A SURVEY ON CERTAIN PATTERN PROBLEMS

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Abstract. The paper contains all the definitions and notations needed to understand the results concerning the field dealing with occurrences of patterns in permutations and words. Also, this paper includes a historical overview on the results obtained in this subject. The authors tried to collect all the currently existing references to the papers directly related to the subject. Moreover, a number of basic approaches to study the pattern problems are discussed.

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Contents

1. Introduction

This paper introduces the reader to a new, but rapidly growing, branch of combinatorics, namely counting occurrences of patterns. This topic is very young, with its roots in the works by Rotem, Rogers, and Knuth in the 1970s and early 1980s. However, the first systematic study was not undertaken until the paper by Simion and Schmidt appeared in 1985. Currently there exist more than two hundred papers on this subject.

The notion of counting occurrences of patterns can be illustrated by the following example. Suppose we have a word, say COMBINATORICS, and assumed we have a lexicographic order on the letters in this word (the letter A is the smallest letter, etc.). Occurrences of, say, the pattern 3421 in this word are, for example, COBA and MTIC, since according to the order of the letters, COBA and MTIC are order-isomorphic to 3421, that is, their letters are in the same relative order as those of 3421. On the other hand, the occurrences of the pattern 1231 are COTC, CORC, and IORI.

There are two basic problems in counting occurrences of patterns. They are the avoidance problem and the counting problem. The former deals with finding the number of words with no occurrence of a given pattern, whereas the latter deals with counting words with a prescribed number of occurrences of a given pattern. In the last decade much attention has been paid to the problem of finding the numbers of permutations containing exactly r occurrences of a given pattern. Most of the authors consider only the case $r = 0$ (the pattern avoidance problem). Only a few papers consider the case $r > 0$ (the pattern counting problem), usually restricting themselves to patterns of length three. Pattern avoidance has proved to be a useful language in a variety of seemingly unrelated problems, from theory of Kazhdan-Lusztig polynomials [29], to singularities of Schubert varieties (see, for example, [30], [31], [32], [33], [34], and [132]), to Chebyshev polynomials (see, for example, [160] and references therein), to rook polynomials for a rectangular board (see [155]), to various sorting algorithms, sorting stacks and sortable permutations (see, for example, [41], [43], [49], [211], [213], and [214]).

On one hand, the present paper contains all the definitions and notations, as well as their generalizations, that are needed to understand the results concerning occurrences of patterns in different kinds of permutations, words and multi-sets. On the other hand, it includes a historical overview on the results obtained in this subject. We tried to collect all the currently existing references to the papers directly related to the subject, and we would like to apologize if we missed any contribution to the field. In any case, we would like to mention the papers from Special Volume on Permutation Patterns in Electronic Journal of Combinatorics (**9:2** (2002–2003)) and references therein, as well as paper [160] and the references therein, as a good source of results in the area. In Section 4 we discuss a number of basic approaches that one can use when studying the patterns. Finally, in Section 5 we mention a generalization of the concept of "pattern" discussed in this paper.

2. Patterns

Let Σ be a totally ordered alphabet. We denote the alphabet $\{1, 2, \ldots, k\}$ by $[k]$. A *word* in the alphabet Σ is a finite sequence of letters of the alphabet. The number of letters in the word σ is called the *length* of σ and denoted by $|\sigma|$. Any ℓ consecutive letters of a word σ generate a *subword* of length ℓ. For example, 1323 is a subword of the word 1132331 over the alphabet [3]. A *subsequence* of length ℓ in a word $\sigma = \sigma_1\sigma_2\ldots\sigma_m$ is the word $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_s}$, where $1 \le i_1 < i_2 < \cdots < i_s \le m$. For example, the word 155243245 over the alphabet [5] has the word 1235 as a subsequence. Let $[k]^n$ denote the set of all the words of length n on the alphabet $[k]$. Clearly, the number of words in $[k]^n$ is $|[k]^n| = k^n$.

A *permutation* of a non-empty finite set A is a one-to-one correspondence between A and itself. In this paper, we take A to be $[n]$, and we write permutations of $[n]$ as words $\pi = \pi_1\pi_2\ldots\pi_n$. For example, there are six permutations of the set [3], namely 123, 132, 213, 231, 312, and 321. We denote the set of all permutations of $[n]$ by $\mathcal{S}_n$ (usually called the symmetric group of order n). Clearly, the number of permutations in $\mathcal{S}_n$ is $|\mathcal{S}_n| = n! = 1 \cdot 2 \cdot \ldots \cdot n$. We denote by $\mathcal{S}$ the set of all permutations of all sizes (including the empty permutation ϵ, that is the permutation of length 0), that is $\mathcal{S} = \cup_{n\ge 0}\mathcal{S}_n$.

Definition 2.1. *Let $\sigma \in [k]^n$ and $\tau \in [\ell]^m$ such that $\ell \le k$, $m \le n$, and τ contains all the letters in $[\ell]$. An* occurrence *of τ in σ is a subsequence $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_m}$ of σ such that $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_m}$ is order-isomorphic to $\tau = \tau_1\tau_2\ldots\tau_m$, that is, $\sigma_{i_p} < \sigma_{i_q}$ (resp. $\sigma_{i_p} = \sigma_{i_q}$) if and only if $\tau_p < \tau_q$ (resp. $\tau_p = \tau_q$). In this context, the word τ is called a* pattern.

For example, the word $\sigma = 1242312 \in [4]^7$ contains three occurrences of the pattern $\tau = 1231$, namely $\sigma_1\sigma_2\sigma_3\sigma_6 = 1241$ and $\sigma_1\sigma_2\sigma_5\sigma_6 = \sigma_1\sigma_4\sigma_5\sigma_6 = 1231$.

2.1. Pattern avoidance in permutations.

The *reduced form* of a permutation σ on a set $\{j_1, j_2, \ldots, j_k\}$, where $j_1 < j_2 < \cdots < j_k$, is a permutation σ' obtained by renaming the letters of the permutation σ so that j_i is renamed i for all $i \in \{1, \ldots, k\}$. For example, the reduced form of the permutations 3651 and 2863 are 2431 and 1432, respectively.

Definition 2.2. *For $k \le n$, we say that a permutation $\sigma \in \mathcal{S}_n$ has an occurrence of the pattern $\tau \in \mathcal{S}_k$ if there exist $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that the reduced form of $\sigma(i_1)\sigma(i_2)\ldots\sigma(i_r)$ is τ.*

For example, the permutation $\pi = 624153$ contains three occurrences of the pattern 213, namely $\pi_2\pi_4\pi_5 = 215$, $\pi_2\pi_4\pi_6 = 213$, and $\pi_3\pi_4\pi_5 = 415$.

Remark 2.3. *Definition 2.2 is a particular case of Definition 2.1, when $n = k$ and $\ell = m$.*

We say that a permutation π *avoids* a pattern τ, or is τ-*avoiding*, if there is no occurrence of τ in π. For example, let $\pi = 83176254$, $\tau = 1234$, and $\tau' = 1243$. Then it is easy to see that π avoids τ, and contains exactly one occurrence of τ', that is π does not avoid τ'. The set of all τ-avoiding permutations in $\mathcal{S}_n$ is denoted $\mathcal{S}_n(\tau)$. For any set T of patterns, we define $\mathcal{S}_n(T) = \cap_{\tau\in T}\mathcal{S}_n(\tau)$.

Fundamental questions are to determine $|\mathcal{S}_n(T)|$ viewed as a function of n for given T, if $|\mathcal{S}_n(T)| = |\mathcal{S}_n(T')|$ to find an explicit bijection (a one-to-one correspondence) between $\mathcal{S}_n(T)$ and $\mathcal{S}_n(T')$, and it is interesting to find relations between $\mathcal{S}_n(T)$ and other combinatorial structures. By determining $|\mathcal{S}_n(T)|$ we mean finding an explicit formula, or the *ordinary* or *exponential generating functions*.

Example 2.4. *The case $\tau \in \mathcal{S}_2$ is trivial. Clearly, $\mathcal{S}_n(12) = \{n(n-1)\dots 1\}$ and $\mathcal{S}_n(21) = \{12\dots n\}$, that is $|\mathcal{S}_n(12)| = |\mathcal{S}_n(21)| = 1$.*

The first interesting case is $k = 3$. The first explicit solution seems to be Hammersley's enumeration of $\mathcal{S}_n(321)$ in [104]. In [126, Ch. 2.2.1] and [128, Ch. 5.1.4] Knuth shows that for any $\tau \in \mathcal{S}_3$, we have $|S_n(\tau)| = C_n$, where C_n is the nth Catalan number given by $C_n = \frac{1}{n+1}\binom{2n}{n}$ (see [201, Sequence A000108]). Other authors considered restricted permutations in the 1970s and early 1980s (see, for example, [192], [193], and [194]), but the first systematic study was not undertaken until 1985, when Simion and Schmidt [199] solved the enumeration problem for every subset of S_3. As mentioned before, there exist currently more than two hundred papers on this subject.

Now let us define some symmetry arguments, which we will often use later in the paper. There are three symmetries on permutations: the *reverse*, the *complement*, and the *inverse* operations.

Definition 2.5. *For a permutation π, we define the* reverse $r : \mathcal{S}_n \to \mathcal{S}_n$, *the* complement $c : \mathcal{S}_n \to \mathcal{S}_n$, *and the* inverse $i : \mathcal{S}_n \to \mathcal{S}_n$ *to be the permutation β such that*

$$\begin{array}{lll} \beta = r(\pi) & \text{if and only if} & \beta_j = \pi_{n+1-j} \quad \text{for} \quad 1 \le j \le n, \\ \beta = c(\pi) & \text{if and only if} & \beta_j = n+1-\pi_j \quad \text{for} \quad 1 \le j \le n, \end{array}$$

and i is the usual inverse operation on the symmetric group $\mathcal{S}_n$.

For example, if $\pi = 14253 \in \mathcal{S}_5$ then $r(\pi) = 35241$, $c(\pi) = 52413$, and $i(\pi) = 13524$. We call these operations *trivial* bijections of $\mathcal{S}_n$ to itself. We denote the group generated by the trivial bijections on the symmetric group $\mathcal{S}_n$ by $\mathcal{G}_p$. The following proposition follows from the results by Simion and Schmidt [199].

Proposition 2.6. *$\mathcal{G}_p$ is isomorphic to the dihedral group D_8.*

Proof. It is easy to see that $r^2 = c^2 = (r\cdot c)^2 = 1$, $c\cdot r = r\cdot c$, $i^2 = (r\cdot i)^4 = (c\cdot i)^4 = 1$, and $i\cdot r\cdot i = c$. So, $\mathcal{G}_p$ is isomorphic to D_8. □

More generally, for a set of patterns T, we define $g(T) = \{g(\tau)|\tau \in T\}$ for any $g \in \mathcal{G}_p$. For example, if $T = \{123, 132\}$ and $g = r$ then $g(T) = \{321, 231\}$. The following proposition was given by Simion and Schmidt [199].

Proposition 2.7. (Simion and Schmidt [199]) *Let π be a permutation and T be a set of patterns. Then π avoids T if and only if $g(\pi)$ avoids $g(T)$ for any $g \in \mathcal{G}_p$.*

Definition 2.8. *Given a subset of patterns T. We denote by $\overline{T}$ the set*

$$\{U \mid \exists\, g \in \mathcal{G}_p \text{ such that } T = g(U)\},$$

called the symmetry class *of T. For simplicity, we denote by $\overline{\tau}$ the symmetric class of the set $T = \{\tau\}$.*

For example, if $T_1 = \{123\}$ and $T_2 = \{132, 2134\}$ then $\overline{T_1} = \{123, 321\}$ and

$$\overline{T_2} = \{\{132, 2134\}, \{213, 1243\}, \{231, 4312\}, \{312, 3421\}\}.$$

Some other examples are listed in Table 1.

Symmetry class	*Sets of patterns*
$\overline{\{123\}}$	$\{123\}$, $\{321\}$
$\overline{\{132\}}$	$\{132\}$, $\{213\}$, $\{231\}$, $\{312\}$
$\overline{\{123, 132\}}$	$\{123, 132\}$, $\{321, 231\}$ $\{123, 213\}$, $\{321, 312\}$
$\overline{\{123, 231\}}$	$\{123, 231\}$, $\{321, 132\}$ $\{123, 312\}$, $\{321, 213\}$
$\overline{\{123, 321\}}$	$\{123, 321\}$
$\overline{\{132, 213\}}$	$\{132, 213\}$, $\{231, 312\}$
$\overline{\{132, 231\}}$	$\{132, 231\}$, $\{132, 312\}$ $\{231, 213\}$, $\{213, 312\}$

Table 1. Examples of symmetry classes

Definition 2.9. *Two sets of patterns T_1 and T_2 are said to be* Wilf-equivalent*, or in the same* Wilf class*, if $|\mathcal{S}_n(T_1)| = |\mathcal{S}_n(T_2)|$ for all $n \geq 0$.*

For instance, using Example 2.4, it is easy to see that there is only one Wilf class for the patterns from $\mathcal{S}_2$. The first interesting case examined was the case of permutations avoiding a pattern from $\mathcal{S}_3$ (see [126, Ch. 2.2.1] and [128, Ch. 5.1.4]). This result was obtained by using Schensted's correspondence between permutations and *Young tableaux*, as well as MacMahon's earlier result on Young tableaux.

Theorem 2.10. (**Knuth**, [126, Ch. 2.2.1] and [128, Ch. 5.1.4]) *For any $\tau \in \mathcal{S}_3$,*

$$|\mathcal{S}_n(\tau)| = C_n,$$

where C_n is the nth Catalan number.

Set from a Wilf class	*Cardinality of the set*
123	C_n
$\{123, 132\}$	2^{n-1}
$\{123, 231\}$	$\binom{n}{2} + 1$
$\{123, 321\}$	0, if $n \geq 5$
$\{123, 132, 213\}$	F_{n+1}
$\{123, 132, 231\}$	n
$\{123, 321, 213\}$	0, if $n \geq 5$
$\|T\| = 4, 5, T \supset \{123, 321\}$	0, if $n \geq 5$
$\|T\| = 4, 5, T \not\supset \{123, 321\}$	2, if $n \geq 4$
$\mathcal{S}_3$	0, if $n \geq 3$

TABLE 2. Wilf classes for subsets of $\mathcal{S}_3$

Later, Simion and Schmidt [199] found the cardinalities of $\mathcal{S}_n(T)$ for any $T \subseteq \mathcal{S}_3$ (see Table 2). In Table 2, C_n is the nth Catalan number and F_n is the nth Fibonacci number (see [201, Sequence A000045]).

The cardinalities $|\mathcal{S}_n(T)|$ become much more difficult to determine, even for $T \subset \mathcal{S}_4$. This is especially true when $|T|$ is much more less than $k!$, in particular, for $|T| = 1$. In this case, we are interested in determining the Wilf classes. Stankova and West [208] classified the Wilf classes in $\mathcal{S}_n$, where $n = 1, 2, \ldots, 7$ (see Table 3).

Classes	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$	$\mathcal{S}_4$	$\mathcal{S}_5$	$\mathcal{S}_6$	$\mathcal{S}_7$
Symmetry	1	1	2	7	23	115	694
Wilf	1	1	1	3	16	91	595

TABLE 3. Symmetry and Wilf classes in $\mathcal{S}_n$ for $n \leq 7$

In cases when one does not succeed in finding $|\mathcal{S}_n(T)|$, other questions arise. For example, what is the asymptotic behavior for $|\mathcal{S}_n(T)|$, and in particular, for $|\mathcal{S}_n(\tau)|$, where $\tau \in \mathcal{S}_k$. In the case $k = 3$, one can find the asymptotic behavior from Theorem 2.10. In almost all cases of $k = 4$, the asymptotic behavior was given by Bóna (see [36] and [37]). In the case $\tau = 12 \ldots k$, Regev [175] gave a complete answer for the asymptotics (see also Gessel [91]).

Another related question is the existence of a constant c such that $|\mathcal{S}_n(T)| < c^n$.

Conjecture 2.11. (**Stanley, Wilf**) *For any pattern $\tau \in S_\ell$, the limit*

$$\lim_{n\to\infty} |S_n(\tau)|^{\frac{1}{n}},$$

exists and is finite.

In the case $k = 3$, the conjecture follows from Theorem 2.10. In almost all cases of $k = 4$, the conjuncture follows from the results by Bóna (see [36] and [37]). Using the results of Regev [175] and Gessel [91], the conjecture holds for $\tau = 12\dots k$ for any $k \geq 1$. Bóna proved the conjecture of Stanley and Wilf for *layered patterns*. A layered pattern is a pattern which can be partitioned into layers so that the elements are increasing within layers and decreasing between layers (for a formal definition see Section 4). For example, in $\mathcal{S}_4$, the layered patterns are 1234 (= (1234)) and 3421 (= (34, 2, 1)). The layers are the subsequences between two adjacent commas. Alon and Friedgut [5] proved the following weaker statement for the conjecture: there exists a constant $c = c(\tau)$ such that $|\mathcal{S}_n(\tau)| \leq c^{n\gamma^\star(n)}$, where $\gamma^\star$ is an extremely slow growing function, related to the Ackermann hierarchy. The proof converts this problem to one about Davenport-Schinzel sequences [123], about words that avoid patterns of equalities.

We refer to [219] for a more detailed overview on the progress with regard to Conjecture 2.11. In particular, the following result by Arratia [8] is mentioned there:

$$\lim_{n\to\infty} |\mathcal{S}_n(\tau)|^{1/n} = \sup_n |\mathcal{S}_n(\tau)|^{1/n}.$$

The Stanley-Wilf conjecture was finally proved by Marcus and Tardos [162] in 2003. To give some details from that paper we need to define the concept of avoidance of a matrix by another matrix.

Definition 2.12. *Let A and P be 0-1 matrices. We say that A contains the $k \times \ell$ matrix $P = (p_{i,j})$ if there exists a $k \times \ell$ submatrix $B = (b_{ij})$ of A with $b_{ij} = 1$ whenever $p_{ij} = 1$. Otherwise we say that A avoids P.*

For a 0-1 matrix Füredi and Hajnal [88] defined $f(n, P)$ to be the maximum number of 1 entries in an $n \times n$ 0-1 matrix avoiding P, and stated the following conjecture.

Conjecture 2.13. (**Füredi, Hajnal**) *For all permutation matrices P we have $f(n, P) = O(n)$.*

By a result of Klazar [124] the truth of Conjecture 2.13 implies the truth of Conjecture 2.11. Conjecture 2.13 was proven in [162], which settled Conjecture 2.11.

In the cases when the formula for $|\mathcal{S}_n(T)|$ is unknown, and the asymptotics are difficult to find, one can ask another question: Is $|\mathcal{S}_n(T)|$ *P-recursive*? A function $f : \mathbb{N} \to \mathbb{C}$ is called *P-recursive* if there exist polynomials $P_0, P_1, \dots, P_k \in \mathbb{C}[n]$, so that

$$P_k(n)f(n+k) + P_{k-1}(n)f(n+k-1) + \ldots + P_0(n)f(n) = 0$$

for all $n \geq 1$ (see [36, 37, 169]). Table 4 shows the present state of research on permutations avoiding given patterns of length 4.

pattern τ	$s_n(\tau) < c^n$	**formula for** $s_n(\tau)$	P**-recursive**
1234	Regev [175]	Gessel [91]	Zeilberger [221]
1342	Bóna [36]	Bóna [37]	Bóna [37]
1324	Bóna [36]	open	open

Table 4. Patterns from $\mathcal{S}_4$

2.2. **Counting occurrences of patterns in permutations.** We denote by $s_\tau^r(n)$ the number of permutations in $\mathcal{S}_n$ that contain exactly r occurrences of the pattern τ. For example, if $n = 3$, $\tau = 12$, and $r = 1$ then there are two such permutations, namely 231 and 312, that is $s_\tau^r(n) = 2$.

In the last decade much attention has been paid to the problem of finding the numbers $s_\tau^r(n)$ for a fixed $r \geq 0$ and a given pattern τ (see [35, 38, 159, 169, 168, 187]). Most of the authors consider only the case $r = 0$, thus studying permutations avoiding a given pattern (see Subsection 2.1). Only a few papers consider the case $r > 0$, usually restricting themselves to patterns of length three. Using the trivial bijections (the reverse and the complement) on $\mathcal{S}_n$ it is immediate that with respect to being equidistributed, the six patterns of length three fall into the two classes $\{123, 321\}$ and $\{132, 213, 231, 312\}$ (see Table 1).

Noonan [168] proved that $s_{123}^1(n) = \frac{3}{n}\binom{2n}{n-3}$. A general approach to this problem was suggested by Noonan and Zeilberger [169]. They gave another proof of Noonan's result, and conjectured that

$$s_{123}^2(n) = \frac{59n^2 + 117n + 100}{2n(2n-1)(n+5)}\binom{2n}{n-4}$$

and $s_{132}^1(n) = \binom{2n-3}{n-3}$. The first conjecture was proved by Fulmek [86], and the second one by Bóna [38]. Also, Noonan and Zeilberger conjectured the following.

Conjecture 2.14. *For any pattern* τ *and* $r \geq 0$, $s_\tau^r(n)$ *is* P*-recursive in* n.

We observe that Conjecture 2.14 yields Conjecture 2.11. For the pattern $\tau = 132$, conjecture 2.14 was proved by Bóna [35]. Mansour and Vainshtein [159] suggested a new approach to this problem in the case $\tau = 132$, which allows us to get an explicit expression for $s_{132}^r(n)$ for any given r.

This problem can be generalized as follows. Let $\mathbf{r}$ be a vector of occurrences, namely $\mathbf{r} = (r_1, r_2, \ldots, r_m)$, and let L be a list of patterns, namely $L = \{\tau^1, \tau^2, \ldots, \tau^m\}$. We denote by $s_L^{\mathbf{r}}(n)$ the number of permutations in $\mathcal{S}_n$ that contain exactly r_i occurrences of the pattern τ^i, for $i = 1, 2, \ldots, m$. Noonan and Zeilbereger [169] have made the following general conjecture.

Conjecture 2.15. (**Noonan, Zeilberger**, [169]) *For any fixed vector of occurrences* $\mathbf{r} = (r_1, r_2, \dots, r_m)$ *and list of patterns* $L = \{\tau^1, \tau^2, \dots, \tau^m\}$, $s_L^{\mathbf{r}}(n)$ *is P-recursive in n.*

This conjecture holds for many cases (see, for example, [50, 60, 108, 130, 156, 157, 188, 187, 189]). Most of the authors considered the case $m = 2$. More precisely, they considered the case of a vector $\mathbf{r} = (r_1, r_2)$, where $r_1 = 0, 1$ and $r_2 \geq 0$, and $L = \{\tau^1, \tau^2\}$, where $\tau^1 \in \mathcal{S}_3$ and $\tau^2 \in \mathcal{S}_k$ (for the case $k = 3$ see Table 5). For example, Chow and West [60] found the generating function for the number of permutations in $\mathcal{S}_n$ that avoid the patterns 132 and $12\dots k$, which is given in terms of Chebyshev polynomials of the second kind. This result was extended to many different cases. Robertson [187] found the number of permutations in $\mathcal{S}_n$ that avoid the pattern 132 and have exactly one occurrence of the pattern 123 and the number of permutations in $\mathcal{S}_n$ that avoid the pattern 123 and have exactly one occurrence of the pattern 132. These results were generalized by Robertson, Wilf and Zeilberger [189]. They found the number of permutations in $\mathcal{S}_n$ that avoid the pattern 132 and contain exactly r occurrences of the pattern $\tau = 123$. This result was extended by Mansour and Vainshtein [156], by Krattenthaler [130], by Jani and Rieper [108], and by Brändén, Claesson and Steingrímsson [50] to the case of permutations containing the pattern $12\dots k$ exactly r times and avoiding the pattern 132. In [157], the above conjecture was proved for the case $L = \{132, \tau\}$ and $\mathbf{r} = (0, 0)$, where $\tau \in \mathcal{S}_k$.

Set L	Vector $\mathbf{r}$	Cardinality $s_L^{\mathbf{r}}(n)$
$\{123, 321\}$	$(0, 1)$	0 for $n \geq 6$
$\{123, 132\}$	$(0, 1)$	$(n-2)2^{n-3}$ for $n \geq 3$
$\{123, 231\}$	$(0, 1)$	$2n - 5$ for $n \geq 3$
$\{132, 213\}$	$(0, 1)$	$n2^{n-5}$ for $n \geq 4$
$\{132, 231\}$	$(0, 1)$	2^{n-3} for $n \geq 3$
$\{123, 321\}$	$(1, 1)$	0 for $n \geq 6$
$\{123, 132\}$	$(1, 1)$	$\frac{1}{2}(n-3)(n-4)2^{n-4}$ for $n \geq 5$
$\{123, 231\}$	$(1, 1)$	$2n - 5$ for $n \geq 5$
$\{132, 213\}$	$(1, 1)$	$(n^2 + 21n - 28)2^{n-9}$ for $n \geq 7$
$\{132, 231\}$	$(1, 1)$	2^{n-3} for $n \geq 4$

TABLE 5. Cardinalities $s_L^r(n)$ for $L \subset \mathcal{S}_3$, $|L| = 2$, $\mathbf{r} = (0, 1), (1, 1)$

One can prove the following proposition.

Proposition 2.16. *The truth of Conjecture 2.15 implies the truth of Conjecture 2.11.*

2.3. **Patterns in signed permutations.** Signed pattern avoidance has proven to be a useful language in combinatorial statistics defined in type-B noncrossing partitions, enumerative combinatorics, algebraic combinatorics, geometric combinatorics and singularities of Schubert varieties. (see, for example, [29, 32, 164, 81, 45, 198, 180]). In this section we extend the avoidance problem from the symmetric group to the *hyperoctahedral group*.

We will regard the elements of the hyperoctahedral group B_n as signed permutations written as $\pi = \pi_1\pi_2\dots\pi_n$ in which each of the symbols $1, 2, \dots, n$ appears, possibly barred. Clearly, the cardinality of B_n is $2^n n!$. For example, B_2 contains 8 elements, namely 12, $1\overline{2}$, $\overline{1}2$, $\overline{12}$, 21, $2\overline{1}$, $\overline{2}1$, and $\overline{21}$. We define $|\pi_i|$ to be π_i if the symbol π_i is not barred, otherwise $|\pi_i|$ is $\overline{\pi_i}$, where we assume $\overline{\overline{\pi_i}} = \pi_i$. Moreover, for $\pi = \pi_1\pi_2\dots\pi_n$ we define $|\pi| = |\pi_1||\pi_2|\dots|\pi_n|$. For example, if $\pi = 1\overline{2}$ then $|\pi| = 12$.

Now let $\tau \in B_k$, and $\pi \in B_n$; we say that π *contains* a signed pattern τ or is a τ*-containing* signed pattern, if there is a sequence of k indices, $1 \le i_1 < i_2 < \dots < i_k \le n$ such that two conditions hold:

(1) $|\pi_{i_p}| > |\pi_{i_q}|$ if and only if $|\tau_p| > |\tau_q|$ for all $k \ge p > q \ge 1$;

(2) π_{i_j} is barred if and only if τ_j is barred for all $1 \le j \le k$.

For example, $\pi = 21\overline{34} \in B_4$ contains the signed patterns $\overline{12}$ and 21 but does not contain the patterns 12 and $2\overline{1}$. If π does not contain a signed pattern τ, then we say that π *avoids* τ, or is a τ*-avoiding*, and in this context τ is called a *signed pattern*. We denote by $B_n(\tau)$ the set of all the τ-avoiding signed permutations in B_n. More generally, we define $B_n(T) = \cap_{\tau \in T} B_n(\tau)$. We denote by $b_T(n)$ the cardinality of $B_n(T)$.

Example 2.17. *The case $\tau \in B_1$ is trivial. Clearly,*

$$B_n(1) = \{\overline{\pi}_1\overline{\pi}_2\dots\overline{\pi}_n | \pi = \pi_1\pi_2\dots\pi_n \in \mathcal{S}_n\} \text{ and } B_n(\overline{1}) = \mathcal{S}_n,$$

that is $b_1(n) = b_{\overline{1}}(n) = n!$.

As we mentioned above, in the symmetric group $\mathcal{S}_n$, for every 2-letter pattern τ, the number of τ-avoiding permutations is 1, and for every pattern $\tau \in \mathcal{S}_3$, the number of τ-avoiding permutations is given by the Catalan numbers (see Theorem 2.10). Simion [198, Section 3] found out that there are similar results for the hyperoctahedral group B_n. More precisely, she proved that for every 2-letter signed pattern τ the number of τ-avoiding signed permutations in B_n is given by $\sum_{j=0}^{n} \binom{n}{j}^2 j!$. Mansour and West [161] found all the cardinalities $b_T(n)$, where $T \subseteq B_2$. (The exhaustive treatment of cases was suggested by the influential paper of Simion and Schmidt [199], which followed a similar program for the cardinalities $s_T(n)$ where $T \subseteq \mathcal{S}_3$. For example, in the case $|T| = 2$ see Table 6).

Now let us define some symmetry arguments on B_n, which will be used later in this paper. There are three symmetries on signed permutations: the *baring*, the *reverse*, and the *complement* operations.

Definition 2.18. *We define three simple operations on signed permutations: the reverse (i.e., reading the permutation right-to-left: $r : \pi_1\pi_2\cdots\pi_n \mapsto \pi_n\pi_{n-1}\cdots\pi_1$), the barring*

Set of signed patterns T	Cardinality $b_T(n)$
$\{12,21\}$, $\{12,1\overline{2}\}$ $\{2\overline{1},1\overline{2}\}$, $\{2\overline{1},\overline{1}2\}$	$(n+1)!$
$\{12,\overline{12}\}$, $\{12,\overline{21}\}$	$\binom{2n}{n}$
$\{12,\overline{2}1\}$	$n!+n!\sum\limits_{i=1}^{n}\left(\frac{1}{i}\sum\limits_{j=0}^{i-1}\frac{1}{j!}\right)$
$\{1\overline{2},\overline{1}2\}$	$2\sum\limits_{l=1}^{n}\sum\limits_{i_1+i_2+\cdots+i_l=n,\ i_j\geq 1}\prod\limits_{j=1}^{l} i_j!$

TABLE 6. Cardinalities $b_T(n)$, where $T \in B_2$, $|T| = 2$.

(i.e., bar : $\pi_1\pi_2\cdots\pi_n \mapsto \phi_1\phi_2\cdots\phi_n$ where $\phi_j = \overline{\pi_j}$ if π_j is not barred, otherwise $\phi_j = \pi_j$) and the complement (i.e., $c : \pi_1\pi_2\cdots\pi_n \mapsto \beta_1\beta_2\cdots\beta_n$ where $\beta_i = n+1-\pi_i$ if π_i is not barred, otherwise $\beta_i = \overline{n+1-|\pi_i|}$) on B_n.

For example, if $\pi = \overline{1}4\overline{2}53 \in B_5$ then $r(\pi) = 35\overline{2}4\overline{1}$, $c(\pi) = \overline{5}2\overline{4}13$, and $bar(\pi) = 1\overline{4}2\overline{53}$. We call these operations *trivial* bijections of B_n to itself. We denote the group generated by the trivial bijections on the hyperoctahedral group B_n by $\mathcal{H}_p$. More generally, for a set of patterns T, we define $g(T) = \{g(\tau)|\tau \in T\}$ for any $g \in \mathcal{H}_p$. For example, if $T = \{\overline{1}23, 13\overline{2}\}$ and $g = r$ then $g(T) = \{32\overline{1}, \overline{2}31\}$. The following proposition was given by Simion [198].

Proposition 2.19. (Simion [198]) *Let π be a permutation and T be a set of patterns. Then π avoids T if and only if $g(\pi)$ avoids $g(T)$ for any $g \in \mathcal{H}_p$.*

Similarly to Subsection 2.1 one can define the symmetry classes and the Wilf classes on signed permutations. Likewise, one can identify classes that avoid signed permutations in the hyperoctahedral group (the natural analogue of the symmetric group) with enumerative properties analogous to those classes avoiding permutations in the symmetric group see Simion [198]).

2.4. **Patterns in coloured permutations.** The goal of this subsection is to give analogies of the enumerative results on certain classes of permutations characterized by pattern-avoidance in the symmetric group and in the hyperoctahedral group (see Subsections 2.1 and 2.3) for coloured patterns, which were introduced by Mansour[139].

We define the group $\mathcal{S}_n^{(r)} = \mathcal{S}_n \wr C_r$ (the wreath product of the cyclic group of order r, C_r, and $\mathcal{S}_n$), which plays the role of the symmetric group $\mathcal{S}_n$ and of the hyperoctahedral group B_n, respectively. We will view the elements of the set $\mathcal{S}_n^{(r)}$ as coloured permutations $\pi = \pi_1\pi_2\ldots\pi_n$ in which each of the symbols $1,2,\ldots,n$ appears once, and is coloured by one of the colours $1,2,\ldots,r$ (more generally, we denote by $\mathcal{S}_{\{a_1,\ldots,a_n\}}^{\{c_1,\ldots,c_r\}}$ the set of all permutations of the symbols $a_1,\ldots,a_n$, where each symbol appears once and is coloured by one of the colours $c_1, \dots, c_r$). Thus, $\mathcal{S}_n^{(1)}$ is identified as $\mathcal{S}_n$, $\mathcal{S}_n^{(2)}$ is identified as B_n, and the cardinality of $\mathcal{S}_n^{(r)}$ is $n!r^n$. We define $|\pi|$ as the permutation $|\pi_1||\pi_2|\dots|\pi_n|$, where $|\pi_j|$ is the symbol which appears in π at position j. For example, $\pi = 1^{(1)}3^{(2)}2^{(1)}$ is a coloured permutation in $\mathcal{S}_3^{(2)}$ and $|\pi| = 132$.

Let $\hat\tau = \tau_1^{(s_1)}\tau_2^{(s_2)}\dots\tau_k^{(s_k)} \in \mathcal{S}_k^{(r)}$ and $\hat\pi = \pi_1^{(v_1)}\pi_2^{(v_2)}\dots\pi_n^{(v_n)} \in \mathcal{S}_n^{(r)}$; we say that $\hat\pi$ *contains* $\hat\tau$, or is $\hat\tau$-*containing*, if there is a sequence of k indices $1 \le i_1 < i_2 < \dots < i_k \le n$ such that the following two conditions hold:

(1) $\pi_{i_p} > \pi_{i_q}$ if and only if $\tau_p > \tau_q$ for all $k \ge p > q \ge 1$;

(2) $v_{i_j} = s_j$ for all $j = 1, 2, \dots, k$.

Otherwise, we say that $\hat\pi$ *avoids* $\hat\tau$, or is $\hat\tau$-avoiding. The set of all $\hat\tau$-avoiding coloured permutations in $\mathcal{S}_n^{(r)}$ is denoted by $\mathcal{S}_n^{(r)}(\hat\tau)$, and in this context $\hat\tau$ is called a *coloured pattern*. For an arbitrary finite collection of coloured patterns T, we say that $\hat\pi$ avoids T if $\hat\pi$ avoids any $\hat\tau \in T$; the corresponding subset of $\mathcal{S}_n^{(r)}$ is denoted by $\mathcal{S}_n^{(r)}(T)$. As an example, $\hat\pi = 3^{(1)}2^{(2)}1^{(2)} \in \mathcal{S}_3^{(2)}$ avoids $2^{(1)}1^{(1)}$, that is, $\hat\pi \in \mathcal{S}_3^{(2)}(2^{(1)}1^{(1)})$.

Similarly to Subsection 2.1 one can define the trivial bijections, the symmetric classes, and the Wilf classes on coloured permutations.

Coloured pattern $\hat\tau$	Cardinality $\lvert\mathcal{S}_n^{(r)}(\hat\tau)\rvert$
$1^{(1)}2^{(1)}$, $1^{(1)}2^{(2)}$	$\sum_{j=0}^{n} j!(r-1)^j \binom{n}{j}^2$

TABLE 7. Cardinalities $|\mathcal{S}_n^{(r)}(\hat\tau)|$, where $\hat\tau \in \mathcal{S}_2^{(r)}$

As we mentioned above, in the symmetric group $\mathcal{S}_n$, for every 2-letter pattern τ, the number of τ-avoiding permutations is one, and for every pattern $\tau \in \mathcal{S}_3$, the number of τ-avoiding permutations is given by the Catalan numbers (see Theorem 2.10). Also, Simion [198] found out that there are similar results for the hyperoctahedral group B_n.

Mansour [139] proved that for every $\hat\tau \in \mathcal{S}_2^{(r)}$, the number of $\hat\tau$-avoiding coloured permutations in $\mathcal{S}_n^{(r)}$ is $\sum_{j=0}^{n} j!(r-1)^j \binom{n}{j}^2$ (see Table 7). He also proved that if two distinct elements $\hat\tau$ and $\hat\theta$ from $\mathcal{S}_2^{(r)}$ are to be avoided simultaneously, then the number of Wilf classes is one for $r = 1$, four for $r = 2$, and six for $r \ge 3$.

2.5. **Patterns in words.** We recall the definition of patterns in words in Definition 2.1. Also, we denote by $[k]^n(\tau)$ the set of all τ-avoiding words. More generally, we define by $[k]^n(T) = \cap_{\tau \in T} [k]^n(\tau)$.

Up to now, most research on restricted permutations dealt with cases where τ and σ are permutations. Some papers dealt with cases where only τ is a permutation (σ is a word). For example, Albert et al. [3] considered *permutations of a multiset* which avoid certain patterns of length 3. Burstein [52] gave a complete description, in the manner of Simion and Schmidt [199], for the case of words avoiding permutation patterns of length 3. Regev [176] found the asymptotics for the number of words avoiding the pattern $k(k-1)\dots 1$.

Word pattern τ	Generating function $\sum_{k,n\geq 0} \lvert[k]^n(\tau)\rvert x^n y^k$
111	$\frac{1}{1-y\left(1+x+\frac{1}{2}x^2\right)}$
112, 121	$\frac{1}{1-y}\left(\frac{1-y}{1-y-xy}\right)^{1/y}$
123, 132	$1+\frac{y}{1-x}+\dfrac{2y^2}{(1-2x)(1-y)+\sqrt{((1-2x)^2-y)(1-y)}}$

TABLE 8. Wilf classes for words avoiding a 3-letter word

Similarly to Subsection 2.1 one can define the trivial bijections, the symmetric classes, and the Wilf classes on words. There are two symmetry classes of 2-letter patterns with representatives 11 and 12. To avoid 11 means that there are no repeated letters, so the number of words in $[k]^n(11)$ is given by $\binom{k}{n}n!$. A 12-avoiding word is just a non-increasing string, so the number of words in $[k]^n(12)$ is given by $\binom{n+k-1}{n}$. Burstein and Mansour [53] studied the case of subword patterns (patterns where the letters have to be adjacent) in words. For example, they gave an explicit expression in terms of generating functions for the number of words avoiding a 3-letter word with repeated letters (see Table 8).

2.6. **Patterns in other sets.** In this subsection we describe some analogies for occurrences of patterns problem. One can generalize the counting occurrences of patterns as follows.

Definition 2.20. *Let $\mathcal{A}$ be a totally ordered alphabet with linear order $<_{\mathcal{A}}$. An* element *Ξ in the alphabet $\mathcal{A}$ is a finite sequence of letters of the alphabet. The number of letters in the element Ξ is called the* length *of Ξ and denoted by $\ell(\Xi)$. Let $\mathcal{B}=\cup_{n\geq 0}\mathcal{B}_n$, where $\mathcal{B}_n$ is a finite set of elements of length n.*

Let $\tau=\tau_1\tau_2\dots\tau_k$ be an element in the alphabet $\mathcal{A}$ and $\Xi=\Xi_1\Xi_2\dots\Xi_\ell$ be any element in $\mathcal{B}_\ell$. An occurrence *of τ in Ξ is a subsequence $\Xi_{i_1}\Xi_{i_2}\dots\Xi_{i_k}$ of Ξ such that $\Xi_{i_1}\Xi_{i_2}\dots\Xi_{i_k}$ is order- isomorphic to τ, that is, $\Xi_{i_p}<_{\mathcal{A}}\Xi_{i_q}$ if and only if $\tau_p<_{\mathcal{A}}\tau_q$ for all $1\leq p<q\leq k$. In this context, the word τ is called a* pattern.

Particular cases of Definition 2.20 are Definitions 2.1 and 2.2. For example, if $\mathcal{A}$ is the set of all natural numbers $\mathbb{N}$, and $\mathcal{B}_n=\mathcal{S}_n$, we get Definition 2.2. Other particular cases of Definition 2.20 which are discussed by some authors are considered in the following subsubsections.

2.6.1. *Even and odd permutations.* The number of *inversions* of π is given by

$$\mathrm{Inv}(\pi) = |\{(i,j) : \pi_i > \pi_j, i < j\}|.$$

In other words, $\mathrm{Inv}(\pi)$ is equal to the number of occurrences of the pattern 21 in a permutation π. For example, if $\pi = 2431$ then π has inversions $(1,4)$, $(2,3)$, $(2,4)$, and $(3,4)$, that is, $\mathrm{Inv}(\pi) = 4$. The *signature* of π is given by $\mathrm{sign}(\pi) = (-1)^{\mathrm{Inv}(\pi)}$. In the example above, $\mathrm{sign}(\pi) = 1$.

A permutation π is said to be an *even permutation* (resp. *an odd permutation*) if $\mathrm{sign}(\pi) = 1$ (resp. $\mathrm{sign}(\pi) = -1$). We denote by E_n (resp. O_n) the set of all even (resp. odd) permutations in $\mathcal{S}_n$. Clearly, $|E_n| = |O_n| = \frac{1}{2}n!$ for all $n \geq 2$.

Simion and Schmidt [199] found the number of even and odd permutations avoiding a 3-letter pattern from $\mathcal{S}_3$. More precisely, they proved

$$e^0_{132}(n) = \tfrac{1}{2(n+1)}\tbinom{2n}{n} + \tfrac{1}{n+1}\tbinom{n-1}{(n-1)/2},$$

$$e^0_{123}(n) = \tfrac{1}{2(n+1)}\tbinom{2n}{n} + \tfrac{(-1)^{\binom{n}{2}}}{n+1}\tbinom{n-1}{(n-1)/2}$$

$$e^0_{231}(n) = \tfrac{1}{2(n+1)}\tbinom{2n}{n} + \tfrac{(-1)^{\binom{n}{2}}}{n+1}\tbinom{n-1}{(n-1)/2},$$

where $e^r_\tau(n)$ is the number of even permutations containing the pattern τ exactly r times.

Mansour [145] considered the case of counting the occurrences of the pattern 132 in even permutations. He presented an algorithm that computes the generating function for $e^r_{132}(n)$ for any $r > 0$. To get the result for a given r, the algorithm performs certain routine checks for each element of the symmetric group $\mathcal{S}_{2r}$. The algorithm has been implemented in C, and yields explicit results for $0 \leq r \leq 6$. For example, the generating function for the number of even permutations in $\mathcal{S}_n$ containing the pattern 132 exactly once is given by

$$-\frac{1}{2}(1-2x-x^2) + \frac{1-3x}{4}(1-4x)^{-1/2} + \frac{1-3x-4x^2+4x^3}{4}(1-4x^2)^{-1/2}.$$

2.6.2. *Involutions.* An *involution* π is a permutation such that $\pi = \pi^{-1}$. Let $\mathcal{I}_n$ denote the set of all the involutions in $\mathcal{S}_n$. We denote by $\mathcal{I}$ the set of all the involutions of all the sizes including the empty involution, that is, $\mathcal{I} = \cup_{n\geq 0}\mathcal{I}_n$.

Several authors gave enumerations of sets of involutions which avoid certain patterns. In [175] Regev provided an asymptotic formula for $|\mathcal{I}_n(12\ldots k)|$ and showed that $\mathcal{I}_n(1234)$ is enumerated by the nth *Motzkin number* $\sum_{i=0}^{[n/2]}\binom{n}{2i}C_i$ (see [201, Sequence A001006]). The general case, which is enumeration of $\mathcal{I}_n(12\ldots k)$, was studied by Gessel [91]. In the cases $k = 5, 6$, Gouyou–Beauchamps [28] gave exact formulas, which are

$$\mathcal{I}_n(12345) = C_{[(n+1)/2]}C_{[(n+2)/2]},$$

and

$$\mathcal{I}_n(123456) = \sum_{i=0}^{[n/2]} \frac{6n!(2i+2)!}{i!(i+1)!(i+2)!(i+3)!(n-2i)!}.$$

On the other hand, Simion and Schmidt [199] obtained explicit formulas for $|\mathcal{I}_n(\tau)|$, where $\tau \in \mathcal{S}_3$ (see Table 9).

Pattern τ	Cardinality $|\mathcal{I}_n(\tau)|$
123, 132, 213, 321	$\binom{n}{[n/2]}$
231, 312	2^{n-1}

Table 9. Involutions avoiding a 3-letter pattern

Pattern-avoiding involutions have also been linked with other combinatorial structures. Gire [94] established a one-to-one correspondence between 1-2 *trees* having n edges and $\mathcal{S}_n(321, 3\bar{1}42)$ (which is the set of permutations in $\mathcal{S}_n$ avoiding the patterns 321 and 231, except that the latter is allowed when it is a subsequence of the pattern 3142). More generally, Guibert [98] gave bijections between 1-2 trees having n edges and each of the sets $\mathcal{S}_n(231, 4\bar{1}32)$, $\mathcal{I}_n(3412)$, and $\mathcal{I}_n(4321)$ (and therefore with $\mathcal{I}_n(1234)$, by transposing the corresponding Young tableaux obtained by applying the Robinson-Schensted algorithm). In addition, Guibert [98] established a bijection between $\mathcal{I}_n(2143)$ (these involutions are sometimes called *vexillary involutions*) and $\mathcal{I}_n(1243)$. Also, Guibert, Pergola, and Pinzani [102] gave a one-to-one correspondence between 1-2 trees having n edges and vexillary involutions in $\mathcal{I}_n$. Thus, all these sets are enumerated by the nth Motzkin number. A remaining open problem is to prove the conjecture of Guibert (see [98]) that $\mathcal{I}_n(1432)$ is also enumerated by the nth Motzkin number.

Guibert and Mansour [100, 101] found a general approach to studying involutions, even involutions, and odd involutions avoiding 132 (or containing 132 exactly once), and avoiding (or containing exactly once) an arbitrary pattern $\tau \in \mathcal{S}_k$. They established a bijection between 132-avoiding involutions and primitive *Dyck words*. They extended this bijection to *bilateral words*. This bijection allows to determine more parameters, in particular, to consider the number of inversions (ocurrences of the pattern 21) and *rises* (occurrences of the pattern 12) of the involutions onto bilateral words, and to consider the case of even (odd) involutions and statistics of some generalized patterns. For example, the generating function for the number of involutions in $\mathcal{I}_n(132, 12\dots k)$ is given by

$$\frac{1}{xU_k\left(\frac{1}{2x}\right)}\sum_{j=0}^{k-1} U_j\left(\frac{1}{2x}\right),$$

where U_m is the mth Chebyshev polynomial of the second kind (see Section 4 for a definition).

Egge and Mansour [70] enumerated various sets of involutions that avoid the pattern 231 or contain it exactly once. Interestingly, many of these enumerations can be given in terms of k-generalized Fibonacci numbers. In particular, they found that the generating function for the number of involutions which contain the pattern 231 exactly once and contain the pattern $k(k-1)\dots 1$ exactly r times is given by

$$\frac{(r+1)x^{kr+4}}{(1-x-x^2-\dots-x^{k-1})^{r+2}}.$$

2.6.3. *Multisets.* A *permutation of a multiset* $1^{a_1}2^{a_2}\dots k^{a_k}$ is a sequence of length $a_1+a_2+\dots+a_k$ which contains a_i occurrences of the letter i for each $1\leq i\leq k$. For example, there are three permutations of the multiset $1^2 2^1$, namely 112, 121, and 211. We observe that the set of permutations of the multiset $1^{a_1}2^{a_2}\dots k^{a_k}$ is all the words $\sigma\in[k]^{a_1+a_2+\dots+a_k}$ such that σ contains exactly a_i occurrences of the letter i. Thus, one can define an occurrence of a pattern in a permutation of a multiset in the same manner as we did before (see Subsection 2.5).

Similarly to the paper by Simion and Schmidt [199], Albert et. al. in [3] considered the number of permutations of $1^{a_1}2^{a_2}\dots k^{a_k}$ which avoid a set of patterns $T\subset\mathcal{S}_3$. This number is denoted by $s_T(a_1,a_2,\dots,a_k)$. For the case $|T|=4$, see Table 10.

Set of patterns T	$s_T(a_1,a_2,\dots,a_k)$ where $k\geq 3$
$\{123,132,213,231\}$	$\binom{a_1+a_2}{a_2}$
$\{123,132,231,312\}$	a_k+1
$\{132,213,231,312\}$	2
$\{123,132,213,321\}$	a_2+1 if $k=3$, 1 if $k=4$, 0 if $k\geq 5$
$\{123,132,231,321\}$	2 if $k=3$, 0 if $k\geq 4$
$\{123,213,231,321\}$	$\binom{a_1+a_3}{a_1}$ if $k=3$, 0 if $k\geq 4$

TABLE 10. Avoidance of patterns in multisets

2.6.4. *Alternating permutations.* A *rise* (resp. *descent*) in a permutation $\pi=\pi_1\pi_2\dots\pi_n$ is an index i such that $\pi_i<\pi_{i+1}$ (resp. $\pi_i>\pi_{i+1}$). The number of rises (resp. descents) in a permutation π is denoted $\mathrm{ris}(\pi)$ (resp. $\mathrm{des}(\pi)$).

A permutation is said to be *alternating* if it starts with rise and then descents and rises come in turn. In other words, an alternating permutation $\pi=\pi_1\pi_2\dots\pi_n$ satisfies $\pi_{2j-1}<\pi_{2j}>\pi_{2j+1}$ for all $1\leq j\leq[n/2]$, that is to say its rise (resp. descent) is equal to an odd (resp. even) index. We denote the set of all alternating permutations on n letters by A_n. Other names that authors have used for these permutations are *zig-sag permutations* and *up down permutations.* An example of an alternating permutation is 14253. The determination of the number of alternating permutations for the set $\{1,2,\dots,n\}$ (or on n

letters) is known as André's problem (see [6, 7]). The number of alternating permutations on n letters, for $n = 1, 2, \ldots, 10$, is 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521 (see [201, Sequence A000111]). These numbers are known as the Euler numbers and have the exponential generating function $\sec x + \tan x$.

Mansour [144] studied the generating function for the number of alternating permutations on n letters that avoid or contain exactly once the pattern 132 and also avoid or contain exactly once an arbitrary pattern on k letters. In several interesting cases, the generating function depends only on k and is expressed via Chebyshev polynomials of the second kind. For example, the generating function for the number of alternating permutations in $\mathcal{S}_n(132, 12\ldots k)$ is given by

$$\frac{(1+x)U_{k-2}\left(\frac{1}{2x}\right)}{xU_{k-1}\left(\frac{1}{2x}\right)},$$

where U_m is the mth Chebshev polynomials of the second kind (see Section 4 for a definition).

2.6.5. *Dumont permutations.* A permutation π is said to be a *Dumont permutation of the first kind* if each even integer in π must be followed by a smaller integer, and each odd integer is either followed by a larger integer or is the last element of π (see, for example, [224]). In [69] Dumont showed that certain classes of permutations on n letters are counted by the Genocchi numbers [201, Sequence A000366]. In particular, Dumont showed that the $(n+1)$st Genocchi number is the number of Dumont permutations of the first kind on $2n$ letters. Mansour [143] studied the number of Dumont permutations of the first kind on n letters avoiding the pattern 132 and avoiding (or containing exactly once) an arbitrary pattern on k letters. For example, the generating function for the number of Dumont permutations of the first kind in $\mathcal{S}_n(132, 12\ldots k)$ is given by

$$F_k(x) + xF_{k-1}(x),$$

where $F_m(x)$ is the solution of the recurrence $Q_r(x) = 1 + \frac{x^2 Q_{r-1}(x)}{1 - x^2 Q_{r-2}(x)}$ with $Q_0(x) = 0$ and $Q_1(x) = 1$.

2.6.6. *Finite approximations of some sequences.* The most attention in the papers on patterns is paid to counting exact formulas and/or generating functions for the number of words or permutations avoiding, or having k occurrences of, certain patterns. In [116, 120, 121] the authors considered another problem, namely counting the number of occurrences of certain patterns in certain words. These words were chosen to be the set of all *finite approximations* of certain sequences.

Let Σ be an alphabet and $\Sigma^\star$ be the set of all words on Σ. A map $\varphi : \Sigma^\star \to \Sigma^\star$ is called a *morphism*, if we have $\varphi(uv) = \varphi(u)\varphi(v)$ for any $u, v \in \Sigma^\star$. It is easy to see that a morphism φ can be defined by defining $\varphi(i)$ for each $i \in \Sigma$. The set of all rules $i \to \varphi(i)$ is called a *substitution system*. We create words by starting with a letter from the alphabet Σ and iterating the substitution system. Such a substitution system is called a *D0L*

(*Deterministic, with no context Lindenmayer*) *system* [134]. D0L systems are classical objects of formal language theory. They are interesting from a mathematical point of view [84], but also have applications in theoretical biology [133].

Suppose a word $\varphi(a)$ begins with a for some $a \in \Sigma$, and that the length of $\varphi^k(a)$ increases without bound. The symbolic sequence $\lim_{k\to\infty} \varphi^k(a)$ is said to be *generated* by the morphism φ. In particular, $\lim_{k\to\infty} \varphi^k(a)$ is a *fixed point* of φ. However, in this paper we are only interested in the finite approximations of $\lim_{k\to\infty} \varphi^k(a)$, that is in the words $\varphi^k(a)$ for $k \geq 1$.

An example of a sequence generated by a morphism is the following sequence w. We create words by starting with the letter 1 and iterating the substitution system ϕ_w: $1 \to 123$, $2 \to 13$, $3 \to 2$. Thus, the initial letters of w are $123132123213\ldots$. This sequence was constructed in connection with the problem of constructing a nonrepetitive sequence on a 3-letter alphabet, that is, a sequence that does not contain any subwords of the type XX, where X is any non-empty word over a 3-letter alphabet. The sequence w has that property. The question of the existence of such a sequence, as well as the questions of the existence of sequences avoiding other kinds of repetitions, were studied in algebra [1, 109, 129], discrete analysis [59, 65, 76, 111, 172] and in dynamical systems [165]. Kitaev and Mansour [120] gave the number of occurrences of some patterns in the finite approximations of w, which are the iteration stages in the construction of w, that is 1, 123, 123132, etc. They found, for example, that the number of occurrences of the pattern 12 in the finite approximations of the sequence w is given by $\frac{1}{2}(3 \cdot 4^{n-1} + 2^n)$ for $n \geq 2$.

In the direction of counting occurrences of patterns in words, Kitaev [116] considered the *sigma-sequence* that was used by Evdokimov [77] to construct chains of maximal length in the n-dimensional unit cube; Kitaev and Mansour [120] considered two different classes of morphisms, and Kitaev, Mansour and Séébold [121] considered the *Peano Curve* that was studied by the Italian mathematician Giuseppe Peano in 1890 as an example of a continuous space filling curve.

3. Generalized patterns

Starting from now, we refer to the patterns defined in Definitions 2.1, 2.2 and 2.20 as *classical* patterns. In this section we consider *generalized patterns*, which were introduced by Babson and Steingrímsson [12]. The motivation for introducing these patterns was the study of *Mahonian permutation statistics* (see [191, 137]).

3.1. Generalized patterns in permutations.

In [12] Babson and Steingrímsson introduced *generalized permutation patterns* that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. In order to avoid confusion we write a classical pattern, say 231, as 2-3-1, and if we write, say 2-31, then we mean that if this pattern occurs in the permutation, then the letters in the permutation that correspond to 3 and 1 are adjacent. For example, the permutation $\pi = 516423$ has only one occurrence of the pattern 2-31, namely the subsequence 564, whereas the pattern 2-3-1 occurs, in addition, in the subsequences 562 and 563. If we use "[" in a pattern, for example if we write $p = [1\text{-}2)$, we indicate that in an occurrence of p, the letter corresponding to the 1 must be the first letter of the permutation, whereas if we write, say, $p = (1\text{-}2]$, then the letter corresponding to 2 must be the last (rightmost) letter of the permutation. Let us give a formal definition of a generalized pattern.

Definition 3.1. *A* generalized pattern *of length k is a word $\tau = x_1\tau_1x_2\tau_2\dots x_k\tau_kx_{k+1}$, where $\tau_1\tau_2\dots\tau_k \in \mathcal{S}_k$, x_1 (resp. x_{k+1}) is either* "(" (resp. ")") or "[" (resp. "]"), and for $1 < j < k+1$, x_j is either the empty string ϵ or a dash "-". If $x_j =$"-" then in the definition of an occurrence of a classical pattern we allow $i_j > i_{j-1} + 1$; also we allow $i_1 > 1$ if $x_1 =$"(", and $i_k < n$ if $x_{k+1} =$")"; else $i_j = i_{j-1} + 1$; in particular $i_1 = 1$ if $x_1 =$"[", and $i_k = n$ if $x_{k+1} =$"]".

Remark 3.2. *Most of generalized patterns considered in this paper had an implicit dash at the beginning and the end, in the sense that they have been allowed to begin, and end, anywhere in a permutation. To simplify the notation, in case if we have no sign* "[" or "]", we remove the parenthesis "(" and ")" from the pattern. Thus, a pattern with no dashes corresponds to a contiguous subword anywhere in a permutation.

For example, the permutation $\pi = 314265$ has two occurrences of the pattern 2-31-4, namely 3-42-6 and 3-42-5.

A number of interesting results on generalized patterns were obtained in [62]. Relations to several well studied combinatorial structures, such as set partitions (see [105]), Dyck paths (see [163]), Motzkin paths (see [66]) and involutions (see [200]), were shown there. The main results from that paper are given in Table 11, where B_n is the n-th Bell number, C_n is the n-th Catalan number, and $B_n^\star$ is the n-th Bessel number. For some other results on generalized permutation patterns see [63, 64, 113, 114, 115, 117, 118, 142, 146, 147].

patterns P	$\lvert\mathcal{S}_n(P)\rvert$	**description**
1-23	B_n	partitions of $[n]$
1-32	B_n	partitions of $[n]$
2-13	C_n	Dyck paths of length $2n$
1-23, 12-3	$B_n^\star$	non-overlapping partitions of $[n]$
1-23, 1-32	I_n	involutions in $\mathcal{S}_n$
1-23, 13-2	M_n	Motzkin paths of length n

TABLE 11. Avoidance of generalized patterns in permutations

As in the paper by Simion and Schmidt [199], dealing with the classical patterns, Claesson [62], Claesson and Mansour [63] considered a number of cases when permutations have to avoid two or more generalized patterns simultaneously (see Table 11). In [113], Kitaev gave either an explicit formula or a recursive formula for almost all cases of simultaneous avoidance of more than two generalized patterns of length three with no dashes, and listed what was known about double restrictions (the remaining cases were described in [117, 118]).

In [113, 117, 118], the authors considered avoidance of an arbitrary generalized 3-letter pattern with additional restrictions such as beginning or ending with certain patterns. These additional restrictions in fact are equivalent to simultaneous avoidance of several patterns. For example, beginning with the pattern 123 is equivalent to simultaneously avoiding the patterns [132), [213), [231), [312) and [321). A motivation for considering additional restrictions such as beginning or ending with some patterns is their connection to some classes of trees. In [113] it was proven that the number of 132-avoiding permutations in $\mathcal{S}_n$ that begin with the pattern 12 is equal to the number of *increasing rooted trimmed trees* with $n+1$ nodes. In an increasing rooted tree, the nodes are numbered and the numbers increase as we move away from the root. A trimmed tree is a tree where no node has a single leaf as a child (every leaf has a sibling).

The case of counting occurrences of a given generalized pattern is studied in the following cases. Elizalde and Noy [72] found the distribution of the number of occurrences of a single generalized pattern from a certain class of generalized patterns with no dashes among all permutations in $\mathcal{S}_n$. In particular, they found the generating functions for the number of permutations in $\mathcal{S}_n$ containing occurrences of the patterns 123 and 132 exactly r times, which are given by

$$\sum_{\pi\in\mathcal{S}} x^{123(\pi)}\frac{y^{|\pi|}}{|\pi|!} = \frac{1}{1-\int_0^y e^{\frac{1}{2}t^2(x-1)}dt},$$

$$\sum_{\pi\in\mathcal{S}} x^{132(\pi)}\frac{y^{|\pi|}}{|\pi|!} = \frac{2\sqrt{(x-1)(x+3)}e^{\frac{1}{2}y(1-x+\sqrt{(x-1)(x+3)})}}{1+x+\sqrt{(x-1)(x+3)}-(1+x-\sqrt{(x-1)(x+3)})e^{y\sqrt{(x-1)(x+3)}}},$$

where $|\pi|$ is the length of π, and $123(\pi)$ and $132(\pi)$ are the number of occurrences of the pattern 123 and 132 in π, respectively.

Claesson and Mansour [63] studied this problem in the case of a given 3-letter generalized pattern with one dash. In particular, they found the generating function for the number of permutations in $\mathcal{S}_n$ containing the pattern 2-13 exactly r times, which is given by

$$\sum_{\pi\in S} x^{2\text{-}13(\pi)}y^{|\pi|} = \cfrac{1}{1-\cfrac{[1]_x y}{1-\cfrac{[1]_x y}{1-\cfrac{[2]_x y}{1-\cfrac{[2]_x y}{1-\ddots}}}}},$$

where $|\pi|$ is the length of π, $[n]_x = 1+x+\cdots+x^{n-1}$, and $2\text{-}13(\pi)$ is the number of occurrences of the pattern 2-13 in π.

Mansour [146, 147] investigated the case of avoidance of the classical pattern 1-3-2 and counting certain generalized patterns. For example, he found, in terms of continued fractions, the generating function for the number of 1-3-2-avoiding permutations in $\mathcal{S}_n$ which contain the pattern τ exactly r times, where $\tau \in \{1\text{-}2\text{-}\cdots\text{-}k, 12\text{-}3\text{-}4\text{-}\cdots\text{-}k, 21\text{-}3\text{-}4\text{-}\cdots\text{-}k\}$. Also, Mansour [148] considered the case of avoidance of a 3-letter generalized pattern with one dash and containing another 3-letter generalized pattern exactly r times. In particular, he found the generating function for the number of 12-3-avoiding permutations which contain the pattern 23-1 exactly once. This function is given by

$$\sum_{d\geq 0}\left[\frac{x^{2d+2}}{p_{d+1}(x)}\left(\sum_{k\geq 0}\frac{x^{2k}}{p_{k+d-1}(x)p_{k+d+1}(x)}-1\right)\right],$$

where $p_m(x)=\prod_{j=0}^{m}(1-jx)$.

3.2. **Generalized patterns in words.** Burstein's work [52] was extended to several directions. One of these directions is counting occurrences of generalized patterns in words. Burstein and Mansour [54] considered the case of avoidance of a 3-letter generalized pattern with repeated letters. For example, they found the generating function for the number of words in $[k]^n$ that avoid the pattern 212, which is given by

$$\sum_{n\geq 0}\sum_{\pi\in[k]^n(212)}x^n=\frac{1}{1-x\sum_{j=0}^{k-1}\frac{1}{1+jx^2}}.$$

More generally, Burstein and Mansour [55] investigated the case of counting occurrences of certain generalized patterns, in particular 3-letter generalized pattern. For example, they found the generating function for the number of words in $[k]^n$ that contain the pattern 212 exactly r times, which is given by

$$\sum_{n\geq 0}\sum_{\pi\in[k]^n(212)}x^ny^{212(\pi)}=\frac{1}{1-x\sum_{j=0}^{k-1}\frac{1}{1+jx^2(1-y)}},$$

where $212(\pi)$ is the number of occurrences of the pattern 212 in π.

3.3. **Partially ordered generalized patterns.** Kitaev [114] introduced a further generalization of generalized patterns, namely *partially ordered generalized patterns (POGP)*. A POGP is a generalized pattern some of whose letters are incomparable. For instance, if we write $p=1\text{-}1'2'$ then we mean that in an occurrence of p in a permutation π the letter corresponding to the 1 in p can be either larger or smaller than the letters corresponding to $1'2'$. Thus, the permutation 13425 has four occurrences of p, namely 134, 125, 325 and 425.

Kitaev [114] considered two particular classes of POGPs: *shuffle patterns* and *multi-patterns*. A multi-pattern is of the form $p=\sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$ and a shuffle pattern is of the form $p=\sigma_0\text{-}a_1\text{-}\sigma_1\text{-}a_2\text{-}\cdots\text{-}a_k\text{-}\sigma_k$, where for any i and j, the letter a_i is greater than any letter of σ_j and for any $i\neq j$ each letter of σ_i is incomparable with any letter of σ_j. For example, $\tau=1'\text{-}2\text{-}1''$ is a shuffle pattern, and $\phi=12\text{-}2'1'$ is a multi-pattern. Kitaev found the exponential generating functions for the number of permutations avoiding a shuffle-pattern and a multi-pattern, in terms of exponential generating functions for the number of permutations avoiding certain generalized pattern with no dashes. In particular, he found the number of τ-avoiding and ϕ-avoiding permutations, which are given by 2^{n-1} and $(n-2)2^{n-1}+2$, respectively.

Kitaev's work [114] was extended to considering POGPs in words. Kitaev and Mansour [119] investigated the analogue of the shuffle patterns and the multi-patterns in permutations. For example, they found that the generating function for the number of words in $[3]^n$ avoiding the shuffle pattern τ is given by $\frac{1-2x+4x^2-3x^3+x^4}{(1-x)^5}$, whereas the generating function for the number of words in $[k]^n$ avoiding the multi-pattern ϕ is given by $\frac{1-kx-2(1-x)^k}{(1-x)^{2k}}$.

The POGPs allow us to study the distribution of the maximum number of *non-overlapping* occurrences of a pattern τ with no dashes, if we only know the e.g.f. for the number of permutations that *avoid* τ. In many cases, this gives nice generating functions. Let τ and ϕ be two patterns. An occurrence of τ *overlaps* an occurrence of ϕ in a permutation π if these two occurrences share a letter in π. For example, if $\tau = 123$, $\phi = 231$, and $\pi = 623514$ then 235 and 351, being occurrences of the patterns τ and ϕ respectively, overlap. Kitaev [114], and Kitaev and Mansour [119] considered the distribution of the maximum number of non-overlapping occurrences of a pattern τ with no dashes in permutations and words, respectively. In particular, if we consider the maximum number of non-overlapping occurrences of the generalized pattern 132, then the distribution of these numbers in permutations is given by

$$\frac{1}{1-yx+(y-1)\int_0^x e^{-\frac{1}{2}t^2}\,dt}.$$

3.4. **Generalized patterns in other sets.** Similarly to the case of classical patterns (see Subsection 2.6), we consider some analogies for occurrences of generalized patterns problem in other sets, which are involutions, alternating permutations, and finite approximations of some sequences.

3.4.1. *Generalized patterns in involutions.* In [100, 101], Guibert and Mansour studied 1-3-2-avoiding involutions with additional restrictions (for the case of the classical patterns see Subsubsection 2.6.2). For example. they found that the distribution of the number of occurrences of the generalized pattern 12 in 1-3-2-avoiding involutions is given by

$$\sum_{\pi\in\mathcal{I}(1\text{-}3\text{-}2)} x^{|\pi|}y^{12(\pi)} = \frac{2(1+x-xy)}{3-3x^2-2xy+2x^2y^2+\sqrt{1-2x^2+x^4+4x^2y^2}},$$

where $|\pi|$ is the length of π and $12(\pi)$ is the number of occurrences of the generalized pattern 12 in π.

3.4.2. *Generalized patterns in alternating permutations.* Mansour [144] considered the case of 1-3-2-avoiding alternating permutations with additional restrictions. This work extends the results from Subsubsection 2.6.4 to the case of generalized patterns. In particular, he found the generating function for the number of 1-3-2-avoiding alternating permutations that avoid the generalized pattern 12-3-4-$\cdots$-k, which is given by

$$\frac{(1+x)U_{k-1}\left(\frac{1}{2x}\right)}{xU_k\left(\frac{1}{2x}\right)},$$

where U_m is the mth Chebyshev polynomial of the second kind.

3.4.3. *Finite approximations of some sequences.* Kitaev [116, 120, 121] discussed counting occurrences of certain classical patterns (see Subsubsection 2.6.6), as well as of certain generalized patterns in finite approximations of certain sequences. For instance, the number of occurrences of the generalized pattern 12 in the finite approximations of the sequence w defined in Subsubsection 2.6.6 is given by $3 \cdot 2^{n-2}$.

4. Some of the approaches to study the pattern problems

Many authors used the same approaches to obtain different results in the subject. In this section, we discuss some of these approaches, which are using transfer matrices, generating trees, continued fractions, and block decompositions.

To proceed further, we recall that *Chebyshev polynomials of the second kind* (in what follows just Chebyshev polynomials) are defined by

$$U_r(\cos\theta) = \frac{\sin(r+1)\theta}{\sin\theta}$$

for $r \geq 0$. Evidently, $U_r(x)$ is a polynomial of degree r in x with integer coefficients. For example, $U_0(x) = 1$, $U_1(x) = 2x$, $U_2(x) = 4x^2 - 1$, and in general, $U_r(x) = 2xU_{r-1}(x) - U_{r-2}(x)$. Chebyshev polynomials were invented for the needs of approximation theory, but are also widely used in various other branches of mathematics, including algebra, combinatorics, and number theory (see [185]).

4.1. **Transfer matrices.**

Apparently, for the first time the relation between restricted permutations and the transfer matrix approach was discovered by Chow and West in [60]. The main result of [60] can be formulated as follows.

Theorem 4.1. ([60, Theorem 3.1]) *Let $T_1 = \{321, 23\dots k1\}$, $T_2 = \{132, 12\dots k\}$, and $T_3 = \{132, 23\dots k1\}$, then :*

(i) *The generating function for the number of permutations avoiding both the patterns from T_1 is given by $R_k(x)$;*
(ii) *The generating function for the number of permutations avoiding both the patterns from T_2 is given by $R_k(x)$;*
(iii) *The generating function for the number of permutations avoiding both the patterns from T_3 is given by $R_k(x)$,*

where $R_k(x) = \frac{U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_k\left(\frac{1}{2\sqrt{x}}\right)}$, *and U_k is the kth Chebyshev polynomial of the second kind.*

The main idea behind the transfer matrix approach can be described as follows (see [204, Theorem 4.7.2]). Consider a directed multigraph on n vertices $v_1, v_2, \dots, v_n$, and let A denote its weighted adjacency matrix, that is, a_{ij} is the number of edges directed from v_i to v_j. Then the generating function for the number of walks from v_r to v_s is given by

$$\frac{(-1)^{r+s}\det(I - xA; r, s)}{\det(I - xA)} \tag{4.1}$$

where I is the identity matrix and $\det(B; r, s)$ is the minor of B with the rth row and sth column deleted.

SIlvia: Check on whether col and row are interchanged.

To apply this approach, one has to construct a bijection between the permutations in question and walks in an appropriate directed graph. We describe below three bijections of this type: the first one is based on generating trees, the second one on Dyck paths, and the third one on diagrams of permutations.

4.1.1. *Generating trees.* Following [217], a *generating tree* is a rooted labeled tree with the property that if v_1 and v_2 are any two nodes with the same label and l is any label, then v_1 and v_2 have exactly the same number of children with label l. To specify a generating tree it therefore suffices to specify:

(1) the label of the root, and
(2) a set of succession rules explaining how to derive from the label of a parent the labels of all of its children.

Example 4.2. (The complete binary tree) Since all the nodes in the complete binary tree are similar, it is enough to use only one label, which we choose to be 2. So we get the following description:

Root: (2)
Rule: $(2) \to (2)(2)$.

Example 4.3. (The Fibonacci tree) Here we have nodes of two different types, so we use two labels: 1 for a non-breeding pair and 2 for a breeding pair. We thus get:

Root: (1)
Rules: $(1) \to (2), \quad (2) \to (1)(2)$

Given a generating tree, one assigns to it a directed graph whose vertices correspond to the labels. There is an edge from vertex i to vertex j for every occurrence of label j in the succession rule $i \to \cdots$. The graphs corresponding to the above two examples are shown in Figure 1.

FIGURE 1. Directed graphs for the complete binary tree and the Fibonacci tree

Given a permutation τ, one defines a rooted tree as follows. The nodes on level n are precisely the elements of $\mathcal{S}_n(\tau)$. The parent of a permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ is the unique permutation $\pi' = (\pi_1, \ldots, \pi_{j-1}, \pi_{j+1}, \ldots, \pi_n)$ such that $\pi_j = n$. We denote the resulting tree $\mathcal{T}(\tau)$. Similarly, the tree corresponding to the set $\mathcal{S}_n(T)$ is denoted by $\mathcal{T}(T)$.

Chow and West [60] proved that the succession rules for the tree $\mathcal{T}(123,(k-2,\ldots,1,k-1))$ are

$$\begin{aligned}&(l)\rightarrow(2)\cdots(l)(l+1), && l<k-2\\ &(k-2)\rightarrow(2)\cdots(k-1)(k-2)(k-2),\end{aligned}$$

and the label of the root is (2). The corresponding graph is shown in Figure 2.

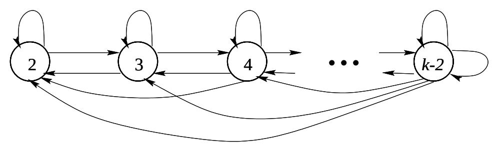

FIGURE 2. The directed graph for $\mathcal{T}(123,(k-2,\ldots,2,1,k-1))$

The corresponding transfer matrix is

$$A_{k-1}=\begin{bmatrix}1&1&0&0&\cdots&0&0&0\\1&1&1&0&\cdots&0&0&0\\1&1&1&1&\cdots&0&0&0\\ \vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\1&1&1&1&\cdots&1&0&0\\1&1&1&1&\cdots&1&1&0\\1&1&1&1&\cdots&1&1&1\\1&1&1&1&\cdots&1&1&2\end{bmatrix}$$

Besides, Chow and West proved that the graphs, and hence the transfer matrices for $\mathcal{T}(213,(k,1,2\ldots,k-1))$ and $\mathcal{T}(213,(1,2,\ldots,k))$ are exactly the same (though the succession rules may vary). The number of permutations in $\mathcal{S}_n(T)$ (in all cases) is thus equal to the number of walks of length n starting from vertex 2. However, since each vertex is connected to vertex 2 by exactly one edge, this number is equal to the number of walks of length $n{+}1$ starting at vertex 2 and ending at the same vertex. The generating function for this number is given by 4.1 with $A=A_k$. It is proved in [60] that the determinants in question satisfy linear recurrences of order two very similar to that for Chebyshev polynomials, which almost immediately yields Theorem 4.1, since $T_1=c(\{123,(k-1,\ldots,2,1,k)\})$, $T_2=r\circ c(\{213,(1,2,\ldots,k)\})$, and $T_3=r\circ c(\{213,(k,1,2,\ldots,,k-1)\})$, where r and c are the reverse and the complement respectively.

4.1.2. *Dyck paths.* A *Dyck path* is a path in the plane integer lattice $\mathbf{Z}^2$, consisting of up-steps $(1,1)$ and down-steps $(1,-1)$, which never passes below the x-axis.

Following [130], we define a bijection Φ between permutations in $\mathcal{S}_n(132)$ and the Dyck paths from the origin to the point $(2n,0)$. Let $\pi=(\pi_1,\ldots,\pi_n)$ be a 132-avoiding permutation. We read the permutation π from left to right and successively generate a Dyck path. When π_j is read, then in the path we adjoin as many up-steps as necessary, followed by a down-step from height h_j+1 to height h_j (measured from the x-axis), where h_j is the number of elements in $\pi_{j+1},\pi_{j+2},\ldots,\pi_n$ which are larger that π_j.

For example, let $\pi = 534261$. The first element to be read is 5. There is one element in 34261 which is larger than 5, therefore the path starts with two up-steps followed by a down-step, thus reaching height 1. Next 3 is read. There are 2 elements in 4261 which are larger than 3, therefore the path continues with two up-steps followed by a down-step, thus reaching height 2. And so on.

Conversely, given a Dyck path starting at the origin and returning to the x-axis, the obvious inverse of the bijection Φ produces a 132-avoiding permutation.

It is proved in [130] that the bijection Φ sends the permutations in $\mathcal{S}_n(132, (1, 2, \ldots, k))$ onto the Dyck paths that never pass above the line $y = k - 1$. Evidently, such paths correspond bijectively to walks of length $2n$ starting at vertex 0 in the graph shown in Figure 3.

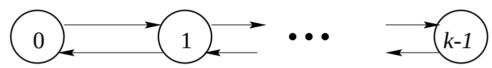

FIGURE 3. The directed graph for Dyck paths in a strip

Using Eq. (4.1) one obtains Theorem 4.1(ii). A further study of the bijection Φ yields part (iii) of the same theorem.

4.1.3. *ECO-method.* In this subsubsection, we present another general method (called ECO) for the enumeration of some classes of combinatorial objects. The main idea of this method is the following: using an operator that performs a "local expansion" on the objects, we give recursive constructions of these classes. Then we introduce some functional equations verified by classes of generating functions. By solving these functional equations, we enumerate the combinatorial objects according to various of parameters (for more details see [17]).

Following [17], the ECO method can be defined as follows. Let $\mathcal{X}$ be a class of combinatorial objects with a *parameter* p, $p : \mathcal{X} \to \mathcal{N}$, and $\mathcal{X}_n = \{x \in \mathcal{X} \,|\, p(x) = n\}$. An operator μ on $\mathcal{X}$ is a function from $\mathcal{X}_n$ to the power set of $\mathcal{X}_{n+1}$. We say that the operator μ is a *recursive operator on* $\mathcal{X}$ if it satisfies the following conditions: (1) for each element $y \in \mathcal{X}_{n+1}$ there exists $x \in \mathcal{X}_n$ such that $y \in \mu(x)$, and (2) if $x_1, x_2 \in \mathcal{X}_n$ and $x_1 \neq x_2$, then $\mu(x_1) \cap \mu(x_2) = \emptyset$.

Proposition 4.4. *If μ is a recursive operator on $\mathcal{X}$, then $\{\mu(x) \,|\, x \in \mathcal{X}_n\}$ is a partition of $\mathcal{X}_{n+1}$.*

Therefore, such an operator on $\mathcal{X}$ gives a recursive description of the class $\mathcal{X}$, that is, the above proposition allows us to construct each object $y \in \mathcal{X}_{n+1}$ from an object $x \in \mathcal{X}_n$ and every $y \in \mathcal{X}_{n+1}$ is obtained by only one $x \in \mathcal{X}_n$. In many cases the recursive description is given by generating trees (see Subsubsection 4.1.1). For example, let us show how the ECO method works on the class of Dyck paths. Let $\mathcal{P}$ be the class of Dyck paths, $\mathcal{P}_n$ be the set of all Dyck paths of length $2n$, and $s : \mathcal{P} \to \mathbb{N}$ be the number of northeast steps. For $p \in \mathcal{P}_n$, we define $\mu(p)$ as the set of all Dyck paths obtained from p by inserting a peak at any point of p's last descent. Then $\mathcal{P}_{n+1}$ is obtained by performing the operator μ on $\mathcal{P}_n$. Now, let p be any Dyck path with k last descents (last descents contains k points), then by writing the rule that characterizes the generating tree on $\mathcal{P}$ we get the rule

$$(k) \to (2) \cdots (k)(k+1).$$

This approach proved to be a good one in many cases of restricted permutations, for example in the case of $\mathcal{S}_n(321)$ (see [17, Section 3.4]) and in the case of Motzkin permutations, $\mathcal{S}_n(321, 3\bar{1}42)$ (see [17, Section 4.4]). (For more examples of classes of combinatorial objects and special classes of restricted permutations see [17, 18, 19, 21, 22, 23, 24, 20].)

4.1.4. *Diagram of a permutation.* The diagram of a permutation is an important tool in theory of Schubert polynomials for permutations. Schubert polynomials were extensively developed by Lascoux and Schützenberger (for more details see [136]).

A *diagram* $D(\pi)$ of a permutation π can be defined as the following: first let π be represented by an $n \times n$-array with a dot in each of the squares (i, π_i) (numbering from the top left hand corner). Shadow all the squares due south or due east of some dot and the dotted cell itself. The diagram $D(\pi)$ is defined as the region left unshaded after this procedure. A square that belongs to the diagram $D(\pi)$ is called a *diagram square* and a row (column) of the array that contains a diagram square is called *diagram row* (*diagram column*).

In [87], Fulton introduced the essential set $\mathcal{E}(\pi)$ of a permutation π together with the rank function used as a tool for an algebraic treatment of Schubert polynomials. By the construction above, each of the connected components of the diagram $D(\pi)$ is called *Young diagram.* We define the *essential set* of a permutation π to be the corners. For any element $(i, j) \in \mathcal{E}(\pi)$, its *rank* is defined to be the number of dots northwest of (i, j), and is denoted by $\rho(i, j)$. Furthermore, we denote the set of all elements of $\mathcal{E}(\pi)$ with rank equals r by $\mathcal{E}_r(\pi)$.

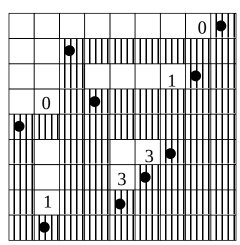

FIGURE 4. The diagram and the ranked essential set for the permutation $\pi = 938417652$.

Fulton [87, Lemma 3.10b] proved a fundamental property of the ranked essential set of a permutation π, that uniquely determines π. An algorithm for retrieving the permutation from its ranked essential set was provided by Erikson and Linusson [74] (see also [73]). The authors gave a complete answer for a question of Fulton, namely, a characterization of all ranked sets of the squares that can arise as ranked essential sets for permutations.

To recover a permutation from its diagram is trivial: row by row, put a dot in the leftmost shaded square in such way that there is exactly one dot in each column.

In [177], Reifegerste used the permutation diagrams to give combinatorial proofs for some enumerative results concerning forbidden subsequences in 132-avoiding permutations and discussed some open problems which have been raised in [160]. By [177, Theorem 2.2], 132-avoiding permutations are exactly those permutations for which the diagram corresponds to a partition, or equivalently, for which the rank of every element of the essential set equal 0. More precisely, the diagram of a permutation in $\mathcal{S}_n(132)$ is a Young diagram fitting in the shape $(n-1, n-2, \ldots, 1)$.

In [178], Reifegerste generalized these bijections to obtain combinatorial proofs for some enumerative results in [71] concerning forbidden subsequences in $\{1243, 2143\}$-avoiding permutations. By [178, Theorem 2.1], the permutations $\pi \in \mathcal{S}_n(1243, 2143)$ are exactly those permutations for which every element of its essential set is of rank at most 1.

4.2. **Continued fractions.** The relation between restricted permutations and continued fractions was discovered by Robertson, Wilf, and Zeilberger in [189]. The main result in [189] can be formulated as follows.

Theorem 4.5. (Robertson, Wilf, and Zeilberger [189, Theorem 1]) *The generating function for the number of permutations in $\mathcal{S}_n(132)$ containing the pattern 123 exactly r times is given by*

$$\sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(132)} x^n z^{123(\pi)} = \cfrac{1}{1-\cfrac{xz^{\binom{0}{2}}}{1-\cfrac{xz^{\binom{1}{2}}}{1-\cfrac{xz^{\binom{2}{2}}}{\ddots}}}}$$

in which the jth numerator is $xz^{\binom{j-1}{2}}$.

To prove this, let π be a permutation avoiding 132. Then each letter in π to the left of n must be greater than any letter to the right of n. Thus, if $\pi = (\pi', n, \pi'')$ (where both π' and π'' must necessarily be 132-avoiding), then

$$(123)\pi = (123)\pi' + (12)\pi' + (123)\pi'',$$

where $(\tau)\pi$ is the number of occurrences of τ in π. It follows that the generating function

$$F(x,y,z) = \sum_{\pi\in S(132)} x^{(1)\pi} y^{(12)\pi} z^{(123)\pi}$$

satisfies the equation $F(x,y,z) = 1 + xF(xy, yz, z)F(x,y,z)$. Equivalently,

$$F(x,y,z) = \frac{1}{1 - xF(xy,yz,z)},$$

and the theorem follows by induction after plugging in $y = 1$.

This result was generalized by Mansour and Vainshtein [156], by Krattenthaler [130], and by Jani and Rieper [108] to the case of permutations containing the pattern $12\dots k$ exactly r times. It turns out that

$$\sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(132)} x^n z^{12\dots k(\pi)} = \cfrac{1}{1-\cfrac{xz^{d_1}}{1-\cfrac{xz^{d_2}}{1-\cfrac{xz^{d_3}}{\dots}}}}, \tag{4.2}$$

where $d_j = \binom{j-1}{k-1}$.

The proof in [156] is a straightforward generalization of the above proof of Theorem 4.5. The proof in [130] is based on the bijection Φ between 132-avoiding permutations and Dyck paths described in the previous section and on the result of Flajolet [79, Theorem 1] presenting the generating function for the Dyck paths in terms of continued fractions. The proof of [108] is based on a bijection between 132-avoiding permutations and rooted ordered trees, which can be obtained from the bijection Φ via the standard bijection between rooted ordered trees and Dyck paths through a depth-first traversal of the trees (see [205, Proposition 6.2.1, Cororllary 6.2.3]). For further generalizations and interesting combinatorial applications see [50].

It was observed in [156] that $R_k(x)$ is the kth approximant for the continued fraction

$$\cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-\ddots}}}.$$

This, together with formula 4.2 applied for $r=0$ immediately gives Theorem 4.1(ii).

Paper [130] contains the description of a bijection Ψ between 123-avoiding permutations and Dyck paths. This bijection, combined with Roblet and Viennot's continued fraction representation of the generating function for Dyck paths [186, Proposition 1] gives the first part of Theorem 4.1.

4.3. **Block decompositions.** The core of this approach initiated by Mansour and Vainshtein [157] lies in the study of the structure of 132-avoiding permutations, and permutations containing a given number of occurrences of 132 [159]. Let us start with the simplest case of 132-avoiding permutations. It was noticed in [157] that if $\alpha\in\mathcal{S}_n(132)$ and $\alpha_t=n$, then $\alpha=(\alpha',n,\alpha'')$ where α' is a permutation of the numbers $n-t+1, n-t+2,\dots,n-1$, α'' is a permutation of the numbers $1,2,\dots,n-t$, and both α' and α'' avoid 132. This representation is called the block decomposition of α, see Figure 4.3.

This simple observation allows to formulate a general result concerning permutations avoiding 132 and an arbitrary pattern $\tau=(\tau_1,\dots,\tau_k)\in\mathcal{S}_k(132)$. Recall that τ_i is said

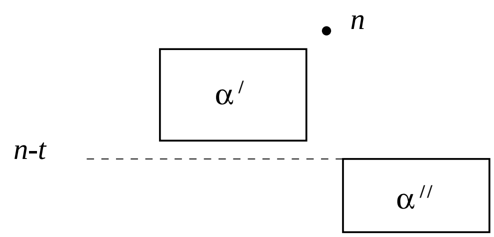

FIGURE 5. The block decomposition for $\alpha \in \mathcal{S}_n(132)$

to be a *right-to-left maximum* if $\tau_i > \tau_j$ for any $j > i$. Let $m_0 = k, m_1, \ldots, m_r$ be the right-to-left maxima of τ written from left to right. Then τ can be represented as

$$\tau = (\tau^0, m_0, \tau^1, m_1, \ldots, \tau^r, m_r),$$

where each of τ^i may be possibly empty, and all the entries of τ^i are greater than all the entries of τ^{i+1}. Define the ith *prefix* of τ by $\pi^i = (\tau^0, m_0, \ldots, \tau^i, m_i)$ for $1 \le i \le r$ and $\pi^0 = \tau^0$, $\pi^{-1} = \varnothing$. Also, the ith *suffix* of τ is defined by $\sigma^i = (\tau^i, m_i, \ldots, \tau^r, m_r)$ for $0 \le i \le r$ and $\sigma^{r+1} = \varnothing$.

Theorem 4.6. ([157, Theorem 1]) *For any $\tau \in \mathcal{S}_k(132)$, the generating function for the number of permutations in $S(132, \tau)$, $F_\tau(x)$, is a rational function satisfying the relation*

$$F_\tau(x) = 1 + x \sum_{j=0}^{r} \big(F_{\pi^j}(x) - F_{\pi^{j-1}}(x)\big) F_{\sigma^j}(x).$$

The proof is rather straightforward. Let $\alpha = (\alpha', n, \alpha'')$ be the block decomposition of $\alpha \in \mathcal{S}_n(132)$. It is easy to see that α contains τ if and only if there exists i, $0 \le i \le r+1$, such that α' contains π^{i-1} and α'' contains σ^i. Therefore, α avoids τ if and only if there exists i, $0 \le i \le r$, such that α' avoids π^i and contains π^{i-1}, while α'' avoids σ^i. We thus get the following relation:

$$f_\tau(n) = \sum_{t=1}^{n} \sum_{j=0}^{r} f_{\pi^j}^{\pi^{j-1}}(t-1) f_{\sigma^j}(n-t),$$

where $f_\tau(n) = |\mathcal{S}_n(132, \tau)|$, and $f_\tau^\rho(n)$ is the number of permutations in $\mathcal{S}_n(\tau)$ containing ρ at least once. To obtain the recursion for $F_\tau(x)$ it remains to observe that

$$f_{\pi^j}^{\pi^{j-1}}(l) + f_{\pi^{j-1}}(l) = f_{\pi^j}(l)$$

for any l and j, and to pass to generating functions. Rationality of $F_\tau(x)$ follows easily by induction.

This technique has been used in many papers for different structures (see [160] and the references therein). For example, it is used in pattern avoidance in involutions (see [100, 101]), coloured permutations (see [139]), generalized patterns (see [142, 146, 147]), and even permutations (see [145]).

4.3.1. *Removing the first or the last element, or removing the greatest or the smallest element.* IIn [169], Noonan and Zeilberger suggested a new approach for the enumeration of permutations with a prescribed number of occurrences of a list of patterns. Many authors have since used this approach directly or indirectly. We now present the method and two applications of it.

To determine the number of permutations in $\mathcal{S}_n$ having exactly r occurrences of the pattern $\tau = \tau_1\tau_2\dots\tau_k$,

(1) Determine the best way to obtain a recurrence for this pattern. There are basically four ways to do this, namely by removing
   (a) the last entry of the permutation;
   (b) the first entry of the permutation;
   (c) the maximal entry of the permutation;
   (d) the minimal entry of the permutation;
(2) Identify the other parameters needed in order to describe the recurrence.

Noonan and Zeilberger [169] used this method to find the number of permutations in $\mathcal{S}_n$ containing the pattern 123 exactly once, which is given by $\frac{3}{n}\binom{2n}{n+3}$; they conjectured that the number of permutations in $\mathcal{S}_n$ containing the pattern 123 exactly twice is given by $\frac{59n^2+117n+100}{2n(2n-1)(n+5)}\binom{2n}{n-4}$ (which was proved by Fulmek [86] using the Dyck paths approach, see Subsubsection 4.1.2) and that the number of permutations in $\mathcal{S}_n$ containing the pattern 132 exactly once is given by $\binom{2n-3}{n-3}$ (which was proved by Bóna in [38], and the general case was suggested by Mansour and Vainstein [159]).

Burstein [52] gives another application of this approach. In his thesis, he presented the number of words avoiding a set of patterns in $\mathcal{S}_3$. For example, he proved that the number of words in $[k]^n$ avoiding a pattern of length three is given by

$$f_{123}(n,k) = f_{132}(n,k) = 2^{n-2(k-2)}\sum_{j=0}^{k-2} a_{k-2,j}\binom{n+2j}{n},$$

where

$$a_{k,j} = \sum_{m=j}^{k} C_m D_{k-m}, \quad D_t = \binom{2t}{t},\ C_m = \frac{1}{m+1}\binom{2m}{m}$$

and the generating function for those words is given by

$$1 + \frac{y}{1-x} + \frac{2y^2}{(1-2x)(1-y)+\sqrt{((1-2x)^2-y)(1-y)}}.$$

4.4. **Young Tableaux.** In 1961, Schensted [196] introduced a bijection (see also [190]) between permutations of the symmetric group $\mathcal{S}_n$ and the pairs of standard Young tableaux (see also [96, 125, 212]). Schensted [196] proved that there is a bijection between the set of Young tableaux having n cells with at most k rows and the set of involutions in $\mathcal{S}_n$ avoiding the increasing pattern of length $k+1$, namely involutions in $\mathcal{S}_n(12\dots k(k+1))$.

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ be a *partition* of a positive integer n, that is, $\sum_{i=1}^{k}\lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. The *Ferrers diagram* F_λ of shape λ is the left-justified rows of cells with λ_i cells in the ith row (reading from top to bottom). A *standard Young tableaux* (more simply, a Young tableaux) of shape λ is a filling of the cells in F_λ with the numbers $1, 2, \ldots, n$ in such a way that the numbers increase in each row (when reading from left to right) and each column (when reading from top to bottom). We denote by Y_n^k the set of Young tableaux having n cells and at most k rows.

The first simple expressions for the number of standard Young tableaux for given a shape were the Frobenius-Young formula [85, 167, 220] and the Frame-Robinson-Thrall hook formula [82]. Since 1954 many proofs of the hook formula have been given using probabilistic (Greene, Nijenhuis and Wilf [97]) or purely combinatorial methods (Remmel [181], Remmel and Whitney [182], Gessel and Viennot [93], Zeilberger [223], Franzblau and Zeilberger [83], and see also [89, 195, 203]).

Let S_n^k be the number of standard Young tableaux with n cells and at most k rows. Schensted [196] proved the number of involutions in $\mathcal{S}_n$ avoiding the increasing pattern of length $k+1$ is given by S_n^k. Regev [175] found the exact formulas for S_n^k where $k = 2, 3$:

$$S_n^2 = \frac{n!}{[n/2]![(n+1)/2]!} \text{ and } S_n^3 = M_n,$$

where M_n is the nth Motzkin number. Gouyou-Beauchamps [95] found another two formulas for S_n^k where $k = 4, 5$ and proved that

$$S_n^4 = C_{[(n+1)/2]} C_{[n/2]+1} \text{ and } S_n^5 = \sum_{i=0}^{[n/2]} \frac{6n!(2i+2)!}{(n-2i)!i!(i+1)!(i+2)!(i+3)!},$$

where C_n is the nth Catalan number. The proofs of these results are purely combinatorial using the bijection between involutions and labelled Motzkin words (see [95, Section 3]).

5. Our final remark

As a final remark, we would like to mention that the idea of the patterns described in this paper can be generalized to the *numbered polyomino patterns* or just *polyomino patterns* (see [122]). The polyomino patterns are two-dimensional, and one can consider occurrences of them in matrices or other two-dimensional shapes. These patterns are interesting, for instance, from a graph theoretic point of view. For example, occurrences of particular polyomino patterns in the adjacency matrix of a graph pose certain restrictions on the set of edges of the graph and on its cycles.

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