(x^a p(x)) = L^*\left(\sum_k p_{nk} x^{a+k}\right) = \sum_k p_{nk}\left(x^a\left(L(x^a) + L(x^k)\right)\right) =$$

$$x^a\left(L^2(1)p(x) + L(p(x))\right). \qquad \square$$

Recall that the coefficients g_n of the generating function $g(x) = \sum_n g_n x^n$ satisfy the recurrence relation:

$$\begin{aligned} g_0 &= 1 \\ g_n &= f_0 g_{n-1} + f_1 g_{n-2} + \ldots + f_{n-1} g_1 = \textstyle\sum_{k=1}^{n-1} f_k g_{n-k}, \qquad n \geq 1. \end{aligned} \qquad (9)$$

From Lemma 4.2 and (9) we have:

Proposition 4.2 For any $n \in \mathbb{N}$, the following identity holds:

$$(L^*)^n(1) = x^a \sum_{k=2}^{n} \left(L^k(1)\left[(L^*)^{n+1-k}\right]_{x=1}\right). \qquad (10)$$

Proof. For $n = 2, 3$ the identity (10) clearly holds. Now, if we suppose it holds for $n \in \mathbb{N}$, we immediately have:

$$\begin{aligned}
(L^*)^{n+1}(1) &= L^*((L^*)^n(1)) = L^*\left(x_a \cdot \frac{(L^*)^n(1)}{x^a}\right) \\
&= x^a\left(L^2(1)\left[L^{*n}(1)\right]_{x=1} + L\left(\frac{L^{*n}(1)}{x^a}\right)\right) \\
&= x^a\left(L^2(1)\left[L^{*n}(1)\right]_{x=1} + \sum_{k=2}^{n} L^{k+1}(1)\left[L^{*n+1-k}(1)\right]_{x=1}\right) \\
&= x^a \sum_{k=2}^{n+1}\left(L^k(1)\left[(L^*)^{n+2-k}\right]_{x=1}\right). \qquad \square
\end{aligned}$$

Corollary 4.2 For any $n \in \mathbb{N}$ we have $\left[L^{*n+1}(1)\right]_{x=1} = g_n$.

Example 4.2 **i)** *The star of Catalan numbers.* The rule operator $L^*_{\mathcal{C}}$ is:

$$L^*_{\mathcal{C}}(1) = x^2$$

$$L^*_{\mathcal{C}}(x^2) = x^4 + x^5$$

$$L^*_{\mathcal{C}}(x^{k+2}) = 2x^4 + 2x^5 + x^6 + x^7 + \ldots + x^k + x^{k+1} + x^{k+2} + x^{k+3}.$$

and it defines the sequence $1, 2, 9, 42, 199, \ldots$, having

$$\frac{1}{1 - \left(\frac{1-2x-\sqrt{1-4x}}{2x^2} - 1\right)}$$

as its generating function.

ii) *The star of Schröder numbers.* Consider the rule operator $L_{\mathcal{S}}$:

$$L_{\mathcal{S}}(1) = x^2$$

$$L_{\mathcal{S}}(x^{2k}) = x^2 + 2x^4 + 2x^6 + \ldots + 2x^{2k} + x^{2k+2}.$$

We get:

$$L^*_{\mathcal{S}}(1) = x^2$$

$$L^*_{\mathcal{S}}(x^2) = x^4 + x^6$$

$$L^*_{\mathcal{S}}(x^{2k+2}) = 2x^4 + 3x^6 + \ldots + 2x^{2k+2} + x^{2k+4}.$$

4.4 Partial sum of a succession rule

Let L be a rule operator and $\{f_n\}_n$ its associated sequence, having $f(x)$ as generating function. The *partial sum* ΣL, is the rule operator leading to the sequence $\{F_n\}_n = \left\{\sum_{j\leq n} f_j\right\}_n$. We can obtain ΣL by means of the product operation, since $F(x) = \sum_n F_n x^n = \frac{1}{1-x} \cdot f(x)$. Thus:

$$\Sigma L = L_1 \otimes L,$$

where L_1 is the rule operator for the sequence $f_n = 1$, for all n, that is:

$$\begin{cases} L(1) = x \\ L(x^k) = kx^k. \end{cases}$$

By applying the product operation we have:

$$\Sigma L(1) = x^{a+1}$$

$$\Sigma L(x) = x$$

$$\Sigma L(x^{h+1}) = x(1 + x^{e_1(h)} + \ldots + x^{e_n(h)})) = x(1 + L(x^h)).$$

This result can also be obtained by proving explicitely the following proposition:

Proposition 4.3 For any $n \in \mathbb{N}$ we have:

$$(\Sigma L)^n(1) = x\left(\sum_{i=1}^{n-1}\left[L^i(1)\right]_{x=1} + L^n(1)\right).$$

For example, the rule operator $L_{\mathcal{C}}$ for Catalan numbers leads to the operator:

$$\Sigma L_{\mathcal{C}}(1) = x^3,$$

$$\Sigma L_{\mathcal{C}}(x) = x,$$

$$\Sigma L_{\mathcal{C}}(x^{h+1}) = x + x^3 + x^4 + \ldots + x^{h+1} + x^{h+2},$$

giving the sequence $1, 3, 8, 22, 64, \ldots$
Moreover, it is easy to prove the following property.

Proposition 4.4 Let L be a rule operator defining the sequence $\{f_n\}_n$. Then a rule operator L' defining a sequence $\{g_n\}_n$, such that $f_n = g_n - rg_{n-1}$, for $n > 1$, exists:

$$\begin{cases} L'(1) = x^{a+r} = x^r L(1) \\ \\ L'(x^r) = rx^r \\ \\ L'(x^{h+r}) = rx^r + x^r L(x^h). \end{cases}$$

Proof. We first prove that for any $n \in \mathbb{N}$,

$$\left(L'\right)^n (1) = x^r \left(\sum_{i=1}^{n-1} r^{n-1} \left[L^i(1)\right]_{x=1} + L^n(1) \right). \tag{11}$$

From (11) we immediately obtain:

$$\left(L'\right)^{n+1} (1) - r(L')^n(1) = x^r \left(r \left[L^n(1)\right]_{x=1} + L^{n+1}(1) - rL^n(1) \right),$$

and then:

$$\left[\left(L'\right)^{n+1} (1) - rL'^n(1)\right]_{x=1} = \left[L^{n+1}(1)\right]_{x=1} = f_n. \qquad \square$$

Proposition 4.5 Let L be a succession rule, defining the sequence $\{f_n\}_n$ and let $L^2(1) - L(1) \in \mathbb{N}[x]$. Then there is a rule operator L' defining the sequence $\{g_n\}_n$ such that $g_0 = 1$, and $g_n = f_n - f_{n-1}$, for $n \geq 1$.

Sketch of proof. Let us consider the following rule operator:

$$L' : \begin{cases} L'(1) = \frac{L(1)}{y} \\ \\ L'\left(\frac{L(1)}{y}\right) = L^2(1) - L(1) \\ \\ L'(x^k) = L(x^k) \end{cases}$$

and let g_n be the sequence described by L'. By applying the sum operation, we easily conclude that:

$$L = L' \oplus xL.$$

Finally, L' defines a sequence for $g_n = \begin{cases} 1 & \text{if } n = 0; \\ f_n - h_n = f_n - f_{n-1} & \text{otherwise.} \end{cases}$ $\square$

Example 4.3 Let $L_{\mathcal{S}}$ be the rule for Schröder numbers:

$$L_{\mathcal{S}}: \quad \begin{cases} L_{\mathcal{S}}(1) = x^2 \\ L_{\mathcal{S}}(x^{2h}) = x^2 + 2x^4 + \ldots + 2x^{2h} + x^{2h+2}. \end{cases}$$

The rule operator L_1,

$$L_1(1) = x^3,$$

$$L_1(x) = x,$$

$$L_1(x^{2h+1}) = x + x^3 + 2x^5 + \ldots 2x^{2h+1}x^{2h+3},$$

defines such a sequence $\{g_n\}_n = \{1, 3, 9, 31, 121, 515, \ldots\}$, that $f_n = g_n - g_{n-1}$, where f_n denotes the nth Schröder number. Moreover, since the rule operator $L_{\mathcal{S}}$ satisfies the hypotheses of Proposition 4.5, there is a rule operator L' defining the sequence k_n such that $k_0 = 1$, and $k_n = f_n - f_{n-1}$ for $n > 0$, that is the sequence $\{1, 1, 4, 16, 68,$
$304, 1412, \ldots\}$ (sequence M3521 in [12]):

$$\begin{cases} L'(1) = x, \\ L'(x) = x^4 \\ L'(x^{2h}) = x^2 + 2x^4 + \ldots + 2x^{2h} + x^{2h+2}. \end{cases}$$

5 Open problems

There are several open problems related to the definition of an algebra of succession rules which, in turn, lead to problems concerning the set of rule operators. Below an overview of the most interesting problems is given:

- **Other operations.**

  Subtraction Let us consider two rule operators L_Ω and $L_{\Omega'}$, defining the sequences $\{f_n\}$ and $\{g_n\}$ respectively. Moreover, let $L_\Omega \ominus L_{\Omega'}$ be the rule operator defining the sequence $\{h_n\}_n$ such that $h_n = \begin{cases} 1 & \text{if } n = 0 \\ |f_n - g_n| & \text{otherwise.} \end{cases}$

The construction of the operator $L_\Omega \ominus L_{\Omega'}$ presents an open problem.

Hadamard product Let L_Ω and $L_{\Omega'}$ be rule operators and, as usual, $\{f_n\}_n$ and $\{g_n\}_n$ be their sequences, with their respective generating functions $f(x)$ and $g(x)$. The *Hadamard product* of L_Ω and $L_{\Omega'}$, denoted as $L_\Omega \odot L_{\Omega'}$, is the rule defining the sequence $\{f_n g_n\}_n$. It is generally quite difficult to determine the generating function $f(x) \odot g(x)$, although the Hadamard product of two $\mathbb{N}$-rational series has been proved to be $\mathbb{N}$-rational [11]. The problem lies in the construction of the rule operator $L_\Omega \odot L_{\Omega'}$. However, we can prove that, in the case of finite rules, it is possible to determine a rule defining the Hadamard product. More precisely we can state that *the Hadamard product of two finite rules is a finite rule.*

Here is an example of our technique: let Ω be the rule for Pell numbers, $\{1, 2, 5, 12, 29, \ldots\}$, and $L_{\Omega'}$ be the rule for the Fibonacci numbers having an odd index, $\{1, 2, 5, 13, 34, \ldots\}$,

$$\Omega : \quad \begin{cases} (2) \\ (2) \rightsquigarrow (2)(3) \\ (3) \rightsquigarrow (2)(2)(3), \end{cases} \qquad \Omega' : \quad \begin{cases} (\bar{2}) \\ (\bar{2}) \rightsquigarrow (\bar{2})(\bar{3}) \\ (\bar{3}) \rightsquigarrow (\bar{2})(\bar{3})(\bar{3}). \end{cases}$$

For each label (h) of Ω and $(\bar{k})$ of Ω', $(h \cdot k)$ is a label of the rule $\Omega \odot \Omega'$, and it is colored only if there is already another label having the same value. The axiom is $(a \cdot b)$, where (a) and $(\bar{b})$ are the axioms of the rules. If the productions of (h) and $(\bar{k})$ are:

$$(h) \rightsquigarrow (c_1) \ldots (c_h)$$

$$(\bar{k}) \rightsquigarrow \overline{(e_1)} \ldots \overline{(e_k)},$$

then the production of $(h \cdot k)$ is:

$$(h \cdot k) \rightsquigarrow (c_1 \cdot e_1) \ldots (c_1 \cdot e_k) \ldots (c_h \cdot e_1) \ldots (c_h \cdot e_k).$$

Referring to our example, the labels of $\Omega \odot \Omega'$ are $(2 \cdot \bar{2}) = (4)$, $(2 \cdot \bar{3}) = (6)$, $(3 \cdot \bar{2}) = (\bar{6})$, $(3 \cdot \bar{3}) = (9)$. For instance, the production for the label (4) is:

$$(4) = (2 \cdot \bar{2}) \rightsquigarrow (2 \cdot \bar{2})(2 \cdot \bar{3})(3 \cdot \bar{2})(3 \cdot \bar{3}) = (4)(6)(\bar{6})(9).$$

In the same way we obtain:

$$\Omega \odot \Omega' : \qquad \begin{cases} (4) \\ (4) \rightsquigarrow (4)(6)(\bar{6})(9) \\ (6) \rightsquigarrow (4)(6)(6)(\bar{6})(9)(9) \\ (\bar{6}) \rightsquigarrow (4)(4)(6)(6)(\bar{6})(9) \\ (9) \rightsquigarrow (4)(4)(6)(6)(6)(6)(\bar{6})(9)(9). \end{cases}$$

The rule $\Omega \odot \Omega'$ has ij labels, i and j being the number of labels of Ω and Ω' respectively.

- **Equivalence.** Is there a criterion whereby we can establish whether two given succession rules are equivalent simply by working on their labels, that is, with no need to determine the corresponding generating functions? Furthermore, given a succession rule, is there a method to obtain some equivalent rules?

- **Inversion.** Let $\{f_n\}_n$ be a non-decreasing sequence of positive integers. Is there a method allowing us to decide whether a succession rule defining the sequence $\{f_n\}_n$ exists and, if it does, to find it? Note that this problem can be solved for finite rules.

- **Colored rules.** Let $\{f_n\}_n$ be a non-decreasing sequence of positive integers defined by a colored succession rule Ω. Is there a criterion to establish whether a non-colored succession rule defining $\{f_n\}_n$ exists? This problem is still open also for finite rules. Regarding the matter, the following facts should be mentioned:

  1. if the sequence $\{f_n\}_n$ has repetitions, that is there exists j such that $f_j = f_{j+1}$, then it is easy to check whether the rule for $\{f_n\}_n$ needs to be colored;
  2. therefore, we can focus exclusively on the case of a strictly increasing $\{f_n\}_n$. The only thing that can be surely stated is that if the sequence $\{f_{n+1} - f_n\}$ is strictly increasing too, then a non-colored succession rule defining $\{f_n\}_n$ must exist, although sometimes it may have a very complicated form:

$$\begin{cases} (f_1) \\ (1) \rightsquigarrow (1) \\ (f_k) \rightsquigarrow (1)^{k-1}(f_{k+1} - f_k + 1). \end{cases}$$

5.1 A Conjecture

Conjecture: *if a succession rule has a rational generating function, then it is equivalent to a finite succession rule.* It is sufficient to prove that each rational generating function of a succession rule satisfies the same properties shared by the generating functions of finite rules, as described in Section 1. If the conjecture proves true, rational functions such as (5) cannot be the generating functions of any succession rule. For example, let Ω be the rule, studied in [1], whose set of labels is the whole set of prime numbers:

$$\Omega : \qquad \begin{cases} (2) \\ (p_n) \rightsquigarrow (p_{n+1})(q_n)(r_n)(2)^{p_n - 3}, \end{cases}$$

where p_n denotes the nth prime number, and q_n and r_n are two primes such that $2p_n - p_{n+1} + 3 = q_n + r_n$ (via Goldbach conjecture). According to our conjecture, as its generating function is rational, $f(x) = \frac{1-2x}{1-4x+3x^2}$, it is possible to find a finite succession rule Ω' equivalent to Ω:

$$\Omega' : \qquad \begin{cases} (2) \\ (2) \rightsquigarrow (2)(3) \\ (3) \rightsquigarrow (2)(3)(4) \\ (4) \rightsquigarrow (2)(3)(4)(4). \end{cases}$$

It should be noticed that the rule Ω' was further exploited in [9], being the 4-approximating rule for Catalan numbers. Furthermore, such a rule describes a recursive construction for Dyck paths whose maximal ordinate is 4.

References

[1] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy and D. Gouyou-Beauchamps, On generating functions of generating trees, *Proceedings of 11th FPSAC, Barcelona* (1999) 40-52.

[2] E. Barcucci, A. Del Lungo, A. Frosini, S. Rinaldi, From rational functions to regular languages, in *Formal Power Series and Algebraic Combinatorics,* D. Krob, A. A. Mikhalev, A. V. Mikhalev Eds., Springer-Verlag, Berlin (2000) 633-644.

[3] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, ECO: a methodology for the Enumeration of Combinatorial Objects, *Journal of Difference Equations and Applications,* Vol.5 (1999) 435-490.

[4] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, Some combinatorial interpretations of q-analogs of Schröder numbers, *Annals of Combinatorics,* 3 (1999) 173-192.

[5] E. Barcucci, E. Pergola, R. Pinzani, S. Rinaldi, Eco method and hill-free paths, *Seminaire Lotharingien de Combinatoire,* B46b (2001), 14pp.

[6] F. R. K. Chung, R. L. Graham, V. E. Hoggatt, M. Kleimann, The number of Baxter permutations, *J. Combin. Theory Ser. A,* 24 (1978) 382-394.

[7] S. Corteel, Problèmes énumératifs issus de l'Informatique, de la Physique et de la Combinatorie, *PhD Thesis, Université de Paris-Sud,* 6082 (2000).

[8] O. Guibert, Combinatoire des permutations à motifs exclus en liason avec mots, cartes planaries et tableaux de Young, *Thèse de l'Université de Bordeaux I,* (1996).

[9] E. Pergola, R. Pinzani, S. Rinaldi, ECO-Approximation of algebraic functions, in *Formal Power Series and Algebraic Combinatorics,* D. Krob, A. A. Mikhalev, A. V. Mikhalev Eds., Springer-Verlag, Berlin (2000) 719-730.

[10] G. Rozenberg, A. Salomaa, *The mathematical theory of L systems*, Academic Press, London (1980).

[11] A. Salomaa and M. Soittola, *Automata-theoretic aspects of formal power series*, Springer-Verlag, New York (1978).

[12] N. J. A. Sloane and S. Plouffe, *The encyclopedia of integer sequences* Academic press, New York (1996).

[13] J. West, Generating trees and the Catalan and Schröder numbers, *Discrete Mathematics,* 146 (1995) 247-262.