TRIGONOMETRIC IDENTITIES, LINEAR ALGEBRA AND COMPUTER ALGEBRA

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Introduction. This is a story about

- Some surprising looking trigonometric and combinatorial sums.
- Some nice applications of elementary linear algebra.
- The way that computer algebra packages can change the way that mathematics is done.

Before progressing we invite the reader to try to prove the following facts.

(1) $$\sum_{m=1}^{99} \frac{\sin\left(\frac{17m\pi}{100}\right)\sin\left(\frac{39m\pi}{100}\right)}{1+\cos\left(\frac{m\pi}{100}\right)} = 1037.$$

(2) If $n \equiv 0 \pmod 2$ and $1 \le j \le k \le n$ then

$$\sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{\cos\left(\frac{m\pi}{n+1}\right)} = (n+1)\sin\left(\frac{j\pi}{2}\right)\sin\left(\frac{(k-1)\pi}{2}\right).$$

(3) If $1 \le j \le k \le n$ and $\beta \ne 2\cos\left(\frac{m\pi}{n+1}\right)$ is rational, then $\displaystyle\sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{2\cos\left(\frac{m\pi}{n+1}\right)+\beta}$ is rational.

(4) if $n \not\equiv 0 \pmod 7$ then

$$\frac{7}{n}\sum_{m=1}^{n} \frac{\cos\left(\frac{2m\pi}{n}\right)}{8\cos^3\left(\frac{2m\pi}{n}\right)+4\cos^2\left(\frac{2m\pi}{n}\right)-4\cos\left(\frac{2m\pi}{n}\right)-1} \equiv n^5 \pmod 7.$$

We came across these types of identities doing some research in Banach space geometry. In looking at certain bases, it became necessary to consider the following $n \times n$ matrix.

$$T_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & \ddots & \\ 0 & 1 & 1 & 1 & \ddots & \\ 0 & 0 & 1 & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ 0 & & & & & 1 \end{pmatrix}.$$

Maple checked the first few of these for invertibility and output the inverses when these existed. The evidence was pretty convincing that

$$\det(T_n) = \begin{cases} -1, & \text{if } n \equiv 0 \pmod 3 \\ 1, & \text{if } n \equiv 1 \pmod 3 \\ 0, & \text{if } n \equiv 2 \pmod 3). \end{cases}$$

and that if $n \not\equiv 2 \pmod 3$ then the entries of T_n^{-1} are all either 0, 1 or -1. Actually, it isn't too hard to write a recurrence for $\det(T_n)$. The point however is that it is much easier to check that a matrix S is the inverse of T_n than it is to algorithmically calculate S. Here the computer algebra package is critical. Given T_3^{-1}, T_6^{-1}, T_9^{-1} and T_{12}^{-1}, it is easy to guess the general form for T_n^{-1} for $n \equiv 0 \pmod 3$. Calculating the small n cases by hand is possible, but in practice most of us would not have had the patience to persist long enough to see the patterns forming. In this particular case, if one lets

$$D = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad U = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and then defines S_{3k} to be the $3k \times 3k$ matrix made up as

$$S_{3k} = \begin{pmatrix} D & U & U & \dots & U \\ U^T & D & U & \dots & U \\ U^T & U^T & D & & U \\ \vdots & \vdots & & \ddots & \vdots \\ U^T & U^T & U^T & \dots & D \end{pmatrix} \tag{1}$$

then it is easy to check that $S_{3k}T_{3k} = I$ and so $S_{3k} = T_{3k}^{-1}$.

Toeplitz matrices and trigonometric identities. Matrices like T_n which are constant on all the diagonals are called Toeplitz matrices. Toeplitz matrices and their infinite dimensional operator analogues appear in many areas of mathematics and in many applications (such as signal processing, communications engineering and statistics). The links between these matrices and trigonometric series are well-known, and so it should have come as no surprise to us that various trigonometric functions soon entered the picture.

You might also try to find the inverses of such matrices using elementary linear algebra. Clearly T_n is self-adjoint for all n and so there exists an orthogonal matrix P_n and a diagonal matrix $E = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ such that $T_n = P_n^* E P_n$. If T_n is invertible, then $T_n^{-1} = P_n^* \mathrm{diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1}) P_n$.

Again, with a little enthusiasm, it is possible to find a recurrence for the characteristic polynomials and solve this to find the eigenvalues and eigenvectors[1]. What we actually did was to get the computer to numerically calculate a few cases, and then stared at them! After recognising that the eigenvalues had something to do with $\cos\left(\frac{j\pi}{n+1}\right)$ it didn't take us long to guess that the eigenvalues are

$$\lambda_j = 1 + 2\cos\left(\frac{j\pi}{n+1}\right), \qquad j = 1, \dots, n,$$

with corresponding eigenvectors

$$v_j = \left(\sin\left(\frac{j\pi}{n+1}\right), \sin\left(\frac{2j\pi}{n+1}\right), \dots, \sin\left(\frac{nj\pi}{n+1}\right)\right).$$

Checking that our guesses were right was easy, since it only depends on the identity

$$\sin((k-1)\theta) + \sin(k\theta) + \sin((k+1)\theta) = (1 + 2\cos(\theta))\sin(k\theta). \tag{2}$$

Having identified n distinct eigenvalues, we have, of course, found all of them. A small calculation shows that $||v_j|| = \sqrt{(n+1)/2}$ for all j. We now have a quite different way of expressing T_n^{-1}. As long as $n \not\equiv 2 \pmod 3$ (in which case one of the eigenvalues is zero), then

$$T_n^{-1} = \frac{n+1}{2} P \,\mathrm{diag}\left(\frac{1}{1+\cos\left(\frac{\pi}{n+1}\right)}, \dots, \frac{1}{1+\cos\left(\frac{n\pi}{n+1}\right)}\right) P \tag{3}$$

where

$$P = P^* = \begin{pmatrix} \sin\left(\frac{\pi}{n+1}\right) & \sin\left(\frac{2\pi}{n+1}\right) & \dots & \sin\left(\frac{n\pi}{n+1}\right) \\ \sin\left(\frac{2\pi}{n+1}\right) & \sin\left(\frac{4\pi}{n+1}\right) & \dots & \sin\left(\frac{2n\pi}{n+1}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \sin\left(\frac{n\pi}{n+1}\right) & \sin\left(\frac{2n\pi}{n+1}\right) & \dots & \sin\left(\frac{n^2\pi}{n+1}\right) \end{pmatrix}.$$

[1]We have tested this statement on some bright undergraduates.

Multiplying out the expression in Equation (3) gives that the (j,k)th element of T_n^{-1} is

$$a_{jk} = \frac{n+1}{2} \sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{2\cos\left(\frac{m\pi}{n+1}\right)+1}.$$

From Equation (1) it is not too hard to check that if $n \equiv 0 \pmod 3$ and we are in the top half of the matrix (that is $j \le k$), then $a_{jk} = \frac{4}{3}\sin\left(\frac{2j\pi}{n+1}\right)\sin\left(\frac{2(k-1)\pi}{n+1}\right)$, and so

$$\sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{2\cos\left(\frac{m\pi}{n+1}\right)+1} = \frac{2n+2}{3}\sin\left(\frac{2j\pi}{n+1}\right)\sin\left(\frac{2(k-1)\pi}{n+1}\right).$$

At this stage we tried to prove this using other techniques, but this seems, to us at least, to be quite difficult.

Having got this far, one obviously looks to see what other sorts of identities might be proved in this way. For example, one can alter Equation (2) slightly to get

$$\sin((k-1)\theta) + \beta\sin(k\theta) + \sin((k+1)\theta) = (\beta + 2\cos(\theta))\sin(k\theta).$$

As above then, we have all the eigenvalues and eigenvectors for the Toeplitz matrices

$$T_{\beta,n} = \begin{pmatrix} \beta & 1 & 0 & \dots & 0 \\ 1 & \beta & 1 & \ddots & \\ 0 & 1 & \beta & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & & & & \beta \end{pmatrix}.$$

To get the process to work, you of course need to be able to guess what $T_{\beta,n}^{-1}$ looks like. For many values of β (such as -2, -1, 0 or 2), a few lines of Maple code were sufficient to have a guess, and, as before, checking that the guess is right is easy. Some of the identities that come from these values of β are given below.

- ($\beta = 0$) If $j \le k \le n$ and $n \equiv 0 \pmod 2$ then
$$\sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{\cos\left(\frac{m\pi}{n+1}\right)} = (n+1)\sin\left(\frac{j\pi}{2}\right)\sin\left(\frac{(k-1)\pi}{2}\right).$$
- ($\beta = -2$) If $j \le k \le n$ then
$$\sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{2\cos\left(\frac{m\pi}{n+1}\right)-2} = \frac{-j(n+1-k)}{2}.$$

$$T_{\beta,1}^{-1} = \begin{bmatrix} \frac{1}{\beta} \end{bmatrix}$$

$$T_{\beta,2}^{-1} = \begin{bmatrix} \frac{\beta}{\beta^2-1} & -\frac{1}{\beta^2-1} \\ -\frac{1}{\beta^2-1} & \frac{\beta}{\beta^2-1} \end{bmatrix}$$

$$T_{\beta,3}^{-1} = \begin{bmatrix} \frac{\beta^2-1}{\beta(\beta^2-2)} & -\frac{1}{\beta^2-2} & \frac{1}{\beta(\beta^2-2)} \\ -\frac{1}{\beta^2-2} & \frac{\beta}{\beta^2-2} & -\frac{1}{\beta^2-2} \\ \frac{1}{\beta(\beta^2-2)} & -\frac{1}{\beta^2-2} & \frac{\beta^2-1}{\beta(\beta^2-2)} \end{bmatrix}$$

$$T_{\beta,4}^{-1} = \begin{bmatrix} \frac{\beta(\beta^2-2)}{\beta^4-3\beta^2+1} & -\frac{\beta^2-1}{\beta^4-3\beta^2+1} & \frac{\beta}{\beta^4-3\beta^2+1} & -\frac{1}{\beta^4-3\beta^2+1} \\ -\frac{\beta^2-1}{\beta^4-3\beta^2+1} & \frac{\beta(\beta^2-1)}{\beta^4-3\beta^2+1} & -\frac{\beta^2}{\beta^4-3\beta^2+1} & \frac{\beta}{\beta^4-3\beta^2+1} \\ \frac{\beta}{\beta^4-3\beta^2+1} & -\frac{\beta^2}{\beta^4-3\beta^2+1} & \frac{\beta(\beta^2-1)}{\beta^4-3\beta^2+1} & -\frac{\beta^2-1}{\beta^4-3\beta^2+1} \\ -\frac{1}{\beta^4-3\beta^2+1} & \frac{\beta}{\beta^4-3\beta^2+1} & -\frac{\beta^2-1}{\beta^4-3\beta^2+1} & \frac{\beta(\beta^2-2)}{\beta^4-3\beta^2+1} \end{bmatrix}$$

TABLE 1. Inverses of $T_{\beta,n}$ for $n = 1, 2, 3, 4$.

- ($\beta = -1$) If $j \le k \le n$ and $n \equiv 0 \pmod 3$ then
$$\sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{2\cos\left(\frac{m\pi}{n+1}\right)-1} = \frac{2(n+1)}{3}\sin\left(\frac{j\pi}{3}\right)\sin\left(\frac{(k-1)\pi}{3}\right).$$
- ($\beta = 2$) If $j \le k \le n$ then
$$\sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{\cos\left(\frac{m\pi}{n+1}\right)+1} = (-1)^{j+k} j(n+1-k).$$

Note that if β is rational and $T_{\beta,n}$ is invertible, then the Gaussian Elimination algorithm implies that the entries in $T_{\beta,n}^{-1}$ are all rational. This proves statement 3 at the beginning of the paper.

Being more optimistic, one might even try to find a formula for general $\beta \in \mathbb{R}$. Maple is of course quite happy to do all the algebra to give you the first few cases of $T_{\beta,n}^{-1}$; for $n \le 4$ these are given in

Table 1. It is pretty clear from these matrices that there are some special polynomials that are occuring in these formulae, the first 6 of which are

$$\begin{aligned} p_1(x) &= x \\ p_2(x) &= x^2 - 1 \\ p_3(x) &= x^3 - 2x \\ p_4(x) &= x^4 - 3x^2 + 1 \\ p_5(x) &= x^5 - 4x^3 + 3x \\ p_6(x) &= x^6 - 5x^4 + 6x^2 - 1 \end{aligned}$$

The coefficients appearing here are binomial coefficients, so for $n \in \mathbb{N}$, we define the polyomial p_n by

$$p_n(x) = \sum_{\ell=0}^{\lfloor n/2 \rfloor} (-1)^\ell \binom{n-\ell}{\ell} x^{n-2\ell}.$$

The evidence from the first few cases is that if $1 \le j \le k \le n$ then (j,k)th entry of $T_{\beta,n}^{-1}$ is

$$\frac{(-1)^{j+k} p_{j-1}(\beta) p_{n-k}(\beta)}{p_n(\beta)}.$$

Since $T_{\beta,n}^{-1}$ must clearly be symmetric, this gives us our candidate formula for this matrix. Again, one can now readily check that our candidate actually does the job. This depends on identities such as

$$\beta(p_{n-1}(\beta) - p_{n-2}(\beta)) = p_n(\beta)$$

which follow easily from properties of binomial coefficients. It is not surprising that the roots of p_n are $\left\{2\cos\left(\frac{m\pi}{n+1}\right)\right\}_{m=1}^n$. We therefore have that if β is not an element of this set of roots, and if $1 \le j \le k \le n$ then

$$\sum_{m=1}^n \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{2\cos\left(\frac{m\pi}{n+1}\right) + \beta} = \frac{(-1)^{j+k}(n+1)p_{j-1}(\beta)p_{n-k}(\beta)}{2p_n(\beta)}.$$

These same techniques will of course work with any matrix for which you can both identify the some function of the matrix, and also identify the eigenvalues and eigenvectors. It is not too hard (at least for small $\ell \in \mathbb{N}$) to write down the entries of $T_{1,n}^\ell$. It is obvious that the entries are all nonnegative integers. Thus, for all $n \ge 1$, $\ell \ge 0$ and $1 \le j,k \le n$,

$$\frac{2}{n+1}\sum_{m=1}^n \sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)\left(2\cos\left(\frac{m\pi}{n+1}\right)+1\right)^\ell \in \mathbb{N}$$

and consequently

$$\frac{2^{\ell+1}}{n+1}\sum_{m=1}^{n}\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)\cos^{\ell}\left(\frac{m\pi}{n+1}\right)\in\mathbb{Z}.$$

With a little imagination one can think up different orthogonal matrices to use. To simpify notation, for an $(2n+1)$-tuple

$$\mathbf{c}=(c_{-n},\dots,c_{-1},c_0,c_1,\dots,c_n),$$

define the Toeplitz matrix $T_{\mathbf{c}}$ to be

$$T_{\mathbf{c}}=\begin{pmatrix} c_0 & c_{-1} & c_{-2} & \dots & c_{-n} \\ c_1 & c_0 & c_{-1} & \ddots & \\ c_2 & c_1 & c_0 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \\ c_n & \dots & & & c_0 \end{pmatrix}.$$

Thus, if $\mathbf{c}=(1,0,\dots,0,1,1,1,0,\dots,0,1)\in\mathbb{R}^{2n+1}$, then knowing that

$$e^{i(k-1)\theta}+e^{ik\theta}+e^{(ik+1)\theta}=(1+2\cos\theta)e^{ik\theta}$$

allows you to identify the eigenvalues for $T_{\mathbf{c}}$ as

$$\lambda_j=1+2\cos\left(\frac{2j\pi}{n}\right),\qquad j=1,\dots,n$$

with corresponding eigenvectors $v_j=\left(e^{2ij\theta},e^{4ij\theta},\dots,e^{2nij\theta}\right)$. Here $T_{\mathbf{c}}$ is invertible if $n\not\equiv 0 \pmod 3$. If $S=(s_{jk})$ is the inverse matrix then (by observing, guessing and checking)

$$s_{jk}=\begin{cases}\dfrac{2(-1)^{n \bmod 3}}{3}, & \text{if } |j-k|+n\equiv 0 \pmod 3,\\[2mm] \dfrac{(-1)^{1+(n \bmod 3)}}{3}, & \text{otherwise.}\end{cases}$$

Using the diagonalization of $T_{\mathbf{c}}$ gives that

$$s_{jk}=\frac{1}{n}\sum_{m=1}^{n}\frac{e^{i(k-j)m\pi/n}}{1+\cos\left(\frac{2m\pi}{n}\right)}$$

so, on taking the real part and writing ℓ for $k-j$, we have that

$$\frac{3}{n}\sum_{m=1}^{n}\frac{\cos(2\ell m\pi/n)}{1+\cos\left(\frac{2m\pi}{n}\right)}=\begin{cases}2(-1)^{n \bmod 3}, & \text{if } |\ell|+n\equiv 0 \pmod 3,\\ (-1)^{1+(n \bmod 3)}, & \text{otherwise.}\end{cases}$$

In particular (on taking $\ell = 1$) we see that for any $n \geq 1$,

$$\frac{3}{n}\sum_{m=1}^{n} \frac{\cos(2m\pi/n)}{1+\cos\left(\frac{2m\pi}{n}\right)} \equiv 2 \pmod 3.$$

Using the fact that

$$e^{i(k-\ell)\theta} + e^{i(k+\ell)\theta} = 2\cos(\ell\theta)e^{ik\theta}$$

we see that this same set of eigenvectors will diagonalise any Toeplitz matrix $T_{\mathbf{c}}$ where $\mathbf{c}$ is of the form

$$(c_1, \dots, c_\ell, 0, \dots, 0, c_2, c_1, c_0, c_1, \dots, c_\ell, 0, \dots, 0, c_\ell, \dots, c_1).$$

Choosing $\ell = 2$ or 3 and $c_j = 1$ for $j \leq \ell$ then gives identities such as

$$\frac{5}{n}\sum_{m=1}^{n} \frac{\cos(2m\pi/n)}{4\cos^2\left(\frac{2m\pi}{n}\right) + 2\cos\left(\frac{2m\pi}{n}\right) - 1} \equiv n^3 \pmod 5$$

for $n \not\equiv 0 \pmod 5$, and identity 4 at the start of this paper. (The danger of guessing formulae is shown by the fact that the obvious guess as to what happens if $\ell = 4$ isn't true!)

Stirling numbers and binomial coefficients. Moving away from trigonometric functions, you can try your luck with other types of matrices. For example, let $P_n(a)$ denote the $n \times n$ matrix whose (j,k)th entry is $\binom{j-1}{k-1}a^{j-k}$. The powers of these 'Pascal matrices' were studied in [1]. Let $B_n = P_n(1)$ be the matrix whose entries are made of binomial coefficiants. This matrix is not diagonalizable, but Maple will quickly find its Jordan form:

$$B_n = U_n \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & 1 \\ 0 & & \dots & 0 & 1 \end{pmatrix} U_n^{-1}.$$

For $n = 6$, the matrices U_6 and U_6^{-1} are

$$U_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 6 & 6 & 1 & 0 \\ 0 & 24 & 36 & 14 & 1 & 0 \\ 120 & 240 & 150 & 30 & 1 & 0 \end{pmatrix}, \quad U_6^{-1} = \begin{pmatrix} 0 & \frac{1}{5} & \frac{-5}{12} & \frac{7}{124} & \frac{-1}{12} & \frac{1}{120} \\ 0 & \frac{-1}{4} & \frac{11}{24} & \frac{-1}{4} & \frac{1}{24} & 0 \\ 0 & \frac{1}{3} & \frac{-1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Actually, staring at the entries of these matrices didn't immediately lead us to any guesses as to the general formulae. Let u_{jk} denote the (j,k)th entry of U_n. It appears that $(n-k)!$ is a factor of u_{jk}, so we looked at what is left. Here we resorted to Neil Sloane's wonderful On-Line Encyclopedia of Integer Sequences [3] which quickly identified the missing factors as Stirling numbers.

Let $s(j,k)$ and $S(j,k)$ denote the Stirling numbers of the first and second kind. There are many fine references on these numbers (see, for example [2] or [4] and [5]). We shall just give a quick definition. For $k \geq 0$ let $(x)_k$ denote the degree k polynomial $x(x-1)\ldots(x-k+1)$. Then the Stirling numbers can be defined in terms of the following generating functions

$$(x)_j = \sum_{k=0}^{j} s(j,k)x^k, \qquad x^j = \sum_{k=0}^{k} S(j,k)(x)_k$$

The conjecture then would be that

$$u_{jk} = (n-k)!\; S(j-1, n-k). \tag{4}$$

Similar explorations concerning $v_{jk} = (j,k)$th element of U_n^{-1} leads one to the belief that

$$v_{jk} = \frac{s(n-j, k-1)}{(n-j)!}.$$

Indeed, one can readily verify that the two matrices with these entries are inverses using standard Stirling number identities.

Since B_n has only a one dimensional eigenspace, it is actually not too hard to generate U. As $B_n - I$ is a lower triangular matrix with positive entries below the diagonal, $(B_n - I)^{n-1}$ will just have a single nonzero entry in the bottom left position. This says that the vector $(1, 0, \ldots, 0)$ is in $\ker((B_n - I)^n) \setminus \ker((B_n - I)^{n-1})$. The orbit of this vector under the matrix $B_n - I$ will then generate a Jordan basis for B_n and hence give the transition matrix U. Let $\mathbf{u}_k$ denote the vector whose jth entry is give by the formula for u_{jk} in (4). To check the above conjecture, one therefore needs to check that for all k, $\mathbf{u}_{k-1} = (B_n - I)\mathbf{u}_k$. In terms of the entries this amounts to checking that for all j, k, n,

$$(n-k+1)!\ S(j-1, n-k+1) = \sum_{\ell=1}^{j-1} \binom{j-1}{\ell-1} (n-k)!\ S(\ell-1, n-k),$$

or, on changing variables to make things look neater,

$$(k+1)S(j, k+1) = \sum_{\ell=0}^{j-1} \binom{j}{\ell} S(\ell, k).$$

This is an easy consequence of the following two standard identities [2, Section 6.1]:

$$\begin{aligned} S(n,m) &= n\, S(n-1,m) + S(n-1,m-1) \\ S(n+1,m+1) &= \sum_{\ell=1}^{n} \binom{n}{\ell} S(\ell, m). \end{aligned}$$

Using that fact that $S(j,\ell) = 0$ for $\ell > j$, then multiplying out $U_n J U_n^{-1}$ would then lead (after a small amount of simplification) to the identity

$$\binom{j}{k} = \sum_{\ell=0}^{j} \left(s(\ell,k) + \ell s(\ell-1,k)\right) S(j,\ell).$$

This is surely already known to those in the field, but it is interesting to see what other identities this leads to. Formulae for powers of B_n were given in [1]: $B_n^m = P_n(m)$. The formula for the inverse of P_n gives that

$$\binom{j}{k}(-1)^{j-k} = \sum_{t=1}^{n}\sum_{\ell=t}^{n}(-1)^{\ell-t}\frac{(n-t)!}{(n-\ell)!}\ s(n-\ell,k)S(j,n-t).$$

Taking mth powers of P_n then gives that (with suitable interpretation of the factorials), for all j, k, m,

$$\binom{j}{k} m^{j-k} = \sum_{\ell=0}^{j}\sum_{t=0}^{m}\binom{m}{t}\frac{\ell!}{(\ell-t)!} s(\ell-t,k)S(j,\ell).$$

Summing over k and putting $m = 1$ or 2 would, for example, give that

$$\begin{aligned} 2^j &= \sum_{k=1}^{j} \sum_{\ell=0}^{j} (s(\ell,k) + \ell s(\ell-1,k)) S(j,\ell) \\ 3^j &= \sum_{k=0}^{j} \sum_{\ell=0}^{j} (s(\ell,k) + 2\ell s(\ell-1,k) + \ell(\ell-1) s(\ell-2,k)) S(j,\ell). \end{aligned}$$

Conclusion. As the examples in this paper show, many standard identities can be interpreted as statements about the eigenvalues (or generalized eigenvalues) of a matrix. Once one has a suitable matrix identity, then elementary linear algebra provides a powerful technique for extracting new and more complicated identities from old ones.

Our little journey of discovery heavily underlined the power of a modern computer algebra package in providing inspiration in mathematical investigations.

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