WHEEL GRAPHS, LUCAS NUMBERS AND THE DETERMINANT OF A KNOT

This is a preprint. I would be grateful for any comments and corrections!

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Abstract. The Kauffman bracket approach is used to give estimates on the size of the determinant (and this way also on the coefficients of the Jones polynomial) of a link of given crossing number, and properties of the knots with maximal determinant are studied. Several number theoretic statements on the determinants of special classes of links are given, leading in particular to elegant proofs of squareness of some arithmetic expressions made up of Lucas and Fibonacci numbers, one of them enumerating spanning trees in wheel graphs.
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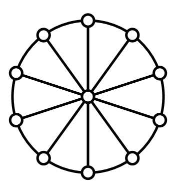
1. Introduction

Consider the wheel graph W_n of $n+1$ vertices.

 $= \quad W_{10}$

The number c_n of spanning trees in W_n can be computed by distinguishing the number of edges of the spanning tree incident to the central vertex of the wheel, and counting the spanning forests of the necklace graph remaining from the spanning tree in W_n after removing the central vertex. This was carried out in [My, p. 469–470]. The resulting sequence is 1 5 16 45 121 ... and can be expressed by the Lucas numbers L_n given by $L_1 = 2$, $L_2 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n > 2$; the relation is $c_n = L_{2n} - 2$. Another occurrence of this sequence is in [Re] as the number of certain unimodular matrices. See also [My2] and [Sl, sequence 004146].

An alternative expression of c_n (whose equivalence to the above one can be shown by elementary generating series arguments, for example) is

$$c_n = F_{2n} + 2\sum_{i=1}^{n-1} F_{2i}\,, \tag{1}$$

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with F_i denoting the Fibonacci numbers (defined by $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$).

A closer look on the numbers c_n reveals that for odd n, c_n is a square. Although there have been, in particular recently, many related results, e. g. [Ch, DF, Du, Du2, Es, MD, Mr], I did not find an explicit statement of this observation. Nonetheless, it is suggestive that this phenomenon should not be the result of an accidental coincidence, and indeed a combinatorial explanation of it is possible by writing down the explicit formula for L_n

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}. \tag{2}$$

However, the same phenomenon occurs with (the odd index members of) some closely related sequences like

$$c'_n = c_n + F_n^2 + 2F_{2n} \quad \text{and} \quad c''_n = c_n + 4F_n^2 + 4F_{2n}, \tag{3}$$

where it is less straightforward to come by.

The aim of this paper is, *inter alia*, to give an explanation of this phenomenon in terms of knot theory (showing how to find further such sequences and prove their squareness in a much easier and more elegant way than via the naive arithmetical approach). It turns out, that the numbers c_n occur as determinants of some (alternating 3-braid) knots and links.

If Δ_L denotes the (1-variable) Alexander polynomial of a link $L \hookrightarrow S^3$ [Al], then $\det(L) = \left|\Delta_L(-1)\right|$ is the order of the homology group $H_1(D_L)$ (over $\mathbb{Z}$) of the double branched D_L cover of S^3 over L (or 0 if this group is infinite) and carries the name “determinant” because of its expression (up to sign) as the determinant of a Seifert [Ro, p. 213] or Goeritz [GL] matrix. This group carries much interesting information on the link (in particular unknotting number estimates [We], sliceness [Ro] and chirality information [HK, St]).

In [St4] we initiated the investigation of the question how much the coefficients of the various link polynomials can grow on knots and links of given number of crossings, and showed how via the Kauffman bracket [Ka2] the problem for the Jones polynomial is equivalent to this of the determinant. We also found that the maximum will be realized by alternating knots/links. The quest for better estimates of this maximum and the properties of the links attaining it led to consider the above mentioned 3-braid links, for which the determinant could be calculated by the method of Krebes [Kr], giving the sequence c_n, and the squareness property is a consequence of work of Hartley and Kawauchi [HK].

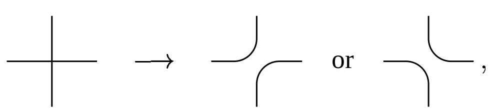
Another, although unrelated, occurence of wheels in knot theory is explored in [BGRT].

2. The determinant of alternating diagrams

Via the relation $\Delta(-1) = V(-1)$ to the Jones polynomial (see [J2, §12]) the determinant provides a bridge between the classical Alexander polynomial and its modern successors [BLM, H, Ka, J], whose nature is rather combinatorial, and it is one of the little topologically understandable information encoded in these invariants. On the other hand, this opens combinatorial approaches for calculating the determinant.

One such approach, which is particularly nice for alternating diagrams, was given by Krebes [Kr] using the Kauffman bracket/state model for the Jones polynomial.

If D is an alternating link diagram, then consider $\hat{D} \subset \mathbb{R}^2$, the (image of) the associated immersed plane curve(s). Then $\det(D)$ is equal to the number of ways to splice the crossings (self-intersections) of $\hat{D}$



so that the resulting collection of disjoint circles has only one component.

In [St4], we showed via the skein relation for the Jones polynomial that for a diagram D of $c(D)$ crossings

$$\left|V(D)\right|_1 := \sum_{2k\in\mathbb{Z}} \left|[V(D)]_{t^k}\right| \le 3^{c(D)-1}.$$

Some experiments reveiled that this bound is not particularly sharp.

The first observation towards an improvement is that using Krebes’s approach, we have

Lemma 2.1 With the above notation,

$$\big|V(D)\big|_1 \le 2^{c(D)-1}.$$

Proof. Let D' be the alternating diagram obtained from D by changing crossings. Then by Kauffman's bracket, we have

$$|V(D)| \le \det(D')$$

If we resolve any $c(D')-1$ crossings in D' in some arbitrary way, then for the last one there is at most one splitting so as the circle picture to have only one component, so that the result follows. □

Although this lemma already gives a sensible improvement of the previous estimate, we can push it even a little further.

Theorem 2.1

1) There exists a constant $C > 0$ such that for any link diagram D of $c(D)$ crossings

$$\det(D) \le C\cdot\delta^{c(D)}, \tag{4}$$

where $\delta \approx 1.83929$ is the inverse of

$$\delta^{-1} = -\frac{1}{3} - \frac{2}{3\left(17+3\sqrt{33}\right)^{1/3}} + \frac{\left(17+3\sqrt{33}\right)^{1/3}}{3} \approx 0.543689,$$

the real positive zero of $f(x) = x^3+x^2+x-1$.

2) If D is an arborescent diagram, then

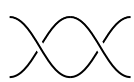
$$\det(D) \le F_{c(D)+1}, \tag{5}$$

and the inequality is sharp (that is, there are relevant diagrams for which equality holds).

Proof. We start by the second part. Let

$$d_n^a := \max\{\det(D) : D \text{ arborescent of } n \text{ crossings}\}$$

An arborescent diagram always has a clusp



whose resolution preserves arborescency. When splicing one of the crossings in the clusp, one of the two resulting diagrams has a kink, so that only one of the splicing of the second crossing can give a circle picture with only one component.

Thus

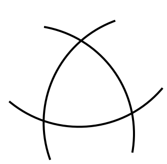
$$d_n^a \le d_{n-1}^a + d_{n-2}^a,$$

which, together with the trivial correctness for $c(D) = 1,2$ by induction establishes the inequality (5). The other inequality follows from considering the rational links $L_n = C(\underbrace{1,1,\dots,1}_{n\text{ times}})$ (here we use Conway's notation [Co]). To see that $\det(L_n) = F_{n+1}$ is an easy calculation with iterated fractions.

The argument for the first part is analogous. Let[1]

$$d_n^\infty := \max\{\det(D) : D \text{ link diagram of } n \text{ crossings}\}$$

Then either D has a clusp, or a triangle



(6)

[1]The strange superscript is used for conformity with notation which will be introduced later.

Then the above argument modifies to show that

$$d_n^\infty \le d_{n-1}^\infty + d_{n-2}^\infty + d_{n-3}^\infty \qquad (n > 2)\,, \tag{7}$$

and thus d_n^∞ can be estimated by (properly scaled) *Tri*bonacci numbers. □

Remark 2.1

1) We have the explicit expression

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]\,, \tag{8}$$

so that for arborescent diagrams (4) holds with the smaller base $\frac{\sqrt{5}+1}{2} \approx 1.61803$ instead of $\delta \approx 1.83929$.

2) The constant C in (4), the way that it comes from the Tribonacci number estimate, can be certainly effectively calculated, but it does not appear appropriate to do so. The standard way is to apply partial fraction decomposition to the generating (rational) function, obtaining a rather nasty expression involving the real and imaginary parts of the zeros of the denominator polynomial, which in the case of a cubic are alredy messy enough. Moreover, C can be successively improved by noting that (7) will hardly be sharp in general. Writing down the first values of d_n^∞ we get

n	0	1	2	3	4	5	6
d_n^∞	1	1	2	3	5	8	16

We see that (7) is sharp for $n = 6$, but not for $n < 6$ (because a diagram of $n < 6$ crossings has a clusp, so that we have the simplified recursion $d_n^\infty \le d_{n-1}^\infty + d_{n-2}^\infty$), and it will certainly not be for high n. Thus one can start the iteration on the right of (7) with higher and higher values of n and smaller initial data, obtaining a sequence of constants C with decreasing numerical value but increasing arithmetical complexity ... However, it is worth remarking that, because of connected sums, in every case $C = 1$ must do the job.

Again it appears appropriate to make an experiment how good the bound is compared to the actual values of d_n. In [St4, §3] we replaced $c(D)$ by $\operatorname{span} V(D) - 1$ giving an inelucidative picture dominated by non-alternating knots. Thus here we consider only alternating knots and link of given crossing number.

For what follows it will be helpful to make some definitions.

Definition 2.1 Let $S \subset \mathbb{N}$. Then define

$$d_n^S = \max\{\det(D) : n(D) \in S, c(D) = n\}\,,$$

where $n(D)$ is the number of components of D, and let K_n^S be a link attaining the maximum. Set $K_n^i := K_n^{\{i\}}$ and $d_n^i := d_n^{\{i\}}$, $K_n^\infty := K_n^{\mathbb{N}}$, $d_n^\infty := d_n^{\mathbb{N}}$, $K_n := K_n^1$, $d_n := d_n^1$.

This definition already contains a question.

Question 2.1 Is K_n^S unique for all S and n?

In all special cases I checked it was so. However, it is not clear in general. For what follows let us avoid any possible ambiguity by choosing one fixed maximizing link for each n and S. In any case we point out the following important fact remarked in [St4].

Theorem 2.2 K_n^S is alternating for each n and S.

ncr	kid	det	fibered	cluspf	flypef	achir	invert	σ	al braid	bind
3	1	3	✔		✔		✔	2	✔	2
4	1	5	✔		✔	$+/-$	✔	0	✔	3
5	2	7			✔		✔	2		3
6	3	13	✔		✔	$+/-$	✔	0	✔	3
7	7	21	✔				✔	0	✔	4
8	18	45	✔	✔	✔	$+/-$	✔	0	✔	3
9	40	75	✔	✔	✔		✔	2	✔	4
10	123	121	✔	✔	✔	$+/-$	✔	0	✔	3
11	266	209	✔	✔	✔		✔	0	✔	4
12	868	377	✔	✔	✔	$-$		0		5
13	3478	663	✔	✔	✔		✔	2	✔	4
14	17895	1145	✔	✔	✔	$-$		0	✔	5
15	82477	2037	✔	✔	✔		✔	0	✔	4
16	361172	3581	✔	✔	✔	$-$		0	✔	5

(9)

Table 1: The knots K_n for $n \leq 16$ and some of their data (from left to right): crossing number, knot identifier, determinant, fiberedness, clusp-freeness, flype-freeness, achirality, invertibility, signature, existence of alternating braid representation, braid index.

As tabulation (up to crossing numbers sufficing to give some more concrete picture) are available only for knots, we made a more serious calculation only for $S = \{1\}$. The knots K_n for $n \leq 16$ reveal many similarities and are listed in table 1, together with the indication of (lack of) some specific properties and, beside their determinants d_n, some other classical invariants (the genera are not included because their behaviour will later be clarified). The last 6 knots, which are not given in Rolfsen's tables [Ro, appendix], are drawn on figure 1. They are numbered according to the tables in [HT].

The meaning of the properties "flype-free" and "clusp-free" is as follows (for the definition of flypes, see [MT]).

Definition 2.2 A knot or link is called *flype-free*, if there is no essential flype applicable on its alternating diagram, that is, by [MT], it has only one alternating diagram (modulo moves in S^2).

Definition 2.3 A knot or link is called *clusp-free*, if there is no (possibly trivial) sequence of flypes making any of its alternating diagrams to have a clusp.

The table reveals some striking coincidences and leads to some (more or less justifiable) conjectures (we defer the discussion of the braid index to the end of the paper, because braids will be considered in more detail subsequently).

Conjecture 2.1

1) K_n is fibered for $n \neq 5$.

2) K_n is clusp-free for $n \geq 8$

3) K_n is flype-free for $n \neq 7$.

4) K_n is invertible for odd n and $-$achiral for even n.

5) $\sigma(K_n) \in \{-2, 0, 2\}$.

6) K_n is (the closure of) an alternating braid except for $n = 5, 12$.

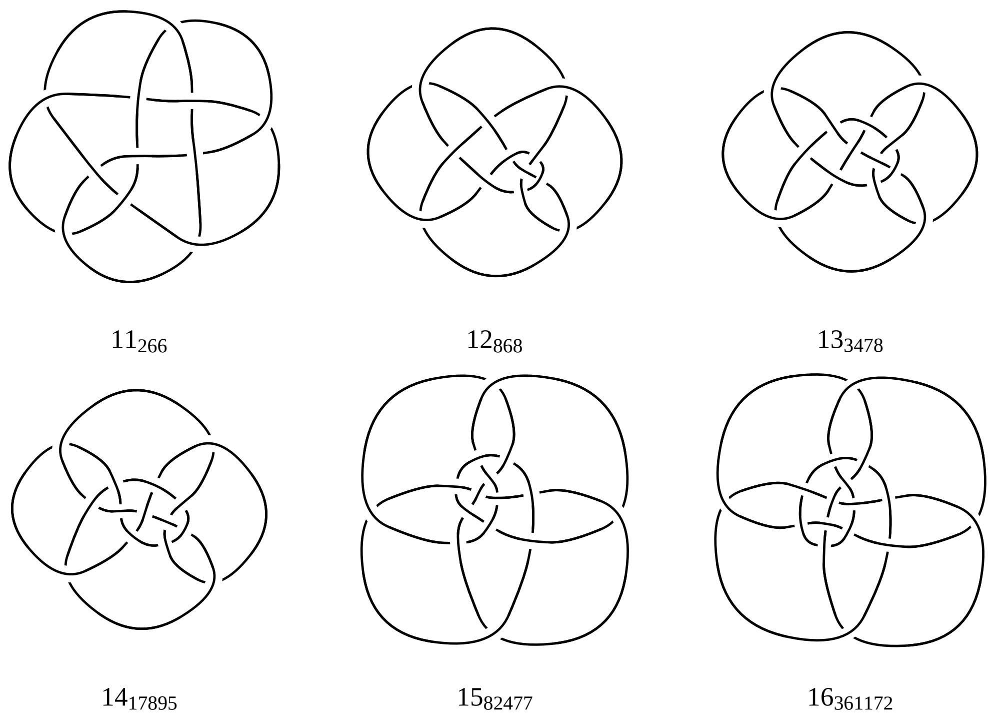


Figure 1: The knots of 11 to 16 crossings with maximal determinant.

7) K_n is prime.

8) K_n is unique.

Although sufficient experimental data is not available for links, it appears that similar phenomena occur there as well. In the following we start the investigation of such phenomena – flype-freeness, clusp-freeness and primality, and give some relations between properties of d_n and such of K_n. The most unconditional statement in this regard we can prove is

Theorem 2.3 Let $S=\{1\}$ or $S=\infty$. Then K_n^S is clusp-free for infinitely many values of n. In fact, every interval $[x,\ x+81]$ contains at least one such n.

The proof of this fact initiates from some weaker properties of K_n following from such of d_n.

Proposition 2.1

a) If $d_n > \max(3d_{n-2}, d_{n-1}+2d_{n-3})$, then K_n is clusp-free.

b) If for $S=\{1\}$ or $S=\infty$ we have $d_n^S > d_l^S d_{n-l}^S$ for any $1<l<n-1$, then K_n^S is prime.

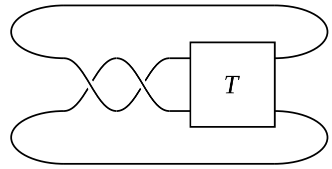
c) If for $S=\infty$ we have $d_n^S > 3d_l^S d_{n-l-1}^S$ for any $1<l<n-2$, then K_n^S is flype-free.

d) If for $S=\infty$, $d_n^S > \min(3d_{n-2}^S, d_{n-1}^S+2d_{n-3}^S)$, then K_n^S is clusp-free.

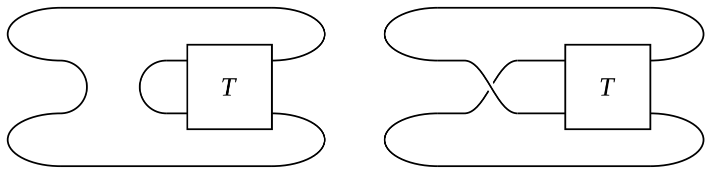
Proof.

a) Assume K_n has a clusp, i.e.

K_n =



Then splicing of the one crossings in the clusp gives a knot and a 2 component link.



(10)

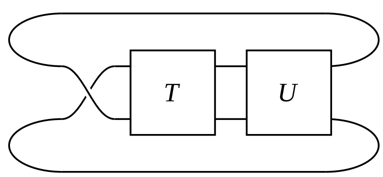
(a) (b)

Case 1. (a) is the knot and (b) is the (2 component) link. Then (a) contributes at most d_{n-2} to d_n and (b) has a mixed crossing (unless it is split in which case it has zero determinant), whose two splicings give again knots, so the contribution is at most $2d_{n-2}$.

Case 2. (b) is the knot and (a) is the link. Then (b) contributes $\le d_{n-1}$ and (a) contributes after splicing a mixed crossing $\le 2d_{n-3}$.

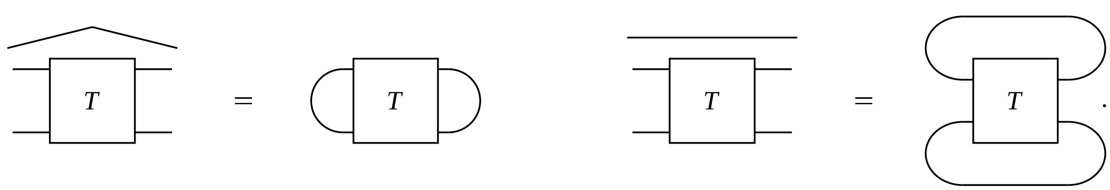
b) This is straightforward from the multiplicativity of the determinant under connected sum and the result of Menasco [Me].

c) Assume that K_n is not flype-free, in particular a diagram of $K = K_n$ is of the form



(11)

with $c(T), c(U) > 1$. Let the two possible closures of a tangle be denoted as follows:



With this notation (11) can be written as

$$K = \overline{1, T, U},$$

where '1' is the 1-tangle and the comma operator denotes tangle sum in the Conway [Co] sense. Then by Krebes' calculus [Kr] for his invariant Kr we have

$$\frac{\det(K_n)}{*} = \mathrm{Kr}(1, T, U) = \frac{\pm 1}{1} \oplus \frac{\pm\det(\overline{T})}{\det(\widehat{T})} \oplus \frac{\pm\det(\overline{U})}{\det(\widehat{U})}$$

so that comparing the numerators we obtain

$$d_n = \det(K_n) = \pm\big(\det(\overline{T}) + \det(\widehat{T})\big) \cdot \det(\widehat{U}) \pm \det(\widehat{T})\det(\overline{U}) \le 3d_{n-l-1}d_l\,,$$

with $l = c(T)$. Here $\oplus$ is the "fraction" addition in $\mathbb{Z} \times \mathbb{Z}/(p,q) \sim (-p,-q)$.

d) Use the inequality $d_n^S \leq d_{n-1}^S + d_{n-2}^S$ following from the clusp and

$$d_{n-1}^S \leq 2d_{n-2}^S\,,$$

following from splicing any arbitrary crossing in an $n-1$ crossing link diagram. □

We now come to the proof of theorem 2.3.

Proof of theorem 2.3. First let $S=\infty$. We use an indirect argument. Assume that K_n^∞ had a clusp for almost all n. Then proposition 2.1.d) shows that $\exists C$:

$$d_n^\infty \leq C\cdot\left(\frac{\sqrt{5}+1}{2}\right)^n\,.$$

Thus it suffices to exhibit links L_n of n crossings with $\limsup_{n\to\infty}\det(L_n)\left(\frac{\sqrt{5}+1}{2}\right)^{-n}=\infty$. For this take the iterated connected sum of some knot K with $\det(K)>\left(\frac{\sqrt{5}+1}{2}\right)^{c(K)}$ with itself. One such knot is $K=13_{3478}$ with det $=663$, while $\left(\frac{\sqrt{5}+1}{2}\right)^{13}\approx 521$.

Now let $S=\{1\}$. If for almost all n the knot K_n had a clusp, then

$$d_n \leq \max\left(3d_{n-2}, d_{n-1}+2d_{n-3}\right)$$

coming from proposition 2.1.a) shows $d_n \leq C\cdot\sqrt{3}^n$ (the zero of $2x^3+x-1$ on $[0,\infty)$ close to $1/2$ is higher than $1/\sqrt{3}$, so that the higher rate of growth comes from the first alternative in the maximum).

Thus again we need to show that there exist knots K_n with $c(K_n)=n$ and

$$\limsup_{n\to\infty}\sqrt[n]{\det(K_n)} > \sqrt{3}\,.$$

By taking again $K_n=\#^nK$ it suffices to find some K with $\det(K)>\sqrt{3}^{c(K)}$. This, unfortunately, is not the case for knots of ≤ 16 crossings, and we need to look at more complicated examples. Luckily, however, the determinant can be computed via the Seifert matrix in polynomial time. A package for this using braid representations was written by S. Orevkov for MATHEMATICA™ [Wo]. Using it I found the closed 81 crossing alternating 10-string braid

$$K=\widehat{\ }\left(\left(\sigma_1\sigma_3\sigma_5\sigma_7\sigma_9\sigma_2^{-1}\sigma_4^{-1}\sigma_6^{-1}\sigma_8^{-1}\right)^9\right),$$

where $\det(K)=24743382596536452489$, and hence $\mu_K := \det(K)\cdot 3^{-c(K)/2}\approx 1.17503$. □

D. Zagier remarked that the inequality $d_{a+b}\geq d_a d_b$ implies that $\lim_{n\to\infty}\sqrt[n]{d_n}$ exists and that

$$\lim_{n\to\infty}\sqrt[n]{d_n} = \sup_K \sqrt[c(K)]{\det(K)}\,,$$

where the supremum is taken over all (alternating) knots K. Thus we have

Corollary 2.1 $\sqrt{3}<\sqrt[81]{24743382596536452489}\approx 1.7355032\leq\lim_{n\to\infty}\sqrt[n]{d_n}\leq\delta\,.$ □

We should also point out that the lower bound $\sqrt{3}$ is of no special importance – in can be successively improved by calculating the determinant of appropriate more and more complicated knots.

Question 2.2 Is $\lim_{n\to\infty}\sqrt[n]{d_n}=\delta$?

Remark 2.2 If we have a knot K of k crossings with $\mu_K > 1$ and know d_n for $n < k$, then we can obtain an explicit (upper) estimate depending on $\varepsilon > 0$ of the smallest number n_0 with $\sqrt[n]{d_n} > \sqrt{3} + \varepsilon$ for any $n > n_0$, which – if sufficiently small – can be used to prove the clusp-freeness of K_n for almost all n. There is little hope to be able to proceed this way, though. Indeed $\mu_K > 1$ occurs only for rather complicated knots, and it does not seem feasible to calculate d_n for n larger than about 20. For example, for

$$K = \hat{}\Big(\left(\sigma_1\sigma_3\sigma_5\sigma_7\sigma_9\sigma_2^{-1}\sigma_4^{-1}\sigma_6^{-1}\sigma_8^{-1}\right)^7\Big),$$

we have $\mu_K = 0.98\ldots$, although it already has crossing number 63.

Another property of the K_i follows from the work we have done in [St5], which rewards us with an easy proof of a growth statement for the *genera* $g(K_n)$ of the K_n.

Theorem 2.4 $g(K_n) \to \infty$.

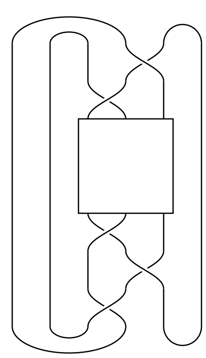
Proof. By [St5, theorem 3.1], $\det(K)$ grows only polynomially in $c(K)$ for alternating knots K of fixed genus. $\square$

3. Recursive sequences and alternating braids

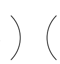
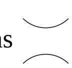
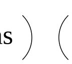
Originally the examples $K_8 = 8_{18}$ and $K_{10} = 10_{123}$ suggested to consider for the proof of theorem 2.3 for $S = 1$ closer the sequence of alternating 3-braids $\widehat{(\sigma_1\sigma_2^{-1})^k}$. Although these braids closely fail in giving the desired examples, they can be used to give an estimate for arbitrary alternating 3-braids and establish the connection to the (modified) Lucas numbers mentioned in the introduction.

Lemma 3.1 $\det\big(\widehat{(\sigma_1\sigma_2^{-1})^k}\big) = c_k$.

Proof. Consider the 2 uppermost crossings of $(\sigma_1\sigma_2^{-1})^k$, the ones from the last factor in the power.



Splicing the uppermost one as gives the rational knot $C(\underbrace{1,1,\ldots,1}_{2k-1})$, whose determinant as we mentioned is F_{2n}.

Splicing the uppermost crossing as  and the second uppermost one as  gives, after deleting the kink from the lowermost crossing, a rational link $C(\underbrace{1,1,\ldots,1}_{2k-3})$ with determinant F_{2k-2}. Finally, splicing both crossings as  gives $(\sigma_1\sigma_2^{-1})^{k-1}$, and then the result follows by induction from (1). $\square$

Corollary 3.1 If β is an alternating 3-braid, then $\det(\hat\beta) \le \left(\frac{\sqrt{5}+1}{2}\right)^{c(\hat\beta)}$, with the inequality in general sharp up to an additive constant.

Proof. Use that any $\beta \in B_3$, except for the ones in the lemma, have a clusp. Splicing one of the crossings in the clusp, we obtain a 3-braid with one crossing less and a rational knot. The contribution of the rational knot of $c(\hat{\beta})-2$ crossings to $\det(\hat{\beta})$ is estimated by theorem 2.1.2) to

$$\frac{1}{\sqrt{5}}\left(\frac{2}{\sqrt{5}+1}\right)\cdot\left(\frac{\sqrt{5}+1}{2}\right)^{c(\hat{\beta})}+C \tag{12}$$

for some fixed constant C, and this of the braid by induction on $c(\hat{\beta})$ by

$$\left(\frac{2}{\sqrt{5}+1}\right)\cdot\left(\frac{\sqrt{5}+1}{2}\right)^{c(\hat{\beta})}.$$

But

$$\frac{2}{\sqrt{5}+1}+\frac{1}{\sqrt{5}}\left(\frac{2}{\sqrt{5}+1}\right)=\frac{2}{\sqrt{5}}<1, \tag{13}$$

so starting the induction for $c(\hat{\beta})$ large enough to gobble the C in (12) by the strict inequality in (13) and checking the initial cases directly, one is done. □

The links of the form $\widehat{(\sigma_1\sigma_2^{-1})^k}$ are not new. They have been considered for a while, notably in [JP] (at least in the knot case $3\nmid k$). There it was observed that for odd k (for which the knots are also called "turks head knots"), the braid $(\sigma_1\sigma_2^{-1})^k$ is of the form $\beta\overline{\beta}$, where $\overline{\beta}$ is obtained from $\beta \in B_n$ by the map $\sigma_i^{\pm 1}\mapsto\sigma_{n-i}^{\mp 1}$, and hence $\widehat{(\sigma_1\sigma_2^{-1})^k}$ is strongly $+$achiral, i. e., admits an embedding into $\mathbb{R}^3$ fixed by the (orientation-reversing) involution $(x,y,z)\mapsto(-x,-y,-z)$, such that this involution additionally preserves the orientation of the knot/link. By the result of [HK] (stated and proved only for knots but true by the same argument also for links[1]), such knots/links have as Alexander module a double $A\oplus A$, so that in particular the Alexander polynomial, and hence the determinant is a square. (Long [Lo] has stronger shown that such knots are algebraically slice.) This, together with lemma 3.1, shows the statement alluded to in the introduction.

Theorem 3.1 c_k is a square number for k odd (hence so is the number of spanning trees in wheel graphs with an odd number of spokes or the by 2 decreased Lucas number L_n with $n\equiv 2 \bmod 4$). □

The fact that the odd index number knots are still at least achiral (in the usual, weak, sense), shows that by [St] c_n for n even is at least the sum of two squares. Unfortunately, contrarily to the result obtained for the odd index parity, there seems no tool available to examine effectively the even index number case. However, the test of the prime decomposition of c_n leads to conjecture even more, namely that these numbers are of the form $c_n=5a_n^2$ for n even, and this can be indeed confirmed from the explicit formula for L_n (2). (This observation seems to fit into a more general pattern conjecturally described at the end of this note.)

On the other hand, for odd k it is clear that now a similar procedure can be applied to more general braids. For example applying the argument to $\beta\overline{\beta}$ with $\beta=\sigma_1(\sigma_1\sigma_2^{-1})^k$ and $\beta=\sigma_1^2(\sigma_1\sigma_2^{-1})^k$ gives the property for c_n' and c_n'' in (3). Considering $\beta\overline{\beta}=(\sigma_1^l\sigma_2^{-l})^k$ gives a more general version of theorem 3.1.

Theorem 3.2 Let $b_0=0$, $b_1=1$ and $b_n=b_{n-2}+lb_{n-1}$. Then

$$l\left(2\sum_{i=1}^{k-1}b_{2i}+b_{2k}\right)$$

is a square for k odd. □

Considering 5-braids may give similar, however, less pleasant statements of this kind.

On the other hand, arithmetic results can have some knot theoretic consequences.

[1] except in the case, when the Alexander module is not completely torsion, which is, however, trivial, as then the Alexander polynomial vanishes

Corollary 3.2 Any rational knot $C(1,1,\dots,1)$ ("twist plat knot" [Ju]) is not algebraically slice.

Proof. Use the result of [Ch] that no odd Fibonacci number > 1 is a square. $\square$

Remark 3.1

1) Of course the same argument shows that $C(1,1,\dots,1)$ is not strongly $+$achiral, but this follows more generally for any rational knot from the result of Hartley-Kawauchi as the 2-branched cover homology group $H_1(D_K)$ is cyclic (and non-trivial), and hence not a double.
2) A similar property could be shown for the rational knots $C(3,1,\dots,1)$ from the result on the Lucas numbers.
3) The knot-theoretic counterpart of the non-squareness of c_k for even k is true also by different arguments. It was remarked in [St3] how the work of Murasugi [Mu] on the Alexander polynomial of periodic knots implies that the Alexander polynomial of any non-trivial knot (and analogously, link), which is the closure of the square of some braid (here $(\sigma_1\sigma_2^{-1})^{k/2}$), is not a square, so that the knot is not strongly $+$achiral (although it is weakly $+$achiral).

The 3-braids are the initial and most elegant member of a picture for more general braids that we describe now by a theorem on the rate of growth of the determinant of braids of given strand number.

Theorem 3.3 If $\beta_i \in B_n$ are alternating braids of fixed strand number.

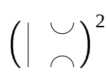
1) Then $\lambda_{\{\beta_i\}} := \limsup\limits_{n\to\infty} \sqrt[c(\beta_i)]{\det(\widehat{\beta}_i)} \le \delta$.
2) Moreover, if $\beta_i = \beta^i$ are powers of some fixed braid β, then $\lambda_\beta := \lambda_{\{\beta^i\}}$ is an algebraic number of degree $\le C_n$, where $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ is the n-th Catalan number.

Proof.

1) This is clearly a consequence of corollary 2.1.
2) Let β_i be n-strand braids and SD_n be the Kauffman algebra of [Ka, definition 3.5] with the special parameter $A = i = \sqrt{-1}$ (so that a separate loop trivializes). It can be shown (see [Ka, theorem 4.3]) that SD_n is (freely?) generated by the C_n loop-free diagrams connecting a pair of $n+n$ points on bottom and on top by n lines. The dimension of SD_n is therefore (at most) C_n. For example for $n = 3$ we have the following 5 elements:

 (14)

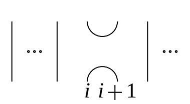
The multiplication is given by stacking up and eventual killing of the resulting diagram if it has a loop. For example

 $= 0$.

Let ϕ_β be the linear operator

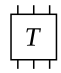
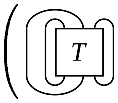
$$SD_n \ni x \overset{\phi_\beta}{\longmapsto} x \prod_{j=1}^{k} (1 + s_{i_j}) \in SD_n$$

with

$s_i =$ 

associated to $\beta = \prod_{j=1}^{k} \sigma_{i_j}^{(-1)^{i_j}} \in B_n$.

It can be decomposed (at least over $\mathbb{C}$) into Eigenvalues λ_i and Jordan box spaces V_i. Fix a Jordan basis of ϕ_β (as it has integer coefficients in some rational basis of DS_n, the Jordan basis can be chosen to lie in some degree $\leq C_n$ extension of $\mathbb{Q}$) and let $\lambda'_1, \dots, \lambda'_l$ be the Eigenvalues λ_i of ϕ_β of maximal norm, whose V_i are not completely killed by the $\mathbb{C}$-linear extension of the map

$DS_n \ni$  $\longmapsto \det\Big($  $\Big) \in \mathbb{N} \subset \mathbb{C}$

Consider the Jordan decomposition of $Id = 1 \in DS_n$

$$\Big| \quad \Big| \quad \Big| \quad \cdots \quad \Big| = \sum_{i=1}^{l} x'_i + x\,, \qquad x'_i \in V_{\lambda'_i} \qquad \exists i_0 : x'_{i_0} \neq 0\,.$$

Then x'_i contributes to $\det\big(\widehat{\beta^n}\big)$

$$\sum_{j=1}^{\dim V_{\lambda'_i}} a_j P_j(n)\, \lambda'^n_i$$

for some $a_j \in \mathbb{C}$ (the coefficients of x'_i in the Jordan basis of $V_{\lambda'_i}$) and $P_j(n) \in \mathbb{Q}[n]$ with $\deg P_j \leq \dim V_{\lambda'_j}$.

Thus

$$\det(\widehat{\beta^n}) = \sum_{j=1}^{l'} \tilde{P}_j(n)\, \lambda'^n_i \tag{15}$$

for some $1 \leq l' \leq l$ and $0 \neq \tilde{P}_j(x) \in \mathbb{Q}[n]$ (discard possible $\tilde{P}_j = 0$) with $\deg \tilde{P}_j \leq \max_{j=1}^{l} \dim V_{\lambda'_j}$. If we show now

$$\limsup \sqrt[n]{\det(\widehat{\beta^n})} = |\lambda'_i|\,,$$

we are through, as λ'_i is the root of a polynomial with rational coefficients of degree C_n.

If $l' = 1$ the claim is straightforward from (15) and for $l' > 1$ this follows from the lemma below by rescaling.

Lemma 3.2 Let $\lambda_1, \dots, \lambda_l$ $(l > 1)$ be distinct unit norm complex numbers and $\{a_{j,n}\}_{n=1}^{\infty}$ for $j = 1, \dots, l$ be sequences with $|a_{j,n}| \geq \varepsilon \forall\, j, n$ and

$$\frac{a_{j,n+1}}{a_{j,n}} \xrightarrow[n\to\infty]{} 1\,.$$

Then the sequence $s_n := \sum_{j=1}^{l} a_{j,n} \lambda_j^n$ does not converge (in particular, not to 0).

Proof. Assume $s_n \to s$ for some $s \in \mathbb{C}$. If

$$M_n := \begin{pmatrix} 1 & \dots & 1 \\ \frac{a_{1,n+1}}{a_{1,n}}\lambda_1 & \dots & \frac{a_{l,n+1}}{a_{l,n}}\lambda_l \\ \frac{a_{1,n+2}}{a_{1,n}}\lambda_1^2 & \dots & \frac{a_{l,n+2}}{a_{l,n}}\lambda_l^2 \\ \vdots & \ddots & \vdots \\ \frac{a_{1,n+l-1}}{a_{1,n}}\lambda_1^{l-1} & \dots & \frac{a_{l,n+l-1}}{a_{l,n}}\lambda_l^{l-1} \end{pmatrix},$$

then

$$M_n \begin{pmatrix} a_{1,n}\lambda_1^n \\ \vdots \\ a_{l,n}\lambda_l^n \end{pmatrix} \longrightarrow \begin{pmatrix} s \\ \vdots \\ s \end{pmatrix},$$

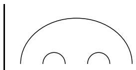
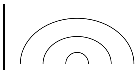
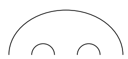
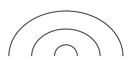
					
	0	0	0	1	0
	0	0	1	0	1
	0	1	0	0	1
	1	0	0	0	0
	0	1	1	0	0

Table 2: The table for the pairing $<,>_3$.

and M_n converge to a Vandermonde matrix, which is not singular, so that $||M_n^{-1}||$ is bounded.

Therefore, in particular $\left\{(a_{1,n}\lambda_1^n,\ldots,a_{l,n}\lambda_l^n)\,:\,n>n_0\right\}$ must lie in some ε'-ball for n_0 large enough. But $|a_{j,n}|\geq\varepsilon$ shows that these components stay outside of some neighborhood of the origin, and

$$\frac{a_{i,n+1}\lambda_i^{n+1}}{a_{i,n}\lambda_i^n}\longrightarrow\lambda_i\neq 1$$

for some i gives a contradiction for ε' small enough. □

This completes the proof of theorem 3.3. □

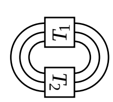
4. Some more problems

As the paper attempts the investigation of a relatively new subject, it is not surprising that it opens many more questions than it can answer. Hoping to whet the interest in further investigations, we conclude by mentioning some of these problems.

The combination of both statements in theorem 3.3 also suggests that if λ is an Eigenvalue of ϕ_β for some β, then $|\lambda|\leq\delta^{c(\beta)}$. Although a dominating Eigenvalue of ϕ_β may have a Jordan space killed by taking the determinant of the usual braid closure, there will often be a (linear combination of) other closure(s) under which not the whole Jordan space is killed (and then for these exotic closures the same argument will apply). The problem is whether indeed one can always find such a closure.

Question 4.1 Define a pairing (or binary quadratic form) on DS_n by

$$\left\langle \boxed{T_1}, \boxed{T_2} \right\rangle_n = \begin{cases} 1 & \text{if } \quad\text{[diagram]}\quad \text{has one loop} \\ 0 & \text{else} \end{cases}$$



Is $<,>_n$ non-degenerate?

Corollary 4.1 If $<,>_n$ is non-degenerate, then any Eigenvalue λ of ϕ_β for any $\beta\in B_n$ has $|\lambda|\leq\delta^{c(\beta)}$. □

Example 4.1 $<,>_3$ is given by the table 2 which shows $<,>_3$ to be non-degenerate. This is by computer check also true for $n=4\ldots10$. It is also an easy exercise to see that $<T_1,.>\not\equiv 0$ if T_1 is a single diagram, as one can always find a diagram T_2 with $\widehat{T_1T_2}=\bigcirc$. However, the argument does not extend in an easy way to arbitrary linear combinations of diagrams.

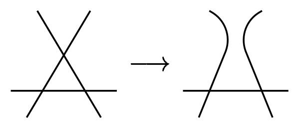
A further problem is that apparently the estimates in theorem 3.3.1) and those of the Eigenvalues of ϕ_β are not sharp. This is related to the following conjecture:

Conjecture 4.1 Only finitely many K_n have the same braid index $b(K_n)$, or alternatively, $\liminf_{n\to\infty} b(K_n) = \infty$.

Unfortunately, solving this conjecture appears to require unimaginable effort at the present state of the art. We should, however, give some rough heuristical motivation for it (although it is far from a rigorous proof).

The reason is that the diagrams $\widehat{\beta}_i$ for braids β_i of fixed strand number have either clusps or triangle regions (6) with bounded (minimal) distance $\le k = k_l$ between two among them, k_l depending only on the strand number l of the β_i (see [St2]; the distance is here the minimal number of intersections of a path from the one region to the other with the plane curve of the diagram).

This means that, even in the case there is no clusp, the sequence of crossings to splice can be chosen so that we splice the corners of a triangle, so that even in the case of



after k steps we obtain a clusp. (This requires to show that there are paths of bounded length between triangles going only through quadrangles.) Therefore, letting

$$\widehat{d_n} := \max\{\det(D) : D \text{ is an } n \text{ crossing diagram obtained by splicing crossings in a } l\text{-braid diagram}\}$$

and applying

$$\widehat{d_n} \le \widehat{d}_{n-1} + \widehat{d}_{n-2} + \widehat{d}_{n-3}$$

recursively on each summand on the right, in depth k of the recursion we can in fact use the simpler formula $\widehat{d}_{n'} \le \widehat{d}_{n'-1} + \widehat{d}_{n'-2}$.

Thus if T_n denote the Tribonacci numbers, $\widehat{d_n} \le \tilde{d_n}$ for a linearly recurrent sequence $\tilde{d_n}$ with

$$\tilde{d_n} = \sum_{i=1}^{k} a_i \tilde{d}_{n-i}\,,$$

and $T_n = \sum_{i=1}^{k} a'_i T_{n-i}$ such that $0 \le a_i \le a'_i$ and $a_i < a'_i$ for at least one i. Writing down the generating series of T_n and $\tilde{d_n}$, the denominator polynomials are $f(x) = \sum_{i=1}^{k} a_i x^i - 1$ and $f_1(x) = \sum_{i=1}^{k} a'_i x^i - 1$ resp. On the positive real line, f and f_1 have unique zeros z_f and z_{f_1}, which are the unique zeros of minimal norm for these functions (use $a_i, a'_i \ge 0$ and apply triangle inequality).

Now $z_{f_1}^{-1} = \limsup_{n\to\infty} \sqrt[n]{T_n} = \delta$ and $z_f^{-1} = \limsup_{n\to\infty} \sqrt[n]{\tilde{d_n}}$ show the result because $f_1(x) > f(x)$ for $x > 0$, so that $z_f > z_{f_1}$.

Thus there will be a sequence $\{\delta_l\}$ with $\delta_l < \delta_{l+1} < \delta$ and $\delta_l \to \delta$ such that 'δ' in theorem 3.3.1) can be replaced by 'δ_l' for l-strand braids $\{\beta_i\}$. This would imply the conjecture, under the (again strong and hard to verify) assumption that the answer to question 2.2 is positive.

We conclude by the remark that, by writing out the endomorphisms ϕ_β as matrices, for appropriate β we obtain squareness properties for some linear combinations of entries of such matrices.

Example 4.2 Consider the matrix

$$A = \begin{pmatrix} 1 & 18 & 18 & 24 & 12 \\ 0 & 13 & 0 & 18 & 0 \\ 0 & 0 & 25 & 0 & 18 \\ 0 & 18 & 0 & 25 & 0 \\ 0 & 0 & 18 & 0 & 13 \end{pmatrix}.$$

Then, writing $A^k = \left(a_{i,j}^{(k)}\right)_{i,j=1}^{5}$, we have that $a_{1,4}^{(2k+1)} + a_{1,5}^{(2k+1)}$ is always a square. This follows again from [HK], as A^T represents the endomorphism ϕ_β for $\beta = \sigma_1\sigma_2^{-2}\sigma_1\sigma_2^{-1}\sigma_1^2\sigma_2^{-1}$ in the basis (14) of DS_3. Interestingly again $a_{1,4}^{(2k)} + a_{1,5}^{(2k)}$ is always of the form $10x^2$, although there is no knot-theoretical explanation of this fact.

The last example, together with some further experiments, leads to the following conjecture.

Conjecture 4.2 If $\beta' \in B_3$ is an alternating braid, and $\beta = \beta'\overline{\beta'}$, then ϕ_β Jordan-decomposes over the quadratic number field $\mathbb{Q}(\sqrt{d})$ (or at least over $\mathbb{Q}(\sqrt{d}, i)$), where $d = \det(\widehat{\beta^2})$, and $\sqrt{\det(\widehat{\beta^{2k}})/d} \in \mathbb{Z}$ for all $k > 0$.

As TL_n has an antiautomorphism (turn around by 180°), for $\beta = \beta'\overline{\beta'}$, ϕ_β is conjugate to its inverse, so that the characteristic polynomial $\chi(\phi_\beta)$ of ϕ_β is self-conjugate, i.e., $\chi(\phi_\beta)(x) \doteq \chi(\phi_\beta)(x^{-1})$ (where $\doteq$ denotes equality up to units in $\mathbb{Z}[x, x^{-1}]$). However, $\chi(\phi_\beta)$ turns out to have (at least in all cases calculated in an experiment) some unexpected properties.

For 3-braids the polynomial $\chi(\phi_\beta)$ had the form $(x-1)P(x)^2$ with a quadratic polynomial P, and in fact ϕ_β decomposes into $Id_1 \oplus \phi'_\beta \oplus \phi'_\beta$ (where Id_1 is the 1-dimensional identity map) under a certain, but not plausible, choice of basis.

For 5-braids $\chi(\phi_\beta) = (x-1)^6 P_1(x)^5 P_2(x)^4$ with $P_{1,2}$ being self-conjugate polynomials of degree 4 with alternating coefficients ($[P_i]_{x^j} \cdot [P_i]_{x^{j+1}} < 0$ for $0 \le j < 4$), which additionally seem related, as always $[P_1]_x + [P_2]_{x^2} = +2$. For example, for

$$\beta' = \sigma_3^{-1}\sigma_4\sigma_1^{-1}\sigma_2\sigma_4\sigma_1^{-1}\sigma_3^{-1}\sigma_2\sigma_4\sigma_3^{-1}$$

(and $\beta = \beta'\overline{\beta'}$) we have

$$\chi(\phi_\beta) = (x-1)^6\,(1 - 26\,166x + 2\,297\,755x^2 - 26\,166x^3 + x^4)^5\,(1 - 1533x + 26\,168x^2 - 1533x + x^4)^4\,.$$

It is interesting to see what phenomena occur for more strands, but for 7-braids the dimension of TL_7 is 429, and this renders experiments rather difficult.

These phenomena motivate and merit some further investigations in the future.

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