TILTING AND COTILTING FOR QUIVERS OF TYPE $\tilde{A}_n$

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Abstract. Tilting and cotilting modules are classified for the completed path algebra of a quiver of type $\tilde{A}_n$ with linear orientation. This classification problem arises naturally in the classification of cotilting modules over certain associative algebras [5]. The combinatorics of the collection of all tilting and cotilting modules is described in terms of Stasheff associahedra.

Introduction

Throughout we fix a field k. We consider the completion $k[\![\Delta]\!]$ of the path algebra of the following quiver.

$$\begin{tikzcd} \Delta\colon & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n \arrow[llll, bend left] \end{tikzcd}$$

More precisely, $k[\![\Delta]\!] = \varprojlim k[\Delta]/\mathfrak{m}^i$ where $\mathfrak{m}$ denotes the ideal of the path algebra $k[\Delta]$ which is generated by all arrows in Δ. In this paper we classify all finitely presented tilting modules and all locally finite cotilting modules over $k[\![\Delta]\!]$. The initial motivation for this project is to complete the classification of all cotilting modules over a tame hereditary algebra [5], which includes the classification of all cotilting modules for quivers of type $\tilde{A}_n$ having non-linear orientation. To this end we are interested in cotilting objects of certain Grothendieck categories which we call tubes.

Let $\mathcal{C}$ be an abelian Grothendieck category which is a k-category and has a generating set of finite length objects. We say that $\mathcal{C}$ is a *tube* if the full subcategory $\operatorname{fin}\mathcal{C}$ formed by the finite length objects has the following properties:

- $\operatorname{Hom}(X,Y)$ and $\operatorname{Ext}^1(X,Y)$ have finite k-dimension for all $X, Y \in \operatorname{fin}\mathcal{C}$;
- $\operatorname{fin}\mathcal{C}$ has *Serre duality*, that is, there is an equivalence $\tau\colon \operatorname{fin}\mathcal{C} \to \operatorname{fin}\mathcal{C}$ and a natural isomorphism $D\operatorname{Ext}^1(X,Y) \cong \operatorname{Hom}(Y,\tau X)$ for all $X, Y \in \operatorname{fin}\mathcal{C}$, where $D = \operatorname{Hom}_k(-,k)$;
- there are only finitely many isomorphism classes of simple objects in $\operatorname{fin}\mathcal{C}$.

Note that the Auslander-Reiten quiver of $\operatorname{fin}\mathcal{C}$ has the shape of a tube [14] provided that $\mathcal{C}$ is connected; this explains the terminology. The number of simple objects in $\mathcal{C}$ is called the *rank* of $\mathcal{C}$. Tubes arise in the category of regular modules over a tame hereditary algebra, but also as subcategories of other abelian categories, see for instance [2, 10]. We shall use that a tube of rank n is equivalent to the category of locally finite $k[\![\Delta]\!]$-modules. Recall that a module is *locally finite* if it is a filtered colimit of finite length modules.

Next we recall the definition of a cotilting object [6] for any Grothendieck category $\mathcal{C}$. To this end we fix an object T in $\mathcal{C}$. We let $\operatorname{Prod} T$ denote the category of all direct summands in any product of copies of T. The object T is called *cotilting object* if the following holds:

(C1) the injective dimension of T is at most 1;
(C2) $\operatorname{Ext}^1(T^\alpha, T) = 0$ for every cardinal α;
(C3) there is an exact sequence $0 \to T_1 \to T_0 \to Q \to 0$ with each T_i in $\operatorname{Prod} T$ for some injective cogenerator Q.

By definition, two cotilting objects T and T' are equivalent if $\operatorname{Prod} T = \operatorname{Prod} T'$. Let us mention a result from [5] which motivates the classification of cotilting objects.

For any locally finite Grothendieck category $\mathcal{C}$, there exists a bijection between the set of torsion pairs $(\mathcal{T}, \mathcal{F})$ for the category $\operatorname{fin} \mathcal{C}$ such that $\mathcal{F}$ generates $\operatorname{fin} \mathcal{C}$, and the set of equivalence classes of cotilting objects in $\mathcal{C}$.

Our first result describes the structural properties of an arbitrary cotilting object in a tube.

Theorem A. *Let T be an object in a tube of rank n satisfying $\operatorname{Ext}^1(T, T) = 0$.*

(1) *T decomposes uniquely into a coproduct of indecomposable objects having local endomorphism rings.*
(2) *T is a cotilting object if and only if the number of pairwise non-isomorphic indecomposable direct summands of T equals n.*

The classification of cotilting objects in a tube of rank n is the same as the classification of locally finite cotilting modules over $k[\![\Delta]\!]$. Note that $k[\![\Delta]\!]$ is a noetherian algebra over a complete local ring which is of *artinian type*, that is, each non-zero locally finite module has a non-zero artinian direct summand. For this class of algebras we have the following.

Theorem B. *Let Λ be a noetherian algebra over a complete local ring which is of artinian type. Then the duality between Λ- and Λ^{op}-modules induces a bijection between the equivalence classes of finitely presented Λ-tilting modules and the equivalence classes of locally finite Λ^{op}-cotilting modules.*

This result extends the bijection between finitely presented tilting and cotilting modules over artin algebras. It would be interesting to see a general correspondence between tilting and cotilting modules which does not depend on finiteness conditions on the algebra.

The second part of this paper is devoted to the classification of all finitely presented tilting modules over $k[\![\Delta]\!]$. It is somewhat surprising that all of them are induced from tilting modules over the path algebra of the following quiver.

$$\begin{tikzcd} \Gamma\colon\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n \end{tikzcd}$$

The collection of all $k[\Gamma]$-tilting modules is best described in terms of the Stasheff associahedron of dimension $n-1$. Another connection between representations of Dynkin quivers and generalized associahedra is discussed in [12].

Theorem C. *The isomorphism classes of faithful and basic partial $k[\Gamma]$-tilting modules correspond bijectively to the faces of the Stasheff associahedron of dimension $n-1$. This correspondence identifies the tilting modules with the vertices, and it identifies the Hasse diagram of the lattice of all tilting modules with the 1-skeleton of the Stasheff associahedron. Therefore the lattice of tilting modules is a Tamari lattice.*

The collection of all faithful partial $k[\![\Delta]\!]$-tilting modules is obtained by glueing together n copies of a Stasheff associahedron of dimension $n-1$. This leads to a combinatorial structure which seems to be new; it is discussed in an appendix which is independent from the rest of this paper. It turns out that the tilting modules are parametrized by integer sequences as follows.

Theorem D. *The map sending a $k[\![\Delta]\!]$-module X to the sequence $(a_1,\dots,a_n)$ where a_i denotes the number of composition factors of $X/\operatorname{rad}X$ isomorphic to the simple with support $i\in\Delta$, induces a bijection between the isomorphism classes of finitely presented basic $k[\![\Delta]\!]$-tilting modules and the sequences $(a_1,\dots,a_n)$ of non-negative integers satisfying $\sum_i a_i = n$.*

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1. Cotilting versus tilting

Let Λ be an associative R-algebra over a commutative ring R. We denote by $\operatorname{Mod}\Lambda$ the category of (right) Λ-modules and $\operatorname{mod}\Lambda$ denotes the full subcategory formed by the finitely presented Λ-modules. In this section we establish a connection between cotilting objects for the category of locally finite Λ-modules and tilting modules over Λ^{op}. We need to fix some notation and terminology.

Recall that a Λ-module is *locally finite* if it is a filtered colimit of finite length modules. The full subcategory formed by the locally finite Λ-modules is denoted by $\operatorname{Fin}\Lambda$. In addition, we consider the full subcategories given by the noetherian Λ-modules (written as $\operatorname{noeth}\Lambda$), the artinian Λ-modules (written as $\operatorname{art}\Lambda$), and the finite length Λ-modules (written as $\operatorname{fin}\Lambda$).

Next we recall the definition of a finitely presented tilting module. A module $T\in\operatorname{mod}\Lambda$ is a *tilting module* if

(T1) the projective dimension of T is at most 1;
(T2) $\operatorname{Ext}^1_\Lambda(T,T)=0$;
(T3) there is an exact sequence $0\to\Lambda\to T_0\to T_1\to 0$ with each T_i in $\operatorname{add} T$.

A tilting module is called *basic* if each indecomposable direct summand occurs exactly once in a direct sum decomposition. Two finitely presented tilting modules T, T' are *equivalent* if $\operatorname{add} T=\operatorname{add} T'$.

Throughout this section we assume that Λ is a noetherian R-algebra and that R is a complete local ring R. Let I be the injective envelope of $R/\operatorname{rad} R$. The functor $D=\operatorname{Hom}_R(-,I)\colon \operatorname{Mod} R\to\operatorname{Mod} R$ induces functors between $\operatorname{Mod}\Lambda$ and $\operatorname{Mod}\Lambda^{\mathrm{op}}$ which become dualities on appropriate subcategories.

Lemma 1.1. *The functor D induces inverse dualities* $\operatorname{noeth}\Lambda\to\operatorname{art}\Lambda^{\mathrm{op}}$ *and* $\operatorname{art}\Lambda^{\mathrm{op}}\to\operatorname{noeth}\Lambda$.

We do not give the proof of this lemma but refer instead to [1, Section I.5] for basic facts about algebras over complete local rings.

The following characterization of a tilting module is classical. Bongartz proved it for finite dimensional algebras [3], but the same proof works in our setting. We denote for any module X by $\delta(X)$ the number of pairwise non-isomorphic indecomposable direct summands of X.

Lemma 1.2. *A finitely presented Λ-module T is a tilting module if and only if the following holds:*

(1) *the projective dimension of T is at most* 1*;*
(2) $\operatorname{Ext}^1_\Lambda(T,T)=0$*;*
(3) $\delta(T)=n$ *where n denotes the number of simple Λ-modules.*

Moreover, each module satisfying (1) and (2) is a direct summand of a tilting module.

Next recall from [8] that an object X in a locally finite Grothendieck category is *endofinite* if $\operatorname{Hom}(C,X)$ has finite length as $\operatorname{End}(X)$-module for each finite length object C. All we need to know about endofinite objects is collected in the following lemma.

Lemma 1.3. (1) *Every endofinite object decomposes into indecomposable objects with local endomorphism rings.*
(2) *A finite coproduct of endofinite objects is endofinite, and all coproducts of a fixed endofinite object are endofinite.*
(3) *If X is indecomposable and endofinite, then* $\operatorname{Add} X=\operatorname{Prod} X$.

Proof. See [7, Section 3] and [8, Section 3.6] □

Lemma 1.4. *Each artinian Λ-module is an endofinite object in* $\operatorname{Fin}\Lambda$.

Proof. Let X be artinian and C of finite length. One checks that $\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(DX,DC)$ has finite length as a $\operatorname{End}_{\Lambda^{\mathrm{op}}}(DX)$-module, for instance by induction on the composition length of C. Then apply the duality, to see that $\operatorname{Hom}_\Lambda(C,X)$ is of finite length over $\operatorname{End}_\Lambda(X)$. □

We say that the algebra Λ is of *artinian type* if each non-zero locally finite Λ-module has a non-zero direct summand which is artinian. Note that 'artinian type' is equivalent to 'finite representation type' in case Λ is artinian.

Proposition 1.5. *Suppose Λ is of artinian type. Let X be a locally finite Λ-module satisfying* $\operatorname{id} X \leq 1$ *and* $\operatorname{Ext}^1_\Lambda(X,X)=0$.

1. *X decomposes into a coproduct of indecomposable modules with local endomorphism rings.*
2. *$\delta(X) \leq n$ where n is the number of simple Λ-modules.*
3. $\operatorname{Ext}^1_\Lambda(X',X)=0$ *for every product $X'=X^\alpha$ taken in* $\operatorname{Fin}\Lambda$.

Proof. Up to isomorphism, X has only a finite number of indecomposable artinian direct summands. This follows from Lemma 1.2, using the duality D. Label the indecomposables $X_1,\dots,X_p$. Using Zorn's lemma, we find a maximal direct summand X' of X which is a coproduct of modules in $\{X_1,\dots,X_p\}$. Clearly, $X'=X$ since Λ is of artinian type, and X is endofinite by Lemmas 1.3 and 1.4. Now all assertions follow from the properties of endofinite objects. □

Lemma 1.6. *Suppose Λ is of artinian type. Let T be a cotilting object in* $\operatorname{Fin}\Lambda$. *Then there exists an exact sequence $0\to T_1\to T_0\to D(\Lambda^{\mathrm{op}})\to 0$ such that T and $T_0\amalg T_1$ are equivalent cotilting objects and each T_i belongs to* $\operatorname{art}\Lambda\cap\operatorname{Prod}T$.

Proof. We write $Q=D(\Lambda^{\mathrm{op}})$ and note that Q is an injective cogenerator for the category $\operatorname{Fin}\Lambda$. Next observe that for indecomposable objects X and Y in $\operatorname{Fin}\Lambda$, we have that $\operatorname{Hom}_\Lambda(X,Y)$ is finitely generated as $\operatorname{End}_\Lambda(X)$-module. This is because X and Y are artinian by our assumption, and we have the duality $\operatorname{art}\Lambda\to\operatorname{noeth}\Lambda^{\mathrm{op}}$.

Now choose an exact sequence $0\to U_1\to U_0\to Q\to 0$ with $U_i\in\operatorname{Prod}T$. We know from Proposition 1.5 that T decomposes into a coproduct of indecomposable objects and only finitely many isoclasses occur. We find therefore a map $f\colon T_0\to Q$ such that $U_0\to Q$ factors through f and T_0 decomposes into finitely many indecomposables from $\operatorname{Prod}T$. In particular, $T_0\in\operatorname{art}\Lambda$. We may assume that f is minimal, that is every endomorphism $g\colon T_0\to T_0$ with $f\circ g=f$ is an isomorphism. Note that f factors through $U_0\to Q$ since $\operatorname{Ext}^1_\Lambda(T_0,U_1)=0$. Thus $U_0\cong T_0\amalg V_0$ for some object V_0, and we obtain the following commutative diagram.

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
& V_1 \arrow[r, "\sim"] \arrow[d] & V_0 \arrow[d] & & \\
0 \arrow[r] & U_1 \arrow[r] \arrow[d] & U_0 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T_1 \arrow[r] \arrow[d] & T_0 \arrow[r, "f"] \arrow[d] & Q \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd}$$

We conclude that $U_1 \cong T_1 \amalg V_1$. In particular, each T_i belongs to $\operatorname{art} \Lambda \cap \operatorname{Prod} T$. It remains to show that $T_0 \amalg T_1$ is a cotilting object which is equivalent to T. However, this follows from our construction, using for instance Proposition 3.1 in [5]. ☐

Lemma 1.7. *Let $Y \in \operatorname{Mod} \Lambda$ be artinian. Then the class of modules X satisfying $\operatorname{Ext}^1_\Lambda(X, Y) = 0$ is closed under taking products.*

Proof. We can decompose $Y = Y' \amalg Y''$ such that Y' is injective and $Y'' = D \operatorname{Tr} Z$ for some $Z \in \operatorname{mod} \Lambda^{\mathrm{op}}$. Now use the Auslander-Reiten formula $\operatorname{Ext}^1_\Lambda(-, D \operatorname{Tr} Z) = D \underline{\operatorname{Hom}}_\Lambda(Z, -)$ (see [1, Proposition I.3.4]). Note that every map $Z \to \prod_i P_i$ into a product of projectives factors through a projective since $\prod_i P_i$ is flat. ☐

Lemma 1.8. *Let $T \in \operatorname{mod} \Lambda^{\mathrm{op}}$ be a tilting module. Then DT is a Λ-cotilting module.*

Proof. Let $T \in \operatorname{mod} \Lambda^{\mathrm{op}}$ be a tilting module. The conditions on T for a tilting module translate via the duality D into the conditions on DT for a cotilting module. More precisely, (C1) and (C3) follow immediately from (T1) and (T3). Condition (C2) follows from (T2), using Lemma 1.7. Thus DT is a cotilting module. ☐

Lemma 1.9. *Let $\mathcal{A}$ be any abelian Grothendieck category and $\mathcal{A}'$ be a localizing subcategory. If T is a cotilting object in $\mathcal{A}$ and belongs to $\mathcal{A}'$, then T is also a cotilting object in $\mathcal{A}'$.*

Proof. We use the well-known fact that in any Grothendieck category, T is a cotilting object if and only if $\operatorname{id} T \leq 1$ and $\operatorname{Cogen} T = {}^\perp T$, where $\operatorname{Cogen} T$ is the class of subobjects of products of copies of T, and ${}^\perp T$ is the class of objects X satifying $\operatorname{Ext}^1(X, T) = 0$.

Now assume that T is a cotilting object in $\mathcal{A}$. Clearly, $\operatorname{id} T \leq 1$ holds in $\mathcal{A}'$ because this is equivalent to $\operatorname{Ext}^2(-, T) = 0$. The inclusion functor $\mathcal{A}' \to \mathcal{A}$ has a right adjoint which preserves products. This implies that the condition $\operatorname{Cogen} T = {}^\perp T$ carries over from $\mathcal{A}$ to $\mathcal{A}'$ as well. In fact, $\operatorname{Cogen}_{\mathcal{A}'} T = \mathcal{A}' \cap \operatorname{Cogen}_{\mathcal{A}} T$. Thus T is a cotilting object in $\mathcal{A}'$. ☐

Theorem 1.10. *Let Λ be of artinian type. Then the following conditions are equivalent for a locally finite Λ-module X:*

(1) *X is a cotilting object in* $\operatorname{Mod}\Lambda$.
(2) *X is a cotilting object in* $\operatorname{Fin}\Lambda$.
(3) $\operatorname{Prod} X = \operatorname{Prod} DT$ *in* $\operatorname{Mod}\Lambda$ *for some finitely presented Λ^{op}-tilting module T.*
(4) $\operatorname{Prod} X = \operatorname{Prod} DT$ *in* $\operatorname{Fin}\Lambda$ *for some finitely presented Λ^{op}-tilting module T.*

Moreover, the assignment $T \mapsto DT$ induces a bijection between the equivalence classes of finitely presented Λ^{op}-tilting modules and the equivalence classes of locally finite Λ-cotilting modules.

Proof. (1) $\Rightarrow$ (2): First observe that the locally finite Λ-modules form a localizing subcategory in $\operatorname{Mod}\Lambda$. Now apply Lemma 1.9.

(2) $\Rightarrow$ (3): Let X be a cotilting object for the category $\operatorname{Fin}\Lambda$. Then X is equivalent to an artinian cotilting object by Lemma 1.6, which is of the form DT for some tilting module $T \in \operatorname{mod}\Lambda^{\mathrm{op}}$. The proof shows that every indecomposable direct summand of DT is a direct summand of X. Thus $\operatorname{Prod} DT \subseteq \operatorname{Prod} X$. On the other hand, $\operatorname{Ext}^1_\Lambda(X^\alpha, X) = 0$ for every product X^α taken in $\operatorname{Mod}\Lambda$, by Lemma 1.7, since X decomposes into a coproduct of artinian objects. We know from Lemma 1.8 that DT is a cotilting Λ-module, and combining this with $\operatorname{Prod} DT \subseteq \operatorname{Prod} X$, we obtain $\operatorname{Prod} DT = \operatorname{Prod} X$, for instance by Proposition 3.1 in [5].

(3) $\Rightarrow$ (4): This follows from the fact that the right adjoint of the inclusion $\operatorname{Fin}\Lambda \to \operatorname{Mod}\Lambda$ preserves products.

(4) $\Rightarrow$ (1): The module DT is a cotilting module by Lemma 1.8. The assumption on X implies that it decomposes into indecomposables, and the isomorphism classes which appear are precisely those appearing in a decomposition of DT. This follows essentially from Proposition 1.5. Thus X is a cotilting module since we know it for DT. ☐

Remark 1.11. The category of locally finite Λ-modules is usually not closed under taking products. However, one checks easily for two locally finite tilting modules T and T', that $\operatorname{Prod} T = \operatorname{Prod} T'$ in $\operatorname{Mod}\Lambda$ if and only if $\operatorname{Prod} T = \operatorname{Prod} T'$ in $\operatorname{Fin}\Lambda$.

2. TUBES

Let $\mathcal{C}$ be a tube of rank n and suppose that $\mathcal{C}$ is connected, that is, any decomposition $\mathcal{C} = \mathcal{C}_1 \amalg \mathcal{C}_2$ into abelian categories implies $\mathcal{C}_1 = 0$ or $\mathcal{C}_2 = 0$. Note that any tube decomposes into finitely many connected tubes. In this section we exhibit some basic properties of $\mathcal{C}$ and establish an equivalence between $\mathcal{C}$ and the category of locally finite $\tilde{\Lambda}_n$-modules.

First we recall the classification of finite length objects which is well-known: each indecomposable object is uniserial and uniquely determinded by its socle and its composition length. For each simple object S and each $n \in \mathbb{N}$, we denote by $S[n]$ the object with socle S and composition length n. We obtain a chain of monomorphisms

$$S = S[1] \longrightarrow S[2] \longrightarrow \cdots$$

and denote by $S[\infty]$ the *Prüfer object* $\varinjlim S[n]$ which is independent of the choice of maps. Note that each Prüfer object is indecomposable injective.

Lemma 2.1. *Every non-zero object in $\mathcal{C}$ has an indecomposable direct factor, and every indecomposable object is of the form $S[n]$ for some simple S and some $n \in \mathbb{N} \cup \{\infty\}$.*

Proof. We use the fact that for each simple S and each $n \in \mathbb{N}$ the natural map

$$S[n] \longrightarrow S[n+1] \amalg S[n]/S$$

is left almost split. Now let X be a non-zero object, and fix a non-zero map $f\colon S \to X$ for some simple S. Let $n \geq 1$ be the maximal number such that there is a factorization

$$f\colon S \longrightarrow S[n] \xrightarrow{f'} X$$

so that f' is a monomorphism. We claim that f' splits. If $n = \infty$, then this is clear since $S[\infty]$ is injective. Assume $n < \infty$ and f' does not split. Then f' factors through the left almost split map starting in $S[n]$. The composite $S[n] \to S[n]/S \to X$ kills S. Therefore f factors through the natural map $S \to S[n+1]$. The corresponding map $S[n+1] \to X$ kills S by our choice of n and this is a contradiction. We conclude that f splits. ☐

Denote by $\tilde{\Lambda}_n$ the completion of the path algebra of the following quiver.

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n \arrow[llll, bend left] \end{tikzcd}$$

The center of $\tilde{\Lambda}_n$ contains a copy of the ring $k[\![t]\!]$ of power series. The generator of this copy corresponds to the sum $\sum_{i=1}^{n} \gamma_i$ where γ_i is the path of length n starting and ending in the vertex i. Note that $\tilde{\Lambda}_n$ is finitely generated over R so that $\tilde{\Lambda}_n$ is a noetherian algebra over a complete local ring.

Lemma 2.2. *The endomorphism ring of $\coprod_{S \text{ simple}} S[\infty]$ is isomorphic to $\tilde{\Lambda}_n$.*

Proof. Number the simples $S_1, \ldots, S_n$ such that there are epimorphisms $\pi_i\colon S_i[\infty] \to S_{i+1}[\infty]$ with simple kernel for each i modulo n. The π_i generate the endomorphism ring of $S_1 \amalg \ldots \amalg S_n$ and we get an isomorphism onto $\tilde{\Lambda}_n$ by sending π_i to the arrow $i \to i+1$. ☐

Proposition 2.3. *The category $\mathcal{C}$ is equivalent to the category of locally finite $\tilde{\Lambda}_n$-modules.*

Proof. The category $\operatorname{art}\mathcal{C}$ is abelian and $Q = \coprod_{S \text{ simple}} S[\infty]$ is an injective cogenerator. Moreover, each object $X \in \operatorname{art}\mathcal{C}$ admits an injective copresentation $0 \to X \to I_0 \to I_1$ with each $I_i \in \operatorname{add} Q$. It follows that the opposite category is equivalent to the category of finitely presented modules over $\operatorname{End}(Q)^{\mathrm{op}}$ via the functor $\operatorname{Hom}(-, Q)$. Composing this functor with the duality $\operatorname{noeth}\tilde{\Lambda}_n^{\mathrm{op}} \to \operatorname{art}\tilde{\Lambda}_n$ induces an equivalence $F\colon \operatorname{art}\mathcal{C} \to \operatorname{art}\tilde{\Lambda}_n$. This induces an equivalence $\mathcal{C} \to \operatorname{Fin}\tilde{\Lambda}_n$ by sending $X = \varinjlim X_\alpha$ to $\varinjlim FX_\alpha$ since every object in $\mathcal{C}$ is a filtered colimit of finite length objects. ☐

Using the equivalence between $\mathcal{C}$ and the category of locally finite $\tilde{\Lambda}_n$-modules, we obtain from Theorem 1.10 the following correspondence between tilting and cotilting objects.

Corollary 2.4. *The algebra $\tilde{\Lambda}_n$ is of artinian type. Therefore there are, up to equivalence, canonical bijections between*

(1) *cotilting objects in a tube of rank n,*
(2) *locally finite cotilting modules over $\tilde{\Lambda}_n$,*
(3) *finitely presented tilting modules over $\tilde{\Lambda}_n$.*

Proof. The bijections are established in Theorem 1.10. All we need to show is that $\tilde{\Lambda}_n$ is of artinian type. However, this follows from Lemma 2.1. $\square$

3. TILTING FOR QUIVERS OF TYPE A_n

We fix a quiver of type A_n with linear orientation

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n \end{tikzcd}$$

and denote by Λ_n its path algebra over the field k. For each $i \in \{1, \dots, n\}$, let P_i be the indecomposable projective Λ_n-module having as a k-basis all paths ending in the vertex i. Let $\mathcal{I}(n)$ denote the set of intervals $[i, j]$ in $\mathbb{Z}$ with $0 \leq i < j \leq n$. Each indecomposable Λ_n-module is of the form $M_{[i,j]} = P_j / \operatorname{rad}^{j-i} P_j$, and we write $M_X = \coprod_{I \in X} M_I$ for any $X \subseteq \mathcal{I}(n)$. It easy to compute $\operatorname{Ext}^1_{\Lambda_n}(-,-)$ and we obtain the following.

Lemma 3.1. $\operatorname{Ext}^1_{\Lambda_n}(M_I, M_J) = 0 = \operatorname{Ext}^1_{\Lambda_n}(M_J, M_I)$ *if and only if the intervals I and J are compatible, that is, $I \subseteq J$ or $J \subseteq I$ or $I \cap J = \emptyset$.*

We denote for each module M by $\operatorname{top} M$ the factor $M/\operatorname{rad} M$, and $\dim M$ denotes the sequence $(a_1, \dots, a_n)$ where a_i is the number of composition factors of M isomorphic to the simple $P_i/\operatorname{rad} P_i$. The classification of the Λ_n-tilting modules is well-known [4].

Proposition 3.2. *The map sending a Λ_n-module M to* $\dim(\operatorname{top} M)$ *induces a bijection between the set of isomorphism classes of basic tilting modules over Λ_n and the set of sequences $(a_1, \dots, a_n)$ of non-negative integers such that $\sum_i a_i = n$ and $\sum_{i \leq p} a_i \leq p$ for all $1 \leq p \leq n$.*

Proof. Lemma 3.1 reduces the classification of tilting modules to the classification of subsets $X \subseteq \mathcal{I}(n)$ of cardinality n such that all elements in X are pairwise compatible. Now everything follows from Lemma A.1 since we have for $X \subseteq \mathcal{I}(n)$ that $\operatorname{top} X = \dim(\operatorname{top} M_X)$. $\square$

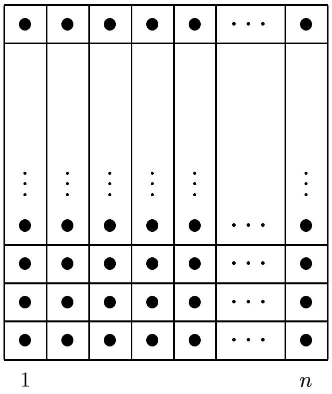
4. TILTING FOR QUIVERS OF TYPE $\tilde{A}_n$

We fix a quiver of type $\tilde{A}_{n-1}$ with linear orientation

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n \arrow[llll, bend left] \end{tikzcd}$$

and denote by $\tilde{\Lambda}_n$ the completion of its path algebra over the field k. In this section the finitely presented tilting modules over $\tilde{\Lambda}_n$ are classified. We present two approaches: a reduction to the classification of the Λ_n-tilting modules, and a reduction to the classification of the $\tilde{\Lambda}_{n-1}$-tilting modules.

For each $i \in \{1, \dots, n\}$, we denote by P_i the indecomposable projective $\tilde{\Lambda}_n$-module corresponding to the vertex i. The following figure shows the indecomposable $\tilde{\Lambda}_n$-modules. The dots in the ith column represent the factors of P_i, with $P_i/\operatorname{rad}^{j+1} P_i$ sitting on top of $P_i/\operatorname{rad}^j P_i$ for $1 \leq j < \infty$.



The cyclic group C_n of order n acts in an obvious way on $\tilde{\Lambda}_n$ and therefore on $\operatorname{mod} \tilde{\Lambda}_n$. For each $g \in C_n$ and $X \in \operatorname{mod} \tilde{\Lambda}_n$, we denote by X^g the translate of X.

4.1. Classification via tilting modules over Λ_n.

Consider the embedding of quivers $A_n \to \tilde{A}_{n-1}$ which sends the vertex $i \in A_n$ to $i \in \tilde{A}_{n-1}$. This induces an embedding $\Lambda_n \to \tilde{\Lambda}_n$.

Lemma 4.1. *The functor*

$$F\colon \operatorname{mod} \Lambda_n \longrightarrow \operatorname{mod} \tilde{\Lambda}_n, \quad X \mapsto X \otimes_{\Lambda_n} \tilde{\Lambda}_n$$

has the following properties:

1. *F is faithful, exact, and preserves indecomposability.*
2. $\operatorname{Ext}^1_{\Lambda_n}(X, Y) \cong \operatorname{Ext}^1_{\tilde{\Lambda}_n}(FX, FY)$ *for all* $X, Y \in \operatorname{mod} \Lambda_n$.
3. $X \in \operatorname{mod} \tilde{\Lambda}_n$ *belongs to the image of F iff* $\operatorname{Ext}^1_{\tilde{\Lambda}_n}(X, P_n) = 0$.
4. $X \in \operatorname{mod} \Lambda_n$ *is tilting iff FX is tilting.*

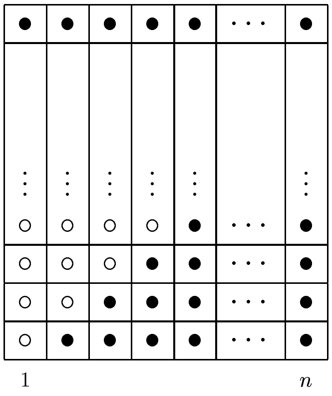
Proof. (1) This is straightforward. Note that $\tilde{\Lambda}_n$ is projective as a Λ_n-module.

(2) Use the Auslander-Reiten formula $\operatorname{Ext}^1_\Lambda(X, Y) = D \operatorname{Hom}_\Lambda(Y, D \operatorname{Tr} X)$. Note that F commutes with the Auslander-Reiten translate $D \operatorname{Tr}$.

(3) An indecomposable module belongs to the image if and only if it is of the form $P_i/\operatorname{rad}^j P_i$ with $j < i$. Now use again the Auslander-Reiten formula.

(4) A Λ-module X is a tilting module if and only if $\operatorname{Ext}^1_\Lambda(X, X) = 0$ and X has n pairwise non-isomorphic indecomposable summands. Thus (4) follows from (1) and (2). □

The following figure describes the image of F.



Proposition 4.2. *A $\tilde{\Lambda}_n$-module is a tilting module if and only if it is isomorphic to $(FT)^g$ for some $g \in C_n$ and some Λ_n-tilting module T. For fixed $g \in C_n$, two Λ_n-tilting modules T and T' are equivalent if and only if $(FT)^g$ and $(FT')^g$ are equivalent.*

Proof. We apply Lemma 4.1. There it is shown that F preserves tilting modules. Now suppose that $T \in \operatorname{mod} \tilde{\Lambda}_n$ is a tilting module. Then T has at least one indecomposable projective summand because every module X of finite length satisfying $\operatorname{Ext}^1_{\tilde{\Lambda}_n}(X, X) = 0$ has at most $n - 1$ pairwise non-isomorphic indecomposable summands. Let $T = T' \amalg T''$ and choose $g \in C_n$ such that $(T')^g \cong P_n$. Then T^g belongs to the image of F by Lemma 4.1, since $\operatorname{Ext}^1_{\tilde{\Lambda}_n}(T^g, P_n) = 0$. Let $T^g = FX$. Then X is a tilting module, again by Lemma 4.1, and $T = (FX)^{g^{-1}}$. This completes the proof. ☐

Corollary 4.3. *The map sending a $\tilde{\Lambda}_n$-module M to* $\dim(\operatorname{top} M)$ *induces a bijection between the set of isomorphism classes of basic tilting modules over $\tilde{\Lambda}_n$ and the set of sequences $(a_1, \dots, a_n)$ of non-negative integers such that $\sum_i a_i = n$.*

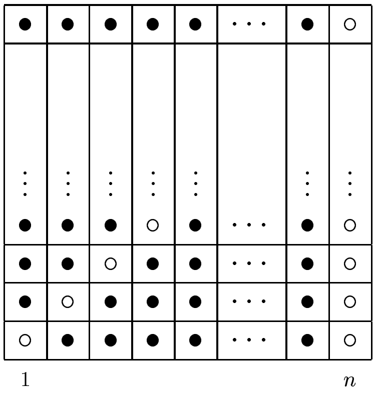
Proof. This follows from the classification of tilting modules over Λ_n in Proposition 3.2, using that $\operatorname{top}(FM) \cong F(\operatorname{top} M)$. ☐

4.2. **Classification via tilting modules over $\tilde{\Lambda}_{n-1}$.** Let

$$\tilde{\Lambda}_n = P_1 \amalg \dots \amalg P_{n-1} \amalg P_n \longrightarrow P_1 \amalg \dots \amalg P_{n-1} \amalg P_1 = P$$

be the map sending $(x_1, \dots, x_{n-1}, x_n)$ to $(x_1, \dots, x_{n-1}, \rho(x_n))$ with $\rho \colon P_n \to P_1$ being the monomorphism with simple cokernel. The composition of the induced map $\operatorname{Hom}_{\tilde{\Lambda}_n}(\tilde{\Lambda}_n, \tilde{\Lambda}_n) \to \operatorname{Hom}_{\tilde{\Lambda}_n}(\tilde{\Lambda}_n, P)$ with the inverse of the isomorphism $\operatorname{Hom}_{\tilde{\Lambda}_n}(P, P) \to \operatorname{Hom}_{\tilde{\Lambda}_n}(\tilde{\Lambda}_n, P)$ induces a ring homomorphism $\phi \colon \tilde{\Lambda}_n \to \operatorname{End}_{\tilde{\Lambda}_n}(P)$. Clearly, $\operatorname{End}_{\tilde{\Lambda}_n}(P)$ is Morita equivalent to $\tilde{\Lambda}_{n-1}$, and restriction of scalars along ϕ induces a fully faithful functor $\phi_* \colon \operatorname{mod} \tilde{\Lambda}_{n-1} \to \operatorname{mod} \tilde{\Lambda}_n$ with inverse $\phi^* \colon \operatorname{mod} \tilde{\Lambda}_n \to \operatorname{mod} \tilde{\Lambda}_{n-1}$ induced by $P \otimes_{\tilde{\Lambda}_n} -$. Note that ϕ is a universal localization in the sense of Schofield [15], making the arrow $n \to 1$ in $\tilde{\Lambda}_n$, hence the map $\rho \colon P_n \to P_1$ in $\operatorname{mod} \tilde{\Lambda}_n$, invertible. In particular, the image of ϕ_* is the full subcategory of modules X in $\operatorname{mod} \tilde{\Lambda}_n$ with $\operatorname{Hom}_{\tilde{\Lambda}_n}(S_1, X) = 0$

and $\mathrm{Ext}^1_{\tilde{\Lambda}_n}(S_1, X) = 0$, since $S_1 = \mathrm{Coker}\,\rho$. The following figure illustrates the image of ϕ_*.



The embedding $\mathrm{mod}\,\tilde{\Lambda}_{n-1} \to \mathrm{mod}\,\tilde{\Lambda}_n$ via ϕ_* is not appropriate for our purpose; we need a slight modification. To this end we consider the full subcategory $\mathcal{X}$ of modules X in $\mathrm{mod}\,\tilde{\Lambda}_n$ satisfying $\mathrm{Ext}^1_{\tilde{\Lambda}_n}(X, S_1) = 0$ and $\mathrm{Ext}^1_{\tilde{\Lambda}_n}(S_1, X) = 0$, which in addition have no direct summand isomorphic to S_1. We denote by $I \colon \mathcal{X} \to \mathrm{mod}\,\tilde{\Lambda}_n$ the inclusion functor.

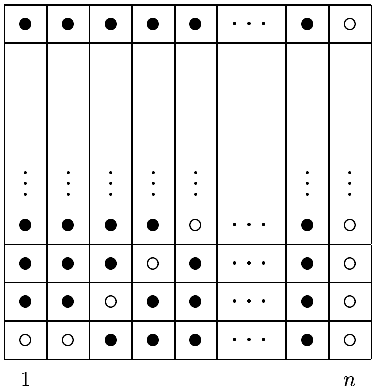
Lemma 4.4. *The functor* $\phi^* \circ I \colon \mathcal{X} \to \mathrm{mod}\,\tilde{\Lambda}_{n-1}$ *is an equivalence.*

Proof. The functor ϕ^* is a left adjoint of the embedding ϕ_*. Denoting by $\mathcal{Y}$ the image of ϕ_*, we see that the composite $\phi_* \circ \phi^*$ leaves almost all indecomposables in $\mathcal{X}$ unchanged, except the indecomposables $X \in \mathcal{X}$ with $\mathrm{soc}\,X = S_1$, which are sent to $X/\mathrm{soc}\,X$. Thus the following diagram commutes.

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\sim"] \arrow[d, "I"] & \mathcal{Y} \arrow[d] \\
\mathrm{mod}\,\tilde{\Lambda}_n \arrow[r, "\phi_* \circ \phi^*"] & \mathrm{mod}\,\tilde{\Lambda}_n
\end{tikzcd}$$

The assertion follows by composing $\phi_* \circ \phi^*$ with ϕ^*, since $\phi^* \circ \phi_* = \mathrm{id}_{\mathrm{mod}\,\tilde{\Lambda}_{n-1}}$. $\square$

We denote by $G = I \circ (\phi^* \circ I)^{-1}$ the composite of I with an inverse of $\phi^* \circ I$. The following figure illustrates the image of G.



Lemma 4.5. *The functor* $G\colon \operatorname{mod} \tilde{\Lambda}_{n-1} \to \operatorname{mod} \tilde{\Lambda}_n$ *has the following properties:*

(1) G *is fully faithful.*

(2) $\operatorname{Ext}^1_{\tilde{\Lambda}_{n-1}}(X, Y) \cong \operatorname{Ext}^1_{\tilde{\Lambda}_n}(GX, GY)$ *for all* $X, Y \in \operatorname{mod} \tilde{\Lambda}_{n-1}$.

(3) $X \in \operatorname{mod} \tilde{\Lambda}_n$ *belongs to the image of* G *iff* $\operatorname{Ext}^1_{\tilde{\Lambda}_n}(X, S_1) = 0 = \operatorname{Ext}^1_{\tilde{\Lambda}_n}(S_1, X)$ *and no direct summand of* X *is isomorphic to* S_1.

Proof. (1) and (3) follow immediately from the definition of G and Lemma 4.4. To prove (2) one uses the Auslander-Reiten formula. □

Proposition 4.6. *Let* $n > 1$. *A* $\tilde{\Lambda}_n$*-module is a tilting module if and only if it is either projective or isomorphic to* $(S \amalg GT)^g$ *for some* $g \in C_n$, *some* $\tilde{\Lambda}_{n-1}$*-tilting module* T, *and some non-zero* $S \in \operatorname{add} S_1$. *For fixed* $g \in C_n$, *two* $\tilde{\Lambda}_{n-1}$*-tilting modules* T *and* T' *are equivalent if and only if* $(S_1 \amalg GT)^g$ *and* $(S_1 \amalg GT')^g$ *are equivalent.*

Proof. Let $T \in \operatorname{mod} \tilde{\Lambda}_n$ be a tilting module and suppose for simplicity that T is basic. Let $\dim(\operatorname{top} T) = (a_1, \dots, a_n)$. Suppose first $a_i \neq 0$ for all i. We claim that in this case T is projective. In fact, T has a projective indecomposable direct summand, say P_i, since there is no tilting module of finite length. We have $\operatorname{Ext}^1_{\tilde{\Lambda}_n}(P_{i+1}/U, P_i) \neq 0$ for all proper factors P_{i+1}/U of P_{i+1}. Thus P_{i+1} is a summand of T. Proceeding by induction, we see that T is projective. Now assume $a_n = 0$ and $a_1 \neq 0$. It is easily checked that this implies $\operatorname{Ext}^1_{\tilde{\Lambda}_n}(T, S_1) = 0$ and $\operatorname{Ext}^1_{\tilde{\Lambda}_n}(S_1, T) = 0$. Thus T has a decomposition $T = T' \amalg S_1$ with $T' = GU$ for some module $\tilde{\Lambda}_{n-1}$-module U, by Lemma 4.5. Moreover, U is a tilting module. Thus any non-projective $\tilde{\Lambda}_n$-tilting module is of the form $(S \amalg GU)^g$ for some $g \in C_n$, some $\tilde{\Lambda}_{n-1}$-tilting module U, and some non-zero $S \in \operatorname{add} S_1$. The converse of this statement is an immediate consequence of Lemma 4.5. This completes the proof. □

5. The collection of all tilting modules

In this section we study the collection of all tilting modules over a fixed algebra Λ. We assume that $\operatorname{mod} \Lambda$ is a Krull-Schmidt category. Thus it is sufficient to study basic tilting modules. Recall that an object is *basic*, if each indecomposable direct summand occurs exactly once in a direct sum decomposition. Let T and U be finitely presented tilting modules. One defines

$$T \leq U \iff T^{\perp} \subseteq U^{\perp}$$

where $T^{\perp} = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_{\Lambda}(T,X) = 0\}$. This defines a partial ordering on the set of isomorphism classes of basic tilting modules which we denote by $\mathcal{T}(\Lambda)$.

There is an alternative description of this partial ordering because $T^{\perp} = \operatorname{Gen} T$ where $\operatorname{Gen} T$ denotes all factors of finite coproducts of copies of T in $\operatorname{mod}\Lambda$.

Lemma 5.1. *Suppose every indecomposable Λ-module is uniserial, that is, the lattice of submodules forms a chain. Then $T \leq U$ if and only every indecomposable summand of T is a factor of some indecomposable summand of U.*

The poset $\mathcal{T}(\Lambda)$ has been studied by various authors. Recent work of Happel and Unger [11] describes the Hasse diagram of this poset in terms of a graph defined by Riedtmann and Schofield [13].

We are also interested in the set $\mathcal{S}(\Lambda)$ of isomorphism classes of finitely presented Λ-modules which are faithful, basic, and selforthogonal. Recall that a module X is *selforthogonal* if $\operatorname{Ext}^1_{\Lambda}(X,X) = 0$. For $X, Y \in \mathcal{S}(\Lambda)$ we define $X \leq Y$ if X is isomorphic to a direct summand of Y.

From now on we fix $n \geq 1$ and assume that Λ is the completed path algebra of a quiver of type A_n or $\tilde{A}_{n-1}$ with linear orientation. Thus $\Lambda = \Lambda_n$ or $\Lambda = \tilde{\Lambda}_n$. The combinatorial analysis of $\mathcal{T}(\Lambda)$ is based on the description of the indecomposable Λ-modules via intervals. To each interval I in $\mathcal{I}(n)$ or $\tilde{\mathcal{I}}(n)$ we assign the indecomposable M_I. This is by definition the factor of the projective P_i of composition length $l-1$ where $i = \sup I$ and $l = \operatorname{card} I$. Note that the M_I provide a complete list of indecomposable Λ-modules. In order to describe $\mathcal{T}(\Lambda)$, we use the Tamari lattice $\mathcal{C}(n)$ and its variation $\tilde{\mathcal{C}}(n)$, which are defined and discussed in the appendix.

Theorem 5.2. *The assignment $X \mapsto M_X = \coprod_{I \in X} M_I$ induces isomorphisms*

$$\mathcal{C}(n) \xrightarrow{\sim} \mathcal{T}(\Lambda_n) \quad \text{and} \quad \tilde{\mathcal{C}}(n) \xrightarrow{\sim} \mathcal{T}(\tilde{\Lambda}_n)$$

of partially ordered sets.

Proof. The fact that both maps are well-defined bijections follows from the classification of the tilting modules for Λ_n in Proposition 3.2, and for $\tilde{\Lambda}_n$ in Corollary 4.3. For the partial ordering in $\mathcal{T}(\Lambda)$ we use the description given in Lemma 5.1. The lemma given below translates the factor relation between indecomposable Λ-modules into a relation between the corresponding intervals. The relation $I \twoheadrightarrow J$ between intervals is precisely the one used for the definition of the partial ordering on $\mathcal{C}(n)$ and $\tilde{\mathcal{C}}(n)$. Thus both maps respect the poset structure and the proof is complete. □

Lemma 5.3. *Let $I, J \in \mathcal{I}(n)$ or $I, J \in \tilde{\mathcal{I}}(n)$.*

(1) *There is a monomorphism $M_I \to M_J$ if and only if $I \rightarrowtail J$.*

(2) *There is an epimorphism* $M_I \to M_J$ *if and only if* $I \twoheadrightarrow J$.

Proof. Clear. ☐

Next we describe the cover relation in $\mathcal{T}(\Lambda)$. This is based on the analysis of $\mathcal{C}(n)$ and $\tilde{\mathcal{C}}(n)$ in the appendix.

Proposition 5.4. *Let* $T, T' \in \mathcal{T}(\Lambda)$. *Then* T *covers* T' *or* T' *covers* T *if and only if* T *and* T' *have precisely* $n-1$ *indecomposable direct summands in common.*

Proof. For Λ_n apply Lemma A.4, and for $\tilde{\Lambda}_n$ use Proposition B.2 to reduce from $\tilde{\mathcal{C}}(n)$ to $\mathcal{C}(n)$. ☐

Proposition 5.5. *For* $T, T' \in \mathcal{T}(\Lambda)$ *the following are equivalent:*

(1) T *covers* T'.
(2) *There are decompositions* $T = T_0 \amalg X$ *and* $T' = T_0' \amalg X$ *such that* T_0 *and* T_0' *are indecomposable with a monomorphism* $T_0 \to X$ *and an epimorphism* $X \to T_0'$.
(3) *There are decompositions* $T = T_0 \amalg X$ *and* $T' = T_0' \amalg X$ *such that* T_0 *and* T_0' *are indecomposable with a monomorphism* $T_0 \to X_0$ *and an epimorphism* $X_0 \to T_0'$ *for some indecomposable summand* X_0 *of* X.

Proof. Apply Lemma A.5 and Proposition B.2. ☐

We end this section with a description of $\mathcal{S}(\Lambda)$ which is the analogue of our results on $\mathcal{T}(\Lambda)$. We refer to the appendix for the definitions of $\mathcal{B}(n)$ and $\tilde{\mathcal{B}}(n)$.

Theorem 5.6. *The assignment* $X \mapsto M_X = \coprod_{I \in X} M_I$ *induces isomorphisms*

$$\mathcal{B}(n) \xrightarrow{\sim} \mathcal{S}(\Lambda_n) \quad \text{and} \quad \tilde{\mathcal{B}}(n) \xrightarrow{\sim} \mathcal{S}(\tilde{\Lambda}_n)$$

of partially ordered sets.

Proof. First observe that a Λ_n-module is faithful if and only if the indecomposable projective of maximal dimension appears as a direct summand. Thus a subset $X \subseteq \mathcal{I}(n)$ corresponds to a faithful and selforthogonal module M_X if and only if X belongs to $\mathcal{B}(n)$. This follows from Lemma 3.1.

Now let $\Lambda = \tilde{\Lambda}_n$. Observe that a $\tilde{\Lambda}_n$-module is faithful if and only if there is a non-zero projective direct summand. Thus every faithful selforthogonal module lies, up to a cyclic permutation, in the image of F, by Lemma 4.1. Note that $F(M_X) = M_{\pi^*(X)}$ for each $X \in \mathcal{B}(n)$. Thus F commutes with the embedding $\mathcal{B}(n) \to \tilde{\mathcal{B}}(n)$. We conclude that $X \mapsto M_X$ induces an isomorphism $\tilde{\mathcal{B}}(n) \to \mathcal{S}(\tilde{\Lambda}_n)$. ☐

Appendix A. Stasheff associahedra

Fix an integer $n \geq 1$. The *Stasheff associahedron* of dimension $n-1$ is a convex polyhedron whose faces are indexed by the meaningful bracketings of a string of $n+1$ letters [17, 20]. We shall identify the Stasheff associahedron with its poset of faces. This can be described as follows. Let $\mathcal{I}(n)$ be the set of intervals $[i,j] = \{i, i+1, \dots, j\}$ in $\mathbb{Z}$ with $0 \leq i < j \leq n$. Two intervals I, J are said to be *compatible* if $I \subseteq J$ or

$J \subseteq I$ or $I \cap J = \emptyset$. Denote by $\mathcal{B}(n)$ the set of all subsets $X \subseteq \mathcal{I}(n)$ such that $[0, n] \in X$ and all intervals in X are pairwise compatible. The set $\mathcal{B}(n)$ is ordered by inclusion. In fact, $\mathcal{B}(n)$ is a lattice and we identify it with the lattice of faces of the Stasheff associahedron of dimension $n - 1$ by identifying an interval $[i, j]$ with the bracketing $x_0 \dots (x_i \dots x_j) \dots x_n$ of the string $x_0 \dots x_n$. This identification is order reversing, that is, $X \subseteq Y$ in $\mathcal{B}(n)$ if and only if the face corresponding to X contains the face corresponding to Y. In particular, a set $X \in \mathcal{B}(n)$ of cardinality p corresponds to a face of dimension $n - p$. Note that the cardinality of a set in $\mathcal{B}(n)$ is bounded by n.

A *vertex* of $\mathcal{B}(n)$ is by definition an element in $\mathcal{B}(n)$ having cardinality n. The set of vertices of $\mathcal{B}(n)$ is denoted by $\mathcal{C}(n)$. Let us give an alternative description of the set of vertices. To this end define $\operatorname{top} X$ for each $X \subseteq \mathcal{I}(n)$ to be the sequence $(a_1, \dots, a_n)$ with $a_p = \operatorname{card}\{I \in X \mid \sup I = p\}$ for $1 \leq p \leq n$.

Lemma A.1. *The map sending $X \in \mathcal{C}(n)$ to $\operatorname{top} X$ induces a bijection between $\mathcal{C}(n)$ and the set of sequences $(a_1, \dots, a_n)$ of non-negative integers such that $\sum_i a_i = n$ and $\sum_{i \leq p} a_i \leq p$ for all $1 \leq p \leq n$. In particular, the cardinality of $\mathcal{C}(n)$ equals the Catalan number $C(n) = \frac{1}{n+1}\binom{2n}{n}$*

Proof. Identifying $X \in \mathcal{C}(n)$ with a bracketing of a string $x_0 \dots x_n$, the sequence $\operatorname{top} X = (a_1 \dots, a_n)$ represents the positions of the closing brackets. Clearly, $\operatorname{top} X$ satisfies $\sum_i a_i = n$ and $\sum_{i \leq p} a_i \leq p$ for all p. Moreover, each bracketing is determined by this data. □

We define the following relations on the set $\mathcal{I}(n)$ of intervals:

$$\begin{aligned} I \rightarrowtail I' \quad &\iff \quad \inf I = \inf I' \text{ and } \operatorname{card} I \leq \operatorname{card} I'; \\ I \twoheadrightarrow I' \quad &\iff \quad \sup I = \sup I' \text{ and } \operatorname{card} I \geq \operatorname{card} I'. \end{aligned}$$

Given subsets X and X' of $\mathcal{I}(n)$, we define $X \rightarrowtail X'$ if for each $I \in X$ there exists $I' \in X'$ with $I \rightarrowtail I'$. Analogously, $X' \twoheadrightarrow X$ if for each $I \in X$ there exists $I' \in X'$ with $I' \twoheadrightarrow I$.

Lemma A.2. *The set $\mathcal{C}(n)$ is partially ordered via*

$$X' \geq X \iff X' \twoheadrightarrow X.$$

Proof. Transitivity is clear. Now suppose $X \geq X' \geq X$. Both sets have cardinality n. The assumption implies that all intervals in $X \cup X'$ are pairwise compatible. Thus $X = X'$. □

Remark A.3. Let $X, X' \in \mathcal{C}(n)$. Then one can show that $X' \twoheadrightarrow X$ if and only if $X' \rightarrowtail X$.

It turns out that $\mathcal{C}(n)$ is in fact a lattice, which appears as *Tamari lattice* in the literature [18, 16]. The Tamari lattice can be described in many ways via the known bijections between families of Catalan objects. Our description seems to be new. It is related to the usual definition via the covering relation in $\mathcal{C}(n)$. Recall that an element x in a poset covers another element x' if $\{y \mid x \geq y \geq x'\} = \{x, x'\}$.

Lemma A.4. *Let $X, X' \in \mathcal{C}(n)$. Then X covers X' or X' covers X if and only if $X \cap X'$ has cardinality $n-1$.*

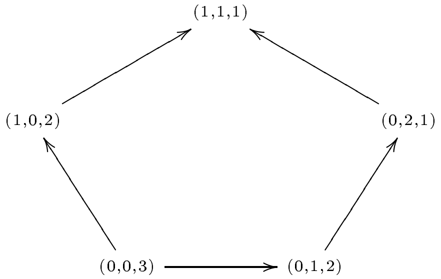
Lemma A.5. *Let $X, X' \in \mathcal{C}(n)$ and $Y = X \cap X'$. Then the following are equivalent:*

1. *X covers X'.*
2. *Y has cardinality $n-1$ and $X \rightarrowtail Y \twoheadrightarrow X'$.*
3. *There are intervals I, I' such that $X = Y \cup \{I\}$ and $X' = Y \cup \{I'\}$. Moreover, $I \cup I' \in Y$ and $I \rightarrowtail (I \cup I') \twoheadrightarrow I'$.*

The proofs of Lemma A.4 and Lemma A.5 are elementary, but rather technical and therefore omitted. A key observation is the following. Given $I \in X \in \mathcal{C}(n)$ with $I \neq [0, n]$, there exists $I' \in X \setminus \{I\}$ such that either $I \rightarrowtail I'$ or $I' \twoheadrightarrow I$.

Corollary A.6. *The Hasse diagram of the Tamari lattice $\mathcal{C}(n)$ equals the 1-skeleton of the Stasheff associahedron $\mathcal{B}(n)$.*

The following figure shows the Hasse diagram of $\mathcal{C}(3)$.



Appendix B. Circular associahedra

Fix an integer $n \geq 1$. We need some notation. Given $X \subseteq \mathbb{Z}$ and $z \in \mathbb{Z}$, we define $X + z = \{x + z \mid x \in X\}$. This definition extends to subsets $X \subseteq 2^{\mathbb{Z}}$ and $X \subseteq 2^{(2^{\mathbb{Z}})}$.

Let $\mathcal{I}$ be the set of possibly infinite intervals $I \subseteq \mathbb{Z}$ with $\sup I < \infty$. Two intervals I and J are said to be *n-equivalent* if there exists $z \in \mathbb{Z}$ such that $J = I + zn$. We denote by $\tilde{\mathcal{I}}(n)$ the set of equivalence classes of n-equivalent intervals from $\mathcal{I}$. Next consider the projection

$$\pi \colon \mathbb{Z} \longrightarrow \{0, 1, 2, 3, \ldots\}, \quad z \mapsto \begin{cases} z & \text{if } z \geq 0, \\ 0 & \text{if } z < 0. \end{cases}$$

This induces an injective map $\pi^* \colon \mathcal{I}(n) \to \tilde{\mathcal{I}}(n)$ which takes $I \in \mathcal{I}(n)$ to the equivalence class of $\pi^{-1}(I)$. We define $\tilde{\mathcal{B}}(n)$ to be the set of subsets of $\tilde{\mathcal{I}}(n)$ which are of the form $\pi^*(X) + z$ for some $X \in \mathcal{B}(n)$ and some $z \in \mathbb{Z}$. Thus we have an injective map

$$\mathcal{B}(n) \longrightarrow \tilde{\mathcal{B}}(n), \quad X \mapsto \pi^*(X),$$

and viewing this as an identification, we get

$$\tilde{\mathcal{B}}(n) = \bigcup_{p=0}^{n-1} \mathcal{B}(n) + p.$$

We note that $\tilde{\mathcal{B}}(n)$ is partially ordered by inclusion.

A *vertex* of $\tilde{\mathcal{B}}(n)$ is by definition an element in $\tilde{\mathcal{B}}(n)$ having cardinality n. The set of vertices of $\tilde{\mathcal{B}}(n)$ is denoted by $\tilde{\mathcal{C}}(n)$. Each $X \in \tilde{\mathcal{B}}(n)$ is a set of equivalence classes of intervals in $\mathbb{Z}$. Thus we can define $\operatorname{top} X = (a_1, \dots, a_n)$ with $a_p = \operatorname{card}\{I \in X \mid p \equiv \sup I \pmod n\}$ for $1 \leq p \leq n$. Note that for each $I \in \tilde{I}(n)$, the values $\inf I$ and $\sup I$ are well-defined modulo n.

Lemma B.1. *The map sending $X \in \tilde{\mathcal{C}}(n)$ to $\operatorname{top} X$ induces a bijection between $\tilde{\mathcal{C}}(n)$ and the set of sequences $(a_1, \dots, a_n)$ of non-negative integers such that $\sum_i a_i = n$. In particular, the cardinality of $\tilde{\mathcal{C}}(n)$ equals $\binom{2n-1}{n-1}$.*

Proof. We use the embedding $\mathcal{C}(n) \to \tilde{\mathcal{C}}(n)$ via π^* and the description of $\mathcal{C}(n)$ via integer sequences in Lemma A.1. Given a sequence $(a_1, \dots, a_n)$, there is a cyclic permutation $(a_k, \dots, a_{k-1})$ such that $\sum_{i=1}^{p} a_{k+i-1} \leq p$ for all $1 \leq p \leq n$. Thus each sequence is of the form $\operatorname{top} X$ for some $X \in \tilde{\mathcal{C}}(n)$. On the other hand, two elements X, X' in $\mathcal{C}(n)$ get identified in $\tilde{\mathcal{C}}(n)$ after a cyclic permutation, that is $\pi^*(X') = \pi^*(X) + p$ for some p, if and only if $\operatorname{top} X'$ is a cyclic permutation of $\operatorname{top} X$. ☐

We define the following relations on the set $\tilde{\mathcal{I}}(n)$ of intervals:

$$\begin{aligned} I \rightarrowtail I' \quad &\iff \quad \inf I = \inf I' \pmod n \text{ and } \operatorname{card} I \leq \operatorname{card} I'; \\ I \twoheadrightarrow I' \quad &\iff \quad \sup I = \sup I' \pmod n \text{ and } \operatorname{card} I \geq \operatorname{card} I'. \end{aligned}$$

As in Section A, this induces relations $X \rightarrowtail X'$ and $X \twoheadrightarrow X'$ for subsets X, X' of $\tilde{\mathcal{I}}(n)$. Moreover, one obtains a partial ordering on the set $\tilde{\mathcal{C}}(n)$ via

$$X' \geq X \iff X' \twoheadrightarrow X.$$

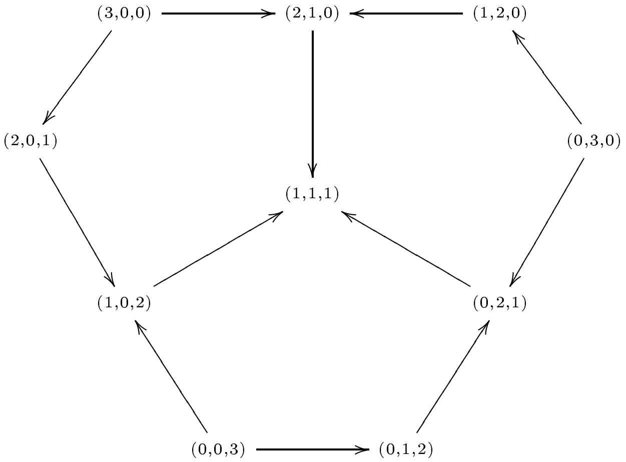
Next we describe the poset structure of $\tilde{\mathcal{C}}(n)$. We use two approaches: a description via $\mathcal{C}(n)$ and a description via $\tilde{\mathcal{C}}(n-1)$. It is convenient to identify each element X in $\mathcal{C}(n)$ or $\tilde{\mathcal{C}}(n)$ with the integer sequence $\operatorname{top} X$. We define $\mathbf{1} = (1, \dots, 1)$ and for each $i \in \{1, \dots, n\}$ we denote by $\mathbf{0}_i$ the sequence $(a_1, \dots, a_n)$ with $a_i = n$ and $a_j = 0$ for $j \neq i$.

Proposition B.2. *The poset $\tilde{\mathcal{C}}(n)$ has the following properties:*

1. *$\mathbf{1}$ is the unique maximal element.*
2. *$\{\mathbf{0}_i \mid 1 \leq i \leq n\}$ is the set of minimal elements.*
3. *Each set of elements has a supremum.*
4. *The natural embedding $\mathcal{C}(n) \to \tilde{\mathcal{C}}(n)$ induces an isomorphism of posets between $\mathcal{C}(n)$ and the interval $[\mathbf{0}_n, \mathbf{1}]$.*

Proof. The assertions follow from some elementary properties of the embedding $\mathcal{C}(n) \to \tilde{\mathcal{C}}(n)$. This embedding sends X to $\pi^*(X)$ and we observe that $\operatorname{top} X = \operatorname{top} \pi^*(X)$. Moreover $X \leq Y$ in $\mathcal{C}(n)$ if and only if $\pi^*(X) \leq \pi^*(Y)$. Finally, we note that each $X \in \tilde{\mathcal{C}}(n)$ contains at least one infinite interval, say I with $i = \sup I$, and this implies $\mathbf{0}_i \leq X$. ☐

The following figure shows the Hasse diagram of $\tilde{\mathcal{C}}(3)$.



Proposition B.3. *Let $n > 1$. The map*

$$\tilde{\mathcal{C}}(n-1) \longrightarrow \tilde{\mathcal{C}}(n), \quad (a_1, \dots, a_{n-1}) \mapsto (a_1 + 1, a_2, \dots, a_{n-1}, 0)$$

induces an isomorphism of posets onto its image. Moreover, the image is interval closed.

Proof. The assertion follows from an explicit description of the embedding $\tilde{\mathcal{C}}(n-1) \to \tilde{\mathcal{C}}(n)$. The map sends $X \in \tilde{\mathcal{C}}(n-1)$ to $\alpha(X) \cup \{S\}$, where S is the n-equivalence class of the interval $[0,1]$, and $\alpha \colon \tilde{\mathcal{I}}(n-1) \to \tilde{\mathcal{I}}(n)$ sends the $n-1$-equivalence class of an interval $I \subseteq \mathbb{Z}$ with $\sup I \in \{1, \dots, n-1\}$ to the n-equivalence class of the interval $I' \subseteq \mathbb{Z}$ with $\sup I' = \sup I$ and

$$\operatorname{card} I' = \begin{cases} \operatorname{card} I & \text{if } \operatorname{card} I \leq \sup I, \\ 1 + \operatorname{card} I & \text{if } \operatorname{card} I > \sup I. \end{cases}$$

Note that $I \twoheadrightarrow J$ if and only if $\alpha(I) \twoheadrightarrow \alpha(J)$. ☐

Corollary B.4. *Viewing the injective maps $\mathcal{C}(n) \to \tilde{\mathcal{C}}(n)$ and $\tilde{\mathcal{C}}(n-1) \to \tilde{\mathcal{C}}(n)$ as identifications, we have*

$$\tilde{\mathcal{C}}(n) = \bigcup_{p=0}^{n-1} \mathcal{C}(n) + p \qquad \textit{and} \qquad \tilde{\mathcal{C}}(n) \setminus \{\mathbf{1}\} = \bigcup_{p=0}^{n-1} \tilde{\mathcal{C}}(n-1) + p.$$

The first equation says that the poset $\tilde{\mathcal{C}}(n)$ is the union of n copies of the Tamari lattice $\mathcal{C}(n)$. Kiyoshi Igusa pointed out to us that this fact can be expressed numerically by the following inclusion-exclusion formula. Note that the cardinality of $\tilde{\mathcal{C}}(n)$ is $\binom{2n-1}{n-1}$, wheras the cardinality of $\mathcal{C}(n)$ is the Catalan number $C(n) = \frac{1}{n+1}\binom{2n}{n}$.

$$\binom{2n-1}{n-1} = \sum_{i=1}^{n} (-1)^{i-1} \frac{n}{i} \sum_{n_1+\ldots+n_i=n} C(n_1)C(n_2)\ldots C(n_i) \tag{B.1}$$

Note that all n_j in this formula are positive integers. We do not know whether the Hasse diagram of $\tilde{\mathcal{C}}(n)$ arises as the 1-skeleton of a polytope.

REFERENCES

[1] Auslander, M.: *Functors determined by objects*, in: Representation theory of algebras, ed. R. Gordon, Marcel Dekker (1978), 1–244.

[2] van den Bergh, M. and Reiten, I.: *Noetherian hereditary abelian categories with Serre duality*, J. Amer. Math. Soc. **15** (2002), 295-366.

[3] Bongartz, K.: *Tilted algebras*, in: Representations of algebras (Puebla, 1980), Lecture Notes in Math. **903**, Springer, Berlin-New York (1981), 26–38.

[4] Bongartz, K. and Gabriel, P.: *Covering spaces in representation-theory*, Invent. Math. **65** (1981/82), 331–378.

[5] Buan, A. and Krause, H.: *Cotilting modules over tame hereditary algebras*, Pacific J. Math., to appear.

[6] Colpi, R.: *Tilting in Grothendieck categories*, Forum Math. **11** (1999), 735-759.

[7] Crawley-Boevey, W. W.: *Modules of finite length over their endomorphism ring*, in: Representations of algebras and related topics, eds. S. Brenner and H. Tachikawa, London Math. Soc. Lec. Note Series **168** (1992), 127–184.

[8] Crawley-Boevey, W. W.: *Locally finitely presented additive categories*, Comm. Algebra **22** (1994), 1644–1674.

[9] Gabriel, P.: *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448.

[10] Happel, D.: *A characterization of hereditary categories with a tilting object*, Invent. Math. **144** (2001), 381–398.

[11] Happel, D. and Unger, L.: *On a partial order of tilting modules*, preprint.

[12] Marsh, R.; Reineke, M. and Zelevinski, A.: *Generalized associahedra and quiver representations*, preprint.

[13] Riedtmann, C. and Schofield, A.: *On a simplicial complex associated with tilting modules*, Comment. Math. Helv. **66** (1991), 70–78.

[14] Ringel, C. M.: *Finite dimensional hereditary algebras of wild representation type*, Math. Z. **161** (1978), 235–255.

[15] Schofield, A.: *Representations of rings over skew fields*, London Math. Soc. Lec. Note Series **92** (1985).

[16] Stanley, R. P.: *Enumerative combinatorics, Volume 2*, Cambridge Univ. Press (1999).

[17] Stasheff, J. D.: *Homotopy associativity of H-spaces I*, Trans. Amer. Math. Soc. **138** (1963), 275–292.

[18] Tamari, D.: *The algebra of bracketings and their enumeration*, Nieuw Arch. Wisk. **10** (1962), 131–146.

[19] The On-Line Encyclopedia of Integer Sequences, `www.research.att.com/~njas/sequences/`

[20] Ziegler, G. M.: *Lectures on Polytopes*, Springer Verlag, New York (1995).

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