SEQUENCES FROM SQUARES OF INTEGERS

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Abstract

This paper presents a number of sequences based on integers arranged in arrays. This approach provides a simple derivation of some well known sequences. In addition, a number of new integer sequences are obtained.

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1. Introduction

This paper begins with a well known combinatorial expression for the sum of the first n natural numbers

$$1 + 2 + 3 + 4 + 5 + \ldots + n = \frac{n(n+1)}{2}. \tag{1}$$

known as the triangular numbers. Arranging the values for this sum in a sequence starting from $n = 1$ gives

$$1, 3, 6, 10, 15, \ldots$$

This is sequence A000217 in the Encyclopedia of Integer Sequences maintained by Sloane [1]. One could view the components of the sums that make up this sequence as lines or 1-dimensional arrays

$$1, \quad 1\,2, \quad 1\,2\,3, \quad 1\,2\,3\,4, \quad 1\,2\,3\,4\,5, \quad \ldots$$

The question then arises, what sequences occur when one considers m-dimensional arrays of integers? For $m = 0$, the result is the trivial sequence

$$1, 1, 1, 1, 1, \ldots$$

More interesting is the case $m = 2$, which gives rise to two-dimensional arrays of integers. This provides connections between seemingly unrelated sequences. The case of sequences from squares is considered in the next section, followed by an investigation of triangles and hexagons.

2. Squares

A square array of integers has the following structure

$$\begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \\ n+1 & n+2 & n+3 & \cdots & 2n \\ \vdots & \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & n^2-n+3 & \cdots & n^2 \end{array} \tag{2}$$

For $n = 1$ to 5, the matrices are

$$1\,, \quad \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}\,, \quad \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}\,,$$

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array}\,, \quad \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{array}\,.$$

One can easily see that the 0-dimensional sequence is located in the upper left hand corner, and the 1-dimensional sequence is given by the first rows. The sequence formed from the sum of the elements in the squares given by

$$s_n = 1 + 2 + 3 + 4 + 5 + \ldots + n^2 = \sum_{i=1}^{n^2} i = \frac{n^2(n^2+1)}{2}, \tag{3}$$

is

$$1, 10, 45, 136, 325, \ldots$$

The following simple sequences

$$\begin{array}{llllll} 1, & 2, & 3, & 4, & 5, & \ldots \\ 1, & 4, & 9, & 16, & 25, & \ldots \\ 1, & 3, & 7, & 13, & 21, & \ldots \end{array}$$

are formed from the elements in the upper right, lower right and lower left corners, respectively. The first of these is just the sequence of natural numbers (A000027)

$$1, 2, 3, 4, 5, \ldots, n, \ldots$$

the second is the sequence of squares of the natural numbers (A000290)

$$1, 4, 9, 16, 25, \ldots, n^2, \ldots$$

while the third is the sequence of central polygonal numbers (A002061)

$$1, 3, 7, 13, 21, \ldots, n^2 - n + 1, \ldots$$

all of which are well known sequences.

By considering shapes in the squares of numbers, many other sequences can be obtained. For example, the sequence of sums of the first column of numbers in each matrix is

$$1, 4, 12, 28, 55, 96, \ldots, \frac{n(n^2 - n + 2)}{2}, \ldots$$

This is sequence A006000 in [1]. which has generating function

$$\frac{(1 + 2x^2)}{(1 - x)^4}$$

The sum of the diagonal elements (upper left to lower right) gives

$$1, 5, 15, 34, 65, \ldots, \frac{n(n^2 + 1)}{2}, \ldots \quad (4)$$

which is sequence A006003, the row sums of an $n \times n$ magic square. Summing these elements with all those above the diagonal gives

$$1, 7, 26, 70, 155, \ldots, \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} in + j + 1 = \frac{n(n + 1)(n^2 + n + 1)}{6}, \ldots \quad (5)$$

which is sequence A006325.

Thus far, only one new sequence has been obtained, namely (3), but considering the three other triangular shapes in the matrix results in the following new sequences. The sum of the diagonal elements and those below it gives

$$1, 8, 34, 100, 235, \ldots, \sum_{i=0}^{n-1} \sum_{j=0}^{i} in + j + 1 = \frac{n(n + 1)(2n^2 - n + 2)}{6}, \ldots \quad (6)$$

The sum of the anti-diagonal elements and those below it gives

$$1, 9, 38, 110, 255, \ldots, \sum_{i=0}^{n-1} \sum_{j=0}^{i} (i + 1)n - j = \frac{n(n + 1)(2n^2 + 1)}{6}, \ldots \quad (7)$$

Finally, the sum of the anti-diagonal elements and those above it gives

$$1, 6, 22, 60, 135, \ldots, \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (i+1)n - j = \frac{n(n+1)(n^2+2)}{6}, \ldots \tag{8}$$

The sum of any two of these sequences yields another sequence, in particular adding (7) and (8) gives

$$1, 15, 60, 170, 390, \ldots, \frac{n(n+1)(n^2+1)}{2}, \ldots \tag{9}$$

This sequence is equivalent to (3) + (4), as can be seen by simplifying

$$\frac{n^2(n^2+1)}{2} + \frac{n(n^2+1)}{2} = \frac{n(n+1)(n^2+1)}{2}$$

Summing just those elements that lie above the diagonal gives

$$0, 2, 11, 36, 90, \ldots, \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} in + j + 1 = \frac{n(n-1)(n^2+2)}{6}, \ldots \tag{10}$$

The sum of the elements below the diagonal results in the sequence

$$0, 3, 19, 66, 170, \ldots, \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} in + j + 1 = \frac{n(n-1)(2n^2+1)}{6}, \ldots \tag{11}$$

The sum of those elements below the anti-diagonal is

$$0, 4, 23, 76, 190, \ldots, \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} (i+1)n - j = \frac{n(n-1)(2n^2+n+2)}{6}, \ldots \tag{12}$$

Finally, the sum of the elements above the anti-diagonal is

$$0, 1, 7, 26, 70, \ldots, \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} (i+1)n - j = \frac{n(n-1)(n^2-n+1)}{6}, \ldots \tag{13}$$

As before, the sum of two of these sequences provides a new sequence, in particular adding (12) and (13) gives

$$0, 5, 30, 102, 260, \ldots, \frac{n(n-1)(n^2+1)}{2}, \ldots \tag{14}$$

This sequence is equivalent to (3) - (4), as can be seen by simplifying

$$\frac{n^2(n^2+1)}{2} - \frac{n(n^2+1)}{2} = \frac{n(n-1)(n^2+1)}{2}.$$

Therefore adding (9) to (14) gives

$$s_n = \frac{n(n-1)(n^2+1)}{2} + \frac{n(n+1)(n^2+1)}{2} = n^2(n^2+1) = 2\sum_{i=1}^{n^2} i.$$

Another interesting combination is (8) + (10) which has elements

$$s_n = \frac{n^2(n^2+2)}{3}, \tag{15}$$

and corresponds to sequence A014820

$$1, 8, 33, 96, \ldots$$

The expression given in [1] for these sequence elements is

$$s_n = \frac{(n^2+2n+3)(n+1)^2}{3},$$

but substituting $n = n-1$ gives the simpler expression (15). A related sequence is (7) + (11) which has elements

$$s_n = \frac{n^2(2n^2+1)}{3}$$

giving

$$1, 12, 57, 176, 425, \ldots$$

For n odd, the elements at the centre of the squares form the sequence

$$1, 5, 13, 25, \ldots, 2n(n-1)+1, \ldots \tag{16}$$

which is sequence A001844 (appropriately named the centered square numbers). If the elements that lie on the diagonal and anti-diagonal of the squares are summed, the sequence is

$$1, 10, 25, 68, 117, \ldots$$

This sequence is equal to twice (4) when n is even, but equal to twice (4) minus the centre element (given by (16)) when n is odd. Thus the n-th sequence element is

$$s_n = \begin{cases} n(n^2+1), & n \text{ even;} \\ (n-\frac{1}{2})(n^2+1), & n \text{ odd.} \end{cases}$$

The final construction for $m = 2$ begins with the four corner elements, each having a sequence which was given previously. Adding these elements together gives

$$4, 10, 20, 34, 52, \ldots, 2(n^2+1), \ldots \tag{17}$$

which is sequence A005893 (without the first element). The sum of the perimeter elements is equal to the sum of the four lines which make up the matrix edges, minus (17). The sum of the two horizontal lines is

$$n(n+1) + n^2(n-1),$$

and the sum of the two vertical lines is

$$n(n^2 - n + 2) + n(n-1).$$

Therefore we have the elements of the perimeter sequence

$$\begin{aligned} s_n &= n(n+1) + n^2(n-1) + n(n^2 - n + 2) + n(n-1) - 2(n^2+1) \\ &= 2(n-1)(n^2+1). \end{aligned}$$

Returning to the sum of all array elements

$$s_n = \sum_{i=1}^{n^2} i = \frac{n^4 + n^2}{2},$$

similar expressions can be obtained for larger m. For $m = 3$ the sequence elements are

$$s_n = \sum_{i=1}^{n^3} i = \frac{n^3(n+1)(n^2-n+1)}{2} = \frac{n^6+n^3}{2}.$$

and for $m = 4$

$$s_n = \sum_{i=1}^{n^4} i = \frac{n^4(n^4+1)}{2} = \frac{n^8+n^4}{2}.$$

For $m = 5$

$$s_n = \sum_{i=1}^{n^5} i = \frac{n^5(n+1)(n^4-n^3+n^2-n+1)}{2},$$

for $m = 6$

$$s_n = \sum_{i=1}^{n^6} i = \frac{n^6(n^2+1)(n^4-n^2+1)}{2},$$

and for arbitrary m

$$s_n = \sum_{i=1}^{n^m} i = \frac{n^m(n^m+1)}{2}.$$

3. Triangles

In this section, integers arranged in a triangle are considered, From the triangular numbers (1), there are obviously $\frac{n(n+1)}{2}$ elements in each triangle. These elements can be arranged in the following form

$$\begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \\ & n+1 & n+2 & \cdots & 2n-1 \\ & & 2n & \cdots & 3n-3 \\ & & & & \vdots \\ & & & & \frac{n(n+1)}{2} \end{array} \tag{18}$$

which for $n = 1$ to 5, gives

$$1\ , \qquad \begin{array}{cc} 1 & 2 \\ & 3 \end{array}, \qquad \begin{array}{ccc} 1 & 2 & 3 \\ & 4 & 5 \\ & & 6 \end{array},$$

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ & 5 & 6 & 7 \\ & & 8 & 9 \\ & & & 10 \end{array}, \qquad \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ & 6 & 7 & 8 & 9 \\ & & 10 & 11 & 12 \\ & & & 13 & 14 \\ & & & & 15 \end{array}.$$

Alternatively, the form can be

$$\begin{array}{ccccc} 1 & & & & \\ 2 & 3 & & & \\ 4 & 5 & 6 & & \\ \vdots & \vdots & \vdots & & \\ \frac{n^2-n+2}{2} & \cdots & \cdots & \cdots & \frac{n(n+1)}{2} \end{array} \tag{19}$$

which for $n = 1$ to 5, gives

$$1\ , \qquad \begin{array}{cc} 1 & \\ 2 & 3 \end{array}, \qquad \begin{array}{ccc} 1 & & \\ 2 & 3 & \\ 4 & 5 & 6 \end{array},$$

$$\begin{array}{cccc} 1 & & & \\ 2 & 3 & & \\ 4 & 5 & 6 & \\ 7 & 8 & 9 & 10 \end{array}, \qquad \begin{array}{ccccc} 1 & & & & \\ 2 & 3 & & & \\ 4 & 5 & 6 & & \\ 7 & 8 & 9 & 10 & \\ 11 & 12 & 13 & 14 & 15 \end{array}.$$

The elements of the sequence formed from the sums of the triangle elements are

$$s_n = 1+2+3+4+5+\ldots+\frac{n(n+1)}{2} = \sum_{i=1}^{\frac{n(n+1)}{2}} i = \frac{n(n+1)(n^2+n+2)}{8}, \quad (20)$$

giving

$$1, 6, 21, 55, 120, \ldots$$

These are the doubly triangle numbers (A002817). The expression in [1] given for the sequence elements is

$$\frac{(n+1)(n+2)(n^2+3n+4)}{8},$$

but substituting $n = n-1$ gives (20).

The sequence formed from the lower left corner of (19) is

$$1, 2, 4, 7, 11, \ldots, \frac{n^2-n+2}{2}, \ldots$$

which is (A000124) and is closely related to the central polygonal numbers (A002061). Note that the sequence elements are given by $n(n+1)/2+1$ in [1], but substituting $n = n-1$ gives the above result.

The sequence formed from the sum of the right column of (18) is

$$1, 5, 14, 30, 55, \ldots, \frac{n(n+1)(2n+1)}{6}, \ldots$$

which are the square pyramidal numbers (A000330). The sequence formed from the sum of the diagonal elements of (18) is

$$1, 4, 11, 24, 45, \ldots, \frac{n(n^2+2)}{3}, \ldots$$

which is A006527. The sequence formed from the sum of the left column of (19) is just the sequence of lower left elements in the squares given in the previous section. The sequence formed from the sum of the diagonal elements of (19) is

$$1, 4, 10, 20, 45, \ldots, \frac{n(n+2)(n+1)}{6}, \ldots$$

which are the tetrahedral numbers (A000292).

4. Hexagons

For $n = 1$ to 4, the hexagons are

$$
1\ ,\quad
\begin{array}{ccccc}
 & 1 & & 2 & \\
3 & & 4 & & 5 \\
 & 6 & & 7 &
\end{array}\ ,\quad
\begin{array}{ccccccccc}
 & & 1 & & 2 & & 3 & & \\
 & 4 & & 5 & & 6 & & 7 & \\
8 & & 9 & & 10 & & 11 & & 12 \\
 & 13 & & 14 & & 15 & & 16 & \\
 & & 17 & & 18 & & 19 & &
\end{array}\ ,
$$

$$
\begin{array}{ccccccccccccc}
 & & & 1 & & 2 & & 3 & & 4 & & & \\
 & & 5 & & 6 & & 7 & & 8 & & 9 & & \\
 & 10 & & 11 & & 12 & & 13 & & 14 & & 15 & \\
16 & & 17 & & 18 & & 19 & & 20 & & 21 & & 22 \\
 & 23 & & 24 & & 25 & & 26 & & 27 & & 28 & \\
 & & 29 & & 30 & & 31 & & 32 & & 33 & & \\
 & & & 34 & & 35 & & 36 & & 37 & & &
\end{array}\ .
$$

In lexicographic order, the sequences formed from the 6 corner elements are

$$
\begin{array}{llllll}
1, & 1, & 1, & 1, & 1, & \ldots, 1, \ldots \\
1, & 2, & 3, & 4, & 5, & \ldots, n, \ldots \\
1, & 3, & 8, & 16, & 27, & \ldots, \frac{3n^2-5n+4}{2}, \ldots \\
1, & 5, & 12, & 22, & 35, & \ldots, \frac{3n^2-n}{2}, \ldots \\
1, & 6, & 17, & 34, & 57, & \ldots, 3n^2-4n+2, \ldots \\
1, & 7, & 19, & 37, & 61, & \ldots, 3n^2-3n+1, \ldots
\end{array}
$$

The first and second sequences are trivial. The fourth sequence corresponds to the pentagonal numbers (A000326) while the last sequence corresponds to the hex numbers (A003215). It is interesting to note that

$$3n^2 - 3n + 1 = (n+1)^3 - n^3,$$

so that

$$\sum_{n=1}^{m} (3n^2 - 3n + 1) = m^3.$$

The sum of the elements on each edge of the hexagon also provide sequences. In lexicographic order, the sequences formed from the 6 edges are

$$1,\quad 3,\quad 6,\quad 10,\quad 15,\quad \ldots, \tfrac{n(n+1)}{2}, \ldots$$

$$1,\quad 4,\quad 13,\quad 32,\quad 65,\quad \ldots, \tfrac{n(2n^2-3n+4)}{3}, \ldots$$

$$1,\quad 7,\quad 22,\quad 50,\quad 95,\quad \ldots, \tfrac{n(n+1)(4n-1)}{6}, \ldots$$

$$1,\quad 9,\quad 38,\quad 102,\quad 215,\quad \ldots, \tfrac{n(14n^2-21n+13)}{6}, \ldots$$

$$1,\quad 12,\quad 47,\quad 120,\quad 245,\quad \ldots, \tfrac{n(7n^2-6n+2)}{3}, \ldots$$

$$1,\quad 13,\quad 54,\quad 142,\quad 295,\quad \ldots, \tfrac{n(6n^2-7n+3)}{2}, \ldots$$

For the final sequence, consider the horizontal and two diagonal lines in the hexagons (which all have the same sum). The sum of the elements on these lines is given by

$$s_n = \frac{(2n-1)(3n^2-3n+2)}{2}$$

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References

[1] N.J.A. Sloane, On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/~njas/sequences/index.html.