Recurring Recurrences in Counting Permutations

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April 23, 2003

Abstract

We present a combinatorial proof of a recurrence that occurs in the sequence enumerating square permutations. We then generalize this result to a class of sequences, which we classify with. Finally, we explore the cause of this shared trait by analyzing the expotential genereating functions of these sequences.

1 Square Permutations

The permutations on n elements form the symmetric group of order $n!$, called $\mathfrak{S}_n$. One interesting question is, which permutations can be written as the square of other permutations? We would like to characterize the set

$$A_n = \{\pi : \pi = \sigma^2, \sigma \in \mathfrak{S}_n\}.$$

Any permutation can be written as a product of disjoint cycles.

Definition 1 (Cycle Type). *The list of the lengths of the cycles of a permutation π is called the cycle type.*

For example, the cycle type of $\pi = (5\,4\,1)(7)(8\,2\,6)(9\,3)$ is $\{3, 3, 2, 1\}$.

We can determine the cycle length of π^2 by composing the squares of all the cycles of π:

1. Odd cycles, when squared, turn into odd cycles of the same length, but with their elements permuted:
$$(1\,2\,3\,4\,5\,6\,7)^2 = (1\,3\,5\,7\,2\,4\,6)$$

2. Even cycles, when squared, split up into two cycles of half the length:
$$(1\,2\,3\,4\,5\,6\,7\,8)^2 = (1\,3\,5\,7)(2\,4\,6\,8)$$

These facts are immediately apparent if we draw the graphs of even and odd permutations. We will present an example to illustrate the process of squaring a permutation:

$$((1\,2\,3\,4\,5)(6\,7)(8\,9\,10\,11)(12\,13)(14))^2 = (1\,3\,5\,2\,4)(6)(7)(8\,10)(9\,11)(12)(13)(14)$$

Each cycle in a square must be generated by one of the above processes. So, in a square permutation, all even cycles must appear in pairs. There are no constraints on odd cycles or their distributions. We can also calculate the square roots of permutations when they fall in this form (although square roots are usually not unique).

2 The Problem

How many square permutations are there in $\mathfrak{S}_n$? A 1974 article by Blum explores this problem [1]. Although there appears to be no simple closed form, we can find an exponential generating function for this sequence. Let a_n be the number of elements of $\mathfrak{S}_n$ which are squares. The exponential generating function is

$$\sum_{n\geq 0} a_n \frac{x^n}{n!} = \sqrt{\frac{1+x}{1-x}} \prod_{k\geq 1} \cosh \frac{x^{2k}}{2k}.$$

The first few terms of the sequence is listed below, and the ratio between each successive a_{2k} and a_{2k+1}:

i	a_i	a_i/a_{i-1}
0	1	
1	1	1
2	1	
3	3	3
4	12	
5	60	5
6	270	
7	1890	7
8	14280	
9	128520	9
10	1096200	
11	12058200	11

One property of this sequence is the recurrence

$$a_{2n+1} = (2n+1)a_{2n}$$

which governs the relationship between any even term and the next term. Analyzing the expoential generating function gives the result after some algebra. However, it gives no insight into the combinatorial structure of the problem. We now present a proof with a combinatorial bijection.

3 A Combinatorial Interpretation

We will prove this identity using a combinatorial argument and strong induction.

In order to prove the identity combinatorially, we will construct a map.

Definition 2 (Square Permutations). *Let A_k be the set of k-element square permutations.*

Definition 3 (Permutations with Inserts). *Let Ω_{2n+1} be a set of permutations of $2n+1$ elements, generated by inserting the last element $2n+1$ into the elements of A_{2n} in every possible location, with the following rules:*

1. *The last element may be spliced into any existing cycle at any location.*
2. *Or, we can insert the element as its own new 1-cycle (i.e. insert it into what was before an imaginary 0-cycle).*

The set Ω_{2n+1} has size $(2n+1)|A_{2n}|$, because we can insert before any element ($2n$ insertion positions per permutation). or we can insert outside any cycle (one additional insertion position per permutation). If we can biject the set onto A_{2n+1} then we will establish the recurrence. We define refinements of Ω and A as follows:

1. Let $\Omega_k(x)$ be the set of permutations in which the last element, k, was inserted into a cycle of length x (that is, now k appears in a cycle of length $x+1$. Define the count analagously, $\omega_k(x) = |\Omega_k(x)|$.
2. Let $A_k(x)$ be the set of permutations in which the last element, k, appears in a cycle of length x. Also define the count analagously, $a_k(x) = |A_k(x)|$.

Before we prove the recurrence, we will define a property which will let us generalize our result.

Property 1 (Odd Cycle Invariance). *Let X be a set of permutations, and X_n be the elements of X which have size n. We will say that X is Odd Cycle Invariant if inserting a disjoint odd cycle of length l into a member of A_k produces an element of A_{k+l}, and removing an odd cycle of length l from a member of A_k produces an element of A_{k-l}.*

That is to say, a set X has this property if adding or removing odd cycles doesn't affect whether a permutation belongs in X. Notice that the set of square permutations is Odd Cycle Invariant. In addition, quite trivially, the set of permutations with odd cycles only also has this property.

Theorem 1 (Recurrence of Square Permutations). *If $a_n = |A_n|$ is the number of permutations in X which have size n, and X is Odd Cycle Invariant then*

$$a_{2n+1} = (2n+1)a_{2n}.$$

In particular, the number of square permutations follows this recurrence.

Proof. We will show these three correspondences by strong induction on n:

$$\omega_{2n+1}(0) = a_{2n+1}(1) \tag{1}$$
$$\omega_{2n+1}(2k+1) = a_{2n+1}(2k+3) \qquad (k \geq 0) \tag{2}$$
$$\sum_k \omega_{2n+1}(2k) = \sum_k a_{2n+1}(2k) \qquad (k > 0) \tag{3}$$

First Part. Recall that the elements of $\Omega_{2n+1}(0)$ are those permutations where $(2n+1)$ was inserted into an empty cycle, i.e., it became a new 1-cycle. So, every member of $\Omega_{2n+1}(0)$ is a member of A_{2n+1}, and, in fact, a member of $A_{2n+1}(1)$, by definition. We biject elements onto themselves, so obviously the two sets are counted by the same number.

Second Part. Here we assume the strong induction hypothesis, namely that the recurrence is true for all previous values. We look at a permutation $\pi \in \Omega_{2n+1}(2k+1)$ and its preimage $\pi' \in a_{2n}$, and consider the cycle which contains the last element (it is of length $2k+2$). We have $\binom{2n}{2k+1}(2k+1)!$ ways to choose the remainder of the cycle. Furthermore, by Odd Cycle Invariance, the remainder of π', and thus the remainder of π, are members of $A_{2n-2k-1}$ (i.e., they are square). We can permute them in $a_{2n-2k-1}$ ways. In all, we can select

$$\binom{2n}{2k+1}(2k+1)!\, a_{2n-2k-1}$$

distinct members of $\Omega_{2n+1}(2k+1)$, and this number is equal to $(2n+1)a_{2n}(2k+1)$.

Now let us count $A_{2n+1}(2k+3)$, the number of square permutations of $2n+1$ elements in which the largest element appears in a cycle of length $2k+3$. We can choose the remaining elements of this cycle in $\binom{2n}{2k+2}(2k+2)!$ ways, and we can permute the rest of the elements into a square (because of Odd Cycle Invariance) of length $2n-2k-2$. The total number is

$$\begin{aligned}
a_{2n+1}(2k+3) &= \binom{2n}{2k+2}(2k+2)!\, a_{2n-2k-2} \\
&= \frac{(2n)!}{(2k+2)!(2n-2k-2)!}(2k+2)!\frac{a_{2n-2k-1}}{2n-2k-1} \\
&= \frac{(2n)!}{(2n-2k-1)!}a_{2n-2k-1} \\
&= \frac{(2n)!}{(2k+1)!(2n-2k-1)!}(2k+1)!a_{2n-2k-1} \\
&= \binom{2n}{2k+1}(2k+1)!a_{2n-2k-1} \\
&= |\Omega_{2n+1}(2k+1)| \\
&= (2n+1)a_{2n}(2k+1),
\end{aligned}$$

and this part of the bijection is complete.

If the set we are counting has only elements with odd cycles, then the first two cases complete the proof (the third only governs inserting into even cycles). In particular, we have shown the result for the set of all permutations with odd cycles, called B. This comes into use later. A quick glance at the first few elements of $b_n = |B_n|$ follows:

i	b_i	b_i/b_{i-1}
0	1	
1	1	1
2	1	
3	3	3
4	9	
5	45	5
6	225	
7	1575	7
8	11025	
9	99225	9

We will use this recurrence of b_n in the third part of the proof for the general case when the set contains even cycles.

Third Part. Suppose that we have now inserted into an even cycle. We consider the permutation as a product of even and odd cycles. Although $\omega_{2n+1}(2k) = a_{2n+1}(2k)$ for each value of k (this is a stronger refinement of our result), we will use a different refinement of a in our proof for convenience. That is to say, we will consider all the cases where the last element is spliced into an even cycle inside a square permutation of size $2n$. This set has cardinality

$$\sum_k |\Omega_{2n+1}(2k)|.$$

We will then biject this set onto the set of square permutations of size $2n+1$ in which the largest element appears in an even cycle. This set has cardinality

$$\sum_k |A_{2n+1}(2k)| = \sum_k a_{2n+1}(2k).$$

Suppose that the number of permutations containing only paired even cycles (that is, the number of square permutations with only even cycles) and of length $2m$ is given by C_{2m}. Consider all the members of $\Omega_{2n+1}(2k)$ but remove the most recently inserted element. What we have left is just a member of A_{2n} (square permutations) which contains some even cycles. Let $2f$ be the total length of the even cycles which remain, and consider the squares which contain exactly $2f$ elements in even cycles, for some fixed f. Call this set $A^*_{2n}(2f)$. This is just another, different, refinement of A.

The number of such squares is C_{2n} times the number of choices we can make for the permutation of the rest of the cycles. Since we have pulled out all the even cycles only the odd ones remain. Let b_n be the number of ways to permute k elements into odd cycles only.

Then

$$|A^*_{2n}(2f)| = \binom{2n}{2f} C_{2f} b_{2n-2f}$$

and when we reinsert the last element, we have

$$2f \binom{2n}{2f} C_{2f} b_{2n-2f}$$

choices.

Now we count the elements on the other side. We can choose $2f-1$ elements (in addition to the largest element) to go inside our even cycles, and then C_{2f} ways to permute them. We can permute the remaining elements into odd cycles in $b_{2n-2f+1}$ ways. So the number of permutations of this type is

$$\binom{2n}{2f-1} C_{2f} b_{2n-2f+1}.$$

These two quantities are equal after (We exploit the recurrence on b_n, that is, $b_{2n-2f+1} = (2n-2f+1)b_{2n-2f}$.) When we sum over all f, the proof is complete.

This completes a purely combinatorial proof of the recurrence with an inductive bijection. For the simpler case of analyzing b_n, the sequence enumerating permutations with only odd cycles, [3] gives a complicated iterative bijective algorithm. A similar approach could probably also be used to solve this problem.

4 Patterns

In the proof of the recurrence, we have never used the fact that even cycles came in pairs. Somehow, there was a fundamental independence of the odd cycles from even cycles. The heart of this fact lies in the recurrence $b_{2n}(2n+1) = b_{2n+1}$. In fact, we propose that this recurrence is satisfied by any sequence which is Odd Cycle Invariant.

A few examples follow, all of which are Odd Cycle Invariant. The ith term of each column is the number of permutations of total length i and following the given property:

1. r_n = Number of permutations of length n.
2. s_n = Number of permutations where the sum of the lengths of the even cycles is 6.
3. t_n = Number of permutations where all even cycles have the same length.
4. u_n = Number of permutations where all even cycles take one of two possible lengths l and k, and at least one such cycle with each length exist.
5. v_n = Number of permutations where even cycles can only have length up to 4.

i	r_i	s_i	t_i	u_i	v_i
0	1	0	0	0	1
1	1	0	0	0	1
2	2	0	1	0	2
3	6	0	3	0	6
4	24	0	15	0	24
5	120	0	75	0	120
6	720	225	405	90	600
7	5040	1575	2835	630	4200
8	40320	6300	22155	7140	28560
9	362880	56700	199395	64260	257040

And it is evident that all of these sequences follow our recurrence. For any of these cases, we can use the same proof as outlined above, replacing C_n with an appropriate squence.

5 Generating Functions

We now present the standard generating function proof, from [2], for the square permutation sequence. From here, we see from another angle the reason why this recurrence occurs.

Definition 4 (Cycle Indicator). *For any permutation π, let us define the cycle indicator monomial to be*

$$C(\pi) = \prod_{p \in \pi} x_p$$

as p ranges over all the cycle lengths of π, counting multiplicities.

For example,

$$C((1,4,5)(2,3,6)(7)) = x_1 x_3^2.$$

Now let $Z_n(x_1, x_2, \ldots, x_n)$ be the polynomial attained by summing the cycle indicator over all permutations of length n:

$$Z_n(x_1, x_2, \ldots, x_n) = \sum_{\pi \in \mathfrak{S}_n} C(\pi).$$

For example,

$$Z_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + 2x_3$$

because among the permutations of three elements, there are one of type $(1,1,1)$, three of type $(2,1)$ and one of type (3). It turns out the exponential generating function of Z_n is

$$\begin{aligned}
\sum_{n=0}^{\infty} Z_n(x_1,x_2,\ldots,x_n)\frac{t^n}{n!} &= \exp\left(\sum_{n=1}^{\infty} x_n \frac{t^n}{n}\right). \\
&= \exp\left(x_1\frac{t}{1}\right)\exp\left(x_2\frac{t^2}{2}\right)\exp\left(x_3\frac{t^3}{3}\right)\cdots \\
&= \left(\sum_{n=0}^{\infty}\frac{x_1^n t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{x_2^n t^{2n}}{2^n n!}\right)\left(\sum_{n=0}^{\infty}\frac{x_3^n t^{3n}}{3^n n!}\right)\cdots.
\end{aligned}$$

This generating function contributes one term for each permutation. What we would really like to do is count one only for permutations which contain an even number of cycles for every even cycle length. To do this for a given cycle length, we define the operator

$$E_n f(x_1,x_2,\ldots,x_n,\ldots) = \frac{1}{2}(f(x_1,x_2,\ldots,x_n,\ldots)+f(x_1,x_2,\ldots,-x_n,\ldots)).$$

Any term which contains x_n to an odd power will be eliminated, because it is an odd function of x_n. All other terms are unchanged. The exponential generating function we want is given by

$$\sum a_n \frac{t^n}{n!} = E_2\,E_4\,E_6\cdots \exp\left(x_1\frac{t}{1}\right)\exp\left(x_2\frac{t^2}{2}\right)\exp\left(x_3\frac{t^3}{3}\right)\cdots$$

when the right-hand side is evaluated at $x_i = 1$ for all i. Since each operator is actually associated with only one factor in the product, we have

$$\begin{aligned}
\sum a_n\frac{t^n}{n!} &= \exp\left(x_1\frac{t}{1}\right)\left(E_2\exp\left(x_2\frac{t^2}{2}\right)\right)\exp\left(x_3\frac{t^3}{3}\right)\left(E_4\exp\left(x_4\frac{t^4}{4}\right)\right)\cdots \\
&= \exp\left(x_1\frac{t}{1}\right)\exp\left(x_3\frac{t}{3}\right)\cdots\cosh\left(x_2\frac{t^2}{2}\right)\cosh\left(x_4\frac{t^4}{4}\right)\cdots
\end{aligned}$$

We then evaluate at x_i:

$$\sum a_n\frac{t^n}{n!} = \exp\left(t+t^3/3+t^5/5+\cdots\right)\prod_{k\geq 1}\cosh\frac{x^{2k}}{2k}$$

Consider the sum $t+t^3/3+t^5/5+\cdots$. It is a power series which we can evaluate:

$$\begin{aligned}
t+t^3/3+t^5/5+\cdots &= (t+t^2/2+t^3/3+t^4/4+t^5/5+\cdots)-(t^2/2+t^4/4+t^6/6+\cdots) \\
&= -\log(1-t)-\frac{1}{2}(-\log(1-t^2)) \\
&= \log\sqrt{\frac{1+t}{1-t}}.
\end{aligned}$$

Plugging back into the original expression, we have

$$\sum_{n\geq 0} a_n\frac{t^n}{n!} = \sqrt{\frac{1+t}{1-t}}\prod_{k\geq 1}\cosh\frac{x^{2k}}{2k}.$$

The desired recursion is a direct consequence of the generating function's composition. However, we can generalize this proof to count other sequences. We start with the generating function for all permutations p_n:

$$\sum_{n\geq 0} p_n \frac{t^n}{n!} = \prod_{k\geq 1} \exp\left(x_k \frac{t^k}{k}\right).$$

There exists some operator E which transforms the left side into $\sum n \geq 0 s_n \frac{x^n}{n!}$, which is the generating function of our sequence. We apply E to both sides, and now we get

$$\sum_{n\geq 0} S_n \frac{t^n}{n!} = E\left(\prod_{k\geq 1} \exp\left(x_k \frac{t^k}{k}\right)\right).$$

However, we know that

$$E\left(\prod_{k\geq 1} \exp\left(x_k \frac{t^k}{k}\right)\right) = \exp\left(x_{2i+1}\frac{t^{2i+1}}{2i+1}\right) E\left(\prod_{k\geq 1,\, k\neq 2i+1} \exp\left(x_k \frac{t^k}{k}\right)\right).$$

by our property.

We then pull out all such terms, and we have:

$$\sum_{n\geq 0} S_n \frac{t^n}{n!} = \prod_{i\geq 1} \exp\left(x_{2i+1}\frac{t^{2i+1}}{2i+1}\right) E\left(\prod_{i\geq 1} \exp\left(x_{2i} \frac{t^{2i}}{2i}\right)\right).$$

As we have shown before, we can convert the product of the terms involving odd powers to $\sqrt{\frac{1+t}{1-t}}$, and get:

$$\sum_{n\geq 0} S_n \frac{t^n}{n!} = \sqrt{\frac{1+t}{1-t}} E\left(\prod_{i\geq 1} \exp\left(x_{2i} \frac{t^{2i}}{2i}\right)\right).$$

Now, we see an interesting property. We notice that $G(t)/(1+t)$ is an even function. Therefore, we have $G(t)/(1+t) = G(-t)/(1-t)$. We equate the coefficients of t^{2n+1}:

$$\begin{aligned} s_{2n+1}\frac{t^{2n+1}}{(2n+1)!} - s_{2n}\frac{t^{2n+1}}{(2n)!} &= s_{2n+1}\frac{-t^{2n+1}}{(2n+1)!} + s_{2n}\frac{t^{2n+1}}{(2n)!} \\ 2s_{2n+1}\frac{t^{2n+1}}{(2n+1)!} &= 2s_{2n}\frac{t^{2n+1}}{(2n)!} \\ s_{2n+1} &= s_{2n}(2n+1) \end{aligned}$$

which is our recurrence.

6 Conclusion

This interesting method of combinatorics may have a more general application, though at the moment it seems difficult to quantify. We have shown that a certain recurrence satisfied by certain sequences can reoccur in more complex sequences. It is also possible that such recurrences share an inate pattern. Classification of these recurring recurrences may be of future research interest.

7 Acknowledgments

First and foremost, we thank Dr. David Ellwood, Ms. Vida Salahi, and the rest of the faculty at the Clay Mathematics Institute for this invaluable research opportunity.

We sincerely thank Dr. Richard Stanley, our mentor, and Dr. Federico Ardila and Mr. Sergi Elizalde, the Academy Fellows, for the background information material and guidance on this research.

Finally, we would like to thank Dr. Roger Howe, Dr. Jeb Willenbring, Dr. Igor Mencattini, and the rest of the participants of the Academy for their support.

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