

**EXPLICIT M/G/1 WAITING-TIME DISTRIBUTIONS FOR A CLASS  
OF LONG-TAIL SERVICE-TIME DISTRIBUTIONS**

by

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### *Abstract*

O. J. Boxma and J. W. Cohen recently obtained an explicit expression for the M/G/1 steady-state waiting-time distribution for a class of service-time distributions with power tails. We extend their explicit representation from a one-parameter family of service-time distributions to a two-parameter family. The complementary cumulative distribution function (ccdf's) of the service times all have the asymptotic form  $F^c(t) \sim \alpha t^{-3/2}$  as  $t \rightarrow \infty$ , so that the associated waiting-time ccdf's have asymptotic form  $W^c(t) \sim \beta t^{-1/2}$  as  $t \rightarrow \infty$ . Thus the second moment of the service time and the mean of the waiting time are infinite. Our result here also extends our own earlier explicit expression for the M/G/1 steady-state waiting-time distribution when the service-time distribution is an exponential mixture of inverse Gaussian distributions (EMIG). The EMIG distributions form a two-parameter family with ccdf having the asymptotic form  $F^c(t) \sim \alpha t^{-3/2} e^{-\eta t}$  as  $t \rightarrow \infty$ . We now show that a variant of our previous argument applies when the service-time ccdf is an undamped EMIG, i.e., with ccdf  $G^c(t) = e^{\eta t} F^c(t)$  for  $F^c(t)$  above, which has the power tail  $G^c(t) \sim \alpha t^{-3/2}$  as  $t \rightarrow \infty$ . The Boxma-Cohen long-tail service-time distribution is a special case of an undamped EMIG.

*Keywords:* M/G/1 queue, waiting-time distribution, Pollaczek-Khintchine formula, long-tail distributions, power-tail distributions, exponential mixture of inverse Gaussian distributions.

## 1. Introduction

The steady-state waiting-time distribution in the M/G/1 queue is available via the classical Pollaczek-Khintchine transform. It can be readily computed by numerical transform inversion, when the service-time Laplace transform is available, e.g., as shown in Abate and Whitt [1]. Nevertheless it is interesting to have explicit formulas. When the service-time distribution has a rational transform, so does the waiting-time distribution, and the transform can be inverted analytically. More generally, the transform can be inverted analytically, yielding the Beneš formula, which is an infinite series containing  $n$ -fold convolutions of the service-time stationary-excess distribution for all  $n$ ; e.g., see 4.82 on p. 255 of Cohen [8]. Because of the complexity of the Beneš formula, however, it is natural to look for more explicit formulas.

A more explicit formula for a non-rational service-time distribution was evidently first obtained for the gamma service-time distribution with shape parameter  $1/2$  in (9.21) of Abate and Whitt [1]. This result was extended in Proposition 8.2 of Abate and Whitt [3] to all exponential mixtures of inverse Gaussian (EMIG) service-time distributions. These service-time distributions have probability densities with asymptotics of the form  $f(t) \sim \alpha t^{-3/2} e^{-\eta t}$  as  $t \rightarrow \infty$ , where  $f(t) \sim g(t)$  as  $t \rightarrow \infty$  means that  $f(t)/g(t) \rightarrow 1$ . Because of the  $e^{-\eta t}$  term, these EMIG distributions do not have a long (a heavy) tail. However, recently, Boxma and Cohen [7] obtained an explicit expression for the M/G/1 waiting-time distribution for a class of long-tail service-time distributions. In this paper, we extend Boxma and Cohen's result to a larger class of long-tail service-time distributions. In particular, we extend our result in [3] to undamped EMIGs, i.e., to distributions with complementary cumulative distribution functions (ccdf's)  $G^c(t) \equiv 1 - G(t) = e^{\eta t} F^c(t)$ , where  $F^c(t)$  is an EMIG cdf. The Boxma-Cohen service-time distributions are a subclass.

Here is how the rest of this paper is organized. In Section 2 we give the explicit solution for the steady-state waiting-time distribution. In Section 3 we show that the service-time distributions used in Section 2 can be represented as undamped EMIGs. In Section 4 we show that both EMIGs and undamped EMIGs are completely monotone (mixtures of exponentials) and give their mixing densities. In Section 5 we give the asymptotic behavior of undamped EMIGs as  $t \rightarrow 0$  and as  $t \rightarrow \infty$ . We apply that result to give the first two terms of the asymptotic expansion for the waiting-time ccdf in Section 2, which agrees with Boxma and Cohen [7]. In Section 6 we discuss the heavy-traffic approximation due to Boxma and Cohen [7]. For the service-time distributions considered here, we derive their limit from the explicit

waiting-time cdf. We conclude in Section 7 by discussing other service-time distributions for which explicit representations of the waiting-time distribution are possible, but the greater complexity make them of dubious value.

## 2. The Explicit Solution

Consider a service-time probability density function (pdf)  $g(t)$  with Laplace transform

$$\hat{g}(s) \equiv \int_0^\infty e^{-st} g(t) dt = 1 - \frac{s}{(\mu + \sqrt{s})(1 + \sqrt{s})}, \quad (2.1)$$

which has mean  $m_1(g) = \mu^{-1}$  and all higher moments infinite. The pdf  $g$  has two-parameters, the displayed  $\mu$  and the scale, which has been omitted. Both can range over the positive reals.

The Pollaczek-Khintchine formula involves the associated stationary-excess pdf  $g_e(t) \equiv \mu G(t)$ ,  $t \geq 0$ . Its Laplace transform has the nice form

$$\hat{g}_e(s) \equiv \frac{1 - g(s)}{sm_1(g)} = \frac{\mu}{(\mu + \sqrt{s})(1 + \sqrt{s})}. \quad (2.2)$$

For  $\mu \neq 1$ ,

$$\hat{g}_e(s) = \left( \frac{\mu}{1 - \mu} \right) \left( \frac{1}{\mu + \sqrt{s}} - \frac{1}{1 + \sqrt{s}} \right), \quad (2.3)$$

so that, by 29.3.37 of Abramowitz and Stegun [6],

$$g_e(t) = \mu G^c(t) = \left( \frac{\mu}{1 - \mu} \right) (\psi(t) - \mu \psi(\mu^2 t)), \quad t \geq 0 \quad (2.4)$$

where

$$\psi(t) \equiv e^t \operatorname{erfc}(\sqrt{t}) \sim \frac{1}{\sqrt{\pi t}} \quad \text{as } t \rightarrow \infty, \quad (2.5)$$

with  $\operatorname{erfc}$  being the complementary error function, i.e.,

$$\operatorname{erfc}(t) \equiv \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2} du \equiv 2\Phi^c(\sqrt{2}t), \quad (2.6)$$

where  $\Phi^c(t) \equiv 1 - \Phi(t)$  is the standard (mean 0, variance 1) normal complementary cumulative distribution function (ccdf); see 7.1.1 and 26.2.29 of Abramowitz and Stegun [6]. We will establish further properties of  $G$  and  $G_e$  in the next section.

The case  $\mu = 1$  was considered by Boxma and Cohen [7]. The case  $\mu = 1$  also corresponds to a subclass of beta mixtures of exponential (BME) pdf's considered by Abate and Whitt [4]; we will discuss this connection further in the next section. Boxma and Cohen show that the service-time cdf when  $\mu = 1$  is

$$G^c(t) = (2t + 1)\psi(t) - 2\sqrt{t/\pi}, \quad t \geq 0, \quad (2.7)$$

for  $\psi$  in (2.5). In the next section we will show that the associated stationary-excess cdf is

$$G_e^c(t) = 2\sqrt{t/\pi} - (2t - 1)\psi(t), \quad t \geq 0. \quad (2.8)$$

We now consider the steady-state waiting-time distribution in the M/G/1 queue with arrival rate  $\lambda$ . It has an atom of  $1 - \rho$  at the origin, assuming that  $\rho \equiv \lambda/\mu < 1$ , but otherwise a pdf. The Laplace transform of the cdf is

$$\hat{W}^c(s) = \frac{\rho}{s}(1 - \hat{w}_\rho(s)), \quad (2.9)$$

where  $\hat{w}_\rho(s)$  is the Laplace transform of the conditional waiting time pdf, given that there is a positive wait, i.e.,

$$\hat{w}_\rho(s) = \frac{(1 - \rho)\hat{g}_e(s)}{1 - \rho\hat{g}_e(s)}. \quad (2.10)$$

Paralleling Proposition 8.2 of Abate and Whitt [3], we can find an explicit expression for  $\hat{W}^c(s)$  and analytically invert it. From (2.2)–(2.10), we deduce the following.

**Theorem 2.1.** *For the service-time pdf  $g(t)$  with Laplace transform  $\hat{g}(s)$  in (2.1),*

$$\hat{w}_\rho(s) = \frac{(1 - \rho)\mu}{\nu_1 - \nu_2} \left( \frac{1}{\nu_2 + \sqrt{s}} - \frac{1}{\nu_1 + \sqrt{s}} \right) \quad (2.11)$$

and

$$\hat{W}^c(s) = \frac{\rho}{\nu_1 - \nu_2} \left( \frac{\nu_1}{\sqrt{s}(\nu_2 + \sqrt{s})} - \frac{\nu_2}{\sqrt{s}(\nu_1 + \sqrt{s})} \right), \quad (2.12)$$

so that

$$W^c(t) = \frac{\rho}{\nu_1 - \nu_2} (\nu_1\psi(\nu_2^2t) - \nu_2\psi(\nu_1^2t)), \quad (2.13)$$

where  $\psi$  is given in (2.5) and

$$\nu_{1,2} = \frac{1 + \mu}{2} \pm \sqrt{\left(\frac{1 + \mu}{2}\right)^2 - (1 - \rho)\mu}. \quad (2.14)$$

**Proof.** Algebra yields (2.11) and (2.12). The Laplace transform (2.12) is easy to invert using 29.3.43 of Abramowitz and Stegun [6]. ■

The case  $\mu = 1$  (with  $\nu_1 = 1 + \sqrt{\rho}$  and  $\nu_2 = 1 - \sqrt{\rho}$ ) was obtained by Boxma and Cohen [7]. They included an atom at the origin in the service-time distribution, which we could do as well. The atom at the origin simply gets absorbed in  $\rho$ , i.e., corresponds to changing the arrival rate  $\lambda$ . This property is most easily seen from the virtual waiting time, which has the same distribution as the actual waiting time in M/G/1. A customer with 0 service time causes no change in the virtual waiting-time process upon its arrival. By the Poisson thinning property,

the arrival process of customers with positive service times is also a Poisson process but with reduced arrival rate  $\lambda(1 - \eta)$ , where  $\eta$  is the atom at 0 in the service-time distribution. Hence, having an atom of mass  $\eta$  at 0 in the service-time distribution is equivalent to changing the arrival rate to  $\lambda(1 - \eta)$  and considering the service-time distribution without the atom, i.e., the conditional service-time distribution given that it is positive.

### 3. Undamped EMIGs

We obtain the service-time transform  $\hat{g}(s)$  in (2.1) by undamping an *exponential mixture of inverse Gaussian* (EMIG) ccdf's. The EMIGs were discussed in Section 8 of [3].

Introducing a slight change of notation, we start with the Laplace transform of an EMIG pdf

$$\hat{f}(s) = \frac{\mu + 1}{\mu + \sqrt{1 + s}}. \quad (3.1)$$

Formula (3.1) is obtained from (8.9) of [3] by first replacing  $\mu$  by  $\mu + 1$  and then introducing the scale parameter  $\omega \equiv 1/2(\mu + 1)$ ; i.e.,  $\hat{f}(s) = \hat{\rho}(s; \omega, \mu + 1) \equiv \hat{\rho}(\omega s, 1, \mu + 1)$  for that  $\omega$ . Paralleling  $\hat{g}(s)$  in (2.1), an extra scale parameter can be added to  $\hat{f}(s)$  in (3.1).

The moments of the pdf with transform in (3.1) can be derived from the inverse Gaussian moments by using (8.3) and (8.10) of [3] ( $r$  should be  $n$  in (8.3)). They are

$$m_1(F) = \frac{1}{2(\mu + 1)}, \quad m_{n+1}(F) = \frac{1}{(2 + 2\mu)^{n+1}} \sum_{k=0}^n \frac{(n + 1 - k)(n + k)!}{k!} \left(\frac{\mu + 1}{2}\right)^k \quad (3.2)$$

and squared coefficient of variation (variance divided by the mean)  $c^2 = \mu + 2$ . For the case  $\mu = 1$ , (3.1) is the BME transform  $\hat{v}(1/2, 3/2; s)$  studied in [4] and the moments in this case are  $m_n = n!\beta_n/(n + 1)$  where  $\beta_n = \binom{2n}{n}4^{-n}$ .

Paralleling (8.13) and (8.14) of [3], the ccdf has the Laplace transform

$$\hat{F}^c(s) = \frac{1 - \hat{f}(s)}{s} = \frac{1}{(\mu + \sqrt{1 + s})(1 + \sqrt{1 + s})} \quad (3.3)$$

$$= \frac{1}{\mu - 1} \left( \frac{1}{1 + \sqrt{1 + s}} - \frac{1}{\mu + \sqrt{1 + s}} \right), \quad \mu \neq 1. \quad (3.4)$$

From (3.4) we see that EMIG stationary-excess pdf is

$$f_e(t) = \frac{\mu + 1}{\mu - 1} v(1/2, 3/2; t) - \frac{2}{\mu - 1} f(t), \quad (3.5)$$

from which we obtain the simple moment recurrence for  $\mu \neq 1$

$$m_{n+1}(F) = \frac{n!\beta_n}{2(\mu - 1)} - \frac{n + 1}{\mu^2 - 1} m_n(F). \quad (3.6)$$

The recurrence formula (3.6) is recommended over (3.2) to calculate the moments. It is noteworthy that the moments  $m_n(F)$  are always integer sequences when  $\mu$  is an integer and they are scaled by the factor  $(2 + 2\mu)^n$ . Except for the cases  $\mu = 0$  and 1, none of these integer sequences are found in Sloane and Plouffe [12]. For example, the moment sequence for  $\mu = 2$  is 1, 5, 51, 807, 17445, 479565, ...

From (3.1) and 29.3.37 of Abramowitz and Stegun [6],

$$f(t) = (\mu + 1) \left( \frac{e^{-t}}{\sqrt{\pi t}} - \mu e^{(\mu^2 - 1)t} \operatorname{erfc}(\mu\sqrt{t}) \right), \quad t \geq 0, \quad (3.7)$$

Going from (3.7) to (3.2) is surprisingly difficult. It can be done by applying the Gosper-Zeilberger algorithm, e.g., see Section 5.8, especially p. 236, of Graham, Knuth and Patashnik [10] or Petkovsek, Wilf and Zeilberger [11]. The associated EMIG pdf in [3], which unfortunately was inadvertently omitted from (8.10) of [3], is

$$\rho(t; 1, \nu) = \frac{\nu e^{-t/2\nu}}{\sqrt{2\pi\nu t}} - 2^{-1}(\nu - 1)e^{(\nu - 2)t/2} \operatorname{erfc}((\nu - 1)\sqrt{t/2\nu}). \quad (3.8)$$

To obtain (3.7) and (3.8), first scale  $t$  by the factor  $2\nu$ , then let  $\nu = \mu + 1$ .

Similarly, from (3.4), we have for  $\mu \neq 1$ ,

$$F^c(t) = \frac{1}{\mu - 1} (\mu e^{(\mu^2 - 1)t} \operatorname{erfc}(\mu\sqrt{t}) - \operatorname{erfc}(\sqrt{t})), \quad t \geq 0, \quad (3.9)$$

whereas for  $\mu = 1$ , we invert  $(1 + \sqrt{1 + s})^{-2}$  to get

$$F^c(t) = (1 + 2t) \operatorname{erfc}(\sqrt{t}) - 2\sqrt{\pi/t} e^{-t}, \quad t \geq 0. \quad (3.10)$$

In the case  $\mu = 1$ , the pdf  $f(t)$  in (3.7) coincides with the beta mixture of exponentials (BME) pdf  $v(1/2, 3/2; t)$  in Abate and Whitt [4], which in turn coincides with the RBM first-moment pdf  $h_1(t)$ ; see Table 3 in [4]. The associated cdf in (3.10) is  $v(3/2, 3/2; t)/4$ . (See the next section for further discussion.)

For all  $\mu > 0$ , the asymptotic expansion for  $F^c(t)$  is

$$F^c(t) \sim \frac{e^{-t}}{\sqrt{\pi t}} \sum_{n=1}^{\infty} (-1)^{n+1} k_n(\mu) n! \beta_n t^{-n} \quad \text{as } t \rightarrow \infty, \quad (3.11)$$

where  $\beta_n = \binom{2n}{n} 4^{-n}$  is the moment sequence of the gamma pdf  $\gamma(t) = e^{-t}/\sqrt{\pi t}$  as in Table 3 of [4] and

$$k_n(\mu) = \sum_{k=0}^{2n-1} \mu^k = \frac{1}{\mu - 1} \left( 1 - \frac{1}{\mu^{2n}} \right), \quad (3.12)$$

drawing on 7.1.23 of Abramowitz and Stegun [6]. Note that  $k_n(1) = 2n$ .

As in our construction of B<sub>2</sub>ME cdf's from BME cdf's in [4], we define the cdf  $G^c$  associated with  $\hat{g}(s)$  in (2.1) by undamping the cdf  $F^c(t)$ , i.e., by letting

$$G^c(t) = e^t F^c(t), \quad t \geq 0. \quad (3.13)$$

Combining (3.3) and (3.13), we obtain

$$\hat{G}^c(s) = \hat{F}^c(s-1) = \frac{1}{(\mu + \sqrt{s})(1 + \sqrt{s})} \quad (3.14)$$

and

$$\hat{g}(s) = 1 - s\hat{G}^c(s) = 1 - \frac{s}{(\mu + \sqrt{s})(1 + \sqrt{s})}, \quad (3.15)$$

just as in (2.1). Moreover,

$$\hat{G}_e^c(s) \equiv \frac{1 - \hat{g}_e(s)}{s} = \left(\frac{\mu+1}{\mu}\right) \frac{1}{\sqrt{s}(1+\sqrt{s})} + \left(\frac{1}{\mu(1-\mu)}\right) \frac{1}{1+\sqrt{s}} - \left(\frac{1}{\mu(1-\mu)}\right) \frac{1}{\mu+\sqrt{s}}, \quad (3.16)$$

so that, by 29.3.37 and 29.3.43 of Abramowitz and Stegun [6],

$$G_e^c(t) = \frac{\mu}{1-\mu} (\mu^{-1}\psi(\mu^2 t) - \psi(t)), \quad t \geq 0, \quad (3.17)$$

for  $\psi$  in (2.5).

In the case  $\mu = 1$ , we can apply the BME and B<sub>2</sub>ME calculus in [4], in particular, (1.20), (1.7) and Table 3, to get

$$\begin{aligned} g_e(t) = G^c(t) &= V_2^c(1/2, 3/2; t) = e^t V(1/2, 3/2; t) \\ &= (1/4)e^t v(3/2, 3/2; t) \\ &= (2t+1)\psi(t) - 2\sqrt{t/\pi} \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} G_e^c(t) &= V_2^c(3/2, 1/2; t) = e^t V^c(3/2, 1/2; t) \\ &= (3/4)e^t v(5/2, 1/2; t) \\ &= 2\sqrt{t/\pi} - (2t-1)\psi(t), \end{aligned} \quad (3.19)$$

as given in (2.8).

#### 4. Representation as a Mixture of Exponentials

We now show that EMIGs and undamped EMIGs are both completely monotone; i.e., can be expressed as mixtures of exponentials. As a consequence, they can be approximated arbitrarily closely by hyperexponential (finite mixtures of exponential) distributions; see Feldmann



and Whitt [9]. Of course, the hyperexponential approximations never match the asymptotic tail behavior. Nevertheless, the associated M/G/1 waiting-time distributions are also matched arbitrarily closely; see [9].

**Theorem 4.1.** *An EMIG is completely monotone; in particular, the cdf can be expressed as*

$$F^c(t) = \int_0^1 e^{-t/y} w(y) dy, \quad (4.1)$$

where

$$w(y) = \frac{\mu + 1}{\pi\sqrt{y}} \left( \frac{\sqrt{1-y}}{1 + (\mu^2 - 1)y} \right), \quad 0 \leq y \leq 1. \quad (4.2)$$

**Proof.** We regard the Laplace transform  $\hat{F}^c(s)$  in (3.4) as the Stieltjes transform of the spectral density; i.e., initially assuming that

$$F^c(t) = \int_0^\infty e^{-xt} \phi(x) dx, \quad (4.3)$$

we obtain

$$\hat{F}^c(s) = \int_0^\infty \frac{1}{s+x} \phi(x) dx. \quad (4.4)$$

We can then calculate the alleged spectral density  $\phi(x)$  by inverting its Stieltjes transform, p. 126 of Widder [14]; i.e.,

$$\phi(x) = -\frac{\text{Im} \hat{F}^c(-x)}{\pi} = \frac{1}{\pi(\mu - 1)} \left( \frac{\sqrt{x-1}}{x} - \frac{\sqrt{x-1}}{x + \mu^2 - 1} \right) = \frac{(\mu + 1)\sqrt{x-1}}{\pi x(x + \mu^2 - 1)}, \quad x > 1. \quad (4.5)$$

The mixing density  $w(y)$  is related to the spectral density  $\phi(x)$  by  $w(y) = y^{-2}\phi(y^{-1})$ . Hence, from (4.5) we obtain (4.2). ■

We can combine (3.13) and Theorem 4.1 to obtain a corresponding result for undamped EMIGS.

**Corollary 1.** *An undamped EMIG is also completely monotone, i.e.,*

$$G^c(t) = \int_0^1 e^{-t(1-y)/y} w(y) dy \quad (4.6)$$

$$= \int_0^\infty e^{-t/z} w(z/(z+1))(1+z)^{-2} dz \quad (4.7)$$

for  $w(y)$  in (4.2).

In two special cases the EMIG is a beta mixture of exponentials (BME), as considered in [4]. Recall that the beta density is

$$b(p, q; y) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} y^{p-1}(1-y)^{q-1}, \quad 0 \leq y \leq 1. \quad (4.8)$$

**Corollary 2.** For  $\mu = 0$ ,  $w(y) = b(1/2, 1/2; y)$ ; for  $\mu = 1$ ,  $w(y) = b(1/2, 3/2; y)$ .

Hence, in the notation of [4], the EMIG in (3.1) is  $\nu(1/2, 1/2; t)$  when  $\mu = 0$  and  $\nu(1/2, 3/2; t)$  when  $\mu = 1$ . For those cases additional properties are given in [4]. Recall that the special case considered by Boxma and Cohen [7] is  $\mu = 1$ . Thus their case is the B<sub>2</sub>ME pdf  $\nu_2(1/2, 3/2; t)$ . By Theorem 8 of [4], it can also be expressed as a gamma mixture of Pareto distributions.

More generally, we can express the mixing pdf  $w(y)$  in (4.2) as a linear combination of beta pdf's. To do so, we expand  $(1 + (\mu^2 - 1)y)^{-1}$  in (4.2) in a power series.

**Theorem 4.2.** For  $\mu > 0$  with  $\mu \neq 1$ ,

$$w(y) = \frac{\mu + 1}{2} \sum_{n=0}^{\infty} (1 - \mu^2)^n \frac{\beta_n}{n + 1} b\left(\frac{2n + 1}{2}, 3/2; y\right). \quad (4.9)$$

where  $\beta_n \equiv \binom{2n}{n} 4^{-n}$ , the moments of  $b(1/2, 1/2; y)$ .

## 5. Time Asymptotics

Combining (3.9) and (3.13), we obtain the undamped EMIG cdf  $G^c(t)$ . From that form, we can obtain the asymptotics as  $t \rightarrow 0$  and as  $t \rightarrow \infty$ . In particular, from (3.11),

**Theorem 5.1.** For the undamped EMIG distribution,

$$G^c(t) \sim 1 - 2(\mu + 1)\sqrt{t/\pi} \quad \text{as } t \rightarrow 0, \quad (5.1)$$

$$G^c(t) \sim \left(\frac{\mu + 1}{2\mu^2}\right) \frac{1}{\sqrt{\pi t^3}} \quad \text{as } t \rightarrow \infty, \quad (5.2)$$

and

$$G_e^c(t) \sim \left(\frac{\mu + 1}{\mu}\right) \frac{1}{\sqrt{\pi t}} \quad \text{as } t \rightarrow \infty. \quad (5.3)$$

Similarly, we obtain the large-time asymptotics for  $W^c(t)$  from (2.13). For other M/G/1 waiting-time asymptotics, see Willekens and Teugels [15], Abate, Choudhury and Whitt [5] and Boxma and Cohen [7].

**Theorem 5.2.** with the undamped EMIG service-time pdf transform  $\hat{g}(s)$  in (2.1),

$$W^c(t) \sim \frac{\rho}{1 - \rho} G_e^c(t) \left[ 1 - \frac{(1 + \mu)^2 - 2(1 - \rho)\mu}{2(1 - \rho)^2 \mu^2 t} \right] \quad \text{as } t \rightarrow \infty. \quad (5.4)$$

Formula (5.4) here agrees with formula (3.12) of Boxma and Cohen [7] for the case  $\mu = 1$ .

## 6. Heavy-Traffic Asymptotics

Boxma and Cohen [7] establish general heavy-traffic limits and approximations as  $\rho \rightarrow 1$ . We obtain their result for our special case directly from the explicit representation in Section 2.

**Theorem 6.1.** *If  $\rho \rightarrow 1$ , then  $\nu_1 \rightarrow 1 + \mu$ ,  $\nu_2/(1 - \rho) \rightarrow \mu/(1 + \mu)$  and*

$$W^c(t/\alpha)\psi(t) \tag{6.1}$$

for  $\psi(t)$  in (2.5), where

$$\alpha = \frac{(1 - \rho)^2}{\rho^2} \left( \frac{\mu}{1 + \mu} \right)^2. \tag{6.2}$$

Based on (6.1), we would use the approximation

$$W^c(t) \approx \psi(\alpha t) = e^{\alpha t} \operatorname{erfc}(\sqrt{\alpha t}) \tag{6.3}$$

for  $\alpha$  in (6.2). Since  $\rho^2 \rightarrow 1$  as  $\rho \rightarrow 1$ , the factor  $\rho^2$  in (6.2) plays no role in the heavy-traffic limit. However, it makes the heavy-traffic approximation (6.3) asymptotically correct as  $t \rightarrow \infty$  for each  $\rho$  as well. We could further simplify the right side of (6.3) by replacing  $\operatorname{erfc}(\sqrt{\alpha t})$  by its asymptotic form as  $\alpha \rightarrow 0$ , but the numerics performed by Boxma and Cohen [7] show that it is better to keep the error function. This phenomenon very closely parallels our asymptotic normal approximation for the M/G/1 busy-period distribution in Abate and Whitt [2]. Indeed, the same approximating functions are involved.

## 7. Other Explicit Expressions

Smith [13] first observed that if the service-time distribution has rational Laplace transform, then so does the M/G/1 steady-state waiting-time distribution, so that at least in principle it can be inverted analytically. This is easy to see in two steps: (1) going from the service-time cdf  $G$  to its associated stationary-excess cdf  $G_e$  and (2) going from  $G_e$  to the waiting-time cdf exploiting the Pollaczek-Khintchine formula. The other explicit representations obtained so far can be viewed as generalizations of this result. If the service-time distribution has a Laplace transform that is a rational function of  $s^{1/n}$ , then it is easy to see that so does the M/G/1 steady-state waiting-time distribution. For general  $n$ , this property seems difficult to exploit, but for  $n = 2$ , we can exploit it, because we can relate the transform involving  $\sqrt{s}$  to the error function.

For example, at least in principle, we can obtain the explicit  $M/G/1$  waiting-time distribution when the service-time distribution is a mixture of  $k$  undamped EMIGs. By the usual partial fraction expansion (assuming no multiple roots), we can represent the waiting-time distribution as a linear combination of undamped EMIGs. However, the additional complexity seems to make this approach unattractive.

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