Metric properties of the Tower of Hanoi graphs and Stern's diatomic sequence

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Abstract

It is known that in the Tower of Hanoi graphs there are at most two different shortest paths between any fixed pair of vertices. A formula is given that counts, for a given vertex v, the number of vertices u such that there are two shortest u,v-paths. The formula is expressed in terms of Stern's diatomic sequence $b(n)$ $(n \geq 0)$ and implies that only for vertices of degree two this number is zero. Plane embeddings of the Tower of Hanoi graphs are also presented that provide an explicit description of $b(n)$ as the number of elements of the sets of vertices of the Tower of Hanoi graphs intersected by certain lines in the plane.

Key words: Tower of Hanoi, shortest paths, Stern's diatomic sequence, plane embeddings

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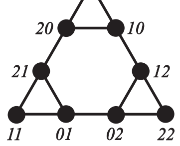

1 Introduction

It is fascinating that the Tower of Hanoi (TH) still attracts the interest of mathematicians 120 years after its invention by the French number theorist Édouard Lucas (1842–1891). This stems from the rich inherent mathematical structure of the problem which can be described as follows. Three vertical pegs hold a certain number of discs of mutually different diameters in such a way that no larger disc lies on a smaller one (*divine rule*). A state obeying this divine rule is called *regular*. The topmost disc on a peg may be moved to the top on another peg, provided that the divine rule is obeyed, i.e. if this move leads from one regular state to another. In the original setting, all discs lie on the same peg (this is a *perfect state* of the puzzle), and the task is to transfer them (in the least possible number of moves) to a preassigned other peg. In trying to solve this problem, called Problem 0 of the TH, one finds oneself readily in a situation where one still has the goal in mind, but has lost the track from the initial configuration. Problem 1 is therefore to get from an arbitrary regular state to a perfect one. But then one can, of course, also ask for a shortest path from a regular to another regular state, which is called Problem 2. (This latter problem seems not to have been posed explicitly before 1976; cf. [33].)

For Problem 1, including Problem 0, it can easily be shown that the largest disc moves at most once in a shortest path and that therefore, by induction, the shortest path is uniquely determined (cf. [9, Theorem 3]). The *assumption* that the largest disc moves only once also in the case of Problem 2, and therefore the uniqueness of the shortest path, can be found in literature as late as about ten years ago, cf. [35]. On the other hand, it was pointed out in the psychological literature by Klahr already in 1978 that uniqueness of the shortest path does *not* hold in some cases (cf. [15, p. 209]). He shows this by looking at what is now called the *Hanoi graph* for three pegs (cf. [15, Figure 7.3]). The latter was named so by Lu [24], but introduced much earlier by Scorer, Grundy and Smith [31]. Hanoi graphs are an efficient mathematical model for the TH: if the pegs are labelled 1, 2 and 3 and if $n \in \mathbb{N}$ is the number if discs, the regular states form the set of vertices $\{1,2,3\}^n$, and an edge is a legal move of one disc; here a regular state is represented by $r = r_1 \dots r_n \in \{1,2,3\}^n$ with r_i being the peg where disc i (numbered from small to large) is currently lying. These graphs H_n can be constructed recursively: H_1 is the complete graph on three vertices, and the step from n to $n+1$ can be taken from Figure 1 which shows H_2.

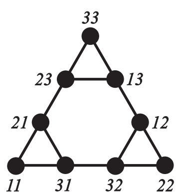

Figure 1: The Hanoi graph H_2

Clearly, solutions to instances of Problem 2 correspond to shortest paths between the corresponding vertices in H_n. Already H_2 shows non-uniqueness of shortest paths by looking, e.g., at the pair $(21, 12)$, so it is quite surprising that this fact had been overlooked for such a long time. The two distinct optimal solutions differ by the number of moves of the largest disc and in fact, by recourse to the graph H_3, Stone [36] remarked in 1982 that in some cases the optimal solution *requires* two moves of the largest disc (cf. also [16, p. 139]). A complete theory of Problem 2 was finally given by Lu [24], Hinz [9, Section 1.3.0] and van Zanten [38]. However, the *decision problem*, if given a pair of states the largest disc moves once or twice in a shortest path or if both alternatives are optimal, had not been solved in a satisfactory way. Of course, it can be done by calculating and comparing the lengths of both paths (cf. [11, Section 2.2]), but a recent result of Romik [30] shows that one can do much better: the decision can be made by a finite automaton after looking at the positions of only the $63/38$ largest pairs of discs on the average.

The relations between the TH, the Sierpiński triangle and Pascal's arithmetical triangle have been investigated in [10, 27, 35] with the astonishing conclusion that the average distance on the Sierpiński gasket is $466/885$ [6, 14]. Other recent results on Hanoi graphs show that they are hamiltonian (cf. [17, Proposition 3], [13, Theorem 1]) and deal with planarity [13, Theorem 2] and error correcting codes [7, 18, 20].

The goal of the present paper is to take a closer look at those pairs of states where the optimal solution is not unique. In Section 3 we present (Theorems 3.5 and 3.8) a formula for the number of states connected to a fixed state by two optimal paths. The formula in particular implies that only for perfect states this number is zero. Our approach essentially uses labelings introduced in [17] for a two parametric generalization of the Hanoi graphs. It is appealing that our enumerative results are expressed by means of Stern's diatomic sequence. (The relations between the TH and the so-called Stern-Brocot array became evident from the work by Parisse [26, Proposition 1 of Section 2]), cf. also [12].) Moreover, for any non-negative integer n, we establish an explicit bijection between the hyperbinary representations of n, and a certain set of vertices of the Sierpiński graphs. These results are complemented by a geometrical interpretation of Hanoi graphs in Section 4. More specifically, we construct plane embeddings of the graphs H_n in which the sets of vertices corresponding to the terms of Stern's diatomic sequence lie on parallel lines. This result may be viewed as a more precise rephrasing of Carlitz's results on Stern's diatomic sequence and the binary Pascal triangle [4, 5].

2 Preliminaries

In this section we introduce the key concepts needed in our approach—the abovementioned labelings of Sierpiński graphs and Stern's diatomic sequence.

Graphs $S(n, k)$ were introduced in [17] as a two parametric generalization of the Hanoi graphs and named *Sierpiński graphs* in [18]. Their introduction was motivated by topological studies of certain generalizations of the Sierpiński gasket [22, 23, 25]. For our purposes we recall that for any $n \in \mathbb{N}$, the graph $S_n := S(n, 3)$ is isomorphic to the graph H_n (cf. [17, Theorem 2]) and is defined as follows. Its vertices are all strings of length n over the alphabet $\{1,2,3\}$, vertices $u = u_1u_2\dots u_n$ and $v = v_1v_2\dots v_n$ being adjacent if and only if there exists an index $h \in \{1,2,\dots,n\}$ such that

(i) $u_t = v_t$, for $t = 1,\dots,h-1$;
(ii) $u_h \neq v_h$;
(iii) $u_t = v_h$ and $v_t = u_h$, for $t = h+1,\dots,n$.

The graph S_4, together with the introduced labeling, is drawn in Fig. 2.

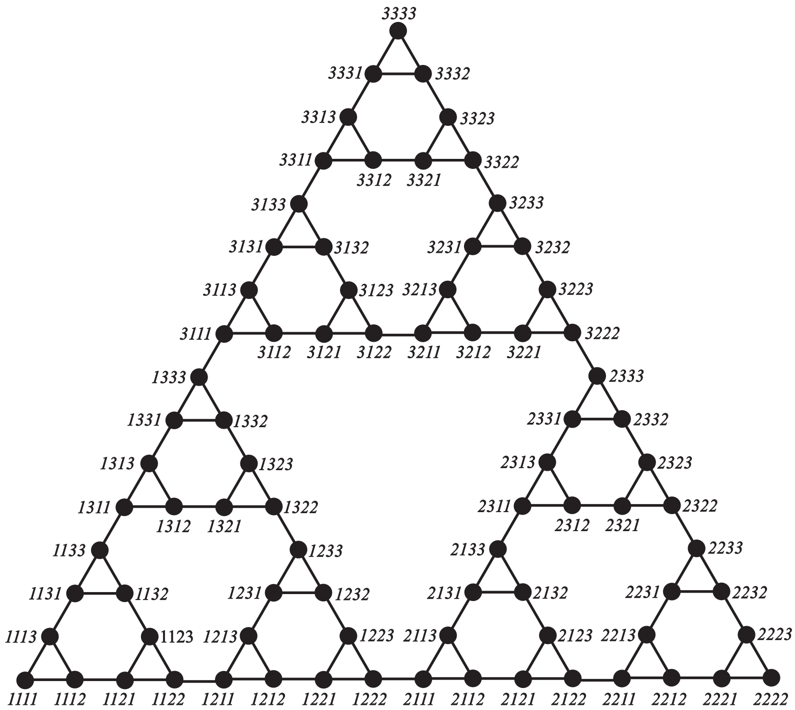

Figure 2: The Sierpiński graph S_4 and its labeling

The vertices of S_n labeled $ii\dots i$, for $i = 1,2,3$, will be called *extreme vertices* of S_n (for obvious reasons—see Fig. 2). Note that the extreme vertices of S_n are precisely the vertices of degree 2. Moreover, these vertices correspond to the perfect states of the TH with n disks.

Let

$$\rho_{i,j} = \begin{cases} 1; & i \neq j, \\ 0; & i = j. \end{cases}$$

(The symbol ρ typographically resembles Kronecker's delta symbol put upside down.) Set in addition

$$\mathcal{P}^{i}_{j_1j_2\dots j_m} \;=\; \rho_{i,j_1}\rho_{i,j_2}\dots\rho_{i,j_m}\,,$$

where the right-hand side term is a binary number, rhos representing its digits. The following result given in [38] (and extended in [17] to all Sierpiński graphs $S(n,k)$) will be used in the sequel.

Proposition 2.1 *Let $u_1u_2\dots u_n$ be a vertex of S_n. Then*

$$d(u_1u_2\dots u_n, ii\dots i) = \mathcal{P}^i_{u_1u_2\dots u_n}.$$

□

Stern's diatomic sequence $b(n)$ is defined recursively by $b(0)=0$, $b(1)=1$, $b(2n)=b(n)$ and $b(2n+1)=b(n)+b(n+1)$, for $n\geq 1$. This sequence is `A002487` in Sloane's online database of integer sequences [32].

Motivated by an idea of Eisenstein [8, p. 710], Stern [34, p. 194] considered an array, each row of which is constructed by mediation from the previous one, starting from two initial values p and q. This so-called *Stern-Brocot array* $(p,q)_n$, $n\in\mathbb{N}_0$ (cf. [2]), in the special case $p=1=q$, was later named *Stern's Diatomic Series* by Lehmer [19]. De Rham [29] seems to have been the first to extract the one-dimensional sequence $b(n)$ from this array (or rather the one with "atoms" $p=0$ and $q=1$). (Unfortunately, there is a misprint in the recurrence relation for b in [29, p. 95].) The connection between the two is that the nth row of Stern's diatomic series consists of the block of terms $b(2^n), b(2^n+1), \dots, b(2^{n+1})$ of what we call Stern's diatomic sequence. Among the many other mathematicians in several different areas of mathematics, who later studied properties of this sequence, let us just mention Carlitz [4, 5] and Lind [21]. A particularly rich source of information on the history of the sequence and of new results is [37]. We just add here that this sequence has been shown to be 2-regular (cf. [1, Example 7]) and represents the 3rd binary partition function (cf. [28, Theorem 5.2]).

A *hyperbinary representation* of a non-negative integer n is a representation of n as a sum of powers of 2, each power being used at most twice. We will employ the notation $(a_1a_2\dots a_m)_{[2]}$ to describe the hyperbinary representation $\sum_{i=1}^m a_i2^{m-i}$, $a_i\in\{0,1,2\}$. Let $\mathcal{H}(n)$ denote the set of all hyperbinary representations of n, where any two representations of the same integer differing only in zeros on the left-hand side are identified. For instance, $(1)_{[2]}$ is the same representation of 1 as $(01)_{[2]}$. It is well-known, cf. [3], that $b(n)$ counts the number of hyperbinary representations of $n-1$. In fact, it is not difficult to see that this is indeed the case: the recursive formulas are established by noting that when $x=(a_1a_2\dots a_m)_{[2]}$ is odd, then a_m must be 1, and if x is even, a_m may be 0 or 2, but not 1. Hence:

Theorem 2.2 *For any $n\in\mathbb{N}$, $b(n)=|\mathcal{H}(n-1)|$.* □

For an n digit binary number $b=b_1b_2\dots b_n$, $b_i\in\{0,1\}$, we will write $\overline{b}$ for the complementary binary number, that is, $\overline{b}=\overline{b}_1\overline{b}_2\dots\overline{b}_n$, where $\overline{b}_i=1-b_i$. For instance, if $b=0001101$, then $\overline{b}=1110010$. Clearly, $b+\overline{b}=2^n-1$, which we state as a lemma for further reference.

Lemma 2.3 *Let b be an n digit binary number. Then $b+\overline{b}=2^n-1$.* □

3 Stern's diatomic sequence and Hanoi graphs

In [17, Theorem 6] it has been shown that in Sierpiński graphs $S(n,k)$ there are at most two shortest paths between any two vertices of $S(n,k)$, so in particular this holds for the graphs S_n and H_n (cf. also [9, Theorem 4]). We are going to study those pairs of vertices in S_n for which two different shortest paths indeed exist. For $v \in S_n$ set

$$X(v) = \{v' \in S_n \mid \text{ there exist two shortest } v, v' - \text{paths}\}\,.$$

(Here and throughout, $v \in G$ stands for $v \in V(G)$ for a graph G.)

For $i = 1,2,3$ let S_n^i be the subgraph of S_n induced by the vertices of the form $iv_2v_3\ldots v_n$. For a vertex $v = v_1v_2v_3\ldots v_n$ of S_n and $i \neq j$ we also set

$$d_{ij}(v) = \mathcal{P}^i_{v_2v_3\ldots v_n} - \mathcal{P}^j_{v_2v_3\ldots v_n}\,,$$

that is, if $v \in S_n^k$ (note that k is uniquely determined by v), then, since S_n^k is obviously isomorphic to S_{n-1},

$$d_{ij}(v) = d(v_2v_3\ldots v_n, ii\ldots i) - d(v_2v_3\ldots v_n, jj\ldots j)\,;$$

cf. Fig. 3. Note that $d_{ij}(v) \leq 2^{n-1} - 1$; cf. [24, Lemma 2].

We wish to determine, for a given vertex v, the size of $X(v)$ and give an explicit description of it. For this purpose we prove the following lemma.

Lemma 3.1 *Let $v \in S_n^i$ and $\{i,j,k\} = \{1,2,3\}$ such that $d_{jk}(v) \geq 0$. Then*

$$\left\{v' \in X(v) \;\middle|\; v' \notin S_n^i\right\} = \left\{v' \in S_n^j \;\middle|\; d_{ik}(v') = 2^{n-1} - d_{jk}(v)\right\}\,.$$

Moreover, if $d_{jk}(v) = 0$, then these sets are empty.

Proof. Let $v' \in S_n$ with $\ell := v'_1 \neq i$ and such that there are two shortest v, v'-paths. The length of the shortest v, v'-path that contains the edge $(i\ell\ldots\ell, \ell i\ldots i)$ is by Proposition 2.1

$$\mathcal{P}^\ell_{v_2v_3\ldots v_n} + 1 + \mathcal{P}^i_{v'_2v'_3\ldots v'_n}\,,$$

while the shortest path through the edges $(im\ldots m, mi\ldots i)$ and $(m\ell\ldots\ell, \ell m\ldots m)$ with $\{i,\ell,m\} = \{1,2,3\}$ is of length

$$\mathcal{P}^m_{v_2v_3\ldots v_n} + 1 + (2^{n-1} - 1) + 1 + \mathcal{P}^m_{v'_2v'_3\ldots v'_n}\,.$$

Since the above two lengths are equal, we have

$$\mathcal{P}^i_{v'_2v'_3\ldots v'_n} - \mathcal{P}^m_{v'_2v'_3\ldots v'_n} = 2^{n-1} - (\mathcal{P}^\ell_{v_2v_3\ldots v_n} - \mathcal{P}^m_{v_2v_3\ldots v_n})\,,$$

in other words, $d_{im}(v') = 2^{n-1} - d_{\ell m}(v)$.

Suppose $\ell = k$, so that $m = j$. Then $d_{ij}(v') = 2^{n-1} - d_{kj}(v) = 2^{n-1} + d_{jk}(v) \geq 2^{n-1}$, which contradicts $d_{ij}(v') \leq 2^{n-1} - 1$. Therefore $\ell = j$ and $m = k$, hence $v' \in S_n^j$ and

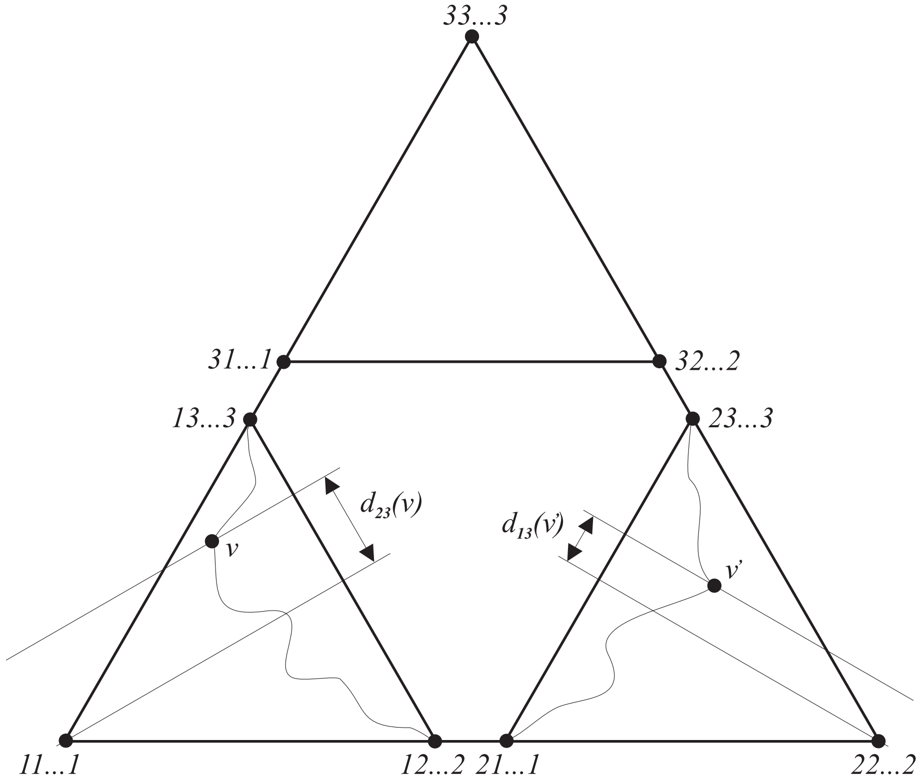

Figure 3: The meaning of the functions d_{ij}

$d_{ik}(v') = 2^{n-1} - d_{jk}(v)$. This proves that the left-hand side set is included in the right-hand side set. The other inclusion is obtained by reversing the order of the argument from the first paragraph of the proof.

Suppose that $d_{jk}(v) = 0$. Then $d_{ik}(v') = 2^{n-1}$, which contradicts the fact that $d_{ik}(v') \leq 2^{n-1} - 1$. Hence the set $\{v' \in X(v) \mid v' \notin S_n^i\}$ is empty. □

In order to write down an explicit description of the set $X(v)$, we introduce the following notation. For $n \in \mathbb{N}$, $k \in \{0, \ldots, n\}$ and $v \in S_n$, let $v^k = v_1 \ldots v_k$ $(\in S_k)$ and $\overline{v}^k = v_{k+1} \ldots v_n$ $(\in S_{n-k})$. Here we consider v^0 to be the empty string. Furthermore, we define mappings ℓ and d as follows. For an arbitrary string x let $\text{first}(x)$ be the first symbol of x, that is, if $x = x_1 x_2 \ldots x_k$, then $\text{first}(x) = x_1$. Now, for any $n \geq 1$ and any $v \in S_n$ let i and j be such that $\{i, j, \text{first}(v)\} = \{1, 2, 3\}$. Then set

$$d(v) = |d_{ij}(v)| \,.$$

Since $|d_{ij}(v)| = |d_{ji}(v)|$, the function d is well-defined. For a vertex $v \in S_n$ and $\overline{v}^k = v_{k+1} \ldots v_n$ we consider $d(\overline{v}^k)$ as a mapping on $V(S_{n-k})$. Note that $\text{first}(\overline{v}^k) = v_{k+1}$. We also set

$$\ell(v) = \begin{cases} i; & d_{ij}(v) > 0, \\ j; & d_{ji}(v) > 0, \\ 0; & d_{ij}(v) = 0\,. \end{cases}$$

Applying Lemma 3.1 to successive depths in S_n, we arrive at:

Theorem 3.2 *Let v be a vertex of S_n. Then*

$$X(v) = \bigcup_{k=0}^{n-2} \left\{ u \in S_n \;\middle|\; u = v^k\, \ell(\overline{v}^k)\, \overline{u}^{k+1},\ d(\overline{u}^k) + d(\overline{v}^k) = 2^{n-k-1} \right\}.$$

□

Note that whenever $\ell(\overline{v}^k) = 0$, the string $v^k\, \ell(\overline{v}^k)\, \overline{u}^{k+1}$ does not represent a vertex of S_n and hence it does not belong to $X(v)$ in accordance with Lemma 3.1 (that is, $\ell(\overline{v}^k) = 0$ means $d_{ij}(\overline{v}^k) = 0$, and by Lemma 3.1 it implies $X(\overline{v}^k) = \emptyset$). In addition, as soon as $\ell(\overline{v}^k) = 0$ for some k, we also have $\ell(\overline{v}^s) = 0$ for all $s \geq k$.

The following lemma will establish the relation between hyperbinary representations and the vertices of Sierpiński graphs.

Lemma 3.3 *Let $k \in \mathbb{Z}$ with $|k| < 2^n$ and $x, y \in \{1, 2, 3\}$, $x \neq y$. Then there is a bijection between the set*

$$\mathcal{H}(2^n - 1 - |k|) = \left\{ (a_1 a_2 \ldots a_n)_{[2]} \;\middle|\; \sum_{i=1}^{n} a_i 2^{n-i} = 2^n - 1 - |k| \right\},$$

and the set

$$\Delta_{x,y}(k) = \{ v \in S_n \mid \mathcal{P}_v^x - \mathcal{P}_v^y = k \}\ .$$

Proof. Using the symmetries of S_n we may without loss of generality assume $x = 1$ and $y = 2$. Note that if $(a_1 a_2 \ldots a_m)_{[2]} = 2^n - 1 - |k|$ with $a_1 \neq 0$, then $2^n - 1 \geq 2^n - 1 - |k| \geq a_1 2^{m-1} \geq 2^{m-1}$. From this it follows that $2^n \geq 2^{m-1} + 1 > 2^{m-1}$, hence $n > m - 1$ and finally $n \geq m$. This remark justifies our use of hyperbinary numbers of fixed length n. Of course, in $(a_1 a_2 \ldots a_n)_{[2]}$ one or more digits a_1, $a_2, \ldots$ may equal 0.

Again by symmetry, we may assume $k \leq 0$. The bijection from $\mathcal{H}(2^n - 1 + k)$ to $\Delta_{1,2}(k)$ will be given by $(a_1 a_2 \ldots a_n)_{[2]} \mapsto v = v_1 v_2 \ldots v_n$ with

$$v_i = \begin{cases} 1; & a_i = 0, \\ 2; & a_i = 2, \\ 3; & a_i = 1. \end{cases}$$

Obviously, $v \in S_n$. We claim that $v \in \Delta_{1,2}(k)$. For this purpose we compute

$$\mathcal{P}_v^1 = \sum_{i=1}^{n} \rho_{1,v_i} 2^{n-i} = \sum_{i:a_i \neq 0} 2^{n-i} = \sum_{i:a_i=1} 2^{n-i} + \sum_{i:a_i=2} 2^{n-i}$$

and

$$\overline{\mathcal{P}_v^2} = \sum_{i=1}^{n} \overline{\rho}_{2,v_i} 2^{n-i} = \sum_{i:a_i=2} 2^{n-i}\,.$$

Combining these two equalities we get

$$\mathcal{P}_v^1 + \overline{\mathcal{P}_v^2} = \sum_{i:a_i=1} 2^{n-i} + 2 \sum_{i:a_i=2} 2^{n-i} = \sum_{i=1}^{n} a_i 2^{n-i} = 2^n - 1 + k\,.$$

By Lemma 2.3, we have $\overline{\mathcal{P}_v^2} = 2^n - 1 - \mathcal{P}_v^2$, whence $\mathcal{P}_v^1 - \mathcal{P}_v^2 = k$ and the claim is proved.

To complete the proof we need to establish bijectivity of the mapping. As it is obviously injective, it remains to show surjectivity. So let $v = v_1 v_2 \ldots v_n \in \Delta_{1,2}(k)$, that is, $\mathcal{P}_v^1 - \mathcal{P}_v^2 = k$. Then set

$$a_i = \begin{cases} 0; & v_i = 1, \\ 1; & v_i = 3, \\ 2; & v_i = 2. \end{cases}$$

Clearly, $(a_1 a_2 \ldots a_n)_{[2]}$ can be viewed as a hyperbinary representation of some number, which is obviously mapped to v, and, moreover, using Lemma 2.3 again,

$$\begin{aligned} \sum_{i=1}^{n} a_i 2^{n-i} &= \left(\sum_{i:a_i=1} 2^{n-i} + \sum_{i:a_i=2} 2^{n-i} \right) + \sum_{i:a_i=2} 2^{n-i} \\ &= \mathcal{P}_v^1 + \overline{\mathcal{P}_v^2} = 2^n - 1 + \mathcal{P}_v^1 - \mathcal{P}_v^2 = 2^n - 1 + k\,, \end{aligned}$$

which completes the argument. □

In Fig. 4, the sets $\Delta_{1,2}(k)$, $k = -15, -14, \ldots, 14, 15$ are given as the intersections of $V(S_4)$ and the corresponding vertical lines. Since in the definition of these sets only the intrinsic metric of S_n has been used, it is quite unusual to observe such regularity with respect to the geometry/metric of the plane in which our copy of S_4 is embedded. We will show later (in Theorem 4.1) that this behaviour is not accidental.

By Theorem 2.2, Lemma 3.3 may be rephrased as follows: for $|k| < 2^n$: $|\Delta_{x,y}(k)| = b(2^n - |k|)$. On the other hand, since H_n is isomorphic to S_n, we have $|\Delta_{x,y}(k)| = z_n(k)$ with the functions z_n defined in [9, p. 305] by

$$z_n(k) = |\{r \in H_n \mid d(r, 1\ldots1) - d(r, 2\ldots2) = k\}|\,.$$

So we also have

$$\forall\, n \in \mathbb{N}_0 \;\forall\, k \in \mathbb{N}_0,\; k \le 2^n :\; b(k) = z_n(2^n - k). \qquad (1)$$

This relationship has an interesting consequence: putting $\mu = 2^{n+1} - k$ with $2^n < k \le 2^{n+1}$ into the three-term recursion relation of [9, Lemma 2.o)], namely

$$z_{n+1}(\mu) = z_n(2^n - \mu) + z_n(\mu) + z_n(2^n + \mu),$$

we arrive at

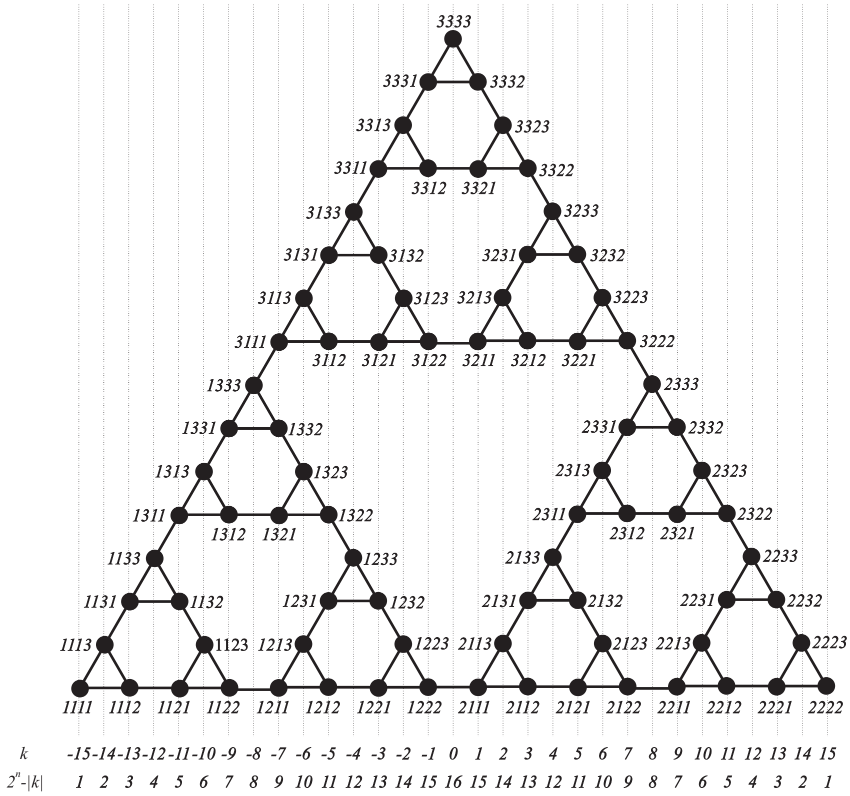

Figure 4: The sets $\Delta_{1,2}(k)$ in S_4, $k = -15, -14, \ldots, 0, \ldots, 14, 15$

Proposition 3.4 $\forall n \in \mathbb{N}_0\ \forall k \in \{2^n + 1, \ldots, 2^{n+1}\} :\ b(k) = b(2^{n+1} - k) + b(k - 2^n)$. □

Together with the two "atoms" $b(0) = 0$ and $b(1) = 1$, this can be chosen as an alternative and more symmetric definition of Stern's diatomic sequence.

Fig. 4 shows that the number of vertices of S_4 belonging to the first 16 lines from the left are indeed $1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1$, i.e. $b(1), b(2), \ldots, b(16)$. In connection with the abovementioned relation between the Hanoi graphs and (the odd entries in) Pascal's triangle, we refer to [10, Proposition 2].

Here is our main result.

Theorem 3.5 *Let v be any vertex of S_n. Then* $|\{u \in X(v) \mid u_1 \neq v_1\}| = b(d(v))$.

Proof. By symmetry we may assume that $v_1 = 1$ and $d_{23}(v) \geq 0$. If $d_{23}(v) = 0$, then by Lemma 3.1, $|\{u \in X(v) \mid u_1 \neq v_1\}| = |\emptyset| = 0 = b(0)$.

If $d_{23}(v) > 0$, applying Lemma 3.1 again, one gets

$$\begin{aligned}\{u \in X(v) \mid u_1 \neq v_1\} &= \{v' \in X(v) \mid v' \notin S_n^1\} \\ &= \{v' \mid v' \in S_n^2,\ d_{13}(v') = 2^{n-1} - d_{23}(v)\} \\ &= \{v' \mid v' \in S_n^2,\ \mathcal{P}^3_{v'_2 v'_3 \ldots v'_n} - \mathcal{P}^1_{v'_2 v'_3 \ldots v'_n} = d_{23}(v) - 2^{n-1}\}\,.\end{aligned}$$

Hence by Lemma 3.3,

$$\begin{aligned}|\{u \in X(v) \mid u_1 \neq v_1\}| &= |\mathcal{H}(2^{n-1} - 1 + (d_{23}(v) - 2^{n-1}))| \\ &= |\mathcal{H}(d_{23}(v) - 1)| \\ &= |\mathcal{H}(d(v) - 1)|\,,\end{aligned}$$

which completes the argument by Theorem 2.2. □

Corollary 3.6 *If v is any vertex of S_n that is not extreme, then*

$$\{u \in X(v) \mid u_1 \neq v_1\} \neq \emptyset\,.$$

In particular, $X(v) \neq \emptyset$.

Proof. Let v be a vertex of S_n^i; note that i is uniquely determined. Now, select j, k, such that $\{i, j, k\} = \{1, 2, 3\}$. Since $v \neq ii \ldots i$, it follows that $d_{jk}(v) \neq 0$. Namely, Proposition 2.1 implies that if $d_{jk}(v) = 0$, then $\rho_{j,v_\ell} = \rho_{k,v_\ell}$ for any $\ell = 1, 2, \ldots, n$. Then $v_\ell = i$ for any ℓ (if $v_\ell = j$ or $v_\ell = k$, then one of these ρ's is 0, and the other 1). By Theorem 3.5, it follows that $|\{u \in X(v) \mid u_1 \neq v_1\}| \geq b(d(v)) = b(|d_{jk}(v)|) > 0$, hence $\{u \in X(v) \mid u_1 \neq v_1\} \neq \emptyset$. □

This has an interesting interpretation in the TH.

Corollary 3.7 *The perfect states of the Tower of Hanoi are the only regular states* **s** *such that for any other regular state* **t** *there is a unique shortest sequence of moves transforming* **s** *to* **t**.

Proof. One direction follows from Corollary 3.6 because extreme vertices correspond to perfect states. The other direction is a consequence of [9, Theorem 3]. □

Applying 3.5 to successive depths in S_n, we arrive at:

Theorem 3.8 *Let $v \in S_n$. Then*

$$|X(v)| = \sum_{k=0}^{n-2} b\left(d(\overline{v}^k)\right)\,.$$

□

As a final remark let us ask for the total number x_n of pairs of vertices in S_{n+1} that are linked by two shortest paths. To start with those ordered pairs where in the model of the TH the largest disc (number $n+1$) *is* moved during the transfer, we have to sum the right-hand side in the formula of Theorem 3.5 for all six possible ordered pairs (i,j) with $i,j \in \{1,2,3\}$, $i \neq j$, and all v as in the assumption of that theorem. The latter amounts to summing $b(\mu)$ over all possible values μ of d_{jk}, namely from 1 to $2^n - 1$, multiplied by the number of vertices with $d_{jk}(v) = \mu$, which is $|\Delta_{jk}(\mu)| = b(2^n - \mu)$. So, making use of (**??**), we get

$$6 \sum_{\mu=1}^{2^n-1} b(\mu)\, b(2^n - \mu) = 6 \sum_{\mu \in \mathbb{N}} z_n(2^n - \mu)\, z_n(\mu),$$

which has been calculated in [9, Proposition 6i] to be equal to $6\left(\Theta_+^n - \Theta_-^n\right)/\sqrt{17}$ with $\Theta_\pm := \left(5 \pm \sqrt{17}\right)/2$. Finally, if we want to take into account those pairs where the largest disc is at rest, observing that there are three choices *where* it can lie, and keeping on performing these steps, we arrive at

$$\begin{aligned} x_n &= \frac{6}{\sqrt{17}} \sum_{k=0}^{n-1} 3^k \left(\Theta_+^{n-k} - \Theta_-^{n-k}\right) \\ &= \frac{3}{4\sqrt{17}} \left\{\Theta_+^{n+1}\left(\sqrt{17}+1\right) - 2 \cdot 3^{n+1}\sqrt{17} + \Theta_-^{n+1}\left(\sqrt{17}-1\right)\right\}, \end{aligned}$$

the first few values being 0, 6, 48, 282, 1476, 7302, 35016, 164850, 767340, 3546366, 16315248, 74837802 for S_1 to S_{12}.

4 Some special embeddings of graphs S_n into $\mathbb{R}^2$

In this section we are going to show that it is not accidental that the sets $\Delta_{x,y}(k)$ are related to the lines in the plane in the way observed on Figure 4. Of course, an arbitrary embedding of S_n into the plane will not work, hence we first define specific embeddings that will be used. These embeddings will provide an explicit description of $b(i)$ as the order of a set of vertices of S_n intersected by a specific line in the plane.

The embeddings $f_n : V(S_n) \longrightarrow \mathbb{R}^2$ of the graphs S_n into the plane $\mathbb{R}^2$ will be defined inductively such that $f_n(11\ldots1) = (-2^n+1, 0)$, $f_n(22\ldots2) = (2^n-1, 0)$, and $f_n(33\ldots3) = (0, (2^n-1)y_0)$ will hold for any n. We use the fixed positive number y_0 in this construction in order to avoid writing $\sqrt{3}$, which would appear if we would restrict ourselves to equilateral triangles. Moreover, the notation also points out that while proving results about $\Delta_{1,2}(k)$ only the axial symmetry with respect to the y-axis will be needed.

The indices in f_n will be used only during the inductive construction—later we will use f for all of these functions.

The function $f_1 : V(S_1) \longrightarrow \mathbb{R}^2$, defined by $f_1(1) = (-1,0)$, $f_1(2) = (1,0)$, and $f_1(3) = (0, y_0)$, is obviously an appropriate embedding of $S_1 = K_3$, see Fig. 5.

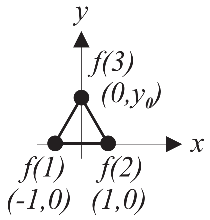

Figure 5: S_1 embedded in $\mathbb{R}^2$ as $f_1(S_1)$

Suppose that the embedding $f_n : V(S_n) \longrightarrow \mathbb{R}^2$, satisfying $f_n(11\dots1) = (-2^n+1, 0)$, $f_n(22\dots2) = (2^n-1, 0)$, and $f_n(33\dots3) = (0, (2^n-1)y_0)$, is given, see Fig. 6.

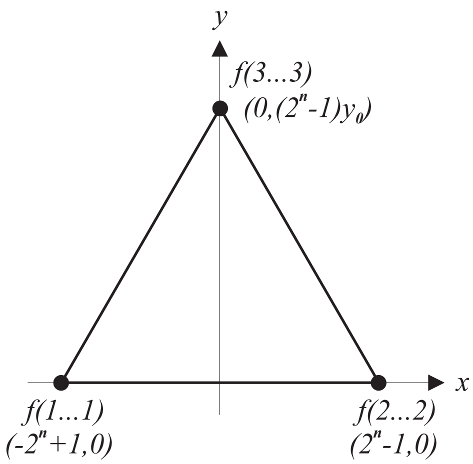

Figure 6: S_n embedded in $\mathbb{R}^2$ as $f_n(S_n)$

Clearly, $V(S_{n+1})$ is the disjoint union of $V(S^1_{n+1})$, $V(S^2_{n+1})$ and $V(S^3_{n+1})$, where each of these sets induces a subgraph isomorphic to S_n. We shall thus define f_{n+1} as f_n followed by a translation, chosen differently for each of these sets.

Let $t_{(a,b)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the following translation:

$$t_{(a,b)}(x, y) = (a, b) + (x, y).$$

Since $t_{(-2^n,0)}(-2^n+1, 0) = (-2^{n+1}+1, 0)$, $t_{(2^n,0)}(2^n-1, 0) = (2^{n+1}-1, 0)$, and $t_{(0,2^n y_0)}(0, (2^n-1)y_0) = (0, (2^{n+1}-1)y_0)$, we define

$$f_{n+1}(i, u_1, \dots, u_n) = \begin{cases} t_{(-2^n,0)}(f_n(u_1, \dots, u_n)); & i = 1, \\ t_{(2^n,0)}(f_n(u_1, \dots, u_n)); & i = 2, \\ t_{(0,2^n y_0)}(f_n(u_1, \dots, u_n)); & i = 3. \end{cases}$$

By the choice of translations we have $f_{n+1}(11\dots1) = (-2^{n+1}+1, 0)$, $f_{n+1}(22\dots2) = (2^{n+1}-1, 0)$, and $f_{n+1}(3\dots3) = (0, (2^{n+1}-1)y_0$. Also, it is easily calculated that for any $i \neq j$, $f_{n+1}(ij\ldots j)$ is as shown on Fig. 7. Note in addition that $f_{n+1}(11\ldots 1)$, $f_{n+1}(13\ldots 3)$, $f_{n+1}(31\ldots 1)$, and $f_{n+1}(33\ldots 3)$ are collinear and so are $f_{n+1}(22\ldots 2)$, $f_{n+1}(23\ldots 3)$, $f_{n+1}(32\ldots 2)$, and $f_{n+1}(33\ldots 3)$. Finally, $f_{n+1}(S^1_{n+1})$, $f_{n+1}(S^2_{n+1})$, and $f_{n+1}(S^3_{n+1})$ are pairwise disjoint, and positioned in the plane as shown on Fig. 7.

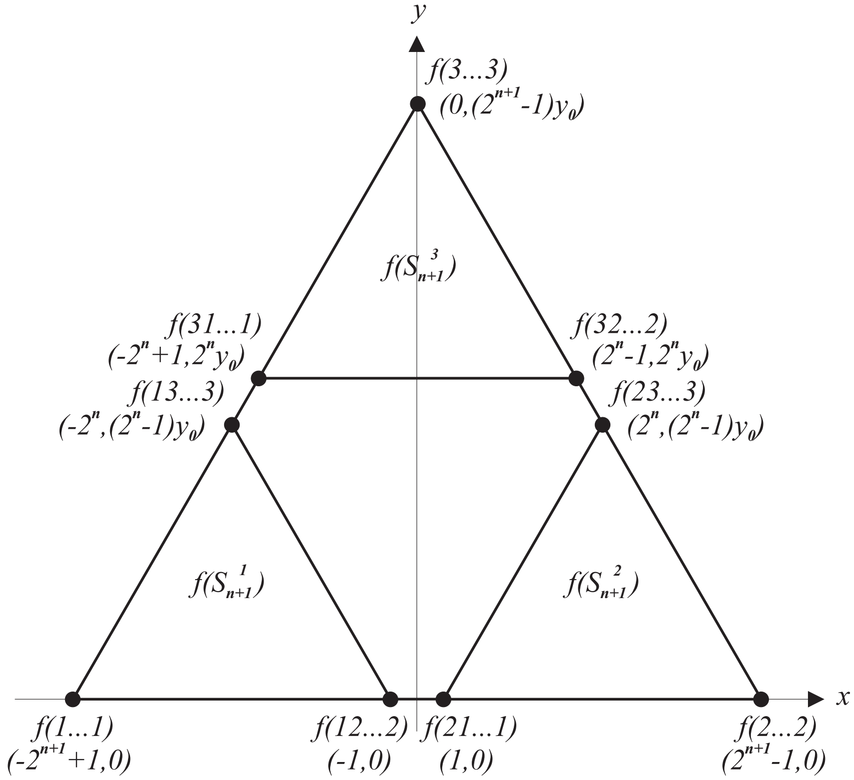

Figure 7: S_{n+1} embedded in $\mathbb{R}^2$ as $f_{n+1}(S_{n+1})$

For the next theorem we introduce the lines ℓ_k as $\ell_k = \{k\}\times\mathbb{R}$. Recall that in Lemma 3.3 we have introduced the sets $\Delta_{x,y}(k)$ as those vertices of S_n for which $\mathcal{P}^x_v - \mathcal{P}^y_v = k$. In the next result we consider $\Delta_{1,2}(k)$ as a set in $f(S_n)$.

Theorem 4.1 *For any n and any k,*

$$\Delta_{1,2}(k) = f(S_n) \cap \ell_k,$$

where $\Delta_{1,2}(k) = \{f(v) \mid v \in S_n, \mathcal{P}^1_v - \mathcal{P}^2_v = k\}$.

Proof. Note that the sets on both sides of the equation may be empty. In fact, from geometric properties of $f(S_n)$ and the lines ℓ_k (as well from Lemma 3.3) it follows that the sets will be simultaneously empty precisely when $|k| \geq 2^n$.

The assertion is clear for $n = 1$, cf. Fig. 5. Suppose now that the statement holds for S_n, $n \geq 1$.

In order to distinguish the sets $\Delta_{1,2}(k)$ defined in S_n and those defined in S_{n+1}, we will denote them by $\Delta_{1,2}^n(k)$, in the former, and by $\Delta_{1,2}^{n+1}(k)$, in the latter case. Then, since for any $u \in S_n$,

$$d_{S_{n+1}}(1u, 11\dots1) - d_{S_{n+1}}(1u, 22\dots2) = d_{S_n}(u, 1\dots1) - d_{S_n}(u, 2\dots2) - 2^n ,$$

we infer that $1u \in \Delta_{1,2}^{n+1}(k)$ if and only if $u \in \Delta_{1,2}^n(k+2^n)$. Using this fact, the definition of the embeddings f, and the way how translations act on the family of lines ℓ_k, we can compute as follows:

$$\begin{array}{rcl} v \in \Delta_{1,2}^{n+1}(k) \cap f(S_{n+1}^1) & \Leftrightarrow & t_{(-2^n,0)}^{-1}(v) \in \Delta_{1,2}^n(k+2^n) \\ & \Leftrightarrow & t_{(-2^n,0)}^{-1}(v) \in f(S_n) \cap \ell_{k+2^n} \\ & \Leftrightarrow & v \in t_{(-2^n,0)}(f(S_n) \cap \ell_{k+2^n}) \\ & \Leftrightarrow & v \in t_{(-2^n,0)}(f(S_n)) \cap t_{(-2^n,0)}(\ell_{k+2^n}) \\ & \Leftrightarrow & v \in f(S_{n+1}^1) \cap \ell_k \, . \end{array}$$

Hence we have shown that

$$\Delta_{1,2}^{n+1}(k) \cap f(S_{n+1}^1) = f(S_{n+1}^1) \cap \ell_k. \tag{2}$$

The proof of

$$\Delta_{1,2}^{n+1}(k) \cap f(S_{n+1}^2) = f(S_{n+1}^2) \cap \ell_k \tag{3}$$

is analogous. To prove that

$$\Delta_{1,2}^{n+1}(k) \cap f(S_{n+1}^3) = f(S_{n+1}^3) \cap \ell_k \tag{4}$$

holds as well, note first that for any $u \in S_n$,

$$d_{S_{n+1}}(3u, 11\dots1) - d_{S_{n+1}}(3u, 22\dots2) = d_{S_n}(u, 1\dots1) - d_{S_n}(u, 2\dots2) \, .$$

In other words, $3u \in \Delta_{1,2}^{n+1}(k)$ if and only if $u \in \Delta_{1,2}^n(k)$. Now we have:

$$\begin{array}{rcl} v \in \Delta_{1,2}^{n+1}(k) \cap f(S_{n+1}^3) & \Leftrightarrow & t_{(0,2^n y_0)}^{-1}(v) \in \Delta_{1,2}^n(k) \\ & \Leftrightarrow & t_{(0,2^n y_0)}^{-1}(v) \in f(S_n) \cap \ell_k \\ & \Leftrightarrow & v \in t_{(0,2^n y_0)}(f(S_n) \cap \ell_k) \\ & \Leftrightarrow & v \in t_{(0,2^n y_0)}(f(S_n)) \cap t_{(0,2^n y_0)}(\ell_k) \\ & \Leftrightarrow & v \in f(S_{n+1}^3) \cap \ell_k. \end{array}$$

Combining (2), (3), and (4), we can conclude the proof as follows:

$$\begin{array}{rcl} \Delta_{1,2}^{n+1}(k) & = & \Delta_{1,2}^{n+1}(k) \cap f(S_{n+1}) \\ & = & \Delta_{1,2}^{n+1}(k) \cap \left(f(S_{n+1}^1) \cup f(S_{n+1}^2) \cup f(S_{n+1}^3)\right) \\ & = & \left(\Delta_{1,2}^{n+1}(k) \cap f(S_{n+1}^1)\right) \cup \left(\Delta_{1,2}^{n+1}(k) \cap f(S_{n+1}^2)\right) \cup \left(\Delta_{1,2}^{n+1}(k) \cap f(S_{n+1}^3)\right) \\ & = & \left(f(S_{n+1}^1) \cap \ell_k\right) \cup \left(f(S_{n+1}^2) \cap \ell_k\right) \cup \left(f(S_{n+1}^3) \cap \ell_k\right) \\ & = & \left(f(S_{n+1}^1) \cup f(S_{n+1}^2) \cup f(S_{n+1}^3)\right) \cap \ell_k \\ & = & f(S_{n+1}) \cap \ell_k \, . \end{array}$$

□

We have formulated Theorem 4.1 for $\Delta_{1,2}(k)$, but from the symmetry it is clear that an analogous conclusion holds for any $\Delta_{x,y}(k)$ with $x \neq y$.

Corollary 4.2 *For any n and any $i = 1, 2, \ldots, 2^n$,*

$$b(i) = |f(S_n) \cap \ell_{i-2^n}| .$$

Proof. Let $i - 2^n = k$. Then k is in the range $-2^n + 1 \leq k \leq 0$, hence Lemma 3.3 is applicable. By Lemma 3.3, $b(i) = |\mathcal{H}(i-1)| = |\mathcal{H}(2^n - 1 + k)| = |\Delta_{1,2}(k)|$. But then it follows by Theorem 4.1 that $|\Delta_{1,2}(k)| = |f(S_n) \cap \ell_k| = |f(S_n) \cap \ell_{i-2^n}|$. □

This can be seen on Fig. 4, where the intersections of S_4 with the lines $x = k$, $k = -15, -14, \ldots, 0$ have $b(1)$, $b(2)$, $\ldots b(16)$ points, respectively. Note that this copy of S_4 is already realized as an $f(S_4)$.

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