A COMPLETE LIST OF FAIR DICE

by

Edward Taylor Pegg, Junior

B.A., University of Colorado at Colorado Springs, 1994

A thesis submitted to the Graduate Faculty of the

University of Colorado at Colorado Springs

in partial fulfillment of the

requirements for the degree of

Master of Science in Applied Mathematics

Department of Mathematics

1997

This thesis for the Master of Science degree by

Edward Taylor Pegg, Junior

has been approved for the

Department of Mathematics

by the following Thesis Committee:

_____

Greg Morrow, Chair of Committee

_____

Rinaldo Schinazi

_____

Robert Carlson

_____

Kenneth Rebman

_____

Date

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

INTRODUCTION

Throughout this document, words in italics are explained in the Glossary, pages 33-35.

What is a fair *die*? If made uniformly, a cube is often considered *fair*. For hundreds of years, the cube has been used in gambling because of its perceived fairness. 'Fair' implies that all *faces* have an equal likelihood of being the 'down' face. Has anyone ever mathematically proven that a cube is fair? Other than by rolling and applying the *law of large numbers*, the answer appears to be 'no.'

Research for this problem started during a lecture, when Dr. Rinaldo Schinazi mentioned 'fair die' after a series of very careful definitions. Similar to the weightless rods found in physics problems, the 'fair die' seemed to be an abstraction. Can a fair die have five sides?

A simple mathematical model suggested 'yes,' and led to the construction a pyramid that was fair within the framework of the model. A second mathematical model suggested 'no,' and gave strong evidence that dice are fair if and only if they are *isohedral*. A polyhedron is isohedral if any face can be mapped directly onto any other face by some rotation or reflection of the polyhedron onto itself. Such a polyhedron is called an *isohedron*.

For the Geometric Model, the five-sided pyramid gives a function that assigns to each height h a value $f(h)$ equal to the probability that a pyramid will land on its unit square base. In this model, when $h \in (1,2)$, the pyramid function is continuous.

The Geometric Model exploits spherical trigonometry to describe how an idealized (i.e. – non-bouncing, non-spinning) object will behave. It maps the *edges* of an object to the surface of a *centroid*-centered sphere. The centroid, also called the center of mass, is the point such that the first moment of every line through the point is zero.

The Geometric Model postulates that if all images on the sphere have equal area, then the object is fair. The model is flawed – unstable faces cause this model to fail. The model, Maple, and **[Wer** 727-736**]** allowed the construction of fair 5-sided prisms and pyramids. I discuss the model further in Chapter Three. The Geometric Model allows for the asymmetrical object constructed in Chapter Four.

Parallel to this research, I built a list of all the dice which are 'obviously' fair – the objects that would work in all of the mathematical models for fairness. The five Platonic polyhedra (tetrahedron, cube, octahedron, dodecahedron, and icosahedron) are 'obviously' fair. Any isohedron is obviously fair. The isohedra are symmetrical, so Group Theory was used to prove that the list of dice is complete. Several proofs exist for the theorem concerning the finite subgroups of SO[3]. The complete list of isohedra led to a new proof of this theorem.

The major result of this thesis was obtained in April 1997. Weird dice are strongly influenced by the way they are tossed. A fair pyramid when dropped on a glass surface from two feet might not be fair anymore when tossed from three feet onto a wooden surface. The Energy State Model takes these factors into account. It can be considered as a convergent Markov Chain, with one matrix for each bounce. Since dice can easily be tossed in a variety of ways, these results show that the isohedra are the **only** fair dice. I discuss the Energy State Model in Chapter Six.

This research touches on and links the following mathematical areas: linear algebra, group theory, graph theory, geometry, mathematical modeling, chaos theory, probability, and analysis. Several times, results were proven by converting the problem into one involving a different field of mathematics.

This thesis explains how dice work, and solves key questions about fairness. The work includes an elementary proof classifying finite subgroups of rotations in 3-space. To quote *Toy Story*'s Buzz Lightyear – "This is falling, with style."

CHAPTER 2

REVIEW OF THE LITERATURE

The initial clue for solving the title problem was inspired by Buffon's Needle Problem:

A board is ruled with a series of equidistant parallel lines, and a very fine needle, which is shorter than the distance between lines, is thrown at random on the board. Denoting by l the length of the needle and by h the distance between lines, the probability that the needle will intersect one of the lines (the other possibility is that the needle will be completely contained within the strip between two lines) is found to be $p = 2l / \pi h$ **[Upe 112].**

The experiment suggested by the above theorem provides a striking application of Bernoulli's law of large numbers. In a trial conducted by R. Wolf, a value for π of 3.1596 was found using 5000 needle tosses. Ambrose Smith reached the value 3.155 using 3204 needle tosses **[Upe 113].** Since a two dimensional object could be analyzed by using a circle, it occurred to me that a three dimensional object could be analyzed using a sphere. From this came the Geometric Model, which I discuss in the next chapter.

One problem with Buffon's model is that it fails to take into account the process of landing. When I tried tossing a needle on a glass surface, the needle bounced around a lot before stopping. On paper, the needle tended to slide or roll. Depending upon the method used for tossing the needle, one might find that certain final angles are more likely than others. This turns out to be a key result of the Energy State Model, which is discussed in Chapter Six.

For three dimensional solids, the method of tossing becomes even more important. In Las Vegas, cubic dice must bounce off the backboard before the roll is considered fair. Without such an element, it is possible, with some practice, to 'cheat' while rolling the dice.

Using the backboard, cubes are considered 'fair dice.' Las Vegas doesn't use dimpled dice – in Chapter Six a good reason for this decision is shown – dimpled dice are not fair. The backboard forces at least three unpredictable bounces.

A key element of fairness seems to be symmetry. Group Theory has powerful tools for analyzing symmetric objects, such as the rotation groups of the sphere.

John Napier, who also analyzed the sphere, invented both the decimal point and logarithms. He developed many of the formulas that are still used for Spherical Trigonometry. After his death in 1617, Euler and Gauss fleshed out a few of the formulas, but Napier's analysis almost exhausted this branch of mathematics. The problems in spherical trigonometry that Napier couldn't solve remain unsolved. For example, if the surface of the sphere is divided into a finite number of congruent pieces, what is the minimal diameter of these pieces? The best known answer corresponds to the 120 triangles of the hexakis icosahedron, but no proof exists **[Cro** 90**].** Another problem, called the problem of Tammes (after a botanist studying pollen grains), asks for optimal arrangements of *n* circles on a unit sphere **[Cro** 114**].**

Many of the problems involving spherical trigonometry lead to page-long formulas that are best evaluated by a computer. Perhaps dice haven't been studied much due to the fact a linkage was needed between computers and an area of mathematics that had been seen as a dead end. Even Euler and Gauss weren't able to improve much upon Napier's results.

Leonhard Euler did, however, add to many other areas of mathematics (as did Gauss); one area he developed was Graph Theory. Euler's formula, *faces (f) - edges (e) + vertices (v) = 2*, applies to any polyhedron topologically equivalent to a sphere – these include all convex polyhedra. There is one basic polyhedron with four faces, the tetrahedron. There are two

basic polyhedra with five sides, the prism (three rectangles and two triangles) and the pyramid (as in Egygt).

Martin Gardner asked how many basic six-sided polyhedra there were in his *Mathematical Games* column in *Scientific American* **[Gar** 233**].** There are seven basic shapes.
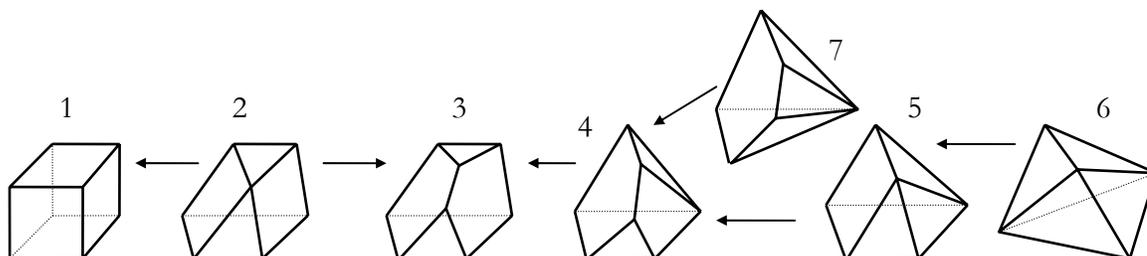
Figure 1. The Seven Basic Hexahedra.

Note the arrows in Figure 1. An arrow indicates a vertex-splitting operation. A vertex is split, the two new vertices are joined by an edge, and the edges incident on the original vertex are apportioned between the two new vertices. The other direction would be considered a vertex-merging operation. Note that, by Euler's formula, the number of faces remains constant with either operation. I rediscovered that all polyhedra of a given number of faces are connected via vertex-splitting and vertex-merging operations. Tutte proved this result in 1961 **[Har** 46**].** Converting the graphs to matrices, it is relatively easy to write a program which will find all of the polyhedra with f faces.

Michael B Dillencourt did just that, and found 1 tetrahedron, 2 pentahedra, 7 hexahedra, 34 heptahedra, 257 octahedra, 2606 enneahedra, 32,300 decahedra, 440,564 hendecahedra, and 6,384,634 dodecahedra **[Dil** 98**]** (Dillencourt's paper was found via the *Encyclopedia of Integer Sequences* **[Slo** M1796**]**).

For unstable polyhedra, Richard K. Guy discovered a 19-sided solid object with the property that 18 of its faces are unstable. When tossed, the object always winds up on the

same face each time. Such an object is called *unistable*. No unistable object with fewer faces has been found. A tetrahedron with two unstable faces exists, with an edge of length 41 opposite an edge of length 4, 24 opposite 20, and 26 opposite 17 **[Cro** 61**].**

The Rigid Tetrahedral Group (❋) is the following collection of 12 3x3 matrices. They form a *group* under multiplication.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \quad H = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$I = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad J = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad K = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Forming the multiplication table helps to verify that this is a group – see Table 1. The group axioms are easily verified. Note that *A* is the *identity element* for this group.

	A	B	C	D	E	F	G	H	I	J	K	L
A	A	B	C	D	E	F	G	H	I	J	K	L
B	B	A	D	C	F	E	H	G	J	I	L	K
C	C	D	A	B	G	H	E	F	K	L	I	J
D	D	C	B	A	H	G	F	E	L	K	J	I
E	E	H	F	G	I	L	J	K	A	D	B	C
F	F	G	E	H	J	K	I	L	B	C	A	D
G	G	F	H	E	K	J	L	I	C	B	D	A
H	H	E	G	F	L	I	K	J	D	A	C	B
I	I	K	L	J	A	C	D	B	E	G	H	F
J	J	L	K	I	B	D	C	A	F	H	G	E
K	K	I	J	L	C	A	B	D	G	E	F	H
L	L	J	I	K	D	B	A	C	H	F	E	G

Table 1. Multiplication Table for a Matrix Group

Two different groups of 24 matrices can be realized by considering the group generated by G above together with $\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, or $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. A group of 48 matrices is generated by adding both matrices and forming all possible products. The original twelve matrices correspond to the rigid *tetrahedral* group, ✳. Adding the first matrix produces the group corresponding to the rigid *octahedral* group, ⌂. Adding the second matrix to either group allows for mirror image reflections. The following are generators for 🖖, the *icosahedral* group:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{5}}{4} & 0 \\ \frac{\sqrt{5}}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

As an example of how groups are useful, the plane $\frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 1$ can be rotated via ✳, producing 12 planes. The pentagon {{1.276, 1.276, 1.276}, {1.442, 3.0, -1.153}, {2.609, 2.609, -2.609}, {3.0, 1.153, -1.442}, {3.0, -1.153, 1.442}} can be isolated from the points where some three of the twelve planes intersect. This pentagon is representable as a matrix which, when multiplied by elements of the matrix group, produces 12 interlocking pentagons. An isohedron results, as shown in Figure 2.
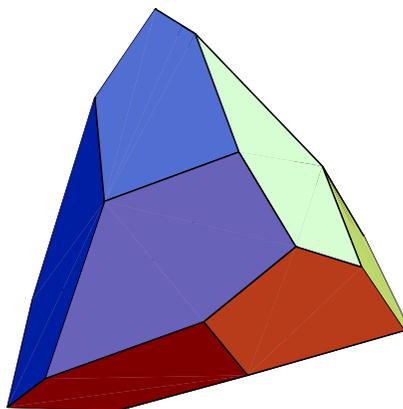
Figure 2. The Tetrahedral Pentagonal Dodecahedron

9

❄ has provided a beautifully symmetric diagram. These groups are rare, unfortunately.

Theorem: ❄🖼, 🛐, and ✋, together with the *cyclic* and *dihedral* groups, form all possible finite subgroups of rotations in three dimensional space.

Proof: See **[Arm** 104**]**, **[Yal** 93**]**, or **[Dub** 260**]** for identical proofs involving actions, orbits, stabilizers, and high-level algebraic theorems. See **[Cox** 53**]** for a different proof involving spherical trigonometry and high-level geometric theorems. See Chapter 5 for a new proof.

CHAPTER 3

THE GEOMETRIC MODEL

Take a cube, and draw its circumsphere. The centroid of the cube coincides with the center of the sphere. Now project the edges of the cube onto the surface of the sphere via the centroid. The sphere's surface will now be divided into six areas. In the figure below, imagine that the other four great circles have been drawn, and consider the segments which correspond to edges.
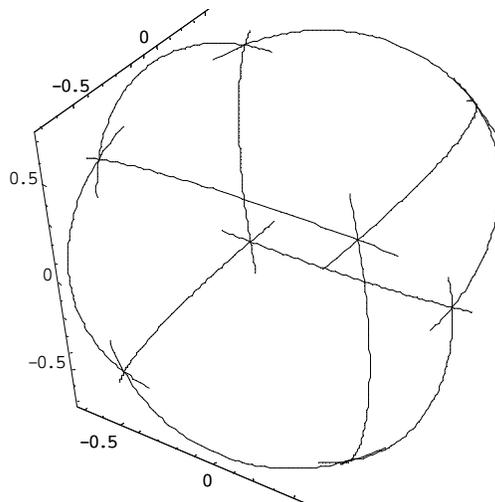
Figure 3. The Geometric Model

If the cube is randomly dropped, and doesn't bounce, the face that winds up on the table is the same as the corresponding face on the sphere, due to gravity. This model explains why loaded dice are *not* fair – moving the center of gravity alters the maps on the sphere.

However, the model falls apart for unstable faces. For me, this was realized when I built one of my first results, set it on one of its faces, and watched in horror as it tipped over

onto another face. Still, it's a pretty good model when all the faces are stable. A knowledge of spherical trigonometry and of spherical triangles is necessary to use the model.

1. For spherical triangles on the unit sphere with spherical coordinates (α_j , δ_j) for point P_j , the arc-length distance, θ (P_1 , P_2), between P_1 and P_2 is given by

$$\cos\theta (P_1 , P_2) = \cos\delta_1 \cos\delta_2 \cos(\alpha_1 - \alpha_2) + \sin\delta_1 \sin\delta_2 .$$

2. The rotation angle, $\Theta_3 = \Lambda(P_1 , P_2 ; P_3)$, from P_1 to P_2 about a third point, P_3 , is given by $\cos\Theta_3 = \cos\Lambda(P_1, P_2; P_3) = \dfrac{\cos\theta(P_1,P_2) - \cos\theta(P_1,P_3)\cos\theta(P_2,P_3)}{\sin\theta(P_1,P_3)\sin\theta(P_2,P_3)}$;

similarly, $\Theta_1 = \Lambda(P_2 , P_3 ; P_1)$ and $\Theta_2 = \Lambda(P_1 , P_3 ; P_2)$.

3. The area of a spherical triangle, Ω_t , equals $\Theta_1 + \Theta_2 + \Theta_3 - \pi$ [1, 2, 3: **Wer** 727].

The full formula for calculating the area of a spherical triangle involves 45 cosines, 42 sines, and 9 arccosines. With Maple, this formula can be graphed.

Figure 4. Area of a Equilateral Spherical Triangle
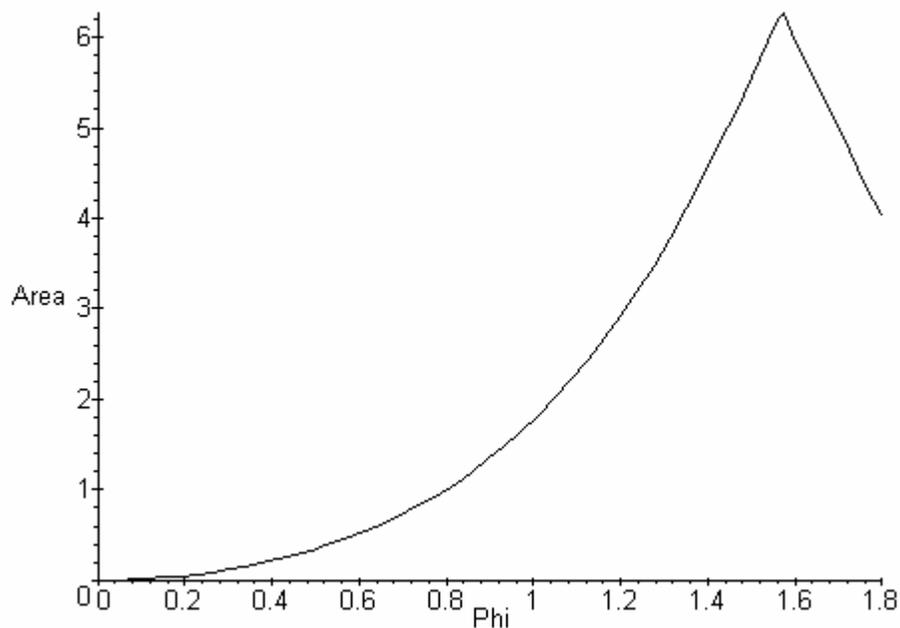
The area of an equilateral spherical triangle is shown in Figure 4, where Φ is the angle from the 'north pole' of the unit sphere to a vertex of the triangle. Note that when Φ is $\pi/2$ radians, the resulting triangle is a great circle, and has an area equal to half the surface area of the sphere. Using an equilateral triangle allows the use of a simpler formula for the area, namely the following:

$$6\arcsin\left(\sqrt{\frac{\sin\left(\frac{1}{2}\left|\arccos\left(\cos(\Phi)^2-\frac{1}{2}\sin(\Phi)^2\right)\right|\right)^2}{\sin\left(\left|\arccos\left(\cos(\Phi)^2-\frac{1}{2}\sin(\Phi)^2\right)\right|\right)^2}}\right)-\pi$$

To solve the problem of the fair 5-sided prism, we need an equilateral spherical triangle with an area of 4π (the surface area of the unit sphere) divided by 5 (the number of sides), or approximately 2.513. An exact solution probably isn't possible – the equation is much more complicated than a fifth order polynomial. However, with Maple, the value of $\Phi = 1.138$ can be approximated. See the graph. With some regular trigonometry, this corresponds to an object made of two equilateral triangles 1.57 on a side, and three .84 x 1.57 rectangles. In an actual trial of this figure, out of 90 tosses, the triangles came up 50 times. If the model worked, the triangles should come up around 36 times. The results of tossing the object have consistently been biased towards the triangles.

For a pyramid, the problem is trickier due to the fact that the center of gravity changes as the height changes. For a solid pyramid with a height of 1, the centroid is at a height of $(1-\sqrt[3]{\frac{1}{2}})$. This is easily calculated by using the cone volume formula ($V = \frac{1}{3}Ah$), the area of an octahedron with unit edges ($\frac{\sqrt{2}}{3}$), and some basic algebra. The angle in radians of .212535 approximates a spherical square with area $4\pi/5$. With the ratio between the height and the

centroid known, the fair solid pyramid has a height of 1, a square base 1.352 on a side, and four isosceles triangles with 1.38344 as the doubled side.

Using the Geometric Model as a basis, many new dice can be created. Some examples are given in the following paragraphs:

Prisms. Let the base be a regular n-gon with unit sides, and let h be the height of the prism. Using the previous methods, for $n = 3$, $h \cong 0.5336$; for $n = 4$, $h = 1$ (the cube); for $n = 5$, $h \cong 1.5060$; for $n = 6$, $h \cong 2.0598$; for $n = 7$, $h \cong 2.6602$; for $n = 8$, $h \cong 3.3049$; for $n = 9$, $h \cong 3.9916$; for $n = 10$, $h \cong 4.7181$; for $n = 11$, $h \cong 5.4824$; for $n = 12$, $h \cong 6.2828$.

Antiprisms. Let the base be a regular n-gon with unit sides, and let h be the height of the antiprism. The height of fair antiprisms is higher than that of fair prisms. The $n = 3$ antiprism is the octahedron, with $h = .8165 = \frac{\sqrt{6}}{3}$. For $n = 4$, $h \cong 1.4953$; for $n = 5$, $h \cong 2.2270$; for $n = 6$, $h \cong 3.0244$; for $n = 7$, $h \cong 3.8864$; for $n = 8$, $h \cong 4.8100$; for $n = 9$, $h \cong 5.7921$; for $n = 10$, $h \cong 6.8297$; for $n = 11$, $h \cong 7.9202$; for $n = 12$, $h \cong 9.0613$.

Snipped Cube. Remove four corners from a unit cube that define a tetrahedron. Let h be the side of one of the resulting equilateral triangles. At some point before the figure becomes a tetrahedron, a fair die of 10 faces will result.

Snipped Octahedron. Remove two opposite corners of a unit octahedron. Let h be a side on the resulting square. At some point before the original faces become unstable, a fair die of 10 faces will result.

In Chapter 5, I'll explain why none of these objects are fair.

CHAPTER 4

AN ASYMMETRICAL FAIR DIE

An application of the previous chapters can produce a die which is asymmetrical. The die is made from cardboard, so it is a shell of sorts. For an example, follow these steps:

1. Find the centroid of a scalene tetrahedron.

2. Move a corner from the centroid while preserving direction.

3. Calculate the new centroid of the figure.

4. Draw a ray from the new centroid through the old centroid. Find the point on the figure where the ray intersects.

5. Calculate how much mass is required at this point to realign the old centroid.

More explicitly, start with the scalene tetrahedron marked by points (5,3,4), (5,-3,-4), (-5,0,5), and (-5,0,-5). Each face of this object is a triangle with edges of length $\sqrt{190}$, $\sqrt{110}$, and 10. All faces are identical, and all faces have identical relationships with each other, so it is an isohedron. Note how the (3,4,5) Pythagorean triangle is exploited. The centroid of this tetrahedron is at the origin.

Move the point (-5,0,-5) to (-6,0,-6). The resulting object is no longer fair under any model. If we use the Geometric model, the die will be fair again if the centroid can be moved back to the origin. To do that, the four triangles of the new figure must be analyzed for both their centroid and their mass. This can be done using Heron's formula, and by exploiting the fact that the centroid is one third of the distance from a midpoint of a side to the opposite vertex. Using these facts, we obtain complete information for all four triangles; see Table 2.

	Vertices	Edges	Centroid	Area
Triangle A	(5,3,4) (5,–3,–4) (–5,0,5)	$\sqrt{190}$ $\sqrt{110}$ 10	(1.666, 0, 1.666)	52. 20
Triangle B	(5,3,4) (5,–3,–4) (–6,0,–6)	$\sqrt{230}$ $\sqrt{134}$ 10	(1.333, 0, –2)	57. 87
Triangle C	(5,3,4) (–5,0,5) (–6,0,–6)	$\sqrt{230}$ $\sqrt{122}$ $\sqrt{110}$	(–2, 1, 1)	57. 92
Triangle D	(5,–3,–4) (–5,0,5) (–6,0,–6)	$\sqrt{190}$ $\sqrt{134}$ $\sqrt{122}$	(–2, –1, –1.666)	61. 76
Counterweight (discussed later)	(5,3,4) (–4.69, .09, 4.97) (5, –2.82, –3.76)	13.363, 10.164, 9.7	(1.77, .09, 1.74)	49. 07

Table 2. The Triangles of an Asymmetrical Fair Die

The edges are found via the *Euclidean distance* formula. The formula for finding the center of mass of two point masses is (Σ mass$_k$ position$_k$ / Σ mass$_k$) **[Tho** 360**]**. Using this, the centroid for triangles A and B is at ((1.49, 0, -.26) mass = 110.072), and the centroid for triangles C and D is at ((-.2543, -.016, -.3186) mass = 119.68). The centroid for the whole figure can be found at ((-.3273, -.0167, .-3210) mass = 229.756).

The equation of the line containing the old (origin) and new (above) centroids is (-.3273t, -.0167t, -.3210t). The plane containing triangle A is $30x - 80y + 60z = 150$. Finding these involved using linear algebra again **[Tho** 733**]**. The point of intersection is at a = (1.77, .09, 1.74). Using the formula above, a mass of 42.5 is needed at this point. To find the counterweight triangle above, draw a line from (5,3,4), which is closest to a, through a, then half that distance again (median property) to find the point b = (.155, -1.365, .61). I want b to be a midpoint of points on the lines (5, 3+6s, 4+8s) and (-5+10t, 3t, 5-t). It turns out that three points are collinear if the matrix formed by those points has determinant zero **[Zwi** 269**]**. Using this matrix and Maple, t =(486 + 517s)/(486+1003s). From there, use the Euclidean distance formula to determine that s = -.97 would make b a midpoint. The counterweight triangle weighs 49.07, so a circle of radius 1.45 must be removed, centered at a. Finished! But, in truth, this figure in only fair in an idealized world.

CHAPTER 5

THE ISOHEDRA

Polyhedra with congruent faces. For which triangles T does there exist a convex polygon with all its faces congruent to T, either with or without reflections allowed? **[**_Unsolved Problems in Geometry_, p 75**]**

Mistake! The above problem is solved, not unsolved. All triangles T can be made into an eight-sided tetragonal scalenohedron, a figure known by crystallographers. I mention this mistake to illustrate the obscurity of isohedra. The overlooked tetragonal scalenohedron is an isohedron. All isohedra (literally, "equal surfaces") share two properties:

P1 -- All faces have identical relationships with the other faces.

P2 -- All faces have identical relationships with the centroid.

Note that either property implies that all faces are identical. Coincidentally, the two properties together guarantee that a die will work in both mathematical models. A loaded die satisfies P1 and not P2. The figure below – a combination of skew prisms and pyramids – satisfies P2 and not P1. Together, they allow for an interesting consequence.

Proposition: An isohedron is a fair die.
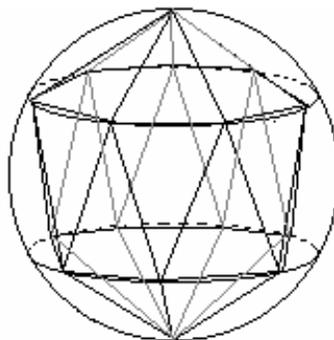
Figure 5. A Polyhedron with Identical Faces which is not Isohedral

Figure 5 suggests how a convex polyhedron can be formed with $4n$ identical isosceles triangles for $n > 3$. Whether other polyhedra exist that satisfy P2 and not P1 is an unsolved question **[Cro** 90**].** I do not know if there exist unloaded figures which satisfy P1 and not P2.

For all faces to have identical relationships with each other, the *vertex degrees* for corresponding vertices on each face need to be identical. An illustration: At the '4' vertex, three more triangles will be found in an isohedral figure, for a total of 4 planes meeting at the '4' vertex, and every face must have vertex degrees 4, 6, and 9.
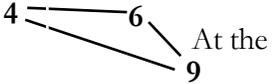
Euler's formula, V+F-E=2, can be used to figure out how many faces such a figure would have. Let F be the number of faces in the figure. The number of '4' vertices will be $F/4$, since every face meets one of the '4' vertices exactly once. The total number of vertices is thus $F/4 + F/6 + F/9$. The number of edges, E, is equal to $\frac{3}{2}F$ (three sides per face, two faces per edge). Substituting for E and V in Euler's formula and solving for F, we obtain the following: $F = \dfrac{2}{(\frac{1}{4}-\frac{1}{2})+(\frac{1}{6}-\frac{1}{2})+(\frac{1}{9}-\frac{1}{2})+1} = 72$.

The above result can be generalized for triangles with vertex degrees a, b, and c:

$$F = \dfrac{2}{(\frac{1}{a}-\frac{1}{2})+(\frac{1}{b}-\frac{1}{2})+(\frac{1}{c}-\frac{1}{2})+1}.$$ It can be extended to isohedra consisting of quadrilateral or pentagons: $$F = \dfrac{2}{(\frac{1}{a}-\frac{1}{2})+(\frac{1}{b}-\frac{1}{2})+(\frac{1}{c}-\frac{1}{2})+(\frac{1}{d}-\frac{1}{2})+1}$$ and

$$F = \dfrac{2}{(\frac{1}{a}-\frac{1}{2})+(\frac{1}{b}-\frac{1}{2})+(\frac{1}{c}-\frac{1}{2})+(\frac{1}{d}-\frac{1}{2})+(\frac{1}{e}-\frac{1}{2})+1}.$$ There are no isohedra based on hexagons or other n-gons, since the denominator would be zero or negative.

We started with a [4,6,9]=72 triangle; that is, a triangle with vertex degrees 4, 6, and 9 leads to a figure with 72 faces. Unfortunately, this is impossible. Consider one of the '9' vertices. Radiating from this point are nine lines, each connecting to a '4' vertex or a '6' vertex. These vertices must alternate, which is impossible with a cyclic arrangement of nine vertices. Either one winds up as an 'odd man out,' so this figure is impossible.

For [a,b,c], if a is odd, then b=c.

Another potentially promising solution is [3,3,4,6] = 24. Can the '3' vertices be adjacent? If so, then two quadrilaterals meet at the edge connecting the '3' vertices. But this leads to a contradiction when we try to add the third quadrilateral to either '3' vertex. So the '3' vertices must not be adjacent. Consider the '6' vertex. It is surrounded by 3 vertices, which connect to '4' vertices. We are then forced to use a quadrilateral with two '4' vertices. This figure is impossible.

For [3,3,a,b], a=b.

Also, *F* must be a positive whole number. Based on the equations and the rules above, an exhaustive list of all possible isohedral vertex degrees follows: [4,4,*X*]=2*X*, [3,3,3,*X*]=2*X*, [3,3,3]=4, [3,6,6]=12, [3,4,3,4]=12, [3,3,3,3,3]=12, [5,5,5]=20, [3,8,8]=24, [4,6,6]=24, [3,4,4,4]=24, [3,3,3,3,4] = 24, [3,5,3,5]=30, [4,6,8]=48, [3,10,10]=60, [5,6,6]=60, [3,4,5,4]=60, [3,3,3,3,5]=60, [4,6,10]=120. As it happens, all of these figures can be constructed, as will be shown. In fact, the same degree sequence may lead to non-isomorphic isohedra, depending upon the symmetry of the faces. A triangle, for example, can be equilateral, isosceles, or scalene; leading to edge codes ααα, ααβ, and αβγ.

Five of the possible figures are Platonic Solids ([3,3,3]=4, [3,3,3,3]=6, etc.). Thirteen more of the possible figures are *duals* of the *Archimedean solids*. The mathematician Catalan was

the first to describe them, in 1865 **[Smi** 58**].** Six more figures, variants of Archimedean duals, were first described by the mathematician Hess in 1883 **[Gru** 76**].** In addition to these twenty four figures, there is an infinite family of isohedral figures with $2n$ sides for each $n > 3$. For n odd, these figures are known as *dipyramids* and *trapezohedrons*, and are duals to the prisms (2 n-gons + n squares) and antiprisms (2 n-gons + $2n$ equilateral triangles). For n even, three variants are possible.

Two more isohedra are the *lens* and the *sphere*. A coin would be isohedral were it not for the edge. A coin is arguably fair if its edge is unstable, but then the edge can be divided according to which face it favors, and a lens-type object results.

This lists *all* of the isohedra. To prove that this list is exhaustive, it's important to understand rotations in three dimensional space. If the point $a = (a,b,c)$ on the unit sphere (the set of points 1 away from the origin) is rotated to the point $d = (d,e,f)$ on the unit sphere, there exist 3x3 matrices A such that $d = A\,a$ **[Zwi** 300**].** This means rotation about some axis through the sphere. The move can be made by first rotating the sphere about the z-axis by an angle of α, then rotating the sphere about the y-axis by θ.

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

The matrix A has the property that it is orthogonal, which means $A^{\mathrm{T}}A = I$. The determinant of this matrix equals 1. If we take the *group* of all 3x3 orthogonal matrices under multiplication we get what is called the orthogonal group, or $O(3)$. The orthogonal matrices with determinant +1 form a subgroup of $O(3)$ which is called the special orthogonal group, or $SO(3)$ **[Dub** 33**].** This group may be identified with the rotations of 3-dimensional Euclidean space; any member of this group will map any point on the unit sphere to another point on the

unit sphere and, conversely, any rotation may be realized by a member of SO(3). For an isohedron to meet the criteria that all faces have identical relationships with the other faces, it must by isomorphic to a *finite subgroup* of *SO*(3). Recall the theorem in Chapter 2. A new proof can be given for this theorem that uses isohedra.

> **Theorem:** ❈⬚, ⿃, and ✋, together with the cyclic and dihedral groups, form all possible finite subgroups of rotations in three dimensional space.

> **Proof:** Let **P** be a plane tangent to the unit sphere. **P** may be represented as a matrix.

Let G be a noncyclic finite subgroup of *SO*(3), containing two elements a and b which have different axes of rotation such that $P \neq a\,P$ and $P \neq b\,P$. Form the set of rotations of this plane, obtaining $X = \{P * g \mid g \in G\}$.

> *Claim 1:* There is no point p on the unit sphere such that $p = a\,p = b\,p$. This is impossible since a and b have different axes of rotation, and only the points on the poles of a rotation are left fixed.

> *Claim 2:* Any ray from the origin passes through a plane in X. Assume this claim is false. For any plane that does not intersect the origin, there is a hemisphere of vectors which do not intersect the plane. Let Q be the intersection of these hemispheres defined by the planes of X. Q cannot contain antipodal points. Else, a line connects them, and all planes in X rotate about this line. $Q = a\,Q = b\,Q$, due to the method of construction. Q has a unique centroid, which leads to a contradiction of the sort seen in Claim 1.

> *Claim 3:* The convex hull of planar intersections visible from the origin is isohedral, if one considers the planes as opaque. With the full *SO*(3), the hull is a solid sphere – for each plane, only the point closest to the origin is visible. But G is finite, so starting from the original

plane **P**, a polygon **p** will be visible from the origin. The hull will be a polyhedron with *n* faces.

This polyhedron is identical when rotated by any element of **G**, giving the claimed isohedron.

Claim 4: The isohedra with vertex degrees and edge codes as given in Table 3 all correspond to ✌, the icosahedral group.

Vertex Degrees	Faces	Edges	Vert	Edge Code	Order Rigid	Order Full	Name
[3,3,3,3,3]	12	30	20	ααααα	60	120	Regular Dodecahedron
[5,5,5]	20	30	12	ααα	60	120	Regular Icosahedron
[3,5,3,5]	30	60	32	αααα	60	120	Rhombic Triacontahedron
[3,10,10]	60	90	32	ααβ	60	120	Triakis Icosahedron
[5,6,6]	60	90	32	ααβ	60	120	Pentakis Dodecahedron
[3,4,5,4]	60	120	62	ααββ	60	120	Trapezoidal Hexecontahedron
[3,3,3,3,5]	60	150	92	αββγγ	60	60	Pentagonal Hexecontahedron
[4,6,10]	120	180	62	αβγ	60	120	Hexakis Icosahedron

Table 3. The Icosahedral Isohedra (see Figure 6)

In all eight possible cases, a rigid rotation group of *order* 60 results. In the case of [4,6,10], only 60 triangles will be considered. Table 4 contains all 13 groups of order 60.

	Number of elements of a given order											
Group	1	2	3	4	5	6	10	12	15	20	30	60
2^2 x 15	1	3	2		4	6	12		8		24	
60	1	1	2	2	4	2	4	4	8	8	8	16
S3 x 10	1	7	2		4	2	28		8		8	
5 x 3:4	1	1	2	6	4	2	4		8	24	8	
A4 x 5	1	3	8		4		12		32			
D10 x 6	1	11	2		4	22	4		8		8	
10.2 x 3	1	1	2	10	4	2	4	20	8		8	
3 x 5:4	1	5	2	10	4	10		20	8			
D60	1	31	2		4	2	4		8		8	
30.2	1	1	2	30	4	2	4		8		8	
S3 x D10	1	23	2		4	10	12		8			
S3 + (5:4)	1	5	2	30	4	10			8			
A5	1	15	20		24							

Table 4. Groups of Order 60

Tables 4, 6, and 8 were all generated with GAP **[GAP].** A table containing the notation used by GAP can be found in the Glossary.

The first twelve groups all contain an element of order 15. Each element needs to be a matrix which accomplishes a rotation. Powers of that matrix will all accomplish the same rotation, and leave some point fixed. Thus, 15 elements will leave the same point fixed – either a vertex, the midpoint of an edge, or the centroid of a face. All other vertices will be rotated around the resulting pole, defining some number of circles. Each circle will contain 15n vertices, for some integer n. There can be 12, 20, 32, 62, or 92 vertices, so two vertices cannot be placed upon these circles, and therefore must be the fixed points defining the poles. Each element will map the original figure to itself, including edges. But no vertex has vertex degree of 15. The last group is *isomorphic* to �, the group of symmetries of an icosahedron.

The above sequence of steps will prove typical in the following claims:

1. Isolate an order of some element, giving a cyclic subgroup of that same order.

2. Examine the number of vertices, then determine the number of fixed points and circles that will be defined by the given element.

3. Any vertices which are fixed points have some number of edges which is equal to the vertex degree for that vertex.

4. Edges from connected to a fixed vertex must be mapped to like edges when operated upon by a member of **G**.

5. Repeat these steps for each order, and determine which orders are impossible.
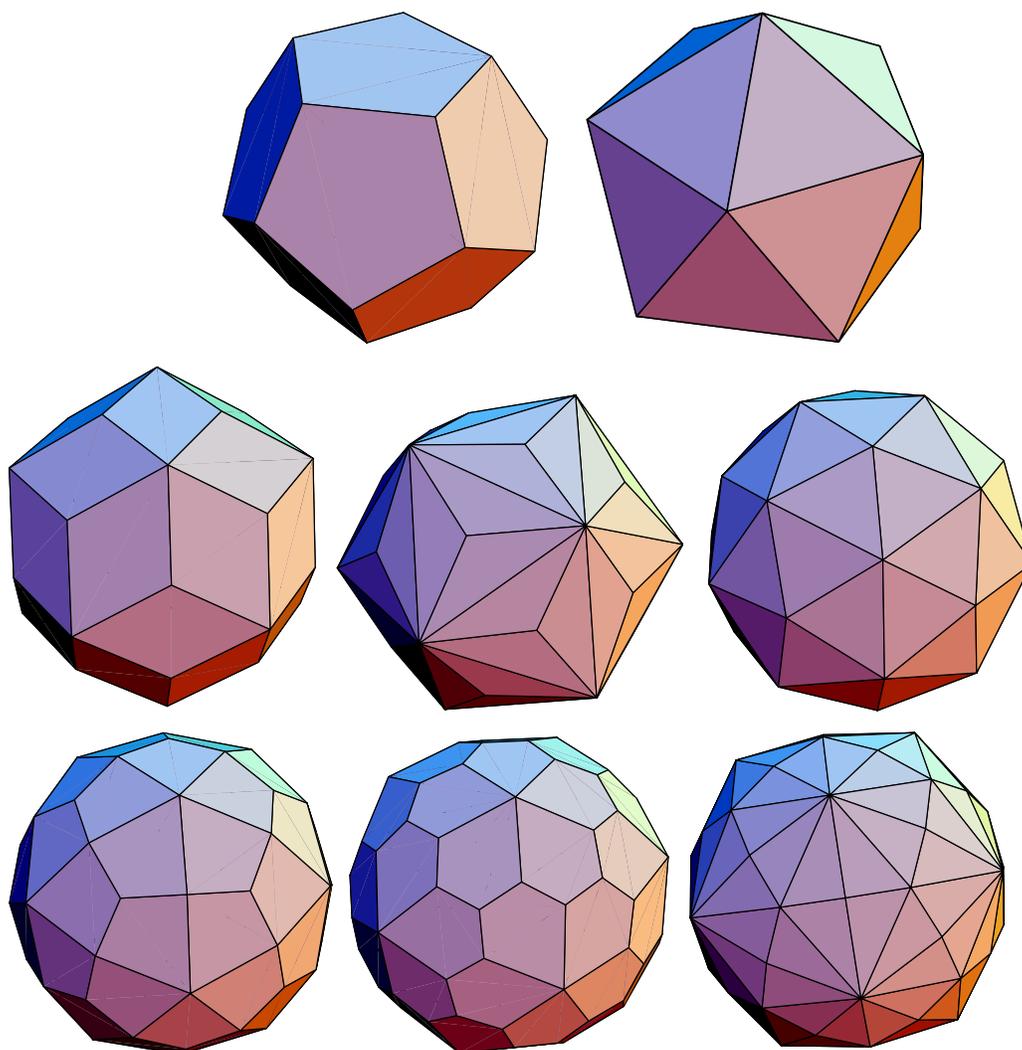
Figure 6. The Icosahedral Isohedra (as given in Table 3)

Claim 5: The isohedra with vertex degrees [4,4,X] and [3,3,3,Y] correspond to a dihedral group of some sort. If X = 4 or Y=3, other groups are possible. In each case, there are 2X or 2Y faces, and thus exactly 2 vertices with vertex degree X or Y. The perpendicular bisecting plane for the segment connecting these two vertices will bisect the isohedron. The cross-section will be one of three different types. In all rotations, the cross-section will be mapped back to the cross-section, giving a dihedral group. For [3,3,3,Y], two variants are possible, depending upon whether the faces are symmetrical. For [4,4,X], only one type of isohedron

exists if X is odd, otherwise three variants are possible. For the first, use 2X isosceles triangles. For the second, move alternate '4' vertices an equal amount toward the origin. For the third, move alternate '4' vertices above and below the bisecting plane.

Claim 6: The isohedra with vertex degrees and edge codes as given in Table 5 all correspond to ⌂, the octahedral group.

Vertex Degrees	Faces	Edges	Vert	Edge Code	Order Rigid	Order Full	Name
[3,3,3,3]	6	12	8	αααα	24	48	Cube
[4,4,4]	8	12	6	ααα	24	48	Octahedron
[3,4,3,4]	12	24	14	αααα	24	48	Rhombic Dodecahedron
[3,3,3,3,3]	12	30	20	αββββ	12	24	Octahedral Pentagonal Dodecahedron
[3,8,8]	24	36	14	αββ	24	48	Triakis Octahedron
[4,6,6]	24	36	14	αββ	24	48	Tetrakis Hexahedron
[3,4,4,4]	24	48	26	ααββ	24	48	Trapezoidal Icositetrahedron
[3,3,3,3,4]	24	60	38	αββγγ	24	24	Pentagonal Icositetrahedron
[4,6,8]	48	72	26	αβγ	24	48	Hexakis Octahedron

Table 5. The Octahedral Isohedra (see Figure 7)

To prove this, please reference Table 6 – The Groups of Order 24. Using previous methods, the first thirteen groups require a vertex degree sequence of X, 6, X; which is impossible. The last two groups are both valid. See Figure 7 for the resulting isohedra.

	# of elements of order …							
Group	1	2	3	4	6	8	12	24
2^3 x 3	1	7	2		14			
12 x 2	1	3	2	4	6		8	
24	1	1	2	2	2	4	4	8
D8 x 3	1	5	2	2	10	4		
Q8 x 3	1	1	2	6	2		12	
S3 x 2^2	1	15	2		6			
S3 x 4	1	7	2	8	2		4	
2 x 6.2	1	3	2	12	6			

	# of elements of order …					
Group	1	2	3	4	6	8	12
12.2	1	1	2	2	2	12	4
SL(2,3)	1	1	8	6	8		
grp(24,11)	1	9	2	6	6		
D24	1	13	2	2	2		4
Q8 + S3	1	1	2	14	2		4
A4 x 2	1	7	8	8			
S4	1	9	8	6			

Table 6. Groups of Order 24.   .
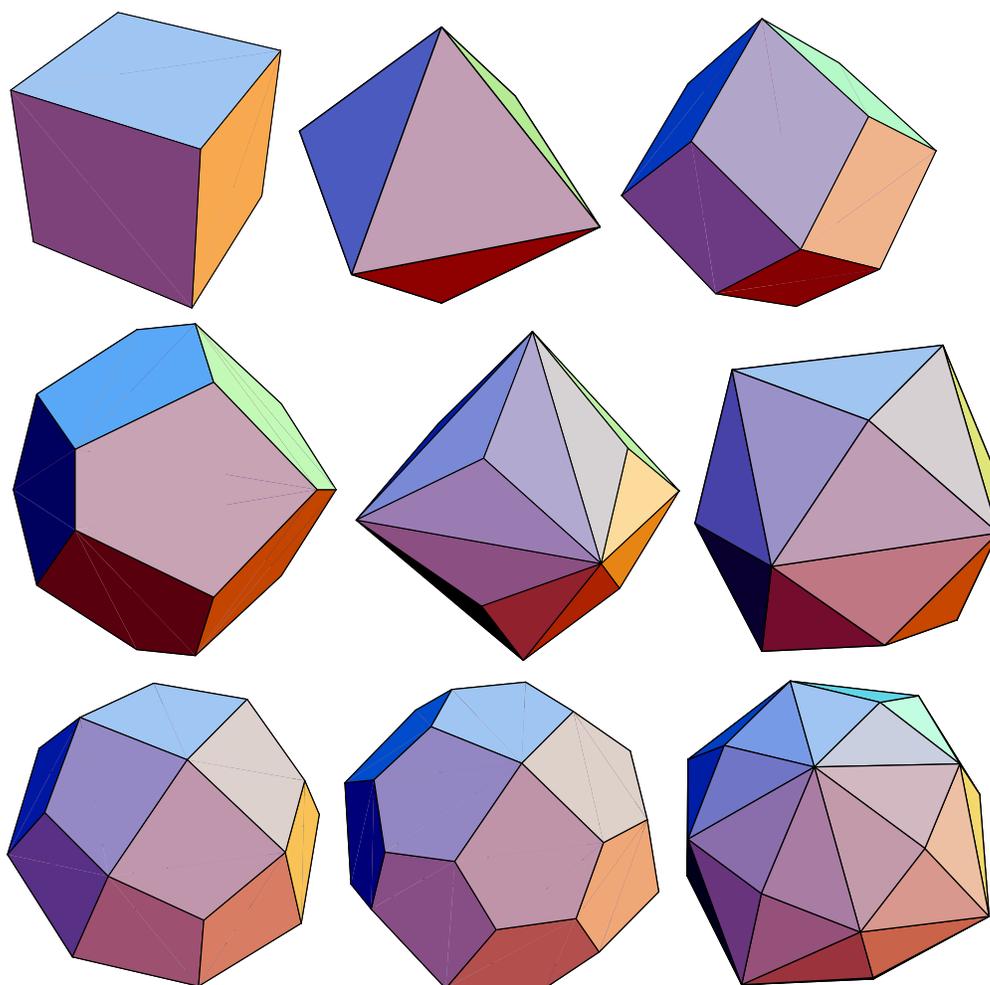
Figure 7. The Octahedral Isohedra (as given in Table 5)

Claim 7: The isohedra with vertex degrees and edge codes as given in Table 7 all correspond to ❋, the tetrahedral group.

Vertex Degrees	Faces	Edges	Vert	Edge Code	Order Rigid	Order Full	Name
[3,3,3]	4	6	4	ααα	12	24	Regular Tetrahedron
[3,6,6]	12	18	8	αββ	12	24	Triakis Tetrahedron
[3,4,3,4]	12	24	14	ααββ	12	24	Trapezoidal Dodecahedron
[3,3,3,3,3]	12	30	20	αββγγ	12	12	Tetragonal Pentagonal Dodecahedron
[4,6,6]	24	36	14	αβγ	12	24	Hexakis Tetrahedron
[3,4,4,4]	24	48	26	ααβγ	12	24	Dyakis Dodecahedron

Table 7. The Tetrahedral Isohedra

As before, looking at all possible groups of order 12 provides the answer (see Table 8). The first four groups contain an element of order 6, which would require a vertex degree sequence of X, 6, X. No such isohedra exists. The last group is the correct one.

Group	1	2	3	4	6	12
6 x 2	1	3	2		6	
12	1	1	2	2	2	4
D12	1	7	2		2	
6.2	1	1	2	6	2	
A4	1	3	8			

Table 8. Groups of Order 12.
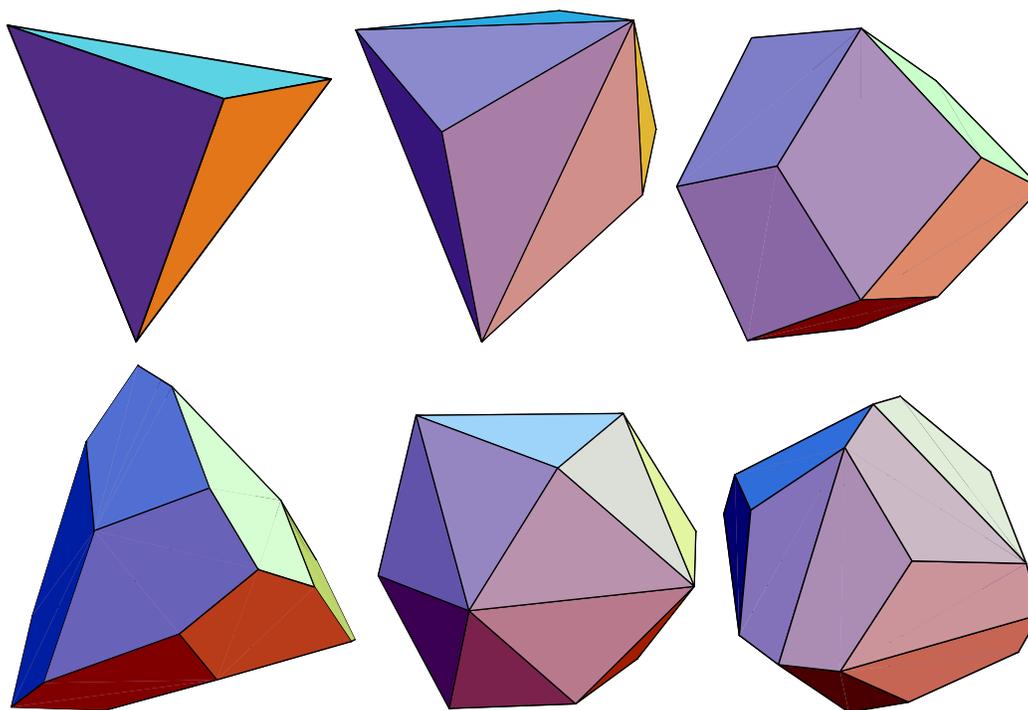
Figure 8. The Tetrahedral Isohedra

In summary, a finite subgroup of SO(3) would require an isohedron, which in turn requires a certain group structure. All other group structures are impossible. This completes the proof. In the process, all possible isohedra have been listed.

CHAPTER 6

THE ENERGY STATE MODEL

Dice bounce. Depending on the type of die, and the surface, the die will bounce a different number of times. Before each bounce, the Geometric Model can be used to determine which face of the die has current influence. During each bounce, a matrix can be set up to determine the relative probabilities that other faces will inherit influence for the next bounce. Thus, a series of matrices can be used to model what is happening to a die. All of these matrices are different, though, because the amount of energy possessed by the die decreases geometrically **[Tho** 556]**. In order for the model to work, some nth bounce must relate to an identity matrix, relating to the state where there isn't enough energy left to shift the die from any face to any other face.

Bouncing dice are somewhat similar to a Markov process. Let b = bounciness, e = kinetic energy, n = bounce number, h_{jk} = Topple height from side j to side k, q_{jk} = (radians swept from point under centroid on face j by edge k)/2π, $f(n)$ = 0 if n $\leq$ 0, 1 if n $\geq$ 1, else n.

$A_{jk} = q_{jk} (1 - f(h_{jk} / e b^n))$ A_{jj} = 1 - (sum of other entries in j column)

A_n = Matrix of the A entries.

The initial state vector x can be determined by the Geometric model. The final probability distribution y is given by $A_6\ A_5\ A_4\ A_3\ A_2\ A_1\ x = y$. As n approaches infinity, A_n approaches the identity matrix. Usually, this happens before n reaches 10. For an initial demonstration, this system can be used to show why nickels rarely land on edge.

A nickel is 2mm thick, with a diameter of 21mm. Using trigonometry, the topple heights are .094mm and 21.095mm. Other values were assigned arbitrarily: $e = 200$, $b = .2$. In order, the rows and columns represent heads, edge, tails.

$$\begin{bmatrix} 1. & 0 & 0 \\ 0 & 1. & 0 \\ 0 & 0 & 1. \end{bmatrix} \begin{bmatrix} 1.00 & .346 & 0 \\ 0 & .308 & 0 \\ 0 & .346 & 1.00 \end{bmatrix} \begin{bmatrix} 1.00 & .461 & 0 \\ 0 & .0778 & 0 \\ 0 & .461 & 1.00 \end{bmatrix} *$$

$$\begin{bmatrix} 1.00 & .484 & 0 \\ 0 & .0316 & 0 \\ 0 & .484 & 1.00 \end{bmatrix} \begin{bmatrix} .530 & .489 & 0 \\ .470 & .0223 & .470 \\ 0 & .489 & .530 \end{bmatrix} \begin{bmatrix} .46 \\ .08 \\ .46 \end{bmatrix} = \begin{bmatrix} .4999 \\ .0003287 \\ .4999 \end{bmatrix}$$

This says that nickels land on edge 3 out of every 10000 tosses. With a nickel, the probability of an edge landing can be somewhat skewed by the limited number of bounces. To counteract this, squaring each A_n will better define the tendency of each bounce. Doing this, we obtain that a nickel will land on edge roughly 15 times out of every hundred million tosses. There are several historical cases where coins really did land on edge (the most famous involved a flip by the composer of *Trumpet Voluntary*), so this set of odds is believable.

When this model is used with the .84 x 1.57 prism from chapter 3, the prism turns out to be progressively more unfair as b increases. When $e = 250$ and $b = .235$, the triangles come up 5 out of every 9 throws ... this correlates perfectly with the experiment using an actual model. This implies I've rigged something, which is true. The odds depend heavily on the force of the toss, so I picked a force that gave the odds I wanted.

$$A1 := \begin{bmatrix} .017 & 0 & .34 & .34 & .34 \\ 0 & .017 & .34 & .34 & .34 \\ .32 & .32 & .014 & .16 & .16 \\ .32 & .32 & .16 & .014 & .16 \\ .32 & .32 & .16 & .16 & .014 \end{bmatrix} \quad A2 := \begin{bmatrix} .075 & 0 & .33 & .33 & .33 \\ 0 & .075 & .33 & .33 & .33 \\ .31 & .31 & .061 & .14 & .14 \\ .31 & .31 & .14 & .061 & .14 \\ .31 & .31 & .14 & .14 & .061 \end{bmatrix}$$

$$A3 := \begin{bmatrix} .31 & 0 & .28 & .28 & .28 \\ 0 & .31 & .28 & .28 & .28 \\ .23 & .23 & .26 & .091 & .091 \\ .23 & .23 & .091 & .26 & .091 \\ .23 & .23 & .091 & .091 & .26 \end{bmatrix} \quad A4 := \begin{bmatrix} 1.0 & 0 & .08 & .08 & .08 \\ 0 & 1.0 & .08 & .08 & .08 \\ 0 & 0 & .83 & 0 & 0 \\ 0 & 0 & 0 & .83 & 0 \\ 0 & 0 & 0 & 0 & .83 \end{bmatrix}$$

The matrices provide a model of a five-sided prism. A_5 is the identity matrix, which means the object has come to rest. Multiplying these matrices together, we obtain that the triangles occur 5 times out of every 9 tosses, which is the rigged result.

This model can be used to show that dimpled dice are unfair. We'll use a six-sided cube with an edge of 10, and use hemispheres with a radius of .984745 to represent the dimples (volume of a dimple is thus 2). Calculate the centroid by removing a solid disk of volume 12 from each face, leaving an isohedral figure of volume 928. Remove the dimples from each disk, then put the remains of each solid disk back to their original faces. The volume of the '1' disk is ten, whereas the volume of the '6' disk is zero. The six solid disks are equivalent to masses of 10 at {1,0,0}, {0,3,0}, and {0,0,5}. This is equivalent to a point mass of 30 at {1/3, 1, 5/3}. The centroid of the dimpled die is at {.0104, .0313, .0522}.

With this data and some basic trigonometry, the Energy State model provides the following set of matrices. I used e=200 and b=.4 in this model. Altering these numbers did not change the final result to any great extent.

$$K \begin{bmatrix} 0.552812 & 0.11265 & 0.112689 & 0.112729 & 0.112768 & 0 \\ 0.112303 & 0.552655 & 0.112381 & 0.112421 & 0 & 0.112499 \\ 0.111951 & 0.11199 & 0.552498 & 0 & 0.112107 & 0.112146 \\ 0.111643 & 0.111682 & 0 & 0.552342 & 0.111799 & 0.111838 \\ 0.111291 & 0 & 0.111369 & 0.111408 & 0.552186 & 0.111486 \\ 0 & 0.111023 & 0.111062 & 0.111101 & 0.11114 & 0.552031 \end{bmatrix} \text{Final bounce}$$

$$K \begin{bmatrix} 0.221125 & 0.19602 & 0.196036 & 0.196051 & 0.196067 & 0 \\ 0.195521 & 0.221062 & 0.195553 & 0.195568 & 0 & 0.1956 \\ 0.19496 & 0.194976 & 0.220999 & 0 & 0.195023 & 0.195038 \\ 0.194477 & 0.194493 & 0 & 0.220937 & 0.19454 & 0.194555 \\ 0.193917 & 0 & 0.193948 & 0.193963 & 0.220874 & 0.193994 \\ 0 & 0.193449 & 0.193465 & 0.19348 & 0.193496 & 0.220812 \end{bmatrix}$$

$$K \begin{bmatrix} 0.0884499 & 0.229368 & 0.229374 & 0.229381 & 0.229387 & 0 \\ 0.228808 & 0.0884248 & 0.228821 & 0.228827 & 0 & 0.22884 \\ 0.228164 & 0.22817 & 0.0883997 & 0 & 0.228189 & 0.228195 \\ 0.227611 & 0.227617 & 0 & 0.0883747 & 0.227636 & 0.227642 \\ 0.226967 & 0 & 0.226979 & 0.226985 & 0.0883498 & 0.226998 \\ 0 & 0.22642 & 0.226426 & 0.226432 & 0.226438 & 0.0883249 \end{bmatrix}$$

$$K \begin{bmatrix} 0.03538 & 0.242707 & 0.24271 & 0.242712 & 0.242715 & 0 \\ 0.242123 & 0.0353699 & 0.242128 & 0.242131 & 0 & 0.242136 \\ 0.241446 & 0.241448 & 0.0353599 & 0 & 0.241456 & 0.241458 \\ 0.240864 & 0.240867 & 0 & 0.0353499 & 0.240874 & 0.240877 \\ 0.240187 & 0 & 0.240192 & 0.240194 & 0.0353399 & 0.240199 \\ 0 & 0.239608 & 0.23961 & 0.239613 & 0.239615 & 0.03533 \end{bmatrix} K \begin{bmatrix} 0.16666 \\ 0.16666 \\ 0.16666 \\ 0.16666 \\ 0.16666 \\ 0.16666 \end{bmatrix} =$$

$$K \begin{bmatrix} 0.1679 \\ 0.1674 \\ 0.1669 \\ 0.1665 \\ 0.1659 \\ 0.1654 \end{bmatrix}$$

This last matrix is extremely interesting! It provides the odds for how often a number on a dimpled die will be the Down face. For the up face, and 10,000 tosses:
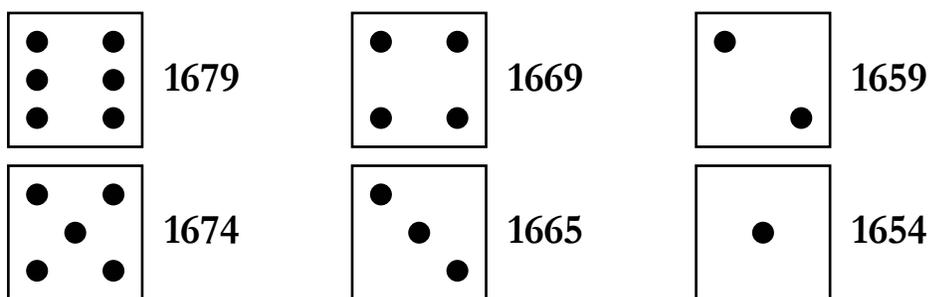
Figure 9. Ten Thousand Tosses of a Dimpled Die

According to the Energy State Model, dimpled dice are not fair. One would need to toss a die many times to see this phenomenon, however.

The only fair dice are the isohedra. This completes the list of fair dice, and this thesis.

BIBLIOGRAPHY

[Arm] Armstrong, M. A. *Groups and Symmetry.* New York: Springer-Verlag, 1988.

[Big] Biggs, Norman. *Algebraic Graph Theory.* Cambridge: Cambridge University Press, 1974.

[Bol] Bollobas, Bela. *Graph Theory: An Introductory Course.* New York: Springer-Verlag, 1979.

[Cox] Coxeter, H. S. M. *Regular Polytopes.* New York: Dover Publications, Inc., 1973.

[Cro] Croft, Hallard T., Falconer, Kenneth J., and Guy, Richard K. *Unsolved Problems in Geometry.* New York: Springer-Verlag, 1991.

[Dil] Dillencourt, Michael B. "Polyhedra of Small Order and Their Hamiltonian Properties." *Journal of Combinatorial Theory* B 66 (1996): 87-122.

[Dub] Dubrovin, B. A., Fomenko, A. T., and Novikov, S. P. *Modern Geometry – Methods and Applications.* New York: Springer-Verlag, 1984.

[Gap] Martin Schönert et.al. GAP -- Groups, Algorithms, and Programming. Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, fifth edition, 1995.

[Gar] Gardner, Martin. *New Mathematical Diversions.* Chicago: University of Chicago Press, 1966.

[Gru] Grunbaum, Branko, and Shepard, G. C. *The Geometric Vein.* New York: Springer-Verlag, 1981.

[Har] Harary, Frank. *Graph Theory.* Menlo Park: Addison-Wesley Publishing Company, 1972.

[Hor] Horn, Roger A., and Johnson, Charles R. *Matrix Analysis.* New York: Cambridge University Press, 1985.

[Pug]   Pugh, Anthony. *Polyhedra: A Visual Approach.* Los Angeles: University of California Press, 1976.

[Slo]   Sloane, N. J. A., and Plouffe, Simon. *The Encyclopedia of Integer Sequences.* New York: Academic Press, 1995.

[Tho]   Thomas, George B. Jr., and Finney, Ross L. *Calculus and Analytic Geometry.* Menlo Park: Addison-Wesley Publishing Company, 1992.

[Upe]   Upensky, J.V. *Introduction to Mathematical Probability.* New York: McGraw-Hill Book Company, Inc., 1937.

[Wel]   Wells, David. *The Penguin Dictionary of Curious and Interesting Geometry.* New York: Penguin Books, 1991.

[Wer]   Wertz, James R. *Spacecraft Attitude Determination and Control.* Boston: D. Reidel Publishing Company, 1985.

[Yal]   Yale, Paul B. *Geometry and Symmetry.* New York: Dover Publications, Inc., 1988.

[Zwi]   Zwillinger, Daniel. *CRC Standard Mathematical Tables and Formulae*, 30th ed. New York: CRC Press, 1996.

GLOSSARY

Abelian. A group G is *abelian* iff for all $a, b \in G$, $a b = b a$.

Antiprism. A polyhedron formed by two *n*-sided polygons and $2n$ isosceles triangles.

Archimedean Solid. A polyhedron where all vertices are identical, and all faces are regular polygons. There are thirteen such polyhedra.

Centroid. The point such that the first moment of a geometric object about every line through the point is zero. Also called the ***center of mass***.

Cyclic. A finite group is *cyclic* if it can be generated from a single element.

Degree. The *degree* deg x of a vertex x in a graph is the number of adjacent vertices.

Die, dice. A rigid solid with *n* stable faces.

Dihedral. A group which is isomorphic to the symmetry group [2, *n*]. It correlates to the group of symmetries of a regular *n*-gon.

Dual. Find the center points of each face of polyhedron *A*. Let these points be the vertices of polyhedron *B*, the *dual* of *A*.

Edge. In a simple graph, an *edge* is an element $e \in E$ (the set of edges) that is incident to two vertices $v, w \in V$ (the set of vertices).

Eigenvector. An *eigenvector* $\boldsymbol{x}$ of a matrix $\boldsymbol{A}$ is a nonzero solution to the system $\boldsymbol{A} \boldsymbol{x} = \lambda \boldsymbol{x}$.

Euclidean Distance. For points $a = \{a_1, a_2, \ldots, a_n\}$ and $b = \{b_1, b_2, \ldots, b_n\}$, the *Euclidean distance* is defined as $\left(\sum_{i=1}^{n} (a_i - b_i)^2 \right)^{1/2}$.

Face. A distinct collection of points that touches the table when the object is at rest. A coin's faces are heads, tails, and edge.

Fair. In probability, a trial is *fair* if for any two possible outcomes E and F, $P(E) = P(F)$. Here, a tossed die will be *fair* if all faces are equally likely to wind up on the table.

Group. A *group* is a non-empty set G and a "product" operation defined on G such that: (1) For any pair of elements a, b in G, the product ab is unique and is in G. (2) There is an element, e, in G, the identity, such that $eb = be = b$ for any b in G. (3) For any a in G, there is an element x in G, called the inverse of a, such that $ax = xa = e$. (4) For all $a, b, c \in G$, $a(bc) = (ab)c$.

Icosahedral. A group which is isomorphic to the symmetry group [3, 5].

Isohedral. A polyhedron is *isohedral* if any face can be mapped directly onto any other face by some rotation or reflection of the polyhedron onto itself. Such a figure is called an *isohedron.*

Isomorphic. Two groups G_1 and G_2 are *isomorphic* if there exists a 1-1 and onto function $\varphi : G_1 \rightarrow G_2$ such that $\varphi(a\,b) = \varphi(a)\,\varphi(b)$ for all $a, b \in G_1$.

Law of Large Numbers. Given by Jakob Bernoulli. Suppose the probability of an event E is p, and in n independent trials the event E occurs n_1 times. Then for an arbitrary positive number ε, $P(|n_1/n - p| < \varepsilon) \geq 1 - 1/(4\varepsilon^2 n)$.

Octahedral. A group which is isomorphic to the symmetry group [3, 4].

Order. The *order* #(*G*) of a group is the number of elements in the group.

Orthogonal. A square matrix is *orthogonal* if $A^{\mathrm{T}}A = I$.

Polyhedral. A graph is *polyhedral* if and only if it is planar and 3-connected.

Polyhedron. A *polyhedron* is a solid figure with many faces.

Rotation Group. A *rotation group* is a set of orthogonal matrices that forms a group of rotations of *n*-dimensional space.

Stable Face. A stable face is one where the minimal Euclidean distance from the center of mass of the solid to the face defines a segment perpendicular to the face (i.e., the center of mass is directly above the face). The object can be placed at rest in *n* distinct ways.

Subgroup. *H* is a *subgroup* of *G* iff *H* is a group, $H \subseteq G$, and for all $a, b \in H$, $ab \in H$ using the binary operation of *G*.

Symmetry Group. A *symmetry group* [m, n] is a group with generator elements *a* and *b* such that $a^{\mathrm{m}} = b^{\mathrm{n}} = (a\,b)^2 = e$ (the identity element).

Tetrahedral. A group which is isomorphic to the symmetry group [3, 3].

m	Cyclic group of order m
Dm	Dihedral group of order m
Qm	Quaternion group of order m
QDm	Quasi-Dihedral group of order m
Sm	Symmetric group on m points
Am	Alternating group on m points
SL(d,q)	Special Linear group
GL(d,q)	General Linear group
K^n	Direct power of n copies of K
K:H	Split extension of K by H
K.H	Non-split extension of K and H
K+H	Subdirect product with identified factor groups of K and H
KYH	Central amalgamated product of the groups K and H
K H	Direct product of K and H

Table 9. Notation of Group Names used by GAP.