Explicit Formula

1. (This solution is due to Euler.) Let us prove another recurrence relation for c_n. (Now it is convenient to think of c_n as of the number of triangulations.) Consider the set of all pairs (a triangulation, a distinguished diagonal of this triangulation). It is clear that the cardinality of this set is $c_n \cdot (n-1)$. On the other hand, there are $n+2$ diagonals which cut an i-gon off our polygon. Thus for a fixed diagonal of this type there are $c_i c_{n-i}$ triangulations containing it. Each pair is counted twice (both for i-gon and $(n-i)$-gon). It follows that

$$(n-1)c_n = (n+2)(c_1 c_{n-1} + c_2 c_{n-2} + \cdots + c_{n-1} c_1)/2.$$

One can easily see that $c_1 c_{n-1} + \cdots + c_{n-1} c_1 = c_{n+1} - 2c_n$, so $(n-1)c_n = (n+2)(c_{n+1} - 2c_n)/2$. This means that

$$\frac{c_{n+1}}{c_n} = \frac{2(2n+1)}{n+2}.$$

Now the explicit formula immediately follows.

This solution is neither the shortest nor the most "natural" one. For the solutions based on the so called "reflection principle" and "Reni lemma" see [1]. G. Esebua & A. Kovtun (Kharkov) proved the explicit formula by induction using the recurrence relation (clear from the interpretation from 2(c₂))

$$c_n + \binom{1}{1} c_{n-1} + \binom{3}{2} c_{n-2} + \cdots + \binom{2n-1}{n} c_0 = \binom{2n}{n}.$$

Mappings

The text below contains several mappings. Most of them are quite "natural"; moreover, they turn out to be bijections. (Most of the proofs are left to the reader.)

Some sets are joined into blocks — it means that there are obvious mappings between these sets; I use the notation $\mathbb{B}_1$ for {2(c₁),2(c₂),2(d),3(l),3(n)}, $\mathbb{B}_2$ for {3(e₁),3(e₂)}, $\mathbb{B}_3$ for {3(i₁),3(i₂),3(k)} and $\mathbb{B}_4$ for {3(a),3(b)}.

Bijections Inside the Blocks

$\mathbb{B}_1$: 2(c₁) $\to$ 2(c₂) Trivial.

2(d) $\to$ 2(c₂) Fix a vertex v. Starting clockwise from v, at each vertex write down 1 if encountering a chord for the first time and -1 otherwise.

2(c₂) $\to$ 3(l) Take partial sums.

2(c₂) $\to$ 3(n) Replace kth 1 by $2k-1$ and kth -1 by $2k$.

$\mathbb{B}_2$: 3(e₁) $\to$ 3(e₂) Restrict the composition of bijections 3(c)$\to$ 2(c₂) $\to$ 2(c₁) on the trees considered.

$\mathbb{B}_3$: 3(k) $\to$ 3(i₁) Take *first differences*: $a_i \mapsto a_{i+1} - a_i$.

3(k) $\to$ 3(i₂) $(a_1, \ldots, a_n) \mapsto (a_n, a_{n-1} - a_n, a_{n-2} - a_{n-1}, \ldots, a_2 - a_1)$

$\mathbb{B}_4$: 3(b) $\to$ 3(a) Remove all endpoints (together with the corresponding edges).

More Bijections

Let T be a tree, v, w be vertices of T. By definition, v is a *brother* of w if they have the same parent.

Define the *depth-first search (DFS)* through a (plane) tree T (it is a walk throughout T passing each edge twice, beginning and ending at the root) as follows.

- If the root has no sons, stop.
- Suppose that e is the leftmost edge starting at the root and it connects the root with v.Then go "down" along e, do *DFS* through the subtree T_0 with root v, go "up" along e and do *DFS* through the tree obtained from T by removing T_0.

Define the *standard numbering* of the vertices. Namely, number the vertices as they appear (for the first time) during *DFS*.

2(a) $\to$ 2(b) Fix a side l of the polygon. Label all sides except l clockwise with factors. Given a triangulation, find a triangle ABC with sides AB and BC labeled by p and q respectively, label AC with (pq) and proceed with it until l gets a label. This label is the corresponding way of multiplication.

time labeling the vertex by the product of labels of its sons.

2(b) $\to$ 2(c_2) *First map.* Erase all right brackets and replace left brackets with '1' and letters (except the last) with '-1'.

Second map. Replace all multiplication signs by '1'and all right brackets with '-1'. (It corresponds to performing multiplications with a "stack calculator".)

3(j_1) $\to$ 2(c_1) Add one step $(1,0)$ and then go upwards to (n,n).

3(f) $\to$ 3(j_1) Replace each step $(1,0)$ and $(1,1)$ by $(1,0)$ and each step $(0,1)$ with $(0,1)$.

3(b) $\to$ 3(p) Consider the standard numbering of vertices. Let v and w be in the same block if v is a right son of w.

3(b) $\to$ 2(c_2) Do a depth-first search, recording '1' when a left edge if encountered for the first time and recording '-1' when a right edge is encountered for the first time.

3(c) $\to$ 2(c_2) Do a depth-first search, recording '1' when going down an edge and recording '-1' when going up an edge.

3(k) $\to$ 2(c_2) Let $b_i = a_i - a_{i+1} + 1$ (set $a_{n+1} = 0$). Replace a_i with 1 $\texttt{-1}^{b_i}$ (where a^n denotes a repeated n times).

3(c) $\to$ 3(i_2) Do a depth-first search, recording one less than the number of v's sons when the vertex v is encountered for the first time (for all vertices except the last).

3(d) $\to$ 2(c_2) For a parallelogram polyomino with columns $C_1, \dots, C_k$ let a_i be the number of (unit) squares in C_i and let b_i be the number of common rows of C_i and C_{i+1}. Consider the following sequence:

$$\texttt{1}^{a_1}\ \texttt{-1}^{a_1-b_1+1}\ \texttt{1}^{a_2-b_1+1}\ \texttt{-1}^{a_2-b_2+1} \dots \texttt{1}^{a_k-b_{k-1}+1}\ \texttt{-1}^{a_k-b_k+1}.$$

Another way to establish the bijection was discovered by almost all participants of the conference who were solving these problems. Namely, draw a parallelogram polyomino as follows: one step of the lower path, one from the upper path etc. For each up (resp. right) step of the upper path write down '1' (resp. '-1') and vice versa for the lower path. Remove the first '1' and the last '-1'.

3(g) $\to$ 2(c_2) Label points $1, 2, \dots, n+1$ from the left to the right. Suppose that i is connected with a_i points greater than i. Consider the following sequence:

$$\texttt{1}^{a_1}\ \texttt{-1}\ \texttt{1}^{a_2}\ \texttt{-1} \dots \texttt{1}^{a_{n+1}}\ \texttt{-1}$$

3(q) $\to$ 2(c_2) Let $a_i = 1$ if i appears in the first row and -1 otherwise (this mapping is quite obvious and therefore 3(q) may be included in $\mathbb{B}_1$).

3(m) $\to$ 2(c_1) Given a permutation $a_1, \dots, a_n$, let b_i be the number of $j > i$ with $a_j < a_i$. Let $p_1 < p_2 < \dots < p_k$ be the list of all p such that $b_p \neq 0$. Define a lattice path as follows: move horizontally to $(b_{p_1} + p_1 - 1, 0)$, then vertically to $(b_{p_1} + p_1 - 1, p_1)$, then horizontally to $(b_{p_2} + p_2 - 1, p_1)$, then vertically to $(b_{p_2} + p_2 - 1, p_2)$ etc. The last part of this path consists of a vertical line from $(b_{p_k} + p_k - 1, p_{k-1})$ to $(b_{p_k} + p_k - 1, p_k)$, then a horizontal line to (n, p_k) and finally a vertical line to (n, n).

Remark. In fact this mapping is the composition 3(m) $\to$ 3(w) $\to$ 2(c_1), where the first mapping is $a_1, \dots, a_n \mapsto b_1, \dots, b_n$. Although this mapping ("the code of a permutation") was introduced during the presentation of the problems, some of the participants preferred another way of including 3(m) to the bijection graph. Namely, they suggested the following mapping 3(m)$\to$ 3(h):

$$a_1, \dots, a_n \mapsto \min_i a_i, \min_{i>1} a_i, \min_{i>2} a_i, \dots.$$

2(c_1) $\to$ 3(h) Given a lattice path L, let a_i be the area bounded by lines $y = -1$, $x = i - 1$, $x = i$ and L.

3(v) $\to$ 3(h) For a vertex v, consider the unique path of length $n - 1$ (meeting n vertices $v_1, \dots, v_n = v$) from the root to v. Let b_i be the number of v_i's sons. Set $a_i = i + 2 - b_i$.

3(m) $\to$ 3(v) Define a tree T_0 as follows. Let the vertices of level $n - 1$ correspond to the permutations from 3(m). Two vertices (from levels j and $j+1$) are connected with an edge if the first of the corresponding permutations is a subsequence of the second one. Then T_0 is isomorphic to the tree from 3(v).

3(v) $\to$ 3(u) Label the root by 0 and its two sons by 0 and 1. Then label the remaining vertices recursively as follows. Suppose that v is one of the vertices at level n and is labeled by j. Suppose also that brothers of v with labels less than j are labeled $l_1, \dots, l_i$. Then v has $i + 2$ children and they get labels $l_1, \dots, l_i, j, n$. Each vertev v at level $n - 1$ corresponds to the sequence of labels on the unique path of length $n - 1$ from the root to v.

Exercise. Prove that for this labeling the number of vertices at level n labeled by j is equal to $c_j c_{n-j}$.

Remark. In fact for almost any type of integer sequences from **3** one can check that step-by-step constructing of sequences of that type is described by the tree from 3(v).

3(b) $\to$ 3(c) For each pair of vertices (v, w) such that v is a right child of w glue together v and w (removing the corresponding edge).

leftmost son (removing the corresponding edge).

2(a) $\to$ 3(a) Fix a side l of the polygon. Given a triangulation, define a tree as follows. Let vertices correspond to the triangles (by the definition, the triangle having side l corresponds to the root) and join two vertices with an edge if corresponding triangles have a common side.

$[n+1]\times$3(s) $\to$ {All lattice paths from $(0,0)$ to (n,n)}
For each n-tuple $A = [a_1, \dots, a_n]$ of the type considered, define its shifts $sh_i(a) = [a_1 + i \bmod n+1, \dots, a_n + i \bmod n+1]$ $(1 \leqslant i \leqslant n)$. As elements of $sh_i(a)$ sum up to $-i$ modulo $n+1$, we see that each n-tuple $[b_1, \dots, b_n]$, where $0 \leqslant b_i \leqslant n$, either belongs to the set considered or is a shift of an n-tuple belonging to this set. Therefore we established a bijection between $n+1$ copies of the set considered and the set of all n-tuples $[b_1, \dots, b_n]$, where $0 \leqslant b_i \leqslant n$. As we deal with unordered tuples, we can assume that $b_1 \leqslant \dots \leqslant b_n$. To obtain a lattice path, recall the inverse mapping to 2(c_1)$\to$ 3(h).

Remark. In other words, the sequence from 3(s) corresponds to a class of sequences $b_1 \leqslant \dots \leqslant b_n$, where $0 \leqslant b_i \leqslant n$, each sequence of this type corresponds to a lattice path and in each class there exists a unique[1] sequence which corresponds to a lattice path from 3(c_1).

3(t)$\to$ 2(c_2) Consider a sequence $1, a_1, \dots, a_n, 1$. (Set $a_0 = a_{n+1} = 1$.) It is clear that $a_i = a_{i+1}$ never holds (otherwise all a_j should be a multiple of a_i). Therefore there exists at least one a_i such that $a_{i-1} < a_i > a_{i+1}$. Since $a_{i-1} + a_{i+1}$ is a multiple of a_i, we see that it is just equal to a_i. When we remove a_i from the sequence, we obtain a sequence with one element less still satisfying the divisibility condition. Conversely, any sequence can be lengthened by adding the term $a_i + a_{i+1}$ between a_i and a_{i+1}. Repeat this operation (starting from the sequence $1, 1$), but now when $a_i + a_{i+1}$ is added between a_i and a_{i+1} insert a mark before a_i and then make changes only to the right from the last mark. The places of n marks completely determine the sequence $a_1, \dots, a_n$ and obviously the marks precede the corresponding numbers. Replace each mark by '`1`'and each a_j $(1 \leqslant j \leqslant n)$ by '`-1`'.

2(a)$\to$ 3(t) (This materialization of the arguments presented above was suggested by R. Travkin.) Fix a side AB of the polygon. Label A and B by 1. Given a triangulation, find a triangle PQR with vertices P and Q labeled by x and y respectively, label R with $x+y$ and proceed with it until all vertices get labels. Read the labels clockwise from A to get a sequence of type 3(t).

3(r) $\to$ 3(i_1) Fix a vertex v. Starting clockwise from v, at each vertex record 1 if it is marked or you encounter a chord for the first time, $-1-p$ if you encounter a chord for the second time and you encountered exactly p marked points belonging to the corresponding arc ω and not belonging to other arcs contained in ω (we consider only arcs that correspond to the chords), and 0 otherwise.

3(c_2) $\to$ 3(r) (Suggested by...) Let us start with a Very Useful Bijection: we identify 3(c_2) with the set of sequences of length $n-1$ of ± 2's and "± 0's" (say coloured in red and blue) with nonnegative partial sums. Namely, remove the first '`1`'and the last '`-1`', split the sequence into subwords of length 2 and replace each subword by the sum of numbers in it. (To distinguish between '`1 -1`' and '`-1 1`', we need to colour zeros.) This mapping is a bijection (to verify that the partial sums are nonnegative, note that the odd partial sums of a sequence from 2(c_2) are positive). After that mark all points that correspond to '`+0`', do nothing with points that correspond to '`-0`' and join the points that correspond to ± 2's as in 2(d).

3(a) $\to$ 3(o) *Hint.* Modify the mapping 3(b)$\to$ 3(p).

3(w) See the discussion of "321-avoiding" permutations of 3(m).

3(x) The sequences $a_n - 1, \dots, a_1 - 1$ are just the codes of the "312-avoiding" permutations from 3(y). Alternatively,define a binary tree $T(a_1, \dots, a_n)$ as follows. Set $T(\text{empty}) = \text{empty}$. If $n > 0$, let the left subtree of the root of $T(a_1, \dots, a_n)$ be $T(a_1, \dots, a_{n-a_n})$ and the right subtree of the root be $T(a_{n-a_n+1}, \dots, a_n)$. This establishes the bijection with 3(a).

See also the remark to the mapping 3(v)$\to$ 3(u).

3(y) *Hint.* Firstly, establish the bijection with "231-avoiding" permutations by the formula $a_1, \dots, a_n \mapsto n+1-a_n, \dots, n+1-a_1$. Introduce the algorithm of *stack-sorting (SS)* of a sequence A as follows:

- $SS(\text{empty}) = \text{empty}$
- $SS(A_1 x A_2) = SS(A_1) SS(A_2) x$, if x is the maximal element in the sequence $A_1 x A_2$

A permutation A (considered as a sequence) is *stack-sortable*, if $SS(A) = 1, 2, \dots, n$. Then "231-avoiding" permutations turn out to be stack-sortable and vice verca.

See also the remark to the mapping 3(v)$\to$ 3(u).

3(q) $\to$ 3(z) Given a standard $2n$-tableaux of the rectangular shape (the objects from 3(q) turn out to be just $2n$-tableaux), define a pair of n-tableaux from 3(z) as follows: take for the shape of the tableaux the part of the rectangular tableaux containing the numbers $1, \dots, n$. Rotate 180° the remaining part of the $2n$-tableaux and replace each entry i from this part by $2n+1-i$ to get the second n-tableaux.

[1] It follows from the Reni lemma [1].

-1' and remove the first (right) step and the last (up) step.

3($\wp$)

3(c$_2$)$\rightarrow$ 3($\mathring{v}$) Firstly, identify paths from 3($\mathring{v}$) with the sequences of ±1's in a usual way. Then recall the Very Useful Bijection from the discussion of 3(r). Replace '+0' $\leftrightarrow$'1 -1', '-0' $\leftrightarrow$'1 1 -1 -1', '+2' $\leftrightarrow$'1 1 -1', and '-2' $\leftrightarrow$'1 -1 -1'.

4. Think of a lattice path as of a sequence of E's (for East, meaning the step $(1,0)$) and N's (for North, meaning the step $(0,1)$). Given a path L of the type considered, define recursively the path $s(L)$ as follows:

- $s(\text{empty}) = \text{empty}$
- $s(L_1X) = L_1s(X)$
- $s(\bar{L}_1X) = Es(X)NL_1^*$,

where L_1 is a path of positive length with endpoints on the diagonal and all other points strictly below the diagonal, $\bar{L}_1$ denotes the path obtained from L_1 by interchanging $E \leftrightarrow N$ and L_1^* is L_1 without the first E and the last N. Then s establishes a bijection between paths considered and paths from 2(c$_1$) (with n replaced by $2n$).

5.

6.

7. For example, 3(r) leads to the Touchard identity[2]

$$\sum_{k\leqslant n/2}\binom{n}{2k}2^{n-2k}c_k = c_{n+1}.$$

References

[1] R. L. Graham, D. E. Knuth, O. Patashnik. Concrete Mathematics.

[2] N. Sloane. On-line encyclopedia of integer sequences.

[3] R. Stanley. Enumerative Combinatorics.

[2] It was already known in 1920's, the bijective proof was discovered by L. W. Shapiro in 1970's.