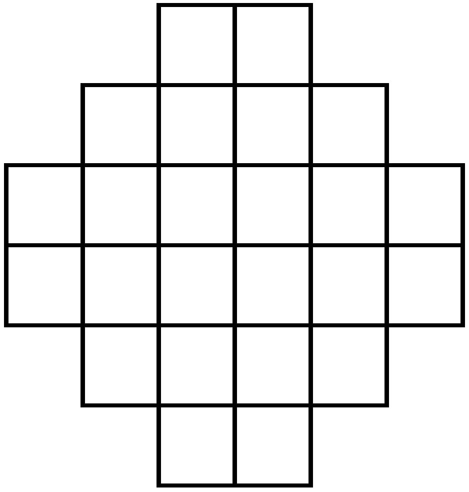
Matchings graphs for Gale-Robinson sequences (featuring Somos-4 and Somos-5)

Mireille Bousquet-Melou (Bordeaux)

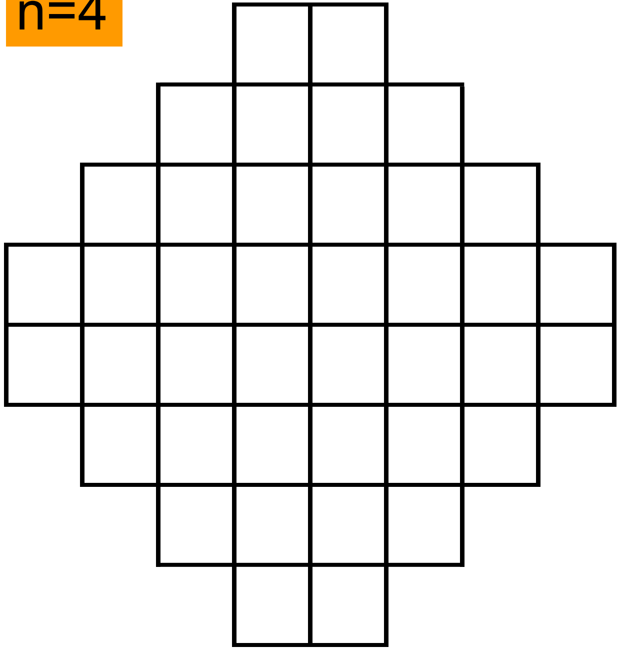
Jim Propp (Wisconsin)

Julian West (Victoria)

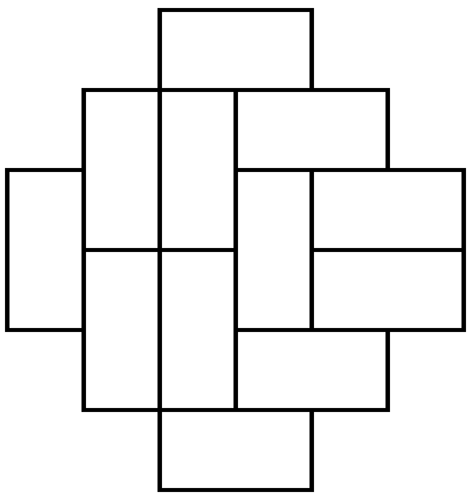
n=3



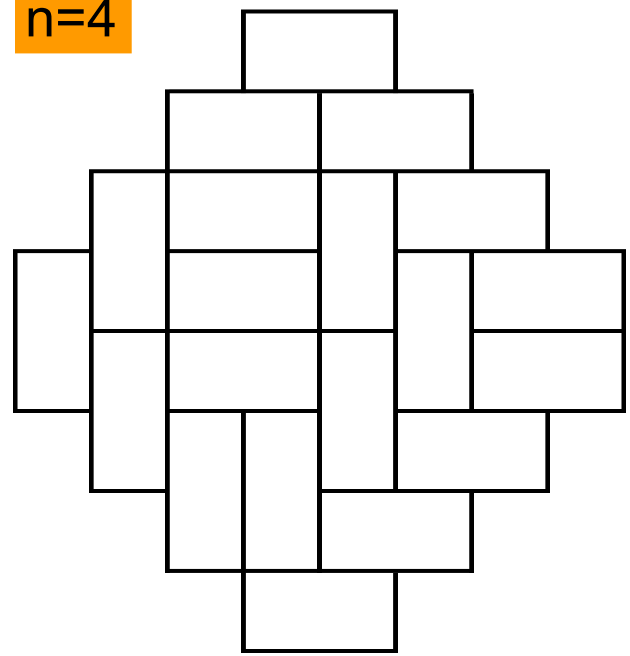
n=4



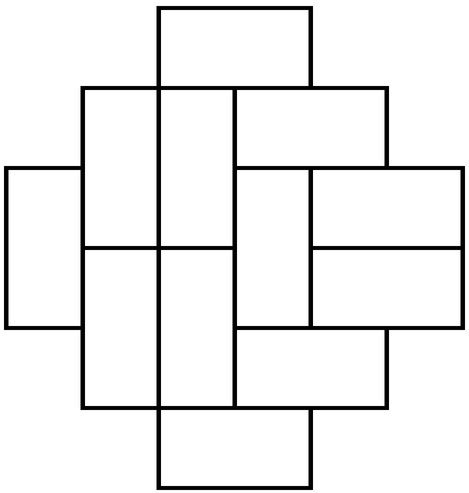
n=3



n=4

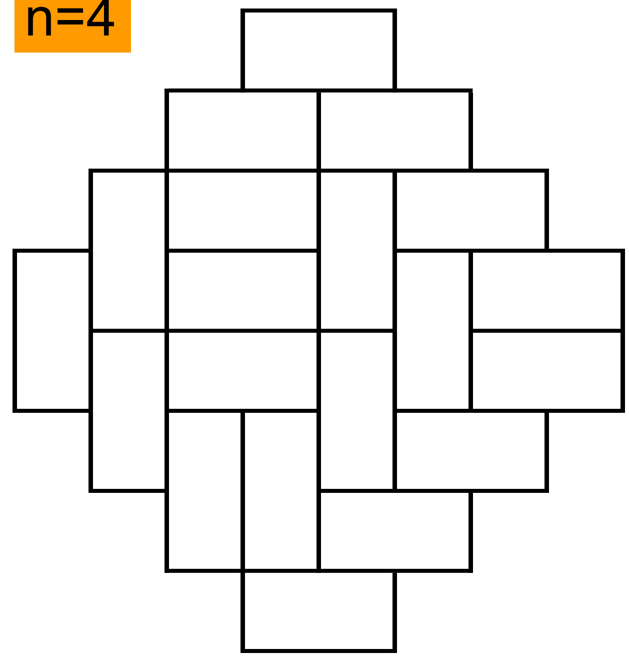


n=3

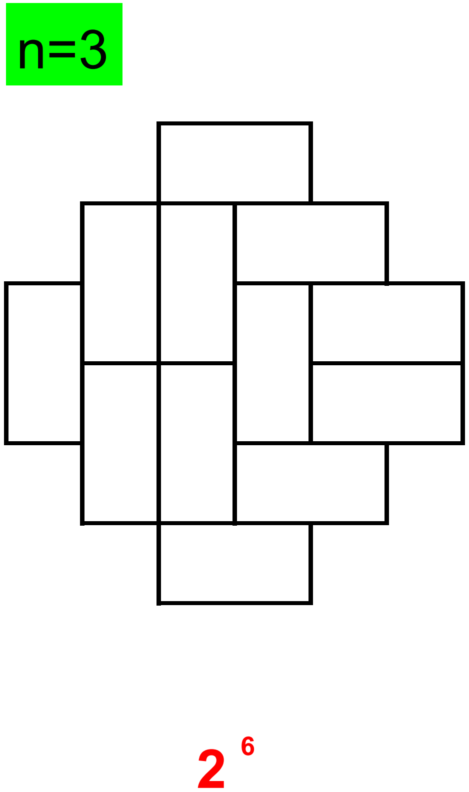


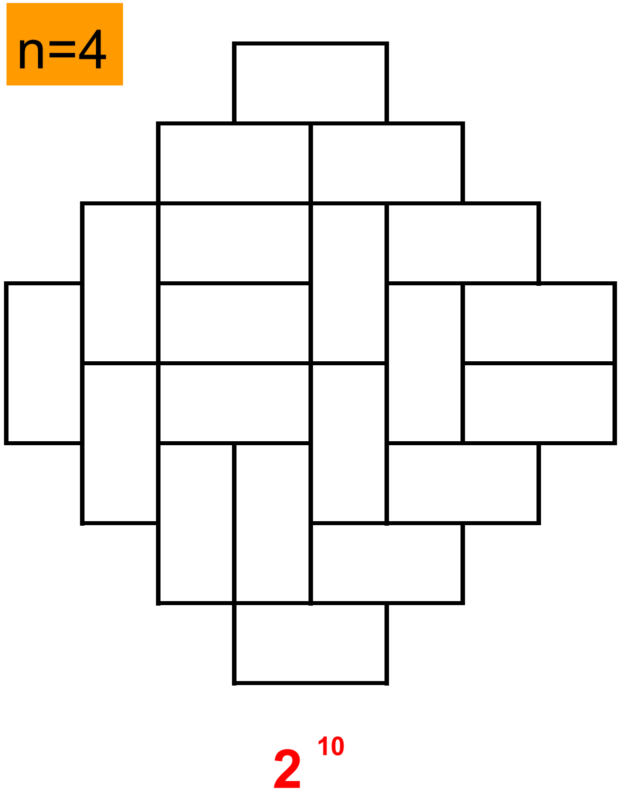
2^6

n=4



2^{10}





$$T(n) = 2^{n(n+1)/2}$$

$$T(n) = 2^{n(n+1)/2}$$

$$T(n) = 2T(n-1)T(n-1)/T(n-2)$$

$$T(n)T(n-2) = T(n-1)T(n-1) + T(n-1)T(n-1)$$

$$T(n) = 2^{n(n+1)/2}$$

$$T(n) = 2T(n-1)T(n-1)/T(n-2)$$

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$$T(n) = 2^{n(n+1)/2}$$

$$T(n) = 2T(n-1)T(n-1)/T(n-2)$$

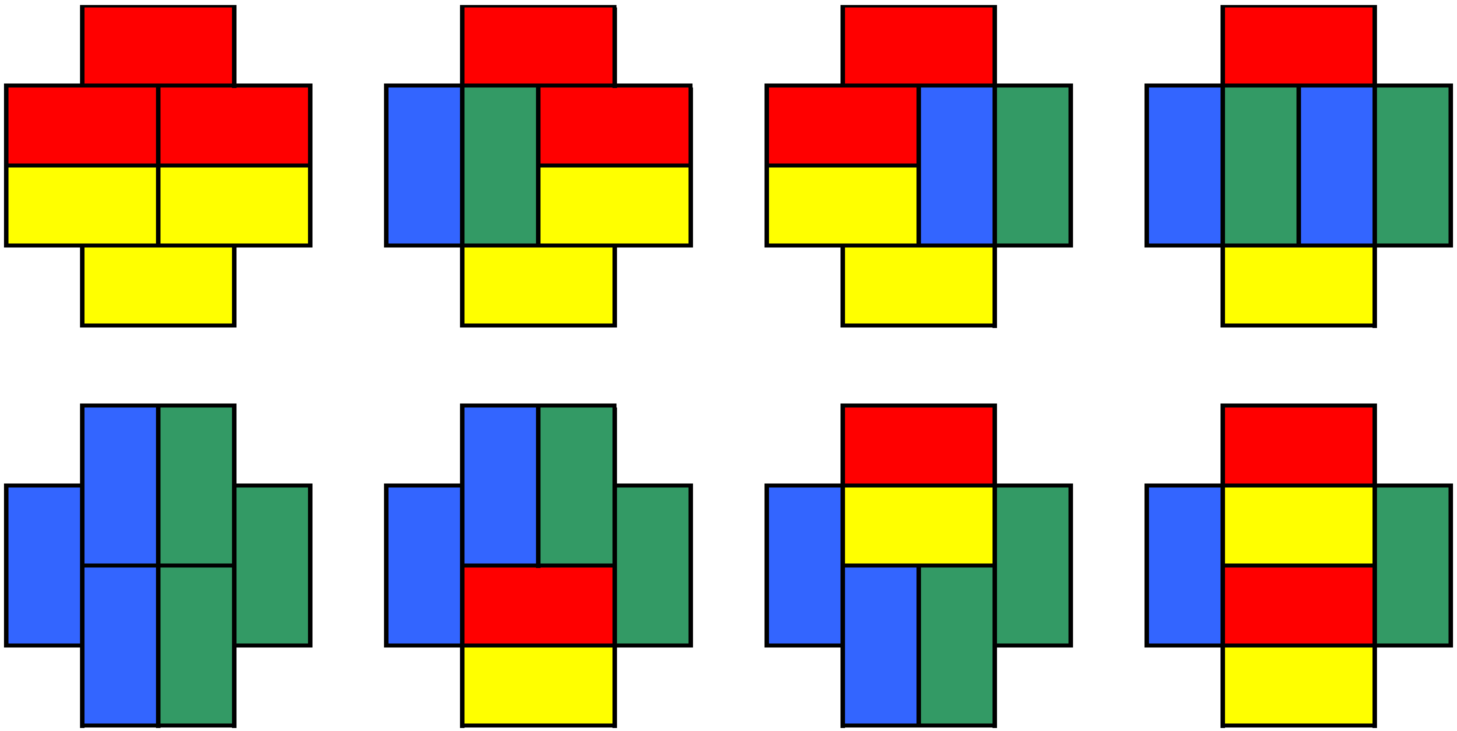
$$T(n)T(n-2) = T(n-1)T(n-1) + T(n-1)T(n-1)$$

<u>Gale-Robinson sequence</u>

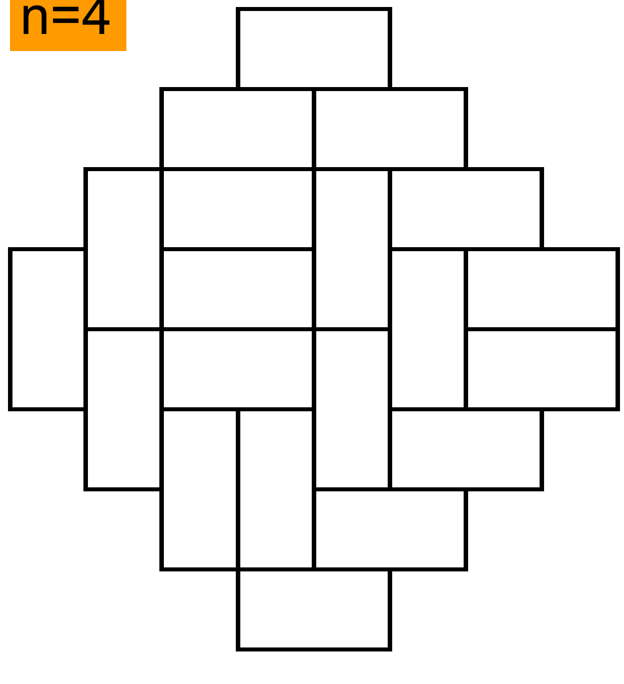
$$T(n)T(n-k) = T(n-i)T(n-j) + T(n-x)T(n-y)$$

Kuo condensation for Aztec diamonds

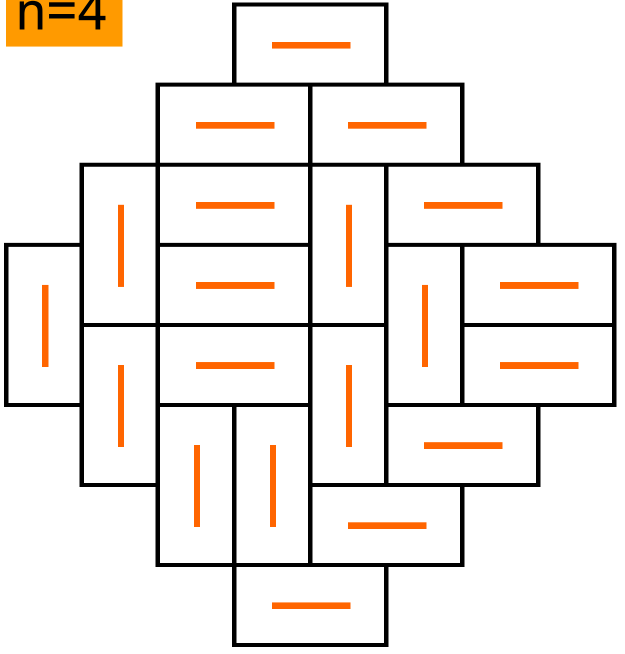
$$T(n)T(n\text{-}2) = T(n\text{-}1)T(n\text{-}1) + T(n\text{-}1)T(n\text{-}1)$$



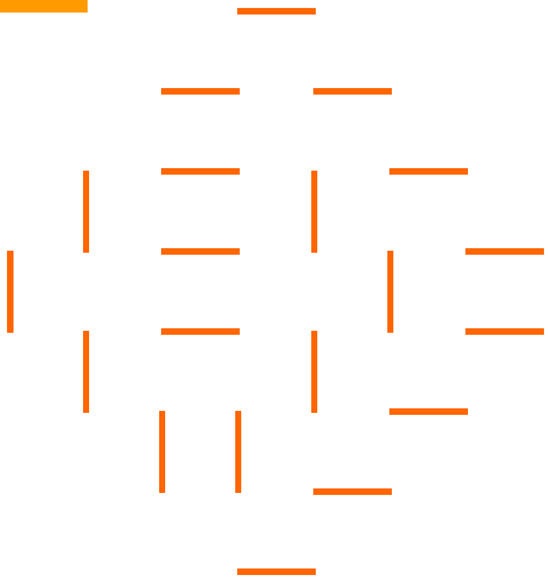
n=4



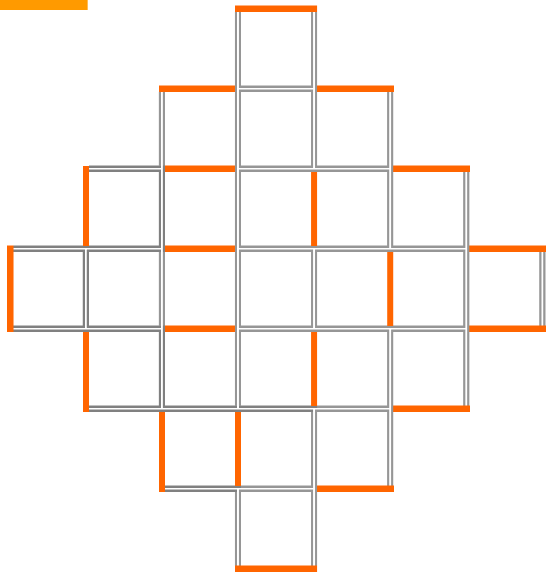
n=4



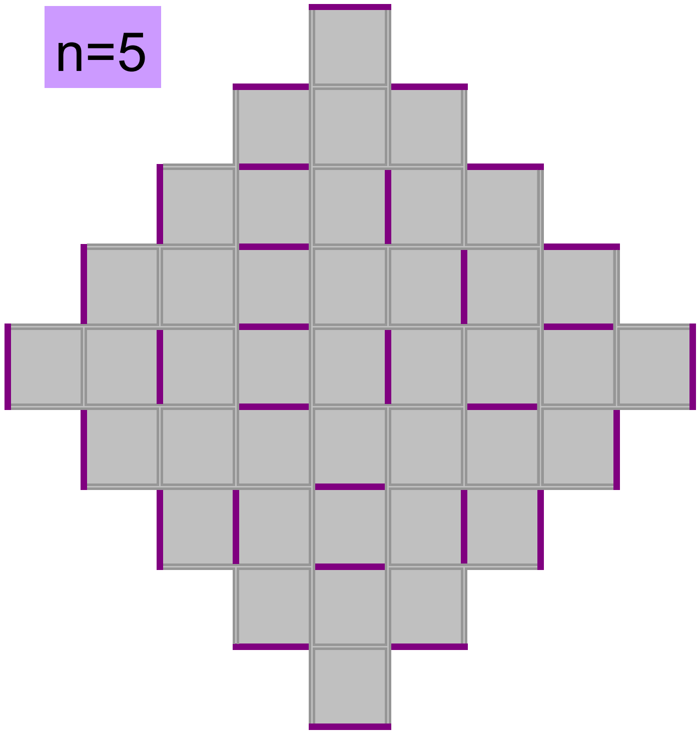
n=4



n=4

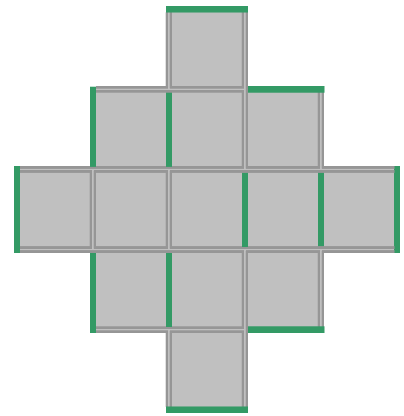


n=5

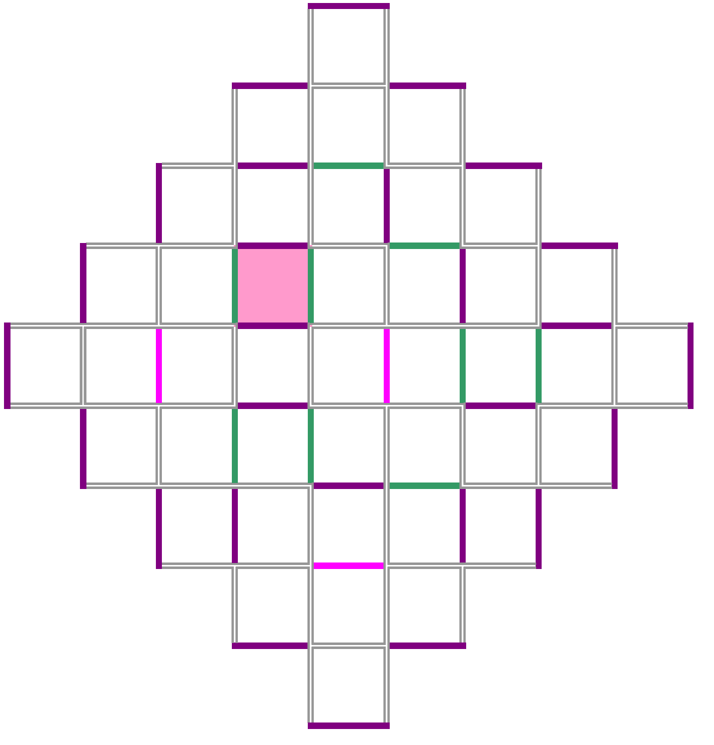


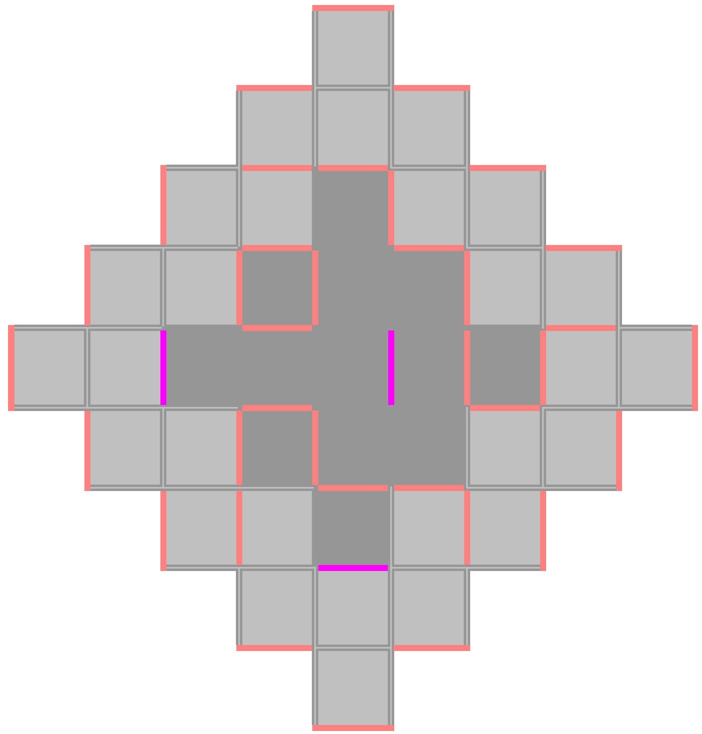
T(5)

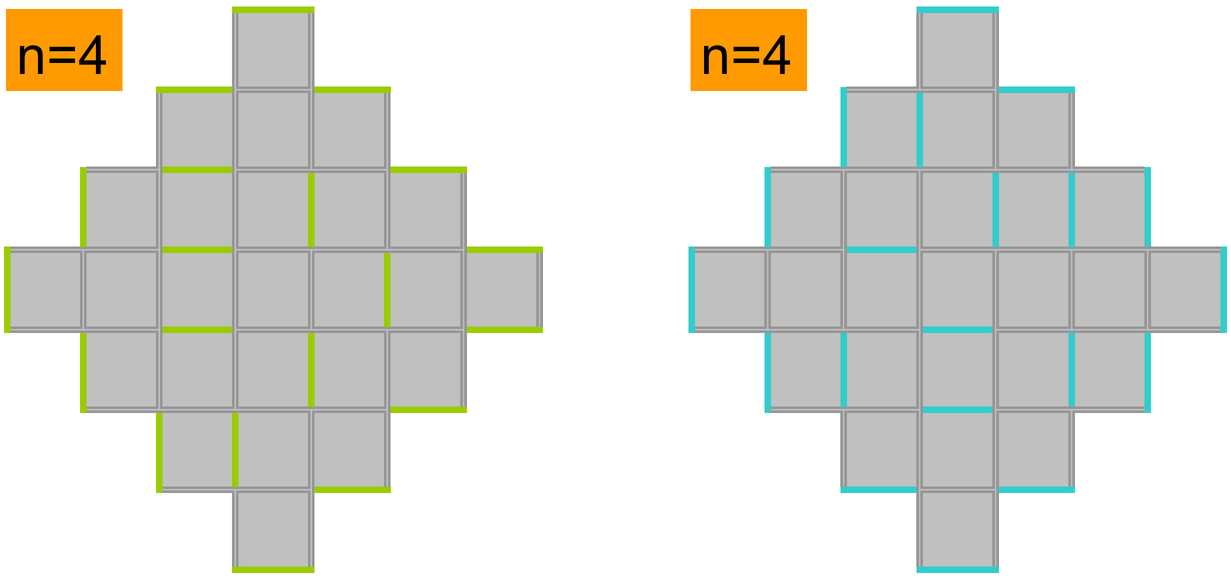
n=3

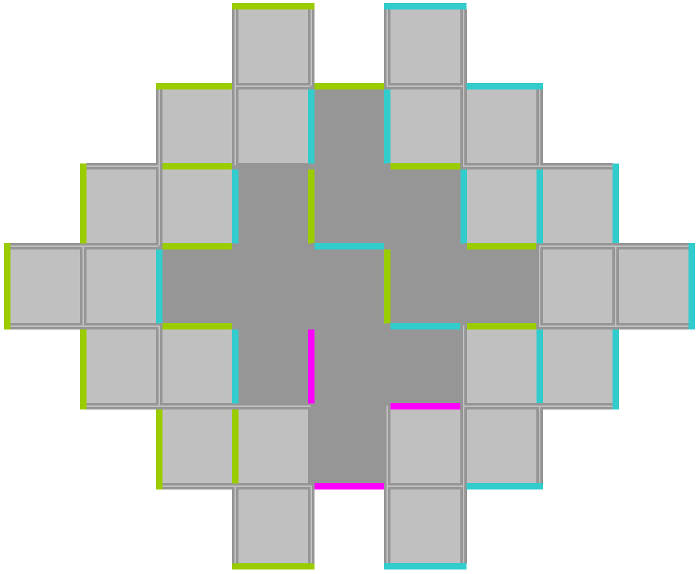


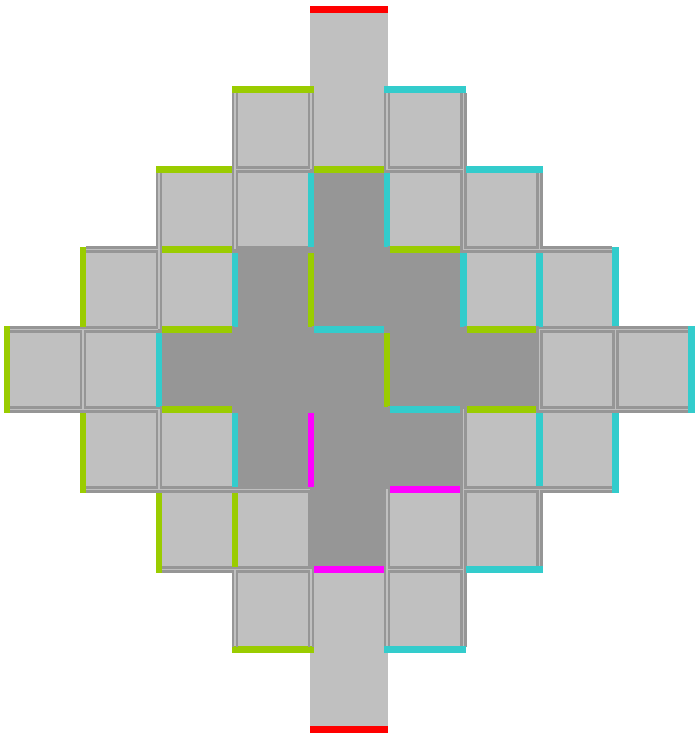
T(3)

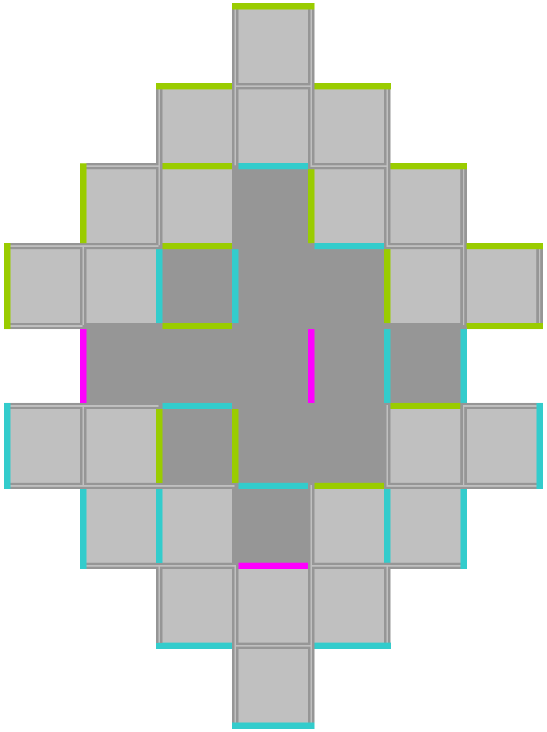


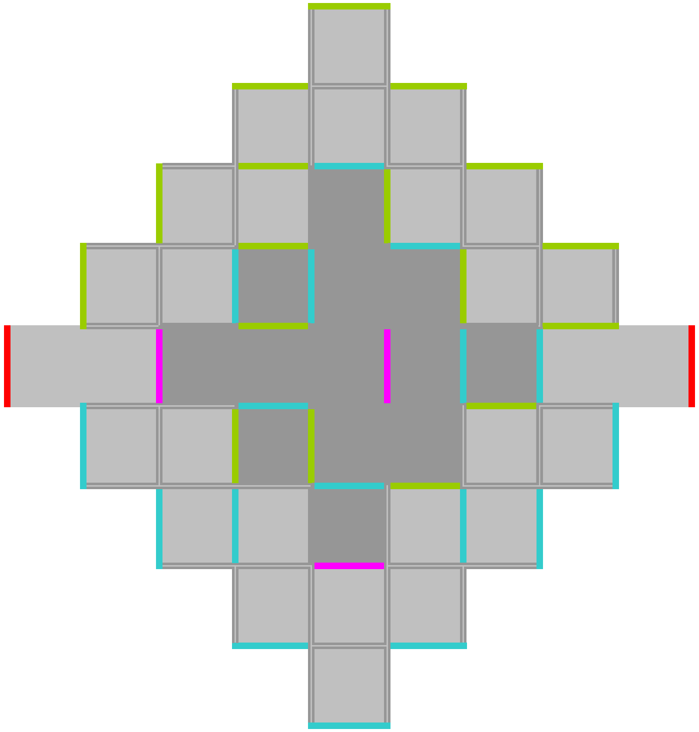


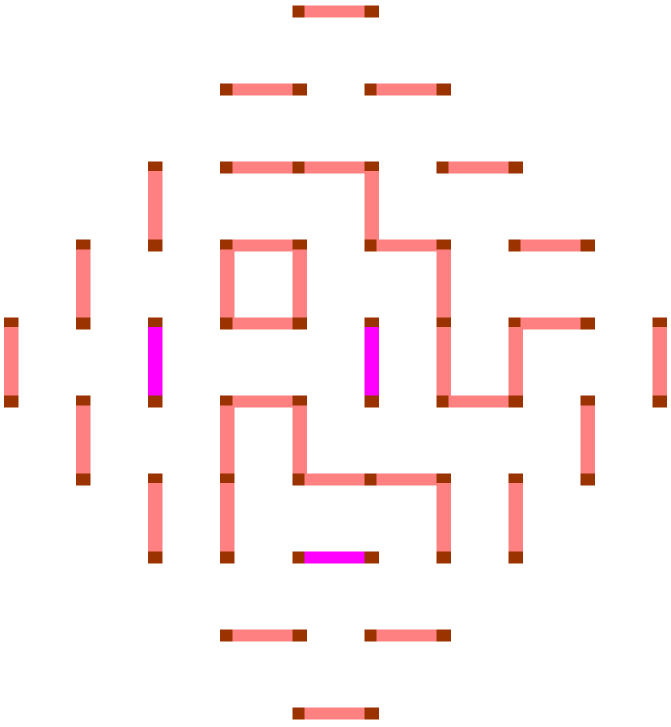

n=4
n=4

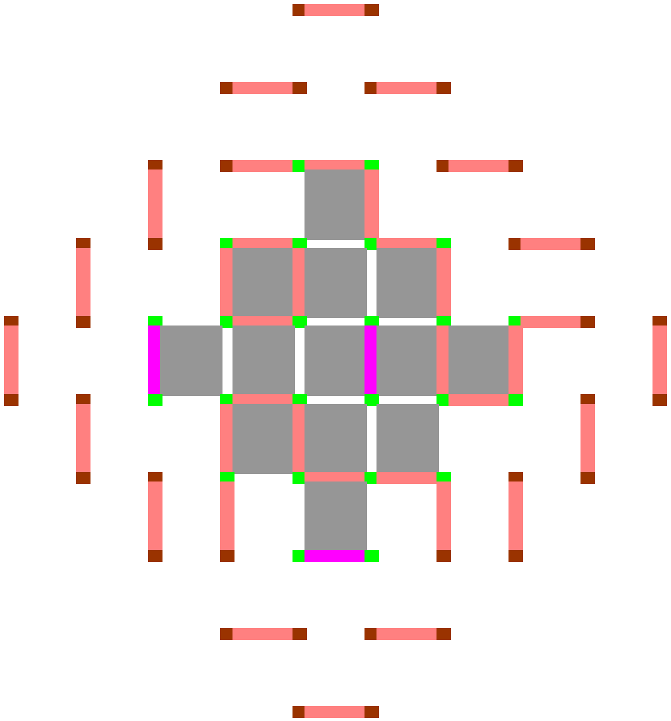


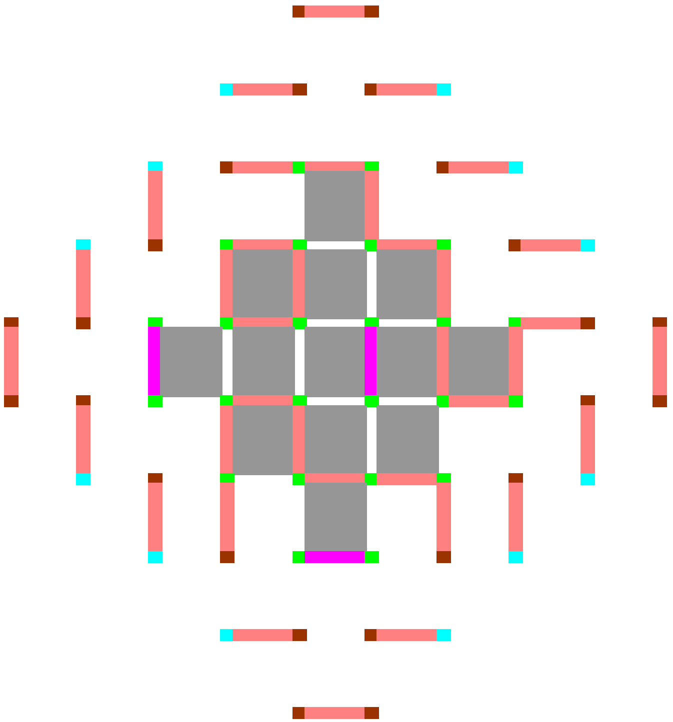


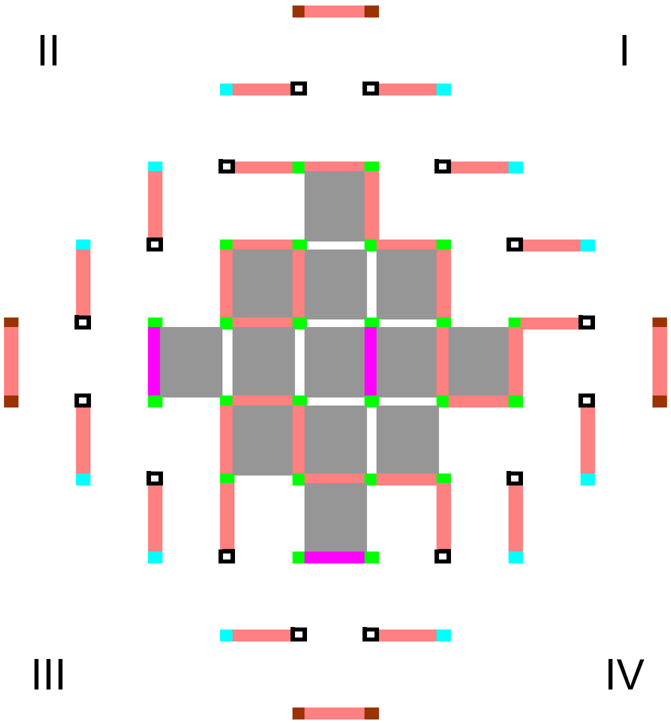


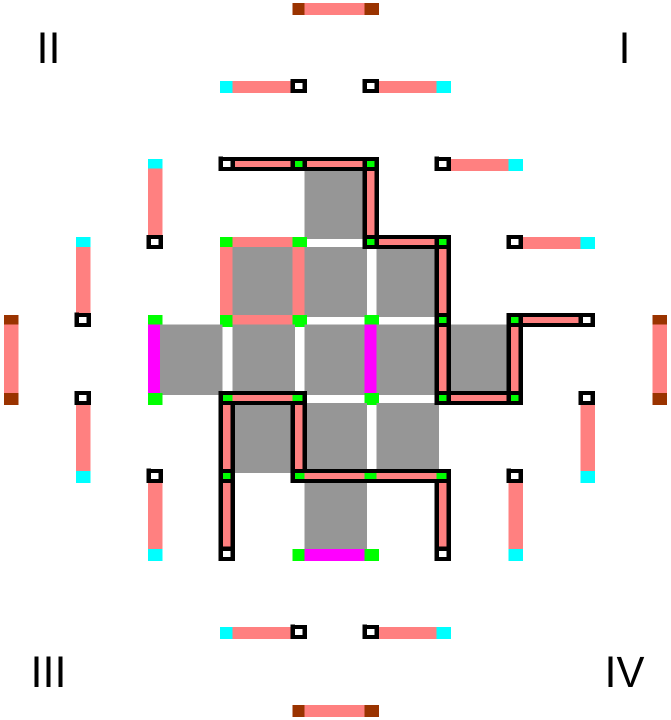


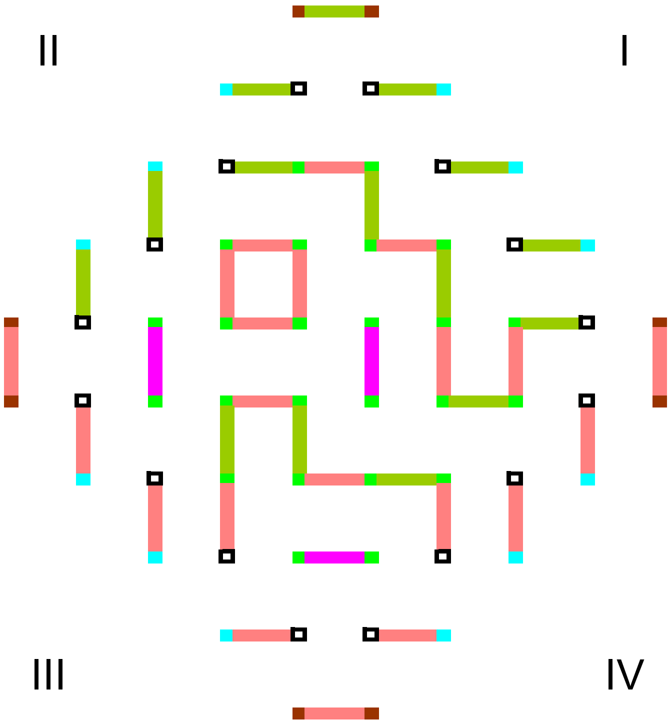


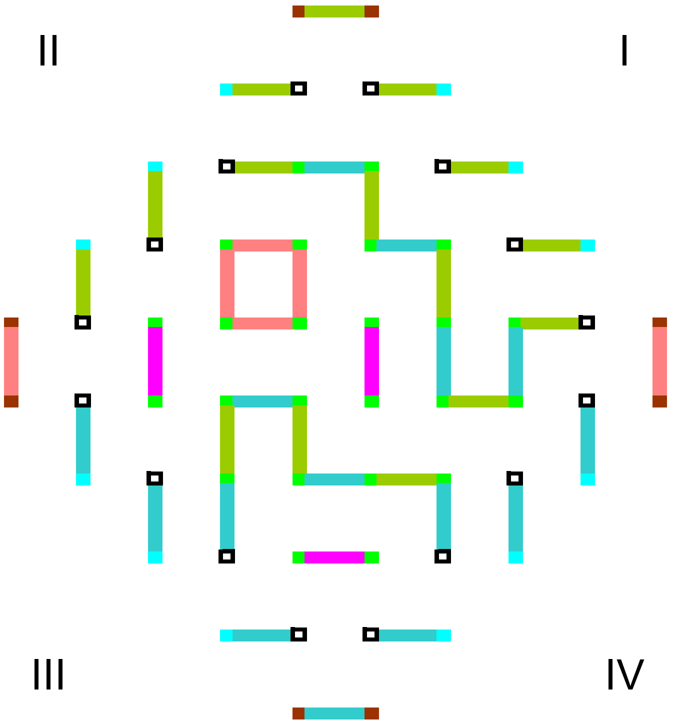


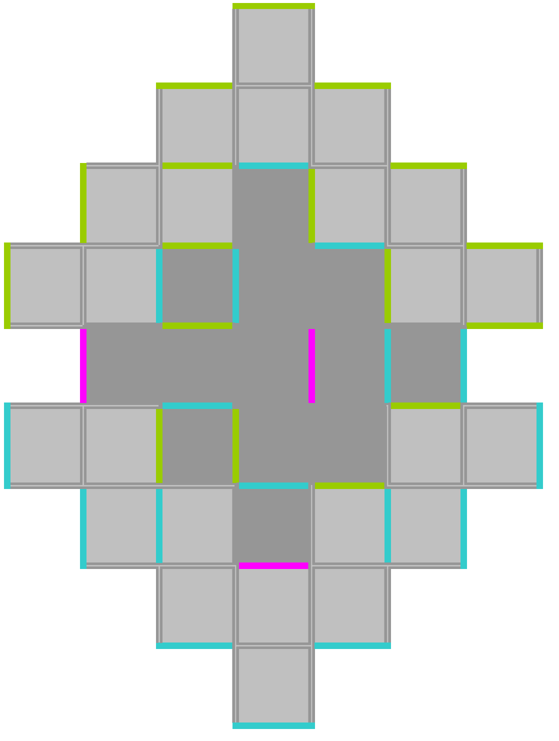


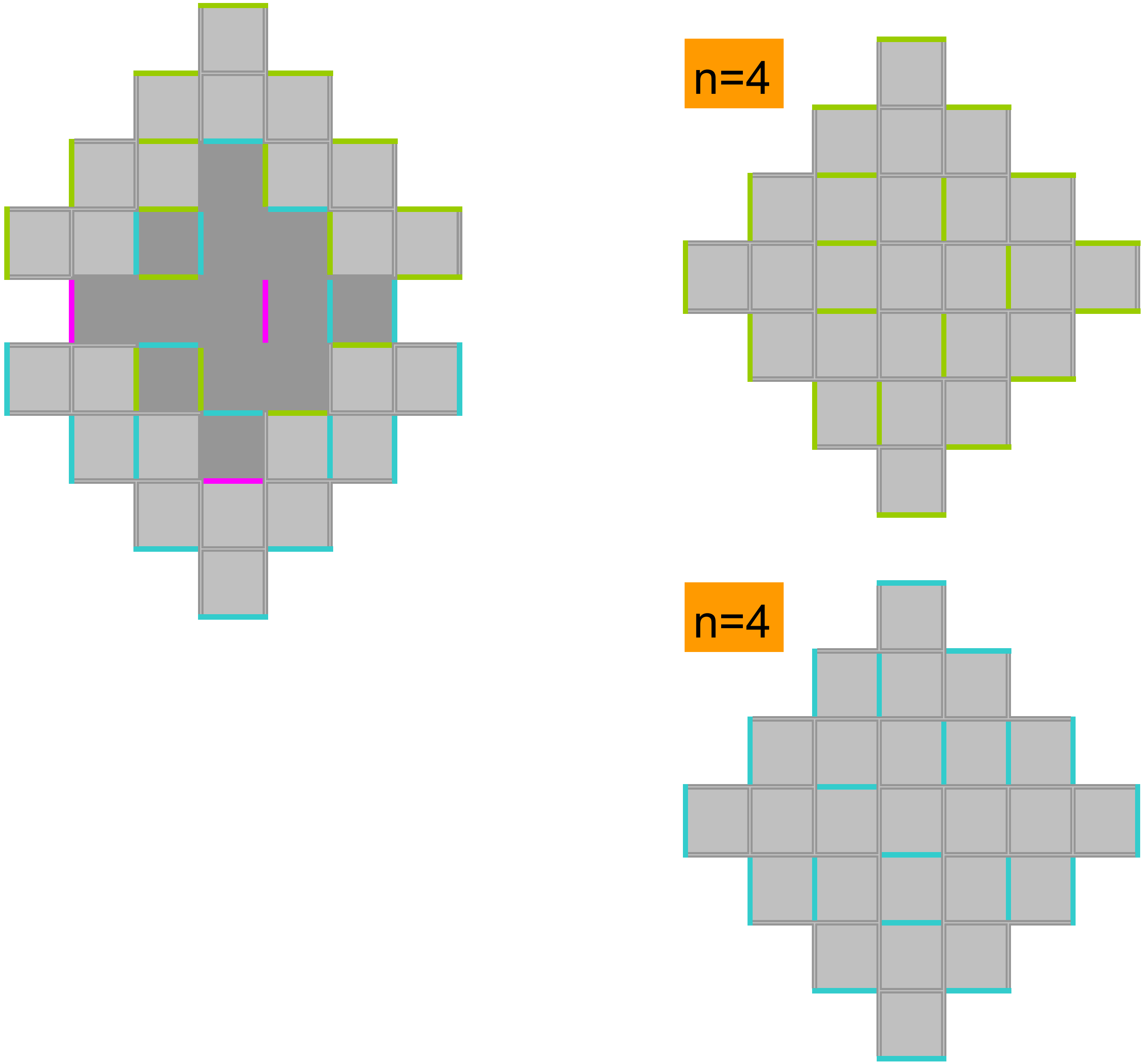

II
I
III
IV


II
I
III
IV

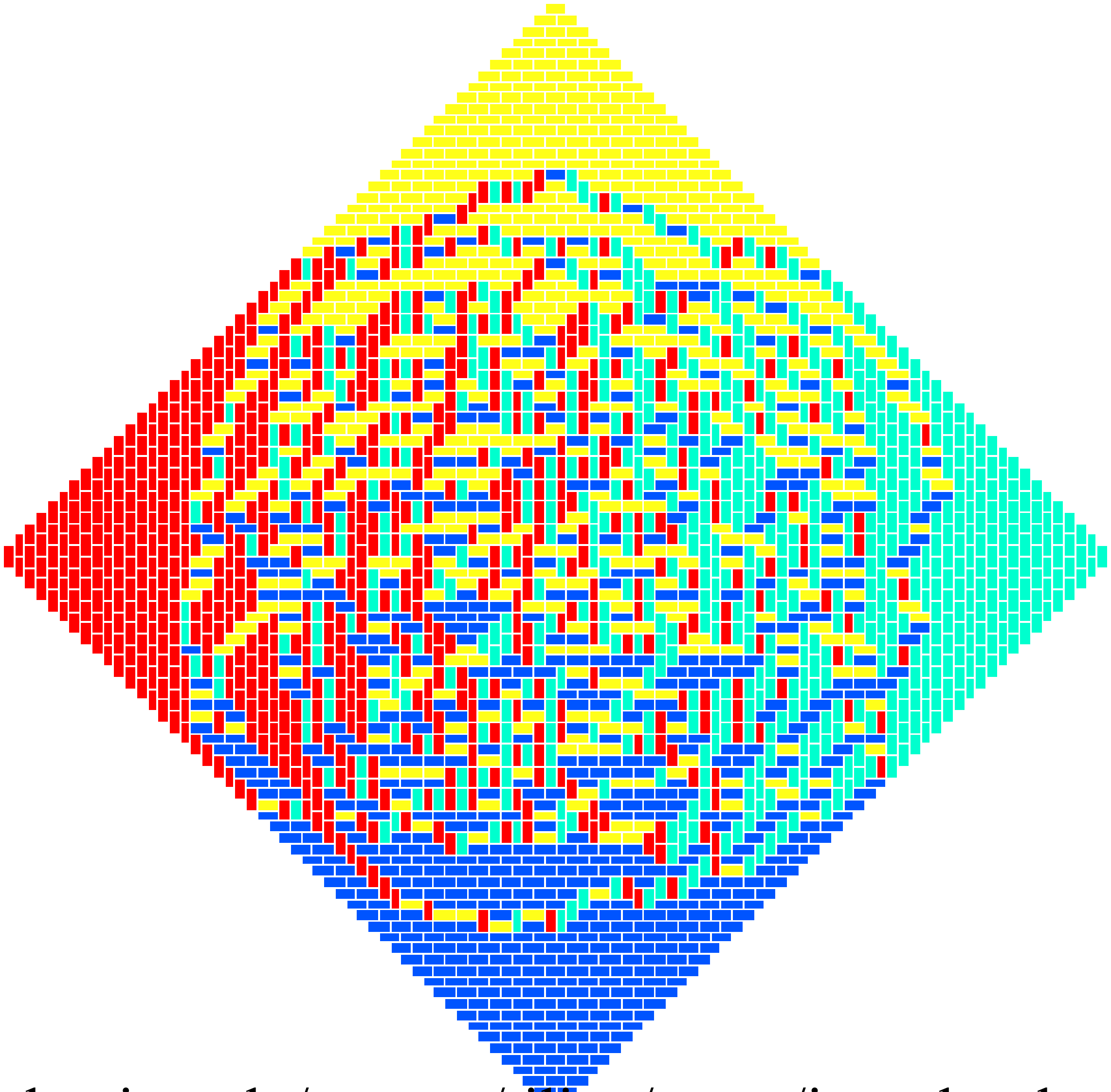

II
I
III
IV


II
I
III
IV




n=4
n=4

Click here for a more rigorous definition of the Aztec diamond. If we take a random tiling from the set of all tilings of the Aztec diamond, we get the following:



http://www.math.wisc.edu/~propp/tiling/www/intro.html http://mathworld.wolfram.com/SomosSequence.html

$$a_n = \frac{\sum_{j=1}^{\lfloor k/2 \rfloor} a_{n-j} a_{n-(k-j)}}{a_{n-k}},$$

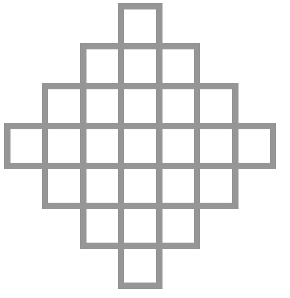
The Somos sequences are a set of related symmetrical recurrence relations which, surprisingly, always give integers. The Somos sequenc of order k is defined by

$$a_n = \frac{\sum_{j=1}^{\lfloor k/2 \rfloor} a_{n-j} a_{n-(k-j)}}{a_{n-k}},$$

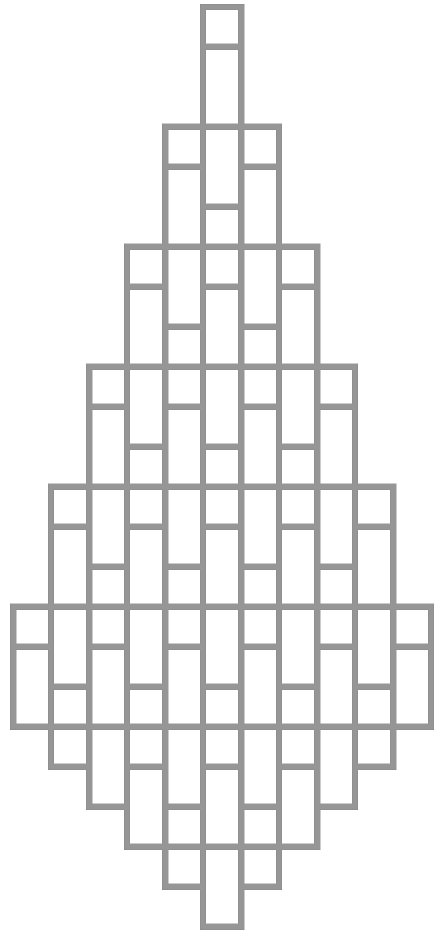
where $\lfloor x \rfloor$ is the floor function and $a_j = 1$ for $j = 0, \ldots, k-1$. Th[illegible] and 3-Somos sequences consist entirely of 1s. The k-Somos sequences for $k = 4, 5, 6$, and 7 are giving 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, ... (Sloane's A006720), 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, 1217, ... (Sloane's A006721), 1, 1, 1, 1, 1, 3, 5, 9, 23, 75, 421, 1103, ... (Sloane's A006722), 1, 1, 1, 1, 1, 1, 3, 5, 9, 17, 41, 137, 769, ... (Sloane's A006723). Gale (1991) gives simple proofs of the integer-only property of the 4-Somos and 5-Somos sequences. Hickerson proved 6-Somos generates only integers using computer algebra, and empirical evidence suggests 7-Somos is also integer-only.

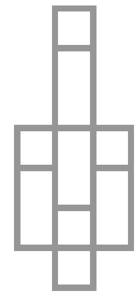
$$T(n)T(n-2)=2T(n-1)T(n-1)$$

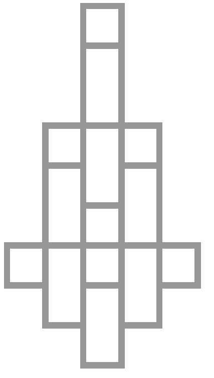
$$T(n)T(n-2)=T(n-1)T(n-1)+T(n-1)T(n-1)$$

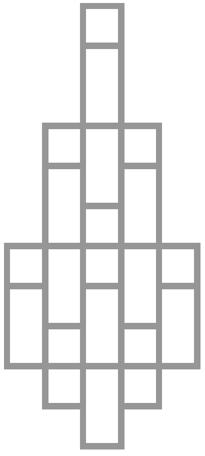


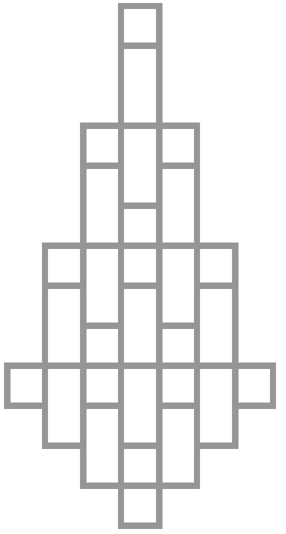
$$T(n)T(n-4)=T(n-1)T(n-3)+T(n-2)T(n-2)$$

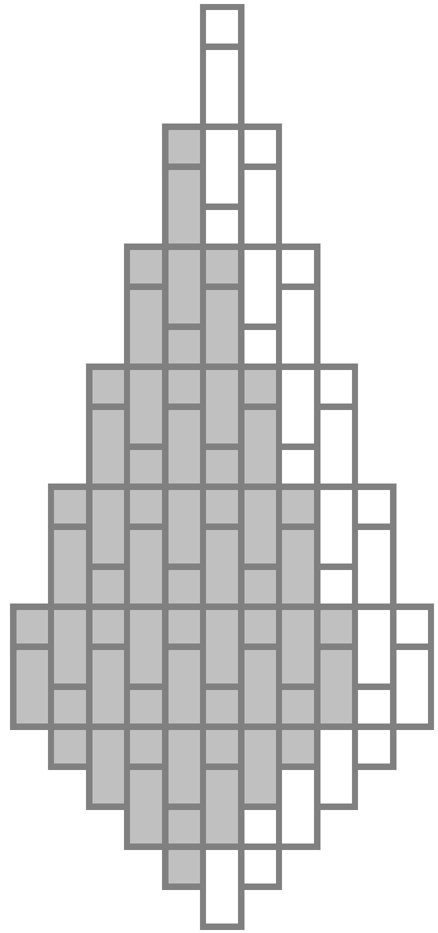


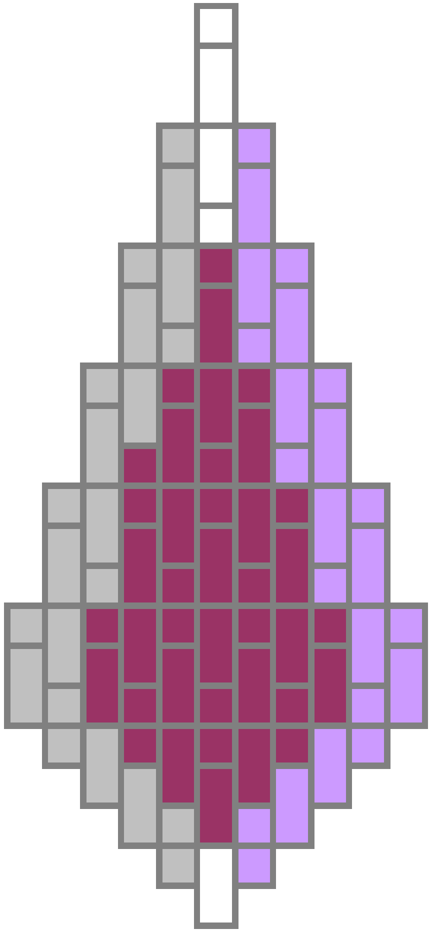


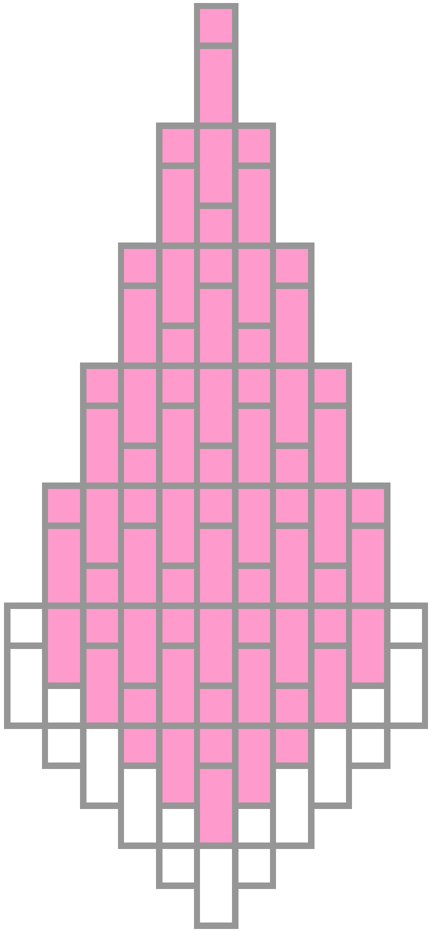


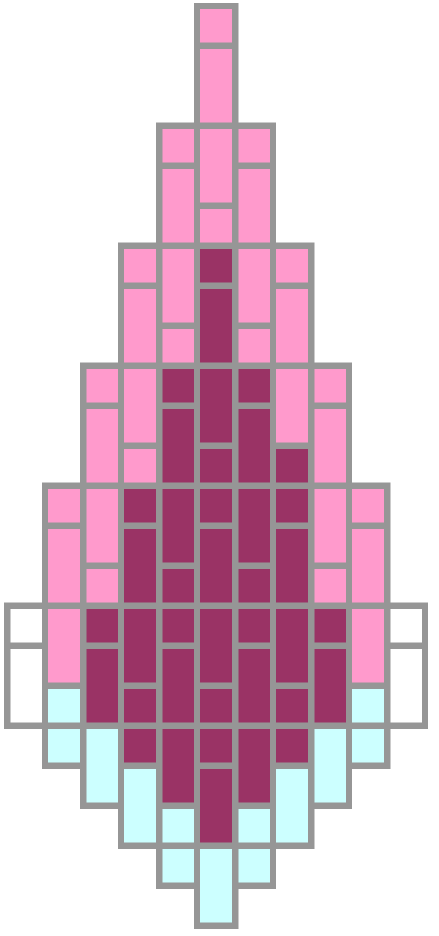


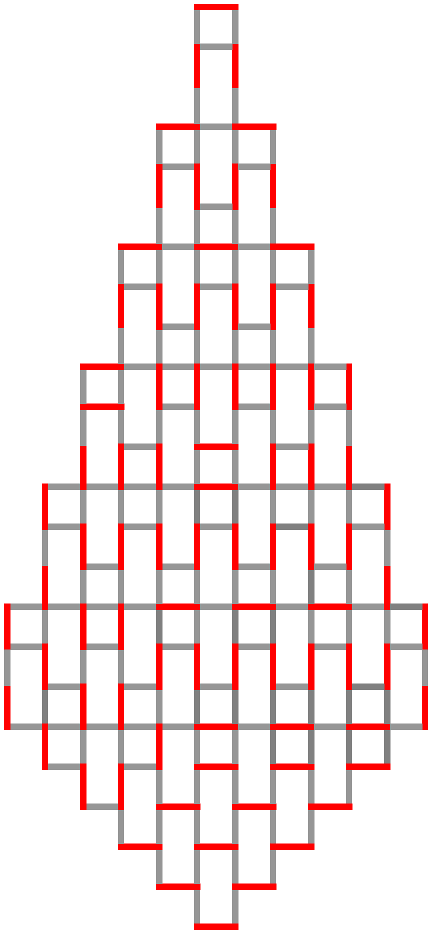


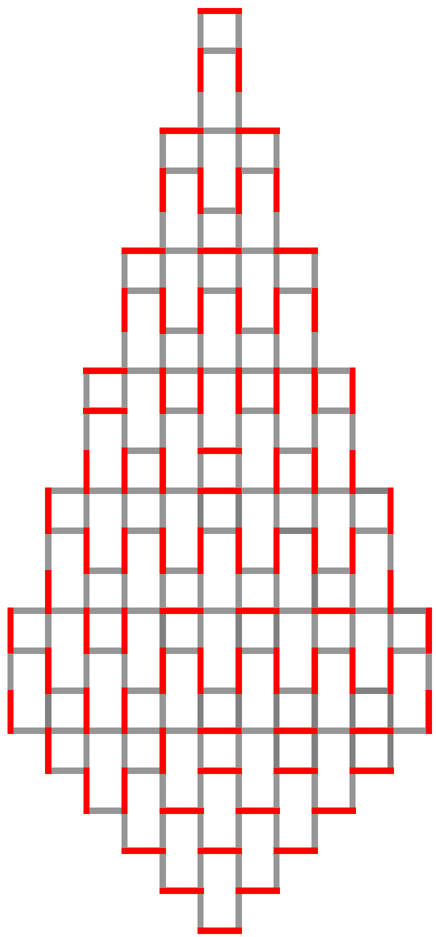


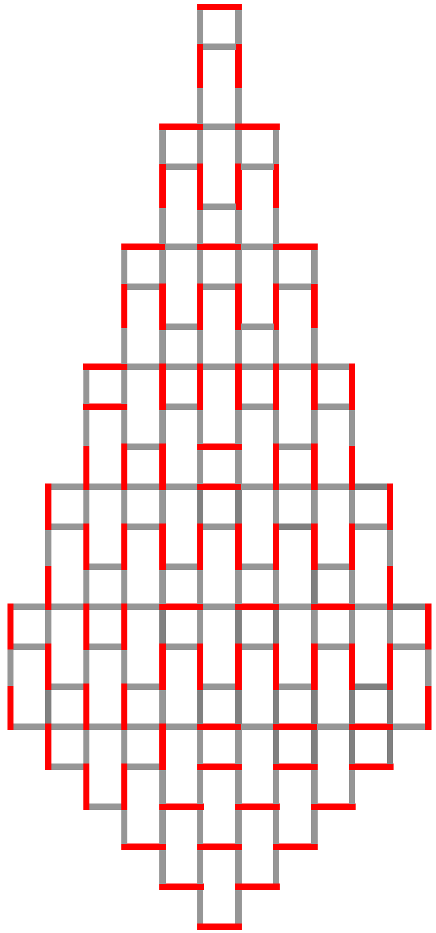
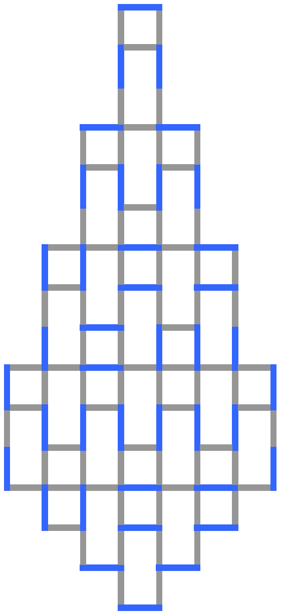


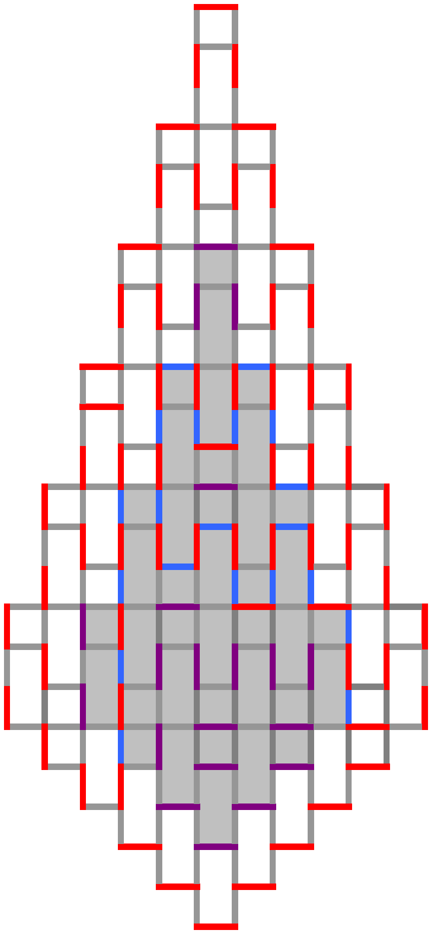


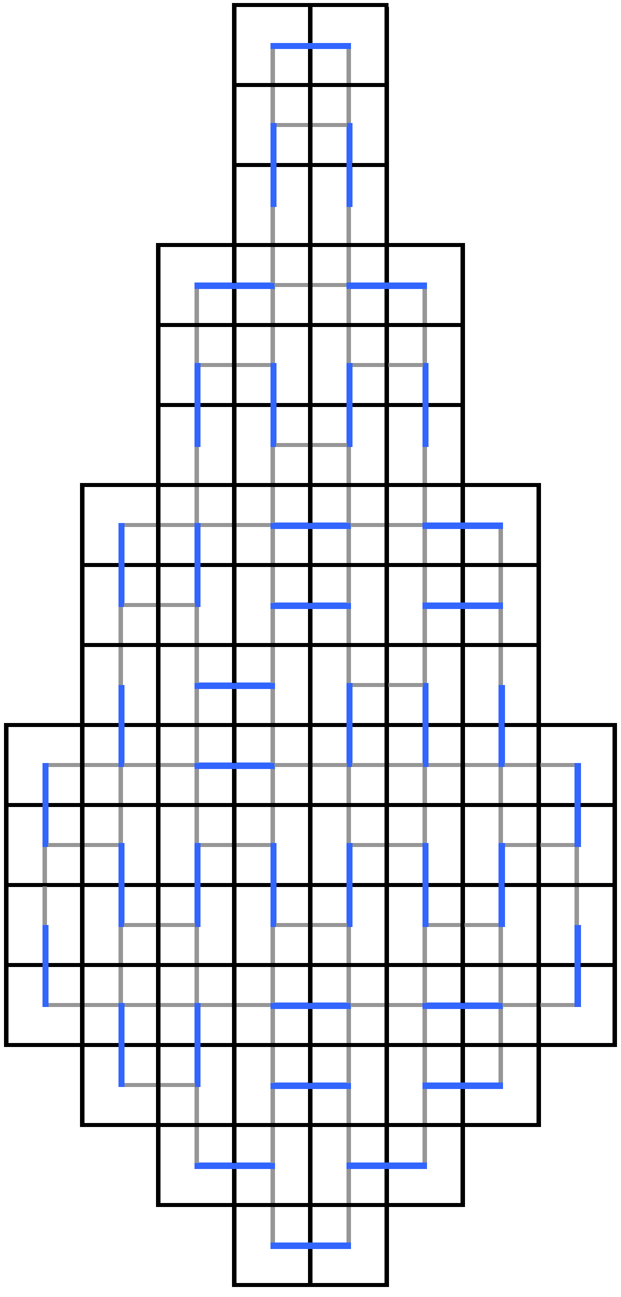


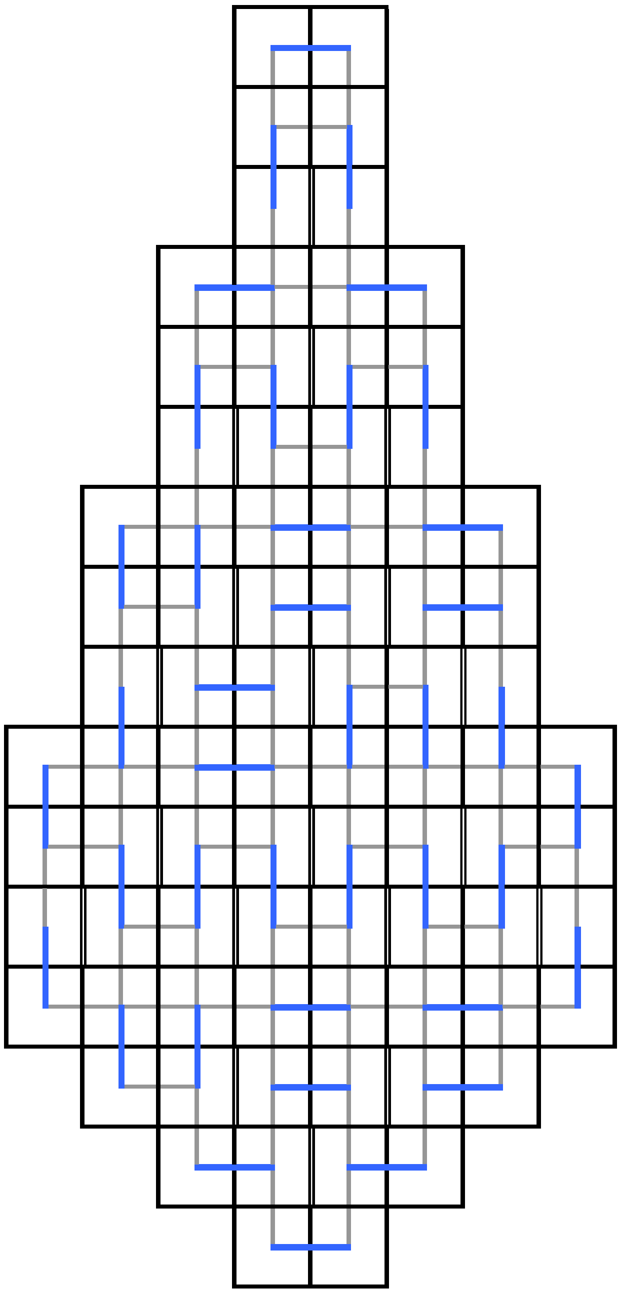


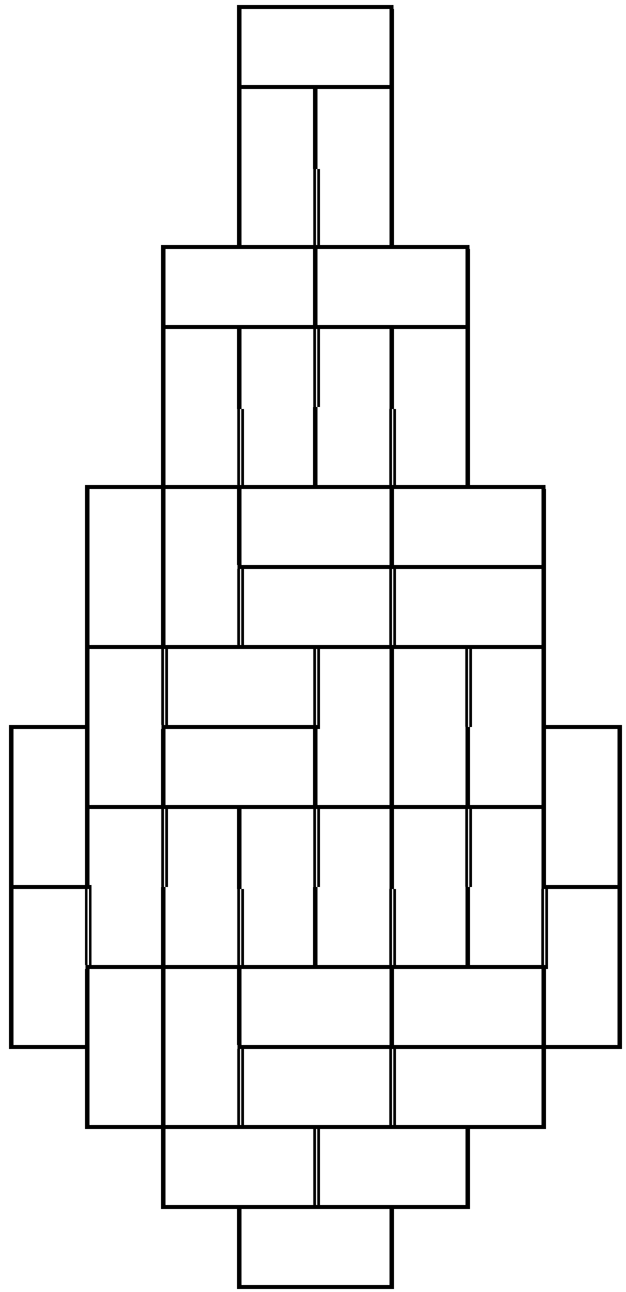


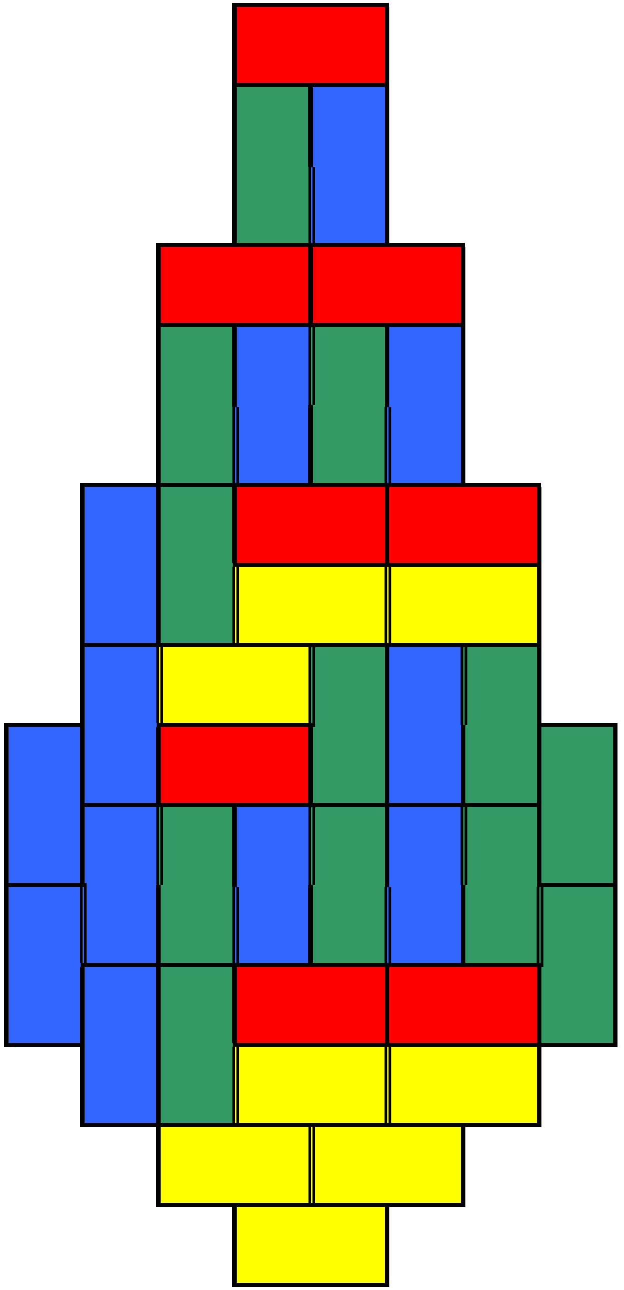



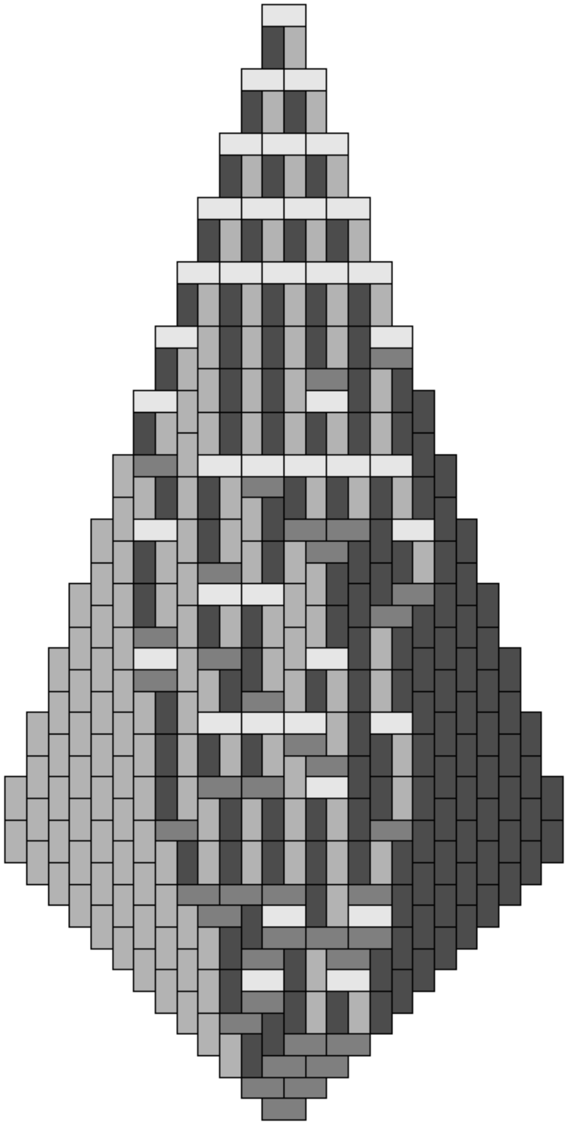


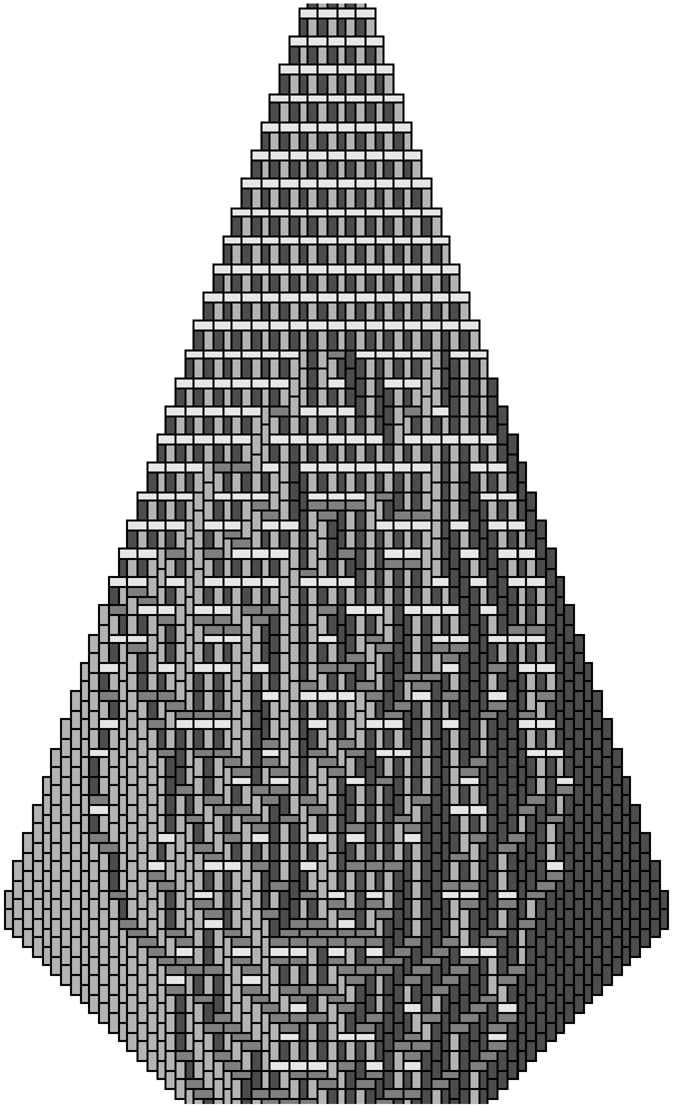


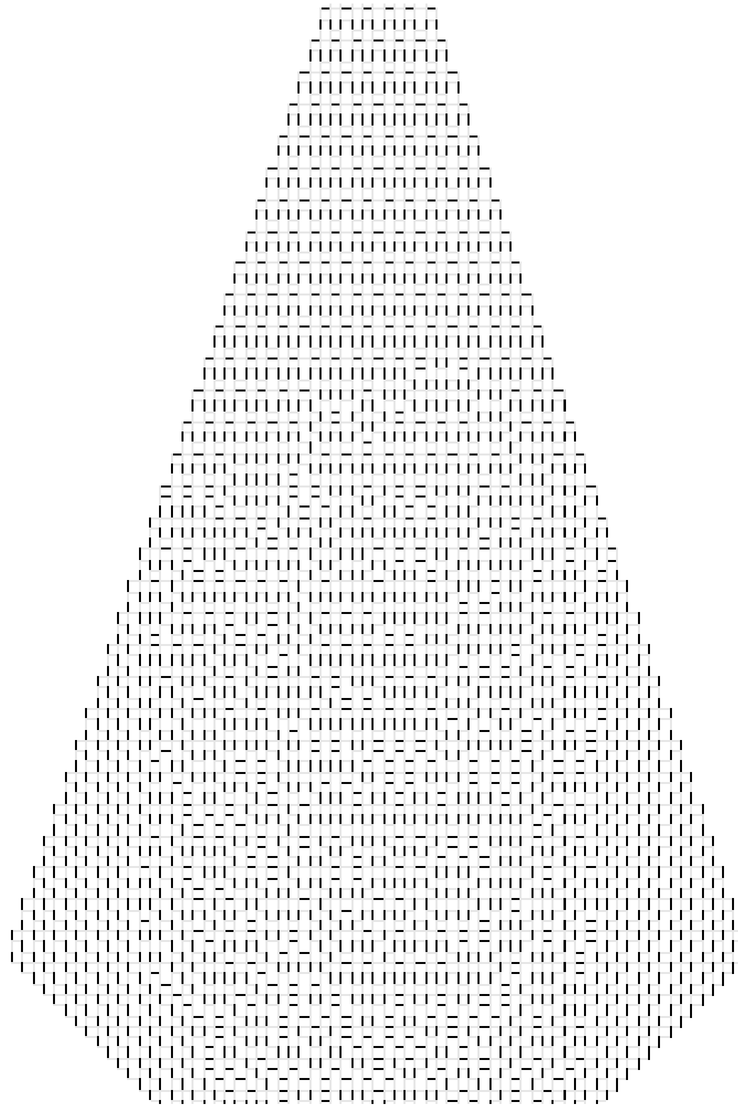












$$B_{ijkl}(n, a, b) =$$

$$a + b - 2\left\lfloor \frac{j-n-1+ax+by}{i} \right\rfloor - 4$$

$$B_{ijkl}(n, a, b) =$$

$$a + b - 2\left\lfloor \frac{j-n-1+ax+by}{i} \right\rfloor - 4$$

$n=30$

$i=2$

$j=5$

$i=2$

$j=5$

$x=3$

$y=4$

$x=3$

$y=4$

$$B_{ijkl}(n, a, b) =$$

$$a + b - 2\left\lfloor \frac{j-n-1+ax+by}{i} \right\rfloor - 4$$

n=30

i=2

j=5

x=3

y=4

1. Calculate values of B(n,a,b)

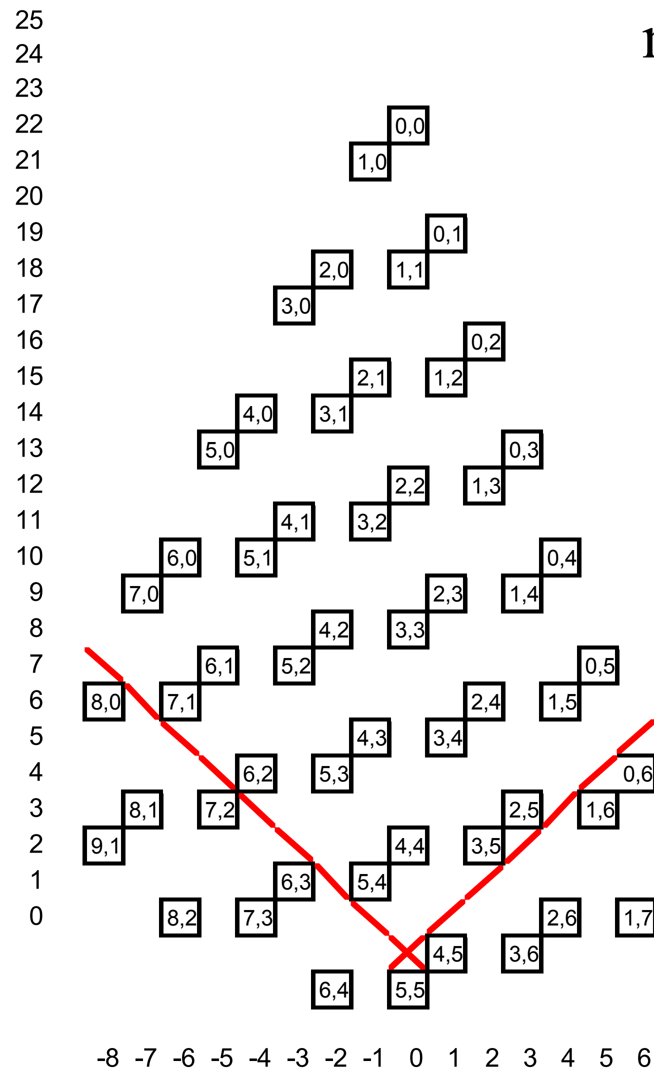
   for a,b >= 0

$$B_{ijkl}(n, a, b) = a + b - 2\left\lfloor \frac{j-n-1+ax+by}{i} \right\rfloor - 4$$

1. Calculate values of B(n,a,b)

   for a,b >= 0

2. Put a box in column (b-a) and row B(n,a,b)



n=30

i=2

j=5

x=3

y=4

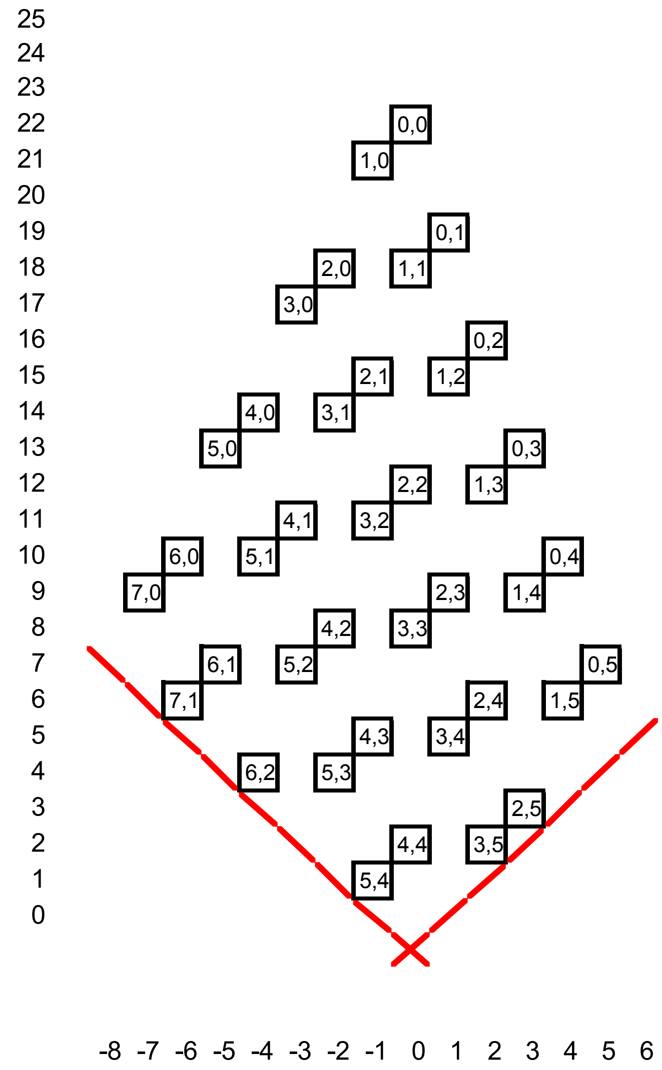
$$B_{ijkl}(n, a, b) = a + b - 2\left\lfloor \frac{j-n-1+ax+by}{i} \right\rfloor - 4$$

1. Calculate values of B(n,a,b)

   for a,b >= 0

2. Put a box in column (b-a) and row B(n,a,b)

3. Keep those boxes for which B(n,a,b)>=|b-a|



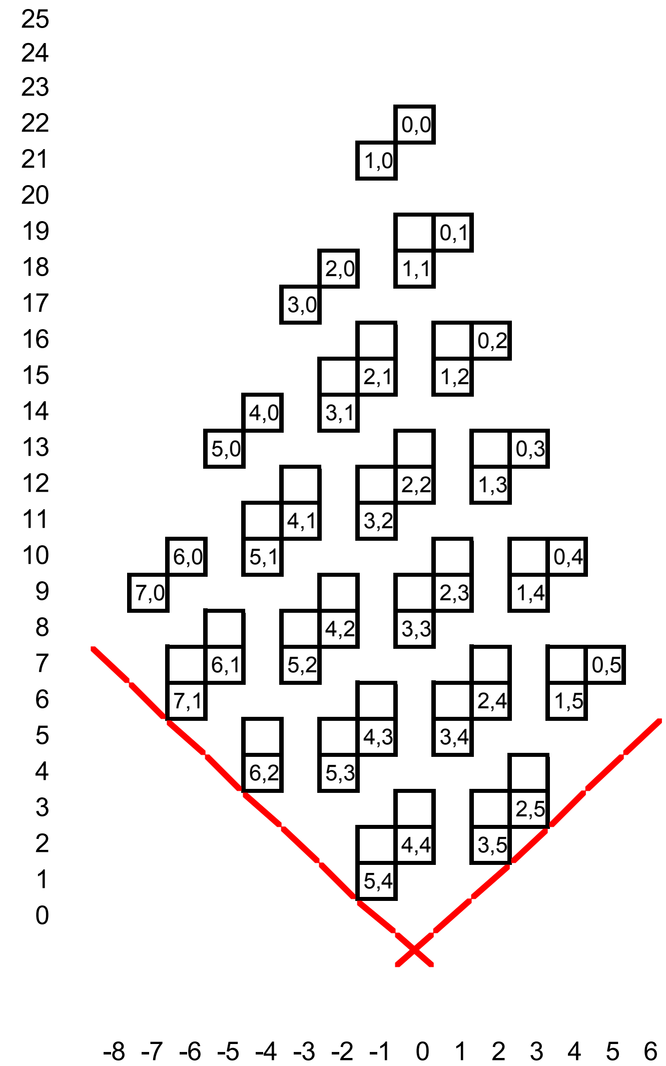
i=2

j=5

x=3

y=4

4. Except for the top box in each column, each box is the bottom half of a pair of squares.



i=2

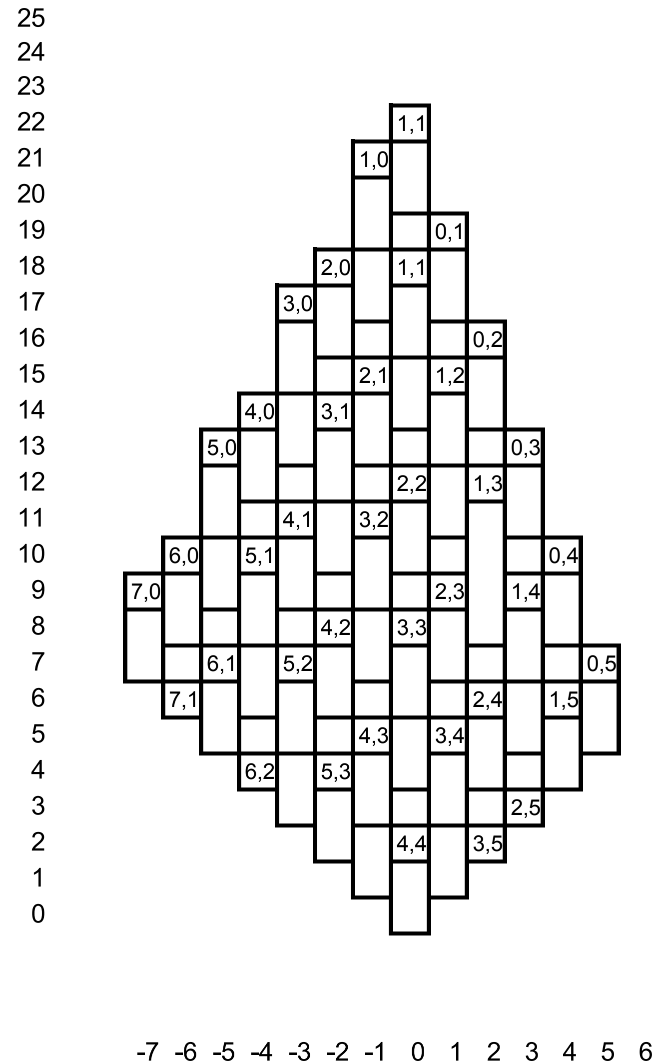
j=5

x=3

y=4

4. Except for the top box in each column, each box is the bottom half of a pair of squares

5. Fill up the remaining space in each column with ‘hexagons’



i=2

j=5

x=3

y=4

$$B_{ijkl}(n, a, b) = a + b - 2\left\lfloor \frac{j-n-1+ax+by}{i} \right\rfloor - 4$$

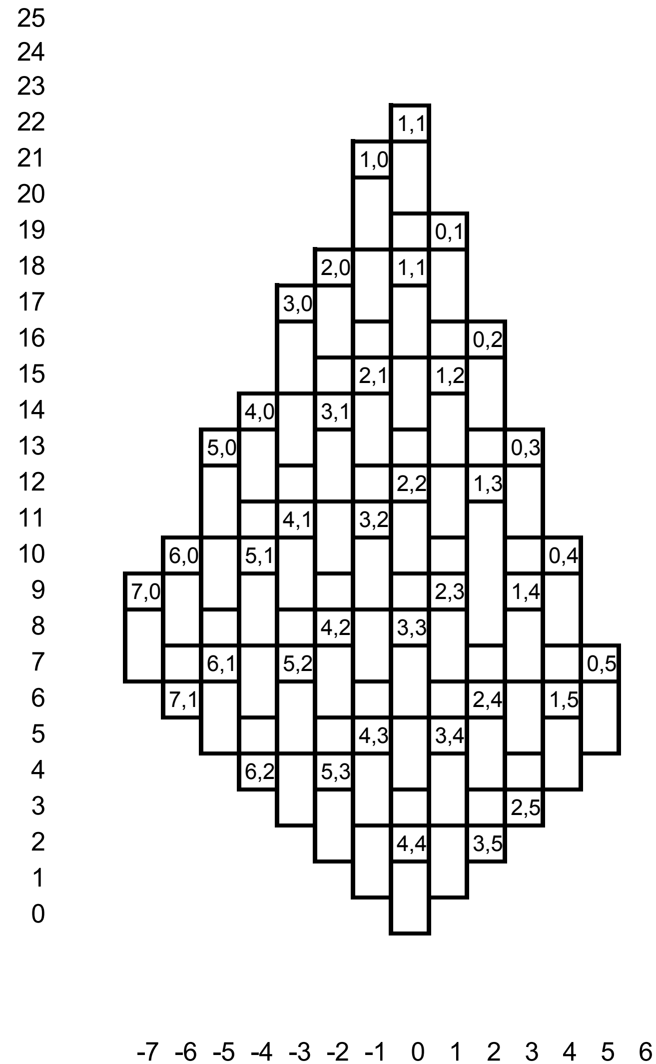
$B(n-x,a-1,b) = B(n,a,b)-1$

$B(n-y,a,b-1) = B(n,a,b)-1$

$B(n-i,a,b) = B(n,a,b)-2$

$B(n-j,a-1,b-1) = B(n,a,b)$

$B(n-k,a-1,b-1 = B(n,a,b)-2$



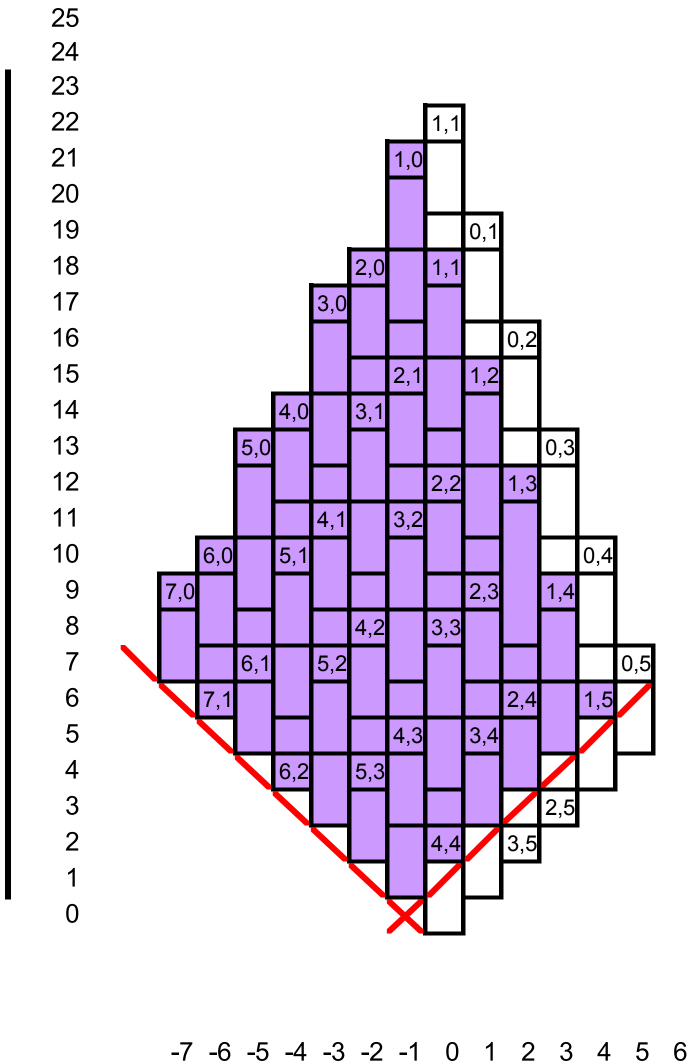
$B(n\text{-}x,a\text{-}1,b) = B(n,a,b)\text{-}1$

$B(n\text{-}y,a,b\text{-}1) = B(n,a,b)\text{-}1$

$B(n\text{-}i,a,b) = B(n,a,b)\text{-}2$

$B(n\text{-}j,a\text{-}1,b\text{-}1) = B(n,a,b)$

$B(n\text{-}k,a\text{-}1,b\text{-}1 = B(n,a,b)\text{-}2$



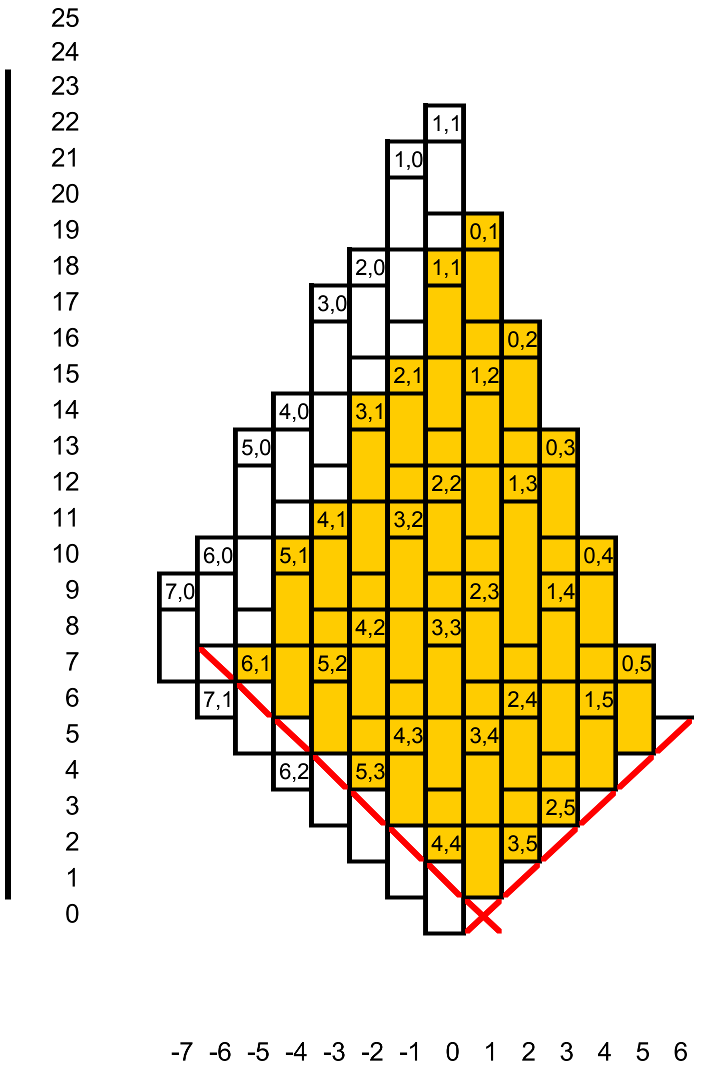
B(n-x,a-1,b) =B(n,a,b)-1

B(n-y,a,b-1) =B(n,a,b)-1

B(n-i,a,b) =B(n,a,b)-2

B(n-j,a-1,b-1)=B(n,a,b)

B(n-k,a-1,b-1 =B(n,a,b)-2



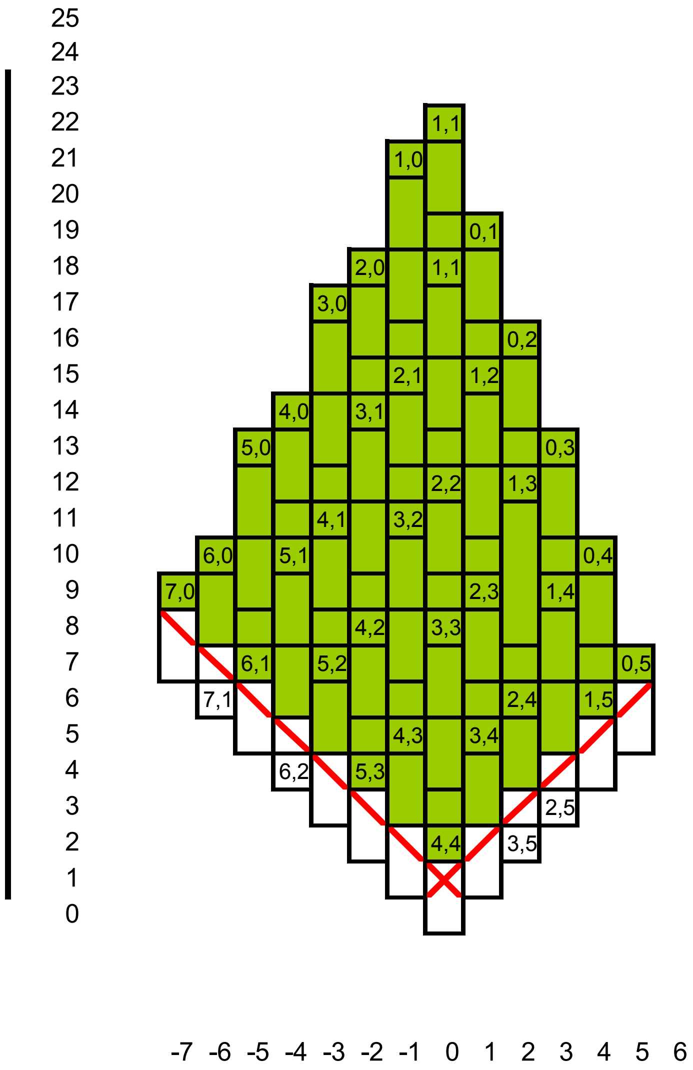
B(n-x,a-1,b) =B(n,a,b)-1

B(n-y,a,b-1) =B(n,a,b)-1

B(n-i,a,b) =B(n,a,b)-2

B(n-j,a-1,b-1)=B(n,a,b)

B(n-k,a-1,b-1 =B(n,a,b)-2



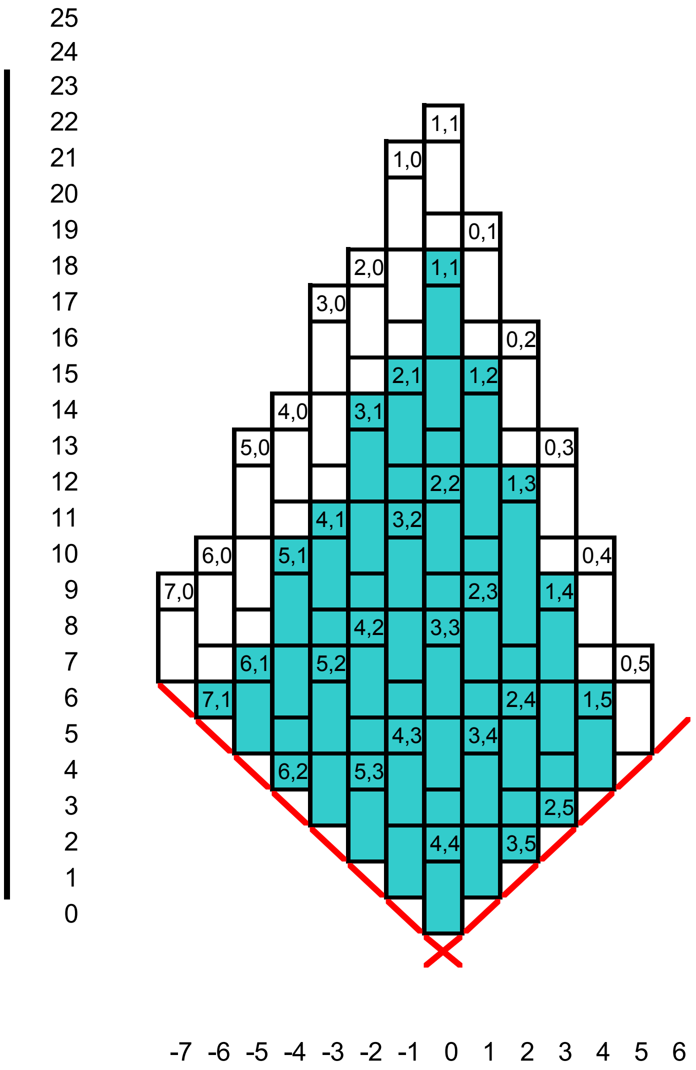
$$B(n-x,a-1,b) \;=B(n,a,b)-1$$

$$B(n-y,a,b-1) \;=B(n,a,b)-1$$

$$B(n-i,a,b) \qquad =B(n,a,b)-2$$

$$B(n-j,a-1,b-1)=B(n,a,b)$$

$$B(n-k,a-1,b-1 =B(n,a,b)-2$$



B(n-x,a-1,b) =B(n,a,b)-1

B(n-y,a,b-1) =B(n,a,b)-1

B(n-i,a,b) =B(n,a,b)-2

B(n-j,a-1,b-1)=B(n,a,b)

B(n-k,a-1,b-1 =B(n,a,b)-2