PATTERN AVOIDANCE IN INVOLUTIONS

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Abstract. This work concerns pattern avoidance in involutions. We give a complete solution for the number of involutions avoiding one or two classical 3-patterns, mainly by relating these to well known combinatorial structures such as Dyck paths and Young tableaux. The results for single 3-patterns were previously obtained by Simion and Schmidt. However, we give new proofs in most cases. We also give some results for the number of involutions avoiding generalised patterns.

Contents

1. Introduction

Classically a k-pattern p is a permutation of $[k] = \{1, 2, \ldots, k\}$ and a permutation π of $[n]$ is said to have an occurrence of p if π has a subword whose letters are in the same relative order as the letters of p. If π has no occurrences of p, we say that π avoids p. For example $\pi = 52134$ avoids $p = 132$ whereas $\pi = 41253$ has two occurrences of p (the subwords 153 and 253).

In the last decades there have been plenty of articles written on the subject of patterns and in particular on pattern avoidance. One of the earliest results worth mentioning is found in Knuth [7], where it is established that for all 3-patterns p, the number of permutations of $[n]$ that avoid p equals the nth Catalan number. In Simion and Schmidt [10], multiavoidance, that is when two or more patterns are simultaneously avoided, was considered and a full solution for the case of double avoidance was given. Simion and Schmidt also treated pattern-avoiding involutions, the topic of this work. Indeed, the results of Section 3, which concern the six classical 3-patterns, are all proven in [10]. However, we give new proofs of some of the results.

As a further development of the concept of patterns, Babson and Steingrímsson [3] introduced generalised patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation for the pattern to occur. Avoidance of generalised patterns has been studied by, for example, Claesson [1], Kitaev [5], [6] and Claesson and Mansour [2]. In Section 4 we give some results for involutions avoiding generalised patterns.

Finally, in Section 5 we investigate double avoidance and give a complete solution for the number of involutions avoiding any two classical 3-patterns.

2. Preliminaries

Before starting the investigation on pattern-avoiding involutions we introduce the main concepts that will be used in this work. To start with, an *alphabet* X is a nonempty set of *letters* and a *word* over X is a finite sequence of letters from X. We denote the *empty word*, that is the word with no letters, by ϵ. Let $x = x_1x_2\cdots x_n$ be a word over X. A *subword* of x is a word $v = x_{i_1}x_{i_2}\cdots x_{i_k}$, where $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$. A *segment* is a word $v = x_ix_{i+1}\cdots x_{i+k}$. We define the *length of* x, denoted by $|x|$, to be the number of elements in x.

2.1. **Permutations.**

Let $[n] = \{1, 2, \ldots, n\}$. A *permutation* π of $[n]$ is a bijection from $[n]$ to $[n]$. However, we sometimes refer to permutations of a subset A of $[n]$. This should be interpreted as a bijection from A to A. There are several different notations for the permutations, suitable for different purposes. A permutation π is usually seen as the word

$$\pi = \pi(1)\pi(2)\cdots\pi(n).$$

Another way of writing the permutation is given by the two line (or French) notation

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & n \\ a_1 & a_2 & \ldots & a_n \end{pmatrix}.$$

This means that $1 \mapsto a_1$, $2 \mapsto a_2$ et cetera, hence the permutation is unaffected by rearrangement of the columns, which makes it easy to find the inverse of π. Indeed

$$\pi^{-1} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 1 & 2 & \dots & n \end{pmatrix}.$$

Rearranging the top line in increasing order gives π^{-1} as a word in the bottom line.

We will also use a third notation, the cycle form, where the letters in $[n]$ are grouped together in cycles. A cycle $(a_1 a_2 \cdots a_k)$ means that $a_i \mapsto a_{i+1}$ for $i < k$ and that $a_k \mapsto a_1$. Fixed points, that is those i for which $i \mapsto i$, are conventionally omitted. As will be shown in the example below, the cycle notation is generally not unique.

We denote the set of permutations of $[n]$ by $\mathcal{S}_n$.

Example 1. Consider the permutation

$$\pi = \begin{cases} 1 \rightarrow 3 \\ 2 \rightarrow 4 \\ 3 \rightarrow 6 \\ 4 \rightarrow 2 \\ 5 \rightarrow 5 \\ 6 \rightarrow 1 \end{cases}$$

We write it as the word

$$\pi = 346251,$$

or in the two line notation;

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 6 & 2 & 5 & 1 \end{pmatrix},$$

from which we get the inverse of π as

$$\pi^{-1} = \begin{pmatrix} 3 & 4 & 6 & 2 & 5 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 1 & 2 & 5 & 3 \end{pmatrix}.$$

The permutation π could be written in cycle form as

$$\pi = (136)(24)$$

but we also have

$$\pi = (24)(136) = (361)(24) = (42)(613).$$

This shows that the cycle notation is not unique. Note that the fixed point 5 is not written out.

2.2. **Involutions.** An *involution* is a permutation that is its own inverse. Thus an involution consists of cycles of length 1 or 2. We let $\mathcal{I}_n$ denote the set of all involutions of $[n]$.

2.3. **Generalised patterns.** A *generalised k-pattern* p is a word of length k consisting of all the elements of $[k]$, in which two letters may or may not be separated by a dash. Consider $\pi = a_1 a_2 \cdots a_n$ in $\mathcal{S}_n$. We say that the subword $v = v_1 v_2 \cdots v_k$ is a *p-subword* of π if the v_i's are in the same relative order as the p_i's and two adjacent letters of v are adjacent in π whenever the corresponding letters of p are not separated by a dash. We also refer to v as an *occurrence of* p. If π has no occurrences of p, we say that π *avoids* p or that π is *p-avoiding.* We define $\mathcal{S}_n(p)$ and $\mathcal{I}_n(p)$ to be the set of p-avoiding permutations and involutions in $\mathcal{S}_n$, respectively, and more generally we let $\mathcal{S}_n(A) = \bigcap_{p\in A} \mathcal{S}_n(p)$, just as $\mathcal{I}_n(A) = \bigcap_{p\in A} \mathcal{I}_n(p)$. It is convenient to regard the pattern p as a function from $\mathcal{S}_n$ to $\mathbb{N}$ where $p\pi$ is defined as the number of p-subwords of π. Thus π is p-avoiding if and only if $p\pi = 0$.

Usually the term pattern refers to the type of patterns $p_1\text{-}p_2\text{-}\cdots\text{-}p_k$ with dashes between each pair of adjacent letters, that is, no attention is paid to whether the letters of the permutation are adjacent or not. Those patters were the first to be defined and studied and we therefore call them classical patterns.

Example 2. Regarded as a permutation statistic (a function from $\mathcal{S}_n$ to $\mathbb{N}$), the pattern (1-2-3) counts the number of increasing subsequences of length 3. For example, the longest increasing sequence of the permutation 21543 is of length two and consequently 21543 avoids (1-2-3).

The pattern (21) counts *descents* in a permutation, that is the number of i's such that $a_i > a_{i+1}$, just as (12) counts the *ascents*, the number of i's such that $a_i < a_{i+1}$.

The pattern $p = (1\text{-}32)$ counts the subwords of the form $a_i\text{-}a_j a_{j+1}$ such that $a_i < a_{j+1} < a_j$. The permutation 25431 has two occurrences of p, namely 254 and 243.

2.4. **Young tableaux.** A *Young tableau P of shape $(n_1, n_2, \ldots n_m)$* is an arrangement of n distinct integers as an array of m left-justified rows, with n_i elements in row i, where $n_1 \geq n_2 \geq n_m \geq 0$ and $n_1 + n_2 + \cdots + n_m = n$. The entries of the rows and the columns must be ordered increasingly from left to right and from top to bottom, respectively. We write $P_{i,j}$ for the element in row i and column j.

Example 3. We have that

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 9 \\ \hline 2 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array}$$

is a Young tableau of shape $(4, 2, 2, 1)$ and that $P_{3,2} = 7$.

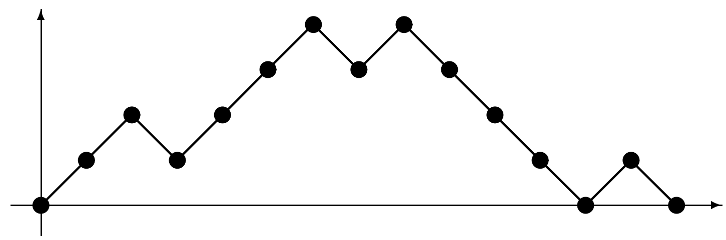

FIGURE 1. The Dyck path in Example 5

2.5. **Inversion tables.** Given a permutation $\pi = a_1a_2\cdots a_n$, we let $t(\pi) = (t_1, t_2, \ldots, t_n)$, where $t_i = |\{j : j > i, a_j < a_i\}|$. That is, the ith entry of t is the number of letters following the ith letter of π that are smaller than the ith letter.

A pair (a_i, a_j) is called an *inversion* of the permutation π if $i < j$ and $a_i > a_j$. Accordingly, t defined above is called the *inversion table* of π, since it gives a measure of the number of inversions that each letter of π causes.

It is easy to see that a permutation is uniquely determined by its inversion table, for a demonstration see for example Stanley [12].

Example 4. Consider $\pi = 1327654$. The corresponding inversion table is $t = (0, 1, 0, 3, 2, 1, 0)$, because there is no element smaller than 1 and there is exactly one element to the right of 3, namely 2, that is smaller than 3 et cetera.

2.6. **Dyck paths.** A *Dyck path of length 2n* is a lattice path from (0,0) to $(0, 2n)$ that consists of steps (1,1) and (1,-1) and that never goes below the x-axis. Denoting the steps (1,1) and (1,-1) by u (for up) and d (for down), a Dyck path can be written as a word over the alphabet $\{u, d\}$. The number of Dyck paths of length $2n$ is the nth *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$. We denote the set of Dyck paths of length $2n$ by $\mathcal{D}_n$.

Example 5. The Dyck path of length $2 \cdot 7$ in Figure 3 is coded by the word *uuduuududddddud*.

3. Pattern avoiding involutions

We start our work on pattern-avoiding involutions by investigating the avoidance of the six classical 3-patterns. For each such pattern we generate and study $\mathcal{I}_n(p)$, when n is small. When counting these involutions we obtain the first elements of the sequences that are presented in Table 1. Our aim is to show that the results are indeed true for all n.

For odd n, when $n/2$ is not an integer, it is natural to consider $\binom{n}{n/2}$ as $\binom{n}{\lfloor n/2 \rfloor}$ or $\binom{n}{\lceil n/2 \rceil}$, since the binomial $\binom{n}{k}$ coefficients are defined only for

p	$\vert\mathcal{I}_n(p)\vert$
(1-2-3)	$\binom{n}{n/2}$
(1-3-2)	$\binom{n}{n/2}$
(2-1-3)	$\binom{n}{n/2}$
(2-3-1)	2^{n-1}
(3-1-2)	2^{n-1}
(3-2-1)	$\binom{n}{n/2}$

TABLE 1. Classical patterns

integer n and k. However, $\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{n-\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}$, so there should be no ambiguities concerning the interpretation. Let $\binom{n}{n/2} := \binom{n}{\lfloor n/2 \rfloor}$.

It is observed that the involutions that avoid (2-3-1) are exactly the same as those that avoid (3-1-2), at least for small n. On the other hand we see that although $|\mathcal{I}_n(p)| = \binom{n}{n/2}$ for four different patterns p, there are no two distinct patterns p and q of these, such that $\mathcal{I}_n(p) = \mathcal{I}_n(q)$. The reader may convince himself of this by studying $\mathcal{I}_n(p)$ for small n.

3.1. **Avoiding p, when p is not an involution.** We consider the case when the pattern p itself is not an involution. As noticed above an involution avoids (2-3-1) if and only if it avoids (3-1-2). In this section this will be shown to follow from the fact that the patterns are inverses of each other. First, however, we show that $\mathcal{I}_n$(2-3-1) is counted by 2^{n-1}.

Proposition 6. *The number of involutions of* $[n]$ *that avoid* (2-3-1) *is* 2^{n-1}.

We give a general description of the elements of $\mathcal{I}_n$(2-3-1). Note that, if $\pi = a_1 a_2 \cdots a_n$ is a permutation of $[n]$, where n is in position k, then π avoids (2-3-1) if and only if it can be written as $\pi = \sigma n \tau$, where $\sigma = a_1 a_2 \cdots a_{k-1}$ is a (2-3-1)-avoiding permutation of $[k-1]$ and $\tau = a_{k+1} a_{k+2} \cdots a_n$ is a (2-3-1)-avoiding permutation of $\{k, \ldots n-1\}$. Furthermore, if π is an involution we see that since n is in position k, the letter k must be in position n, and the only (2-3-1)-avoiding permutation τ of $\{k, \ldots n-1\}$ ending with k is $\tau = (n-1)(n-2)\cdots(k+1)k$, that is, these letters must be in decreasing order. Indeed, all other possible τ's will contain at least one ascent ij, where $i < j$, and ijk will then form a (2-3-1)-subword. Hence every π in $\mathcal{I}_n$(2-3-1) is of the form $\sigma n(n-1)\cdots(k+1)k$ where σ is in $\mathcal{I}_{k-1}$(2-3-1). Such a π can be written explicitly as

$$\pi = k_1 \cdots 1 k_2 \cdots (k_1+1) k_3 \cdots (k_{\ell-1}+1) n \cdots (k_\ell + 1).$$

In other words, the involutions can be considered as divided into segments, such that

(a) each letter in segment i is smaller than every letter in segment $(i+1)$,
(b) the elements in each segment are in decreasing order.

In order to show that $|\mathcal{I}_n(2\text{-}3\text{-}1)| = 2^{n-1}$ we give four proofs, where we construct bijections from $\mathcal{I}_n(2\text{-}3\text{-}1)$ to different sets that are known to be counted by 2^{n-1}.

First proof. Let B_n be the collection of binary strings of length n. Given a binary string $x = x_1x_2\cdots x_{n-1}$ in B_{n-1}, a permutation $\pi = a_1a_2\cdots a_n$ in $\mathcal{S}_n$ is constructed inductively by letting $\pi_0 = 1$ and then, if $\pi_i = \sigma i\tau$, by letting

$$\begin{aligned} \pi_{i+1} &= \sigma i\tau(i+1), \text{ if } x_i = 0 \\ \pi_{i+1} &= \sigma i(i+1)\tau, \text{ if } x_i = 1. \end{aligned}$$

That is, the permutation π is built up by successively placing each of the elements $1,\dots,n$ either as the last element or just before the largest element already placed. This procedure defines a mapping

$$\begin{aligned} \Phi_n : B_{n-1} &\to \mathcal{S}_n, \\ x &\mapsto \pi. \end{aligned}$$

Denote the image of B_{n-1} by A_n. Then A_n consists of all permutations of the form

$$\sigma n(n\text{-}1)(n\text{-}2)\dots(k+1)k, \text{ where } \sigma \in A_{k-1},$$

and is easily seen to coincide with $\mathcal{I}_n(2\text{-}3\text{-}1)$, according to the description above. Since Φ_n is clearly injective we have a one-to-one correspondence between the binary strings of length $(n-1)$ and $\mathcal{I}_n(2\text{-}3\text{-}1)$, hence $|\mathcal{I}_n(2\text{-}3\text{-}1)| = |B_{n-1}| = 2^{n-1}$. □

Example 7. Consider the binary string $x = 010111 \in B_6$. Then Φ_7 maps x to $\pi = 1327654$, via π_i, for $i = 0,\dots,6$, where

$$\begin{aligned} \pi_0 &= 1 \\ \pi_1 &= 12, \text{ since } x_1 = 0 \\ \pi_2 &= 132, \text{ since } x_2 = 1 \\ \pi_3 &= 1324, \text{ since } x_3 = 0 \\ \pi_4 &= 13254, \text{ since } x_4 = 1 \\ \pi_5 &= 132654, \text{ since } x_5 = 1 \\ \pi = \pi_6 &= 1327654, \text{ since } x_6 = 1. \end{aligned}$$

Second proof. In this proof we show the one-to-one correspondence between $\mathcal{I}_n(2\text{-}3\text{-}1)$ and the binary strings of length $(n-1)$ by constructing a mapping Ψ_n from T_n to B_{n-1}. Here T_n is the set of inversion tables $t = (t_1, t_2, \ldots, t_n)$ defined from $\pi = a_1 a_2 \cdots a_n \in \mathcal{I}_n(2\text{-}3\text{-}1)$ as

$$t_i := |\{j : j > i, a_j < a_i\}|.$$

That is, the ith entry of t is the number of letters following the ith letter of π that are smaller than the ith letter. From the appearance of $\mathcal{I}_n(2\text{-}3\text{-}1)$ it follows that the elements in T_n will be of the form

$$(k_1, k_1 - 1, \ldots, 1, 0, \ldots, 0, k_2, k_2 - 1, \ldots, 1, 0, k_\ell, k_\ell - 1, \ldots, 1, 0).$$

For example, a decreasing sequence $a_i a_{i+1} \ldots a_{i+k}$ of length $(k+1)$ will give rise to the segment $(t_i, t_{i+1}, \ldots, t_{i+k}) = (k, (k-1), \ldots, 1, 0)$ in the corresponding inversion table $t(\pi)$.

The mapping

$$\begin{aligned} \Psi_n : T_n &\rightarrow B_{n-1} \\ t = (t_1, t_2, \ldots, t_n) &\mapsto x = x_1 x_2 \cdots x_{n-1} \end{aligned}$$

is now defined by

$$x_i = \begin{cases} 0 \text{ if } t_i = 0, \\ 1 \text{ if } t_i \neq 0. \end{cases}$$

It is easy to see that Ψ_n is invertible, when restricted to (2-3-1)-avoiding involutions. The inverse mapping is given by

$$t_i = \begin{cases} 0, & \text{if } x_i = 0, \\ s, & \text{where } (s-1) \text{ is the number of 1's following } x_i, \text{ if } x_i = 1. \end{cases}$$

A permutation is uniquely determined by its inversion table. Hence there is a one-to-one correspondence between B_{n-1} and $\mathcal{I}_n(2\text{-}3\text{-}1)$ via the inversion tables $\{T_n\}$, and $|\mathcal{I}_n(2\text{-}3\text{-}1)| = 2^{n-1}$. □

Example 8. Consider $\pi = 1327654$ from Example 7. The corresponding inversion table is $t = (0, 1, 0, 3, 2, 1, 0)$, according to Example 4. Now Ψ_7 maps $(0, 1, 0, 3, 2, 1, 0)$ onto 0101110, which is exactly the binary string x, given by the mapping Φ_7 in the first proof.

Third proof. Denote the set of subsets of $[n]$ by $\mathcal{P}_n$. We construct π in $\mathcal{I}_n(2\text{-}3\text{-}1)$ from A in $\mathcal{P}_{n-1}$ by letting the letter i be immediately preceded by a larger letter, if and only if i is in A. Because of the appearance of the elements in $\mathcal{I}_n(2\text{-}3\text{-}1)$ there is only one choice of the larger letter to precede i, namely $(i+1)$, and this algorithm for constructing π from A therefore clearly defines a bijection. Indeed, the segment $(i + k)(i + k - 1) \cdots i$ is contained in π if and only if i, $(i+1)$, $\ldots$, $(i+k)$ are in A. Hence there is a one-to-one correspondence between $\mathcal{P}_{n-1}$ and $\mathcal{I}_n(2\text{-}3\text{-}1)$, so $|\mathcal{I}_n(2\text{-}3\text{-}1)| = |\mathcal{P}_{n-1}| = 2^{n-1}$. □

Example 9. Let $A = \{2, 4, 5, 6\}$. The corresponding π is 1327654. Indeed, the letter 2 is the smallest letter that is in A, and accordingly the smallest letter to be preceded by a larger letter. From this we conclude that 1 is a fixed point and, since 3 is not in A, the decreasing sequence ending with 2 must start with 3. The letter 4 is in A as well as 5 and 6, and hence π must contain the segment 7654.

We also see from this example how to get from π to A. Considering $\pi = 1327654$ we find that exactly the letters 2, 4, 5 and 6 are preceded by larger letters, hence $A = \{2, 4, 5, 6\}$.

Porism 10. *The number of involutions in* $\mathcal{I}_n(2\text{-}3\text{-}1)$ *with exactly* k *descents is* $\binom{n-1}{k}$.

Proof. Consider the bijection from $\mathcal{P}_{n-1}$ to $\mathcal{I}_n(2\text{-}3\text{-}1)$ defined in the third proof above. A (2-3-1)-avoiding involution is constructed from A in $\mathcal{P}_{n-1}$ by letting i be preceded by a larger letter if and only if i is in A. Hence the number of elements in A counts the descents of π. Since there are $\binom{n-1}{k}$ ways of choosing k letters out of $[n-1]$, the result follows. □

Finally, we give a proof by showing a one-to-one correspondence between $\mathcal{I}_n(2\text{-}3\text{-}1)$ and a certain type of Dyck paths, that are easily counted.

Fourth proof (of Proposition 6). Claesson [1] gives a proof of the well-known result that $\mathcal{S}_n(2\text{-}1\text{-}3)$ is counted by the nth Catalan number, in which he defines recursively a bijective mapping Φ from $\mathcal{S}_n(2\text{-}1\text{-}3)$ to the set of Dyck paths of length $2n$. We mimic his proof and construct a mapping Φ from $\mathcal{S}_n(2\text{-}3\text{-}1)$ to the Dyck paths of length $2n$.

Consider $\pi = a_1a_2\cdots a_n$ in $\mathcal{S}_n(2\text{-}3\text{-}1)$ with the letter n in position k. According to the discussion on page 6 we can write $\pi = \sigma n \tau$, where $\sigma = a_1a_2\cdots a_{k-1}$ is a (2-3-1)-avoiding permutation of $[k-1]$ and $\tau = a_{k+1}a_{k+2}\cdots a_n$ is a (2-3-1)-avoiding permutation of $\{k{+}1, k{+}2, \ldots, n{-}1\}$.

Denoting the empty word by ϵ, we define $\Phi(\pi)$ recursively by

$$\Phi(\pi) = \begin{cases} \epsilon, \text{ if } \pi = \epsilon, \\ u\,(\Phi \circ \mathrm{proj})(\sigma)\,d\,(\Phi \circ \mathrm{proj})(\tau), \text{ otherwise.} \end{cases}$$

Here, $\mathrm{proj}(x)$ denotes the *projection* of the word $x = x_1x_2\cdots x_n$, where $x_i \in \mathbb{N}$ and $x_i \neq x_j$, onto $\mathcal{S}_n$, defined by

$$\mathrm{proj}(x) = a_1a_2\cdots a_n, \text{ where } a_i = |\{j \in [n].x_i \geq x_j\}|.$$

For example $\mathrm{proj}(265) = 132$.

It is easy to see that Φ is invertible and hence a bijection.

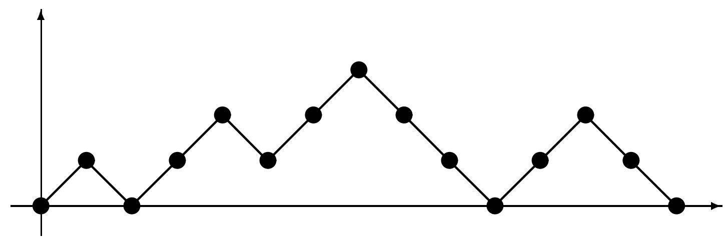

FIGURE 2. The Dyck path in Example 11

Example 11. Consider $\pi = 7312645 \in \mathcal{S}_7(2\text{-}3\text{-}1)$. The corresponding Dyck path is given by:

$$\begin{aligned}\Phi(\pi) &= u\Phi(\epsilon)d\Phi(312645)\\ &= u\epsilon du\Phi(312)d\Phi(12)\\ &= u\epsilon duu\Phi(\epsilon)d\Phi(12)du\Phi(1)d\Phi(\epsilon)\\ &= u\epsilon duu\epsilon du\Phi(1)d\Phi(\epsilon)duu\Phi(\epsilon)d\Phi(\epsilon)d\epsilon\\ &= u\epsilon duu\epsilon duu\Phi(\epsilon)d\epsilon d\epsilon duu\epsilon d\epsilon d\epsilon\\ &= u\epsilon duu\epsilon duu\epsilon d\epsilon d\epsilon duu\epsilon d\epsilon d\epsilon\\ &= uduuduudddduudd.\end{aligned}$$

When restricted to involutions we have that τ, and accordingly $\Phi(\tau)$, is determined by the position of n. In fact, the decreasing sequence starting with n, that is $n\tau$, where $\tau = (n-1)\cdots k$, corresponds to the Dyck path

$$\begin{aligned}\Phi(\tau) &= u\Phi(\epsilon)d\Phi((n-1)(n-2)\cdots k)\\ &= udu\Phi(\epsilon)d\Phi((n-2)(n-3)\cdots k)\\ &\ \vdots\\ &= udud\cdots ud.\end{aligned}$$

The image of $\mathcal{I}_n(2\text{-}3\text{-}1)$, denoted by D_n^*, will therefore be

$$D_n^* = \{uD_{n-k}^* dud\cdots ud\}.$$

That is, a Dyck path in D_n^* ends with a tail of the form $udud\cdots ud$, preceding which, there are no returns to the x-axis. Removing the tail, the down-step just before it and the first up-step of the Dyck path yields a path with the same properties. Note that D_n^* is the set of Dyck paths, in which a down-step is immediately followed by at most one up-step. As a consequence the peaks as well as the valleys are of decreasing height. An illustration of a typical Dyck path in D_n^* is given in Example 12 below.

From the construction, the number of D_n^* satisfies the recursion

$$|D_n^*| = \sum_{i=1}^{n-1} |D_i^*|,$$

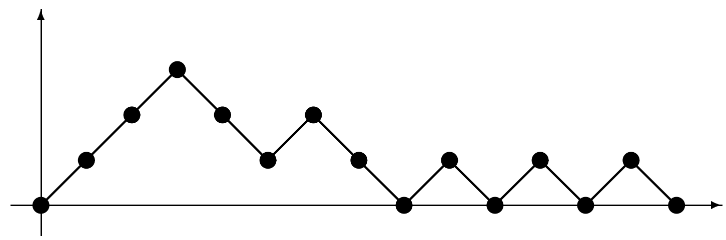

FIGURE 3. The Dyck path in Example 12

and since $D_1^* = \{ud\}$, we have $|D_1^*| = 1$, so $|D_n^*| = 2^{n-1}$. □

Example 12. Let us return to the (2-3-1)-avoiding involution $\pi = 1327654$ from Example 7. We have that π corresponds to the Dyck path:

$$\begin{aligned}\Phi(\pi) &= u\Phi(132)d\Phi(321)\\ &= uu\Phi(1)d\Phi(1)du\Phi(\epsilon)d\Phi(21)\\ &= uuu\Phi(\epsilon)ddu\Phi(\epsilon)ddu\epsilon du\Phi(\epsilon)d\Phi(1)\\ &= uuu\epsilon ddu\epsilon ddu\epsilon du\epsilon du\Phi(\epsilon)d\Phi(\epsilon)\\ &= uuu\epsilon ddu\epsilon ddu\epsilon du\epsilon du\epsilon d\epsilon\\ &= uuudduddududud.\end{aligned}$$

Proposition 13. *The number of involutions of* $[n]$ *that avoid* (3-1-2) *is* 2^{n-1}.

For the proof we need the following lemma.

Lemma 1. *Let* p *be a pattern in* $\mathcal{S}_k$. *Then* $\mathcal{I}_n(p) = \mathcal{I}_n(p^{-1})$.

Proof. Consider the involution π written in two line notation;

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}.$$

Suppose that the subword

$$v = \begin{pmatrix} i_1 & i_2 & \dots & i_k \\ a_{1_1} & a_{i_2} & \dots & a_{i_n} \end{pmatrix}$$

forms an occurrence of p. Then

$$v^{-1} = \begin{pmatrix} a_{1_1} & a_{i_2} & \dots & a_{i_n} \\ i_1 & i_2 & \dots & i_k \end{pmatrix}$$

is a p^{-1}-subword, contained in

$$\pi^{-1} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 1 & 2 & \dots & n \end{pmatrix}.$$

But since π is an involution, we have that $\pi = \pi^{-1}$, so π contains also the p^{-1}-subword v^{-1}. Hence we have an occurrence of p^{-1} if and only if we have an occurrence of p. □

Example 14. Let p be the pattern

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}.$$

Then

$$q = p^{-1} = \begin{pmatrix} 2 & 4 & 1 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

is the inverse of p. Let π be the involution

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 3 & 2 & 9 & 1 & 6 & 8 & 7 & 4 \end{pmatrix}.$$

The letters

$$v = \begin{pmatrix} 2 & 4 & 5 & 8 \\ 3 & 9 & 1 & 7 \end{pmatrix}$$

form an occurrence of the pattern p. Accordingly,

$$v^{-1} = \begin{pmatrix} 3 & 9 & 1 & 7 \\ 2 & 4 & 5 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 7 & 9 \\ 5 & 2 & 8 & 4 \end{pmatrix}$$

forms a q-subword.

Proof of Proposition 13. We have that (3-1-2) is the inverse of (2-3-1). The result then follows immediately from Proposition 6 and Lemma 1. □

We conclude this section by an application of Proposition 6 to a certain set of pattern avoiding permutations. Claesson [1] shows that involutions of $[n]$ are in one-to-one correspondence with permutations of $[n]$ that avoid (1-23) and (1-32). For the proof he constructs a bijection Φ between $\mathcal{I}_n$ and $\mathcal{S}_n(1\text{-}23, 1\text{-}32)$, which we describe below.

The standard form of a permutation π is defined by writing π in cycle notation and requiring that

(a) each cycle is written with its least element first
(b) the cycles are written in decreasing order with respect to their first elements.

The corresponding permutation $\widehat{\pi} = \Phi(\pi)$ is obtained from π in standard form by erasing the brackets separating the cycles. Since involutions consist of cycles of length one or two, each permutation $\widehat{\pi}$ in $\mathcal{S}_n(1\text{-}23, 1\text{-}32)$ is obtained from exactly one involution, and Φ is therefore a bijection.

Corollary 15. *Involutions of* $[n]$ *that avoid* (2-3-1) *are in one-to-one correspondence with permutations in* $[n]$ *that avoid* (1-23), (1-32), (13-2) *and* (3-214)*. Hence*

$$|\mathcal{S}_n(1\text{-}23, 1\text{-}32, 13\text{-}2, 3\text{-}214)| = |\mathcal{I}_n(2\text{-}3\text{-}1)| = 2^{n-1}.$$

Proof. Claesson [1] proves the one-to-one correspondence between $\mathcal{S}_n(1\text{-}23, 1\text{-}32)$ and $\mathcal{I}_n$, so what is left to prove is that, given a (2-3-1)-avoiding involution π we have that $\Phi(\pi)$ avoids (13-2) and (3-214) and vice versa.

To show that $\Phi(I_n(2-3-1)) \subseteq S_n(1-23, 1-32, 13-2, 3-214)$, assume that $\widehat{\pi}$ in $\mathcal{S}_n(1\text{-}23, 1\text{-}32)$ contains a (13-2)-subword. Then there exists a segment of $\widehat{\pi}$ of the form

$$a_1 a_3 \cdots a_2, \quad \text{where } a_1 < a_2 < a_3.$$

Since the cycles of involutions in standard form are of maximum length two and are written with their least element first, $\widehat{\pi}$ necessarily corresponds to an involution π containing the cycle (a_1, a_3). It also follows that the letter a_2 must be contained in a cycle $(\tilde{a}, a_2)$, where $\tilde{a} < a_1$, for otherwise a_2 would precede a_1 in $\widehat{\pi}$. We now have that

$$a_2 \cdots a_3 \cdots \tilde{a} \cdots a_1, \quad \text{where } \tilde{a} < a_1 < a_2 < a_3,$$

is a segment of π, so π contains the (2-3-1)-subword $a_2 a_3 a_1$.

Assume instead that $\widehat{\pi}$ has an occurrence of (3-214), that is $\widehat{\pi}$ contains the segment

$$a_3 \cdots a_2 a_1 a_4, \quad \text{where } a_1 < a_2 < a_3 < a_4.$$

Then $(a_1 a_4)$ must be a 2-cycle of π. The letters a_2 and a_3 can either be fixed points or contained in 2-cycles.

Assuming that a_2 and a_3 both are fixed points implies that π contains a segment of the form

$$a_4 \cdots a_2 \cdots a_3 \cdots a_1, \quad \text{where } a_1 < a_2 < a_3 < a_4.$$

Here $a_2 a_3 a_1$ forms a (2-3-1)-subword.

If a_3 is a fixed point while a_2 is not, then a_2 will be contained in a cycle $(\tilde{a_2}, a_2)$ where $a_1 < \tilde{a_2} < a_2$, once again resulting in the (2-3-1)-subword $a_2 a_3 a_1$ of π.

Finally we assume that a_3 is contained in a 2-cycle $(\tilde{a_3}, a_3)$, where $\tilde{a_3} > a_2$ (or $\tilde{a_3} > \tilde{a_2}$, if there is a cycle $(\tilde{a_2}, a_2)$). We then get the following possible segments of π:

$$\begin{aligned} &a_4 \cdots a_2 \cdots \tilde{a_2} \cdots a_3 \cdots \tilde{a_3} \ldots a_1, \text{ where } \tilde{a_3} < a_3, \\ &a_4 \cdots a_2 \cdots \tilde{a_2} \cdots \tilde{a_3} \cdots a_3 \ldots a_1, \text{ where } a_3 < \tilde{a_3} < a_4, \\ &a_4 \cdots a_2 \cdots \tilde{a_2} \cdots \tilde{a_3} \cdots a_1 \ldots a_3, \text{ where } \tilde{a_3} > a_4. \end{aligned}$$

In all cases we get an occurrence of (2-3-1). Hence it follows that

$$\Phi(I_n(2-3-1)) \subseteq S_n(1-23, 1-32, 13-2, 3-214).$$

To show the converse, that is

$$\mathcal{S}_n(1-23, 1-32, 13-2, 3-214) \subseteq \Phi(I_n(2-3-1)),$$

we consider $\pi \in I_n(2\text{-}3\text{-}1)$. There are essentially two different ways of constructing a (2-3-1)-subword out of three letters a_1, a_2 and $a_3 \in [n]$ such that $a_1 < a_2 < a_3$. Either we get an involution of the form

$$\ldots(a_3b_3)\ldots(a_2b_2)\ldots(a_1b_1)\ldots,$$

where

$$a_1 < a_2 < a_3,\ b_2 < b_3 < b_1,\ a_1 < b_1$$

or an involution of the form

$$\ldots(b_2a_2)\ldots(b_3a_3)\ldots(a_1b_1)\ldots$$

where

$$a_1 < a_2 < a_3,\ b_2 < b_3 < b_1,\ a_2 > b_2,\ a_3 > b_3.$$

Consider the first case. Without loss of generality we let $a_3 \leq b_3$ and $a_2 \leq b_2$. The special cases when a_2 and a_3 are fixed points are given by letting a_2 and a_3 be equal to b_2 and b_3 respectively. Consider the cycle $(ij) = (b_1a_1)$. Clearly $i < j$. Let $(k\ell)$ be the cycle to the left of (ij) ($k = \ell$ denotes the case when k is a fixed point). If $\ell < j = b_3$, then ℓij forms a (3-214)-subword of the corresponding permutation $\Phi(\pi)$, because ℓ is clearly larger than i. Otherwise let $(ij) = (k\ell)$ and repeat the above arguments until a (3-214)-subword is obtained. This is guaranteed to happen, since if we have gone through all cycles between (b_2a_2) and (b_1a_1), then with b_2 as ℓ we have that $\ell = b_2 < b_3$.

Considering the second case, without loss of generality we let $a_1 \leq b$. The case when a_1 is a fixed point is denoted by $a_1 = b_1$. The subword $(b_2a_2a_1)$ will now form an occurrence of (13-2) since $b_2 < a_1 < a_2$.

This proves that

$$S_n(1-23, 1-32, 13-2, 3-214) \subseteq \Phi(I_n(2-3-1)).$$

Hence

$$|S_n(1-23, 1-32, 13-2, 3-214)| = |(I_n(2-3-1)|.$$

□

3.2. **Avoiding (2-1-3) or (1-3-2).** We introduce a couple of results that will be used in the proof of Proposition 18. First we present a well-known property of the patterns in $\mathcal{S}_3$.

Proposition 16. *Let p be a pattern in $\mathcal{S}_3$. Then $|\mathcal{S}_3(p)| = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number.*

One way of proving Proposition 16 is to construct a bijection between the pattern avoiding permutations of $[n]$ and the set of Dyck paths of length $2n$, that are known to be counted by the nth Catalan number. Such a bijection for the case when $p = (2\text{-}3\text{-}1)$ is actually presented in the fourth proof of Proposition 6 on page 9.

Next we consider a consequence of the fact that an involution is its own inverse.

Lemma 2. *Let p be an involution of $[k]$ and π a permutation of $[n]$. Then π avoids the pattern p if and only if π^{-1} avoids p.*

Proof. Consider π written in two line notation:

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}.$$

Suppose that we have an occurrence of p as the subword

$$v = \begin{pmatrix} i_1 & i_2 & \dots & i_k \\ a_{1_1} & a_{i_2} & \dots & a_{i_n} \end{pmatrix}.$$

Since p is an involution, we have that $p^{-1} = p$ and

$$v^{-1} = \begin{pmatrix} a_{1_1} & a_{i_2} & \dots & a_{i_n} \\ i_1 & i_2 & \dots & i_k \end{pmatrix}$$

forms a p-subword contained in

$$\pi^{-1} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 1 & 2 & \dots & n \end{pmatrix}.$$

Hence π avoids p if and only if π^{-1} avoids p. □

Example 17. Let p be the 5-pattern

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}.$$

Clearly p is an involution. Now consider the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 9 & 1 & 2 & 6 & 7 & 3 & 8 \end{pmatrix}.$$

The subword

$$v = \begin{pmatrix} 1 & 3 & 5 & 7 & 8 \\ 5 & 9 & 2 & 7 & 3 \end{pmatrix}$$

forms an occurrence of p and accordingly

$$\pi^{-1} = \begin{pmatrix} 5 & 4 & 9 & 1 & 2 & 6 & 7 & 3 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 8 & 2 & 1 & 6 & 7 & 9 & 3 \end{pmatrix},$$

contains the p-subword

$$v = \begin{pmatrix} 5 & 9 & 2 & 7 & 3 \\ 1 & 3 & 5 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 7 & 9 \\ 5 & 8 & 1 & 7 & 3 \end{pmatrix}.$$

Proposition 18. *The number of involutions of $[n]$ that avoid (2–1–3) is the nth central binomial coefficient $\binom{n}{n/2}$.*

Proof. First we give a general description of the elements in $\mathcal{I}_n(2\text{-}1\text{-}3)$. If $\pi = a_1a_2\cdots a_n$ is a permutation of $[n]$ with the letter 1 in position k, then π avoids (2-1-3) if and only if it can be written as $\pi = \sigma 1\tau$, where $\sigma = a_1a_2\cdots a_{k-1}$ is a (2-1-3)-avoiding permutation of $\{n, (n-1), \ldots, (n-k+2)\}$ and $\tau = a_{k+1}a_{k+2}\cdots a_n$ is a (2-1-3)-avoiding permutation of $\{2, 3, \ldots, (n-k+1)\}$. That is, the letters preceding 1 must all be larger than the ones following 1, and clearly all segments of π must be (2-1-3)-avoiding.

When constructing a (2-1-3)-avoiding involution, π, there are essentially two different ways of positioning the letter 1. Either it can be placed as the first letter a_1, in which case $\sigma = \epsilon$, the empty word, or it can be placed in the second half of the word, that is in position k where $k \geq \frac{n}{2}+1$. Namely, σ, if nonempty, consists of the $(k-1)$ largest letters of $[n]$, in particular k, that is the first letter of π, because 1 is the kth letter, must be one of the $(k-1)$ largest letters, so $k \geq \frac{n}{2}+1$.

Let us now consider the permutation τ. In the first case, when 1 is a fixed point, τ is merely a (2-1-3)-avoiding involution of $\{2, 3, \ldots, n\}$. In the second case though, the letters following 1, in positions larger than k, will all be smaller than k, so an arbitrary permutation of $\{2, 3, \ldots, (n-k+1)\}$ will do as τ as long as it avoids (2-1-3). We notice that the first $(n-k+1)$ letters of π are uniquely determined by τ since the letters of τ must all be contained in 2-cycles (i, a_i), where $i \leq (n-k+1)$. Hence $\pi = a_1a_2\cdots a_n$ can be written as

$$\pi = k\tau^{-1}\rho 1\tau,$$

where τ^{-1} is the inverse of τ seen as a bijection from $\{2, 3, \ldots, (n-k+1)\}$ to $\{k, (k+1), \ldots n\}$ and where $\rho = a_{n-k+2}a_{n-k+3}\cdots a_{k-1}$. To make sure that π is (2-1-3)-avoiding we must check that τ^{-1} avoids (2-1-3) whenever τ does, but this is exactly what is said in Lemma 2. Finally ρ, must be a (2-1-3)-avoiding involution of $\{(n-k+2), (n-k+1), \ldots, (k-1)\}$, on which we recursively repeat the arguments above.

The next step of the proof is to derive an expression for the number of (2-1-3)-avoiding involutions from the above description of them. Let, for the sake of simplicity, $|\mathcal{I}_n(2\text{-}1\text{-}3)|$ be denoted by A_n. With 1 in position k, where $k \geq \frac{n}{2}+1$, the number of possible τ's is the $(n-k)$th Catalan number C_{n-k}, according to Proposition 16. Independently of τ there are A_{2k-n-2} ways of choosing ρ, so the number of (2-1-3)-avoiding involutions with $a_k = 1$ is $A_{2k-n-2}C_{n-k}$. Moreover, there are A_{n-1} possible (2-1-3)-avoiding involutions with 1 as a fixed point. Thus

$$A_n = A_{n-1} + \sum_{k=\lfloor\frac{n}{2}\rfloor+1}^{n} A_{2k-n-2}C_{n-k}$$

and $A_n = 0$ if $n \leq 0$. This recursion is satisfied by the central binomial coefficients [11], thus we conclude that $|\mathcal{I}_n(2\text{-}1\text{-}3)| = A_n = \binom{n}{n/2}$. □

We now turn to avoidance of (1–3–2). For this purpose we introduce the trivial bijections on permutations.

3.2.1. *Trivial bijections.* Let $\pi = a_1a_2 \cdots a_n \in \mathcal{S}_n$. We define the *reverse* of π as $R(\pi) := a_n \cdots a_2a_1$, and the *complement* of π by $C(\pi)(i) = n + 1 - \pi(i)$, where $i \in [n]$. These bijections from $\mathcal{S}_n$ to itself and their composition $C \circ R$ are called *trivial.* Let Φ be a trivial bijection and let π be in $\mathcal{S}_n(p)$. Then the permutation $\Phi(\pi)$ avoids the pattern $\Phi(p)$ and consequently the number of permutations avoiding $R(p)$, $C(p)$ or $R \circ P(p)$ is the same as the number of permutations avoiding the pattern p. Note that the reverse of the generalised pattern (a_1-a_2a_3-a_4a_5) is (a_5a_4-a_3a_2-a_1). Also the dashes are "reversed".

Example 19. Let $p = 534621$. It is clear that p avoids (1–3–2). The reverse of π, $R(\pi) = 125435$, the complement of π, $C(\pi) = 243156$ and their composition, $R \circ C(\pi) = 651342$ then avoid $R(p) =$ (2–3–1), $C(p) =$ (3–1–2) and $R \circ C(p) =$ (2–1–3), respectively.

Lemma 3. *The composition $C \circ R$, restricted to $\mathcal{I}_n$, is a bijection from $\mathcal{I}_n$ to itself.*

Proof. Let π be in $\mathcal{I}_n$. Then π consists of cycles of length 1 and 2, that is, $\pi(j) = k$ whenever $\pi(k) = j$. The case when j is a fixed point is denoted by $k = j$. Let (j, k) be a cycle of π, then

$$\begin{aligned} R(\pi)(n+1-j) &= \pi(n+1-(n+1-j)) = \pi(j) = k, \\ C \circ R(\pi)(n+1-j) &= n+1-R(\pi)(n+1-j) = n+1-k. \end{aligned}$$

Likewise $C \circ R(\pi)(n+1-k) = n+1-j$ which shows that $(n+1-j, n+1-k)$ is a 2-cycle of $C \circ R(\pi)$. Hence $C \circ R(\pi)$ is an involution, so $C \circ R(\mathcal{I}_n) = \mathcal{I}_n$. □

Proposition 20. *The number of involutions of $[n]$ that avoid (1–3–2) is the nth central binomial coefficient $\binom{n}{n/2}$.*

Proof. Let π be in $\mathcal{I}_n$(2–1–3). The permutation $C \circ R(\pi)$ is in $\mathcal{I}_n$ by Lemma 3 and it is clear from above that $C \circ R(\pi)$ avoids $C \circ R$(2–1–3) $=$ (1–3–2). Since $C \circ R$ is a bijection from $\mathcal{I}_n$ to $\mathcal{I}_n$ it follows that

$$C \circ R : \mathcal{I}_n(2\text{–}1\text{–}3) \to \mathcal{I}_n(1\text{–}3\text{–}2)$$

is injective, thus $|\mathcal{I}_n(2\text{–}1\text{–}3)| \leq |\mathcal{I}_n(1\text{–}3\text{–}2)|$. In order to show the converse, note that $C \circ R$ is its own inverse and hence $C \circ R(C \circ R(p)) = p$. An application of the same argument to $C \circ R$(2–1–3) $=$ (1–3–2) implies the desired inequality $|\mathcal{I}_n(1\text{–}3\text{–}2)| \leq |\mathcal{I}_n(2\text{–}1\text{–}3)|$. Thus, it follows that $|\mathcal{I}_n(2\text{–}1\text{–}3)| = |\mathcal{I}_n(1\text{–}3\text{–}2)|$. □

3.3. **Avoiding p, when p is an increasing or decreasing sequence.** This section concerns avoidance of to the two remaining 3-patterns, (1-2-3) and (3-2-1). Although we have not found any direct relation between $\mathcal{I}_n(1\text{-}2\text{-}3)$ and $\mathcal{I}_n(3\text{-}2\text{-}1)$, it is possible to give almost analogous proofs for them being counted by $\binom{n}{n/2}$ by using the RSK algorithm for Young Tableaux, as will be seen below. We start however with a combinatorial proof for (3-2-1)-avoidance, based on work by Kitaev and Claesson.

In Kitaev [5], which concerns multiavoidance of 3-patterns without internal dashes, it is shown that the permutations of $[n]$ that simultaneously avoid (123), (132) and (213) are counted by the central binomial coefficients. We will use this result to conclude that the number of (3-2-1)-avoiding involutions of $[n]$ is $\binom{n}{n/2}$. Thus we have to establish a relation between $\mathcal{I}_n(3\text{-}2\text{-}1)$ and $\mathcal{S}_n(123, 132, 213)$.

Lemma 4. *Involutions of* $[n]$ *that avoid* (3-2-1) *are in one-to-one correspondence with permutations of* $[n]$ *that avoid* (123), (132) *and* (213). *Hence*

$$|\mathcal{I}_n(3\text{-}2\text{-}1)| = |\mathcal{S}_n(123, 132, 213)|.$$

Proof. Claesson [1] gives a proof that there is a one-to-one correspondence between $\mathcal{I}_n$ and $\mathcal{S}_n(1\text{-}23, 1\text{-}32)$ by constructing the bijection Φ, which is described in connection to Corollary 15, on page 12. Furthermore, he observes that the dashes in the patterns are immaterial for the proof and accordingly $\mathcal{S}_n(123, 132) = \mathcal{S}_n(1\text{-}23, 1\text{-}32)$. We show that Φ restricted to the (3-2-1)-avoiding involutions gives exactly the permutations that avoid (123), (132) and (213).

To show that $\mathcal{S}_n(123, 132, 213) \subseteq \Phi(\mathcal{I}_n(3\text{-}2\text{-}1))$, let π be an involution of $[n]$ and let $\widehat{\pi}$ be the corresponding permutation in $\mathcal{S}_n(123, 132)$. Assume that $\widehat{\pi}$ contains a (213)-subword. There then exists a segment of $\widehat{\pi}$ of the form

$$a_2a_1a_3, \text{ where } a_1 < a_2 < a_3.$$

Since the cycles in the standard form are of maximum length two and are written in decreasing order with their least elements first, the only possibility for a_3 to follow a_1 is that (a_1, a_3) is a cycle of π. The letter a_2 is either a fixed point or contained in the 2-cycle $(\widetilde{a_2}, a_2)$, where $a_1 < \widetilde{a_2} < a_2$. Thus π contains either the segment

$$a_3 \cdots a_2 \cdots a_1, \text{ where } a_1 < a_2 < a_3$$

or

$$a_3 \cdots \tilde{a}_2 \cdots a_2 \cdots a_1, \text{ where } a_1 < \tilde{a}_2 < a_2 < a_3,$$

where $a_3a_2a_1$ forms a (3-2-1)-subword in both cases.

In order to show that $\Phi(\mathcal{I}_n(3\text{-}2\text{-}1)) \subseteq \mathcal{S}_n(123, 132, 213)$ we assume that there is an occurrence of (3-2-1) in π, that is, π contains a segment

$$a_3 \cdots a_2 \cdots a_1, \text{ where } a_1 < a_2 < a_3.$$

There are essentially three different ways of constructing this out of a_1, a_2 and a_3.

First, we consider the case when π, written in cycle notation, is of the form

$$\cdots(a_1b_1)\cdots(b_2a_2)\cdots(b_3a_3)\cdots,$$

where

$$a_1 < a_2 < a_3 \text{ and } b_3 < b_2 < b_1.$$

Let $a_1 = b_1$ denote the case when a_1 is a fixed point. Consider the cycle $(ij) = (b_3a_3)$. Clearly $i < j$. Let $(k\ell)$ be the cycle to the left of (ij) ($k = \ell$ denotes the case when k is a fixed point). If $\ell < j = a_3$, then ℓij forms a (213)-subword of the corresponding permutation $\Phi(\pi)$. Otherwise let $(ij) = (k\ell)$ and repeat the above reasoning. We realize that this procedure will cause a (213)-subword to be formed as ℓij. Indeed, if we have gone through all cycles between a_2 and b_3, then with a_2 as ℓ it will be true that $i < \ell < j$, because ℓ is smaller than j ($j \geq a_3 > a_2 = \ell$) and since the cycles are written in decreasing order it follows that ℓ is larger than i.

The next possibility is that π is of the form

$$\cdots(a_1b_1)\cdots(a_2b_2)\cdots(a_3b_3)\cdots,$$

where

$$a_1 < a_2 < a_3 \text{ and } b_3 < b_2 < b_1.$$

Let $a_3 = b_3$ denote the special case when a_3 is a fixed point. By setting $(ij) = (a_2b_2)$, letting $(k\ell)$ be the cycle to the left of (ij) and repeating the arguments from the first case we get an occurrence of (213) in the corresponding permutation $\widehat{\pi}$. Indeed, the fact that b_1 is smaller than b_2 and consequently smaller than every j and also clearly larger than i guarantees that ℓij will form a (213)-subword for some ℓ, i and j.

Finally we consider π, when π is of the form

$$\cdots(a_1b_1)\cdots(a_2b_2)\cdots(b_3a_3)\cdots,$$

where

$$a_1 < a_2 < a_3 \text{ and } b_3 < b_2 < b_1.$$

The special case when a_2 is a fixed point is denoted by $a_2 = b_2$. A (213)-subword is obtained by letting $(ij) = (b_3a_3)$ and once again applying the above arguments.

This proves that $\mathcal{S}_n(123, 132, 213) = \mathcal{I}_n(2\text{-}3\text{-}1)$. □

We are now prepared to conclude the following result.

Proposition 21. *The number of involutions of* $[n]$ *that avoid* $\mathcal{I}_n(3-2-1)$ *is the nth central binomial coefficient* $\binom{n}{n/2}$.

Proof. This follows immediately from Lemma 4 and the fact that $|\mathcal{S}_n(123, 132, 213)| = \binom{n}{n/2}$, shown by Kitaev in [5]. □

3.3.1. *Young tableaux and involutions.* Knuth [8] proves that the number of involutions of $[n]$ is the same as the number of Young tableaux that can be formed from $[n]$. In his proof he constructs a Young tableau from an involution by inserting the letters of the involution into an originally empty Young tableau, using an algorithm I. Together with its inverse D, for deleting elements from a tableau, I is called the *RSK algorithm*, after its creators; Robinson, Schenstedt and Knuth.

Given a Young tableau P and an integer x that is not in P, algorithm I creates a new tableau P' that contains x in addition to its original elements. The tableau P' has the same shape as P except for a new entry added to one of the rows. When inserting the element x into P, it is first compared to the elements in the first row of P. If x is larger than all elements in the first row it is placed as the last element in that row and the algorithm terminates, otherwise it is placed in the position of the smallest element larger than x. This element x' is then inserted into the next row in the same way. The procedure is repeated until an element x' is inserted as the last element of a row.

Example 22. We illustrate the insertion algorithm I by an example. Suppose that we want to insert 4 into the Young tableau P, where

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 7 \\ \hline 2 & 9 \\ \cline{1-2} 5 \\ \cline{1-1} 8 \\ \cline{1-1}\end{array} .$$

First, the 4 will be placed in the entry occupied by 6, since 6 is the smallest element larger than 4 in the first row.

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 & 9 \\ \cline{1-2} 5 \\ \cline{1-1} 8 \\ \cline{1-1}\end{array}$$

Element 6 is then moved down to the second row where it displaces 9.

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} 8 \\ \cline{1-1}\end{array}$$

Finally 9 will be placed as the last element in the third row, since the row contains no element larger than 9, and the procedure terminates. The tableau P has now been transformed into P', where

$$P' = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 5 & 9 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array} .$$

Note that P' has the same shape as P except for the new square, containing 9.

With P' and the position of the entry added when inserting x, it is possible to get back to P by running algorithm I backwards. More generally, given a Young tableau Q and indices (s, t) such that $y = Q_{st}$ is the rightmost element in row s and that column t has no entries below y, algorithm D transforms Q into a Young tableau Q' with no element in position (s, t) but otherwise of the same shape as Q. An element x is then deleted from Q. The method starts by removing the element y from row s and inserting it into row $s - 1$ where it displaces the largest element smaller than y. This element y' is in turn moved up to row $s - 2$. This procedure continues until an element is removed from the first row. If we apply algorithm D to the tableau P' and the indices of the entry that makes the difference in shape between P' and P, we end up with the original tableau P and the element x. Likewise, if we start with a Young tableau Q and indices (s, t) and apply algorithm D we get a tableau Q' and an element z. Inserting z into Q' according to I will get us back to Q. In this sense the algorithms I and D are inverses of each other.

Example 23. We want to transform the Young tableau P' from example 22 back to its original form P. The entry that makes the difference between the shape of P' and that of P has the indices $(3, 2)$, so we start by removing the element in this position, that is 9. The element 9 is inserted into the second row in the position of 6, since 6 is the largest element smaller than 9. Finally 6 replaces element 4 in the first row and we get back to P, with 4 as the deleted element.

$$P' = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 5 & 9 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 & 9 \\ \cline{1-2} 5 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 7 \\ \hline 2 & 9 \\ \cline{1-2} 5 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array} = P \; .$$

By considering a permutation written in two line notation, Knuth constructs a mapping from $\mathcal{S}_n$ to the set of ordered pairs of Young tableaux (P, Q) formed from the elements $\{1, 2, \ldots, n\}$, where P and Q have the same shape. This is done by inserting the elements one by one into an initially empty Young tableau, partly by using algorithm I. This mapping is shown to be invertible, so there is a one-to-one correspondence between $\mathcal{S}_n$ and the set of ordered pairs (P, Q), where P and Q are as above.

Next, Knuth shows that if the permutation

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & n \\ a_1 & a_2 & \ldots & a_n \end{pmatrix}$$

corresponds to the ordered pair of tableaux (P, Q), then the inverse permutation

$$\pi^{-\mathbf{1}} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 1 & 2 & \dots & n \end{pmatrix}$$

corresponds to (Q, P). Hence, since the involutions are the permutations that are their own inverses they correspond to pairs of tableaux (P, P), and therefore the number of tableaux that can be formed from $[n]$ equals the number of involutions of length n. For a detailed proof we refer to [8].

A consequence of the tableau-constructing method based on algorithm I is that the number of rows in the resulting Young tableau P corresponds to the length of the longest decreasing sequence of the permutation. Indeed, for the algorithm not to terminate before the kth row, the element inserted into row i, where $i \leq k$, has to be smaller than the largest element of the row. That is, an element x that causes a movement down to the kth row must have been preceded by a smaller element in the involution (now in the first row), that in turn must have been preceded by an even smaller element (in the second row) et cetera, that is the involution must contain a decreasing sequence of length k. On the other hand, if we let $a_{i_k} \dots a_{i_1}$ denote the lexicographically smallest decreasing sequence of length k, it is easy to realize that when a_{i_1} has been inserted into the first row, element a_{i_j} will be in row j for each j. Hence the Young tableau will have k rows exactly when the longest decreasing sequence is of length k. In particular $\mathcal{I}_n(3\text{-}2\text{-}1)$ will be in one-to-one correspondence with the Young tableaux with at most two rows. It is known that the number of Young tableaux with two or less rows is the nth central binomial coefficient. For a proof see for example Lundin [9]. This therefore gives another proof of Proposition 21.

As the length of the longest decreasing sequence of the involution determines the number of rows, the length of the longest increasing sequence equals the number of columns. This can be seen from the construction by arguments similar to those above. The set of (1-2-3)-avoiding involutions will therefore be in one-to-one correspondence with the Young tableaux with two or less columns. Taking the transpose of a Young tableau; $P_{ij} \mapsto P_{ji}$, that is reflecting in the NW-SE diagonal, clearly gives a bijection from the tableaux with k rows to the tableaux with k columns. Thus the number of Young tableaux with at most two columns is indeed the nth central binomial coefficient. This proves the following proposition.

Proposition 24. *The number of involutions avoiding (*1-2-3*) is the nth central binomial coefficient* $\binom{n}{n/2}$.

4. Involutions avoiding generalised 3-patterns

So far our work has concerned avoidance of classical patterns. In this section we extend the study to include all generalised 3-patterns.

We start our investigation by counting the pattern-avoiding involutions of $[n]$, when n is small ($n \leq 10$). The results are presented in Table 2.

p	$\|\mathcal{I}_n(p)\|$	p	$\|\mathcal{I}_n(p)\|$	p	$\|\mathcal{I}_n(p)\|$	p	$\|\mathcal{I}_n(p)\|$
(1-2-3)	$\binom{n}{n/2}$	(1-23)	A_n	(12-3)	A_n	(123)	B_n
(1-3-2)	$\binom{n}{n/2}$	(1-32)	A_n	(13-2)	$\binom{n}{n/2}$	(132)	C_n
(2-1-3)	$\binom{n}{n/2}$	(2-13)	$\binom{n}{n/2}$	(21-3)	A_n	(213)	C_n
(2-3-1)	2^{n-1}	(2-31)	2^{n-1}	(23-1)	2^{n-1}	(231)	D_n
(3-1-2)	2^{n-1}	(3-12)	2^{n-1}	(31-2)	2^{n-1}	(312)	D_n
(3-2-1)	$\binom{n}{n/2}$	(3-21)	E_n	(32-1)	E_n	(321)	B_n.

Table 2. Generalised patterns

Here:

$$\begin{aligned}
A_n &= 1, 2, 3, 6, 11, 23, 46, 100, 213, 481, \ldots \\
B_n &= 1, 2, 3, 7, 15, 38, 97, 271, 778, 2371, \ldots \\
C_n &= 1, 2, 3, 6, 12, 28, 66, 172, 458, 1305, \ldots \\
D_n &= 1, 2, 4, 8, 17, 39, 94, 241, 646, 1821, \ldots \\
E_n &= 1, 2, 3, 6, 11, 23, 47, 103, 225, 513, \ldots
\end{aligned}$$

Further we consult the On-Line Encyclopedia of Integer Sequences [11] for information about the obtained sequences of $|\mathcal{I}_n(p)|$. However, except for the well-known $\binom{n}{n/2}$ and 2^{n-1}, none of them can be found in [11]. Still the enumeration of $A_n, \ldots, E_n$ is of some interest for comparison reasons. For each row in the table there is a hierarchy amongst the patterns. Namely, an occurrence of a one-dash pattern, $(x\text{-}yz)$ or $(xy\text{-}z)$, is a special case of an occurrence of the classical two-dash pattern $(x\text{-}y\text{-}z)$, and an occurrence of the zero-dash pattern (xyz) implies an occurrence of the one-dash patterns. This hierarchy induces a partial ordering of $\mathcal{I}_n(p)$ with respect to inclusion. Accordingly

$$\begin{aligned}
\mathcal{I}_n(x\text{-}y\text{-}z) &\subseteq \mathcal{I}_n(x\text{-}yz) \subseteq \mathcal{I}_n(xyz), \\
\mathcal{I}_n(x\text{-}y\text{-}z) &\subseteq \mathcal{I}_n(xy\text{-}z) \subseteq \mathcal{I}_n(xyz),
\end{aligned}$$

which implies that

$$\begin{aligned}
|\mathcal{I}_n(x\text{-}y\text{-}z)| &\leq |\mathcal{I}_n(x\text{-}yz)| \leq |\mathcal{I}_n(xyz)|, \\
|\mathcal{I}_n(x\text{-}y\text{-}z)| &\leq |\mathcal{I}_n(xy\text{-}z)| \leq |\mathcal{I}_n(xyz)|.
\end{aligned}$$

Taking a look at the fourth row of Table 2 above, a consequence of $|\mathcal{I}_n(2\text{-}3\text{-}1)| = |\mathcal{I}_n(2\text{-}31)| = |\mathcal{I}_n(23\text{-}1)|$ is seen to be that $\mathcal{I}_n(2\text{-}3\text{-}1) =$

$\mathcal{I}_n(2\text{-}31) = \mathcal{I}_n(23\text{-}1)$, that is an involution avoids (2-3-1) if and only if it avoids (2-31), which is in turn avoided if and only if (23-1) is avoided. However, for $n \geq 5$, the sequence D_n indicates the existence of involutions that avoid (231) even though they may contain (2-3-1)-subwords. This is in fact the case for $\pi = 52431$.

Proposition 25. *The number of involutions that avoid p, when p is equal to* (2-13) *or* (13-2)*, is* $\binom{n}{n/2}$. *Hence*

$$|\mathcal{I}_n(2\text{-}13)| = |\mathcal{I}_n(13\text{-}2)| = \binom{n}{n/2}.$$

For the proof we use a consequence of the proof of Proposition 20.

Porism 26. *[of Proposition 20] For a generalised pattern p we have that* $|\mathcal{I}_n(p)| = |\mathcal{I}_n(C \circ R(p))|$.

Proof. Without loss of generality, the pattern $p = (2\text{-}1\text{-}3)$ in the proof of Proposition 20 could be replaced by any generalised pattern. □

Proof of Proposition 25. In Claesson [1] it is shown that a permutation π avoids (2-13) if and only if it avoids (2-1-3). In particular this is true when π is an involution. Thus, recalling from Proposition 6 that the (2-1-3)-avoiding involutions are counted by $\binom{n}{n/2}$, we obtain the desired result in the first case. An application of Porism 26 to $p = (13\text{-}2) = C \circ R(2\text{-}13)$ then proves the remaining part. □

Proposition 27. *An involution avoids p, where p is one of the patterns* (2-31)*,* (31-2)*,* (23-1) *or* (3-12)*, if and only if it avoids* (2-3-1). *Hence*

$$|\mathcal{I}_n(2\text{-}31)| = |\mathcal{I}_n(31\text{-}2)| = |\mathcal{I}_n(23\text{-}1)| = |\mathcal{I}_n(3\text{-}12)| = 2^{n-1}.$$

Proof. Claesson[1] partitions the twelve one dash patterns into three equidistributed classes, with respect to the patterns considered as permutation statistics. This is done on the basis of their behaviour under actions of the trivial bijections.

As mentioned in the proof of Proposition 25, Claesson [1] shows that a permutation avoids (2-13) if and only if it avoids (2-1-3). Due to the properties of the trivial bijections, the corresponding results are true for all patterns in the (2-13) class, that is (2-31), (13-2) and (31-2). In particular, an involution avoids (2-31) if and only if it avoids (2-3-1) and avoidance of (31-2) is equivalent to avoidance of (3-1-2), which in turn, by Lemma 1, is equal to (2-3-1)-avoidance.

Concerning the two remaining patterns we give a proof by describing the pattern-avoiding involutions. Let $\pi = a_1 a_2 \cdots a_n$ be a (23-1)-avoiding involution with k as the first letter. The initial segment $\sigma = a_1 a_2 \cdots a_k$ of π is easily seen to be determined by k. Indeed, the letter 1 must clearly be in position k and since no ascents are allowed to precede 1, the only possibility is to let σ consist of the k smallest letters in decreasing order. In the same way, the $(k+1)$st letter fixes the subsequent segment, and so forth. This procedure results in an involution of the form that was used to describe $\mathcal{I}_n(2\text{-}3\text{-}1)$ in the proof of Proposition 6. Hence $\mathcal{I}_n(23\text{-}1) \subseteq \mathcal{I}_n(2\text{-}3\text{-}1)$ and since the converse inclusion obviously holds we conclude that $\mathcal{I}_n(23\text{-}1) = \mathcal{I}_n(2\text{-}3\text{-}1)$.

By similar arguments the description of $\mathcal{I}_n(2\text{-}3\text{-}1)$ is easily seen to fit also a (3-21)-avoiding involution, so $\mathcal{I}_n(23\text{-}1) = \mathcal{I}_n(2\text{-}3\text{-}1)$. The details are left to the reader.

We recall from Proposition 6 that the (2-3-1)-avoiding involutions are counted by 2^{n-1}. Thus the second part of the proposition follows accordingly. □

5. Multiavoidance of 3-patterns among involutions

We devote this final section to the case of multiavoidance, that is when two or more patterns are simultaneously avoided. This was first systematically studied for classical 3-patterns by Simion and Schmidt [10] but has recently been extended to generalised patterns, for instance by Claesson [1], Kitaev [5], [6] and Claesson and Mansour [2].

Consider $\mathcal{I}_n(p_1, \ldots p_k)$, where p_i are 3-patterns. Allowing the patterns p_i to be generalised and the number of them, k, to vary, provides us with a huge amount of different restrictions to investigate, even though many of them are not of much interest. Here we limit ourselves to the case of two classical 3-patterns, denoted p and q.

As in the study of generalised patterns we start by counting the involutions of $[n]$ that avoid the pair of patterns p and q, when n is small. The result is presented in Table 3, where a certain cell represents the number of involutions that avoid simultaneously the row and the column pattern.

$p\backslash q$	(1-2-3)	(1-3-2)	(2-1-3)	(2-3-1)	(3-1-2)	(3-2-1)
(1-2-3)		A_n	A_n	n	n	B_n
(1-3-2)			A_n	n	n	C_n
(2-1-3)				n	n	C_n
(2-3-1)					2^{n-1}	D_{n+1}
(3-1-2)						D_{n+1}
(3-2-1)						

Table 3. Double avoidance of classical patterns

Here:

$$\begin{aligned} A_n &= 1,2,2,4,4,8,8,16,16,\ldots \\ B_n &= 1,2,2,2,0,0,0,0,0,\ldots \\ C_n &= 1,2,2,3,3,4,4,\ldots \\ D_n &= 1,1,2,3,5,8,13,21,\ldots \end{aligned}$$

Note the simplicity of the sequences above, compared to those treated earlier in this work. Also note that the sequence D_n is the well known *Fibonacci numbers.*

First we consider the "simplest" sequence $B_n = 1,2,2,2,0,0,0,\ldots$, which counts the involutions that avoid (1-2-3) and (3-2-1). By studying the involutions of length at most 4 it is easy to verify the first 4 B_n's. To realize that an involution of length larger than 4 must have a decreasing or an increasing subsequence of length 3, we recall from the theory behind the proofs of Proposition 21 and 24, that the RSK algorithm gives a bijection between the Young tableaux with n elements and the set of involutions of $[n]$, where the number of rows and columns of the Young tableau equal the length of the longest increasing and decreasing subsequence, respectively. It is easy to see that a Young tableau with $r \cdot c + 1$ elements must have a row containing $r+1$ elements or a column with $c+1$ elements and we conclude that all Young tableaux with $5 = 2 \cdot 2 + 1$ or more elements must have a row or a column with at least 3 elements.

An apparently different approach is to use one of the famous results in combinatorics, proved by Erdős and Szekeres in 1935.

Theorem. *(Erdős-Szekerez) Let $A = (a_1, \ldots, a_n)$ be a sequence of n different real numbers. If $n \geq sr + 1$ then either A has an increasing subsequence of $s+1$ terms or a decreasing subsequence of $r+1$ terms (or both).*

For a proof, see for example [4]. From the theorem it follows immediately that an involution of $[n]$, where $n \geq 5$, must have a decreasing or increasing sequence of length 3. However, to conclude this we use the same argument as above, namely that $5 = 2 \cdot 2 + 1$. In fact, what we implicitly do above is to prove the Erdős-Szekeres Theorem in the case of integer a_i, via the RSK-algorithm.

Let us continue with the case when one of the avoided patterns is (2-3-1) or (3-1-2). As pointed out in Lemma 1, we have that $\mathcal{I}_n(2\text{-}3\text{-}1) = \mathcal{I}_n(3\text{-}1\text{-}2)$, hence $\mathcal{I}_n(p, 2\text{-}3\text{-}1) = \mathcal{I}_n(p, 3\text{-}1\text{-}2)$, and consequently it suffices to consider either of those sets. Also we conclude the obvious result that $|\mathcal{I}_n(2\text{-}3\text{-}1, 3\text{-}1\text{-}2)| = |\mathcal{I}_n(2\text{-}3\text{-}1)| = |\mathcal{I}_n(3\text{-}1\text{-}2)| = 2^{n-1}$.

Proposition 28. *We have that*

$$\begin{aligned} |\mathcal{I}_n(1\text{-}2\text{-}3, 2\text{-}3\text{-}1)| &= |\mathcal{I}_n(1\text{-}2\text{-}3, 3\text{-}1\text{-}2)| = \\ |\mathcal{I}_n(1\text{-}3\text{-}2, 2\text{-}3\text{-}1)| &= |\mathcal{I}_n(1\text{-}3\text{-}2, 3\text{-}1\text{-}2)| = \\ |\mathcal{I}_n(2\text{-}1\text{-}3, 2\text{-}3\text{-}1)| &= |\mathcal{I}_n(2\text{-}1\text{-}3, 3\text{-}1\text{-}2)| = n. \end{aligned}$$

Proof. We recall the description of $\mathcal{I}_n(2\text{-}3\text{-}1) = \mathcal{I}_n(1\text{-}3\text{-}2)$ from the proof of Proposition 6;

$$\mathcal{I}_n(2\text{-}3\text{-}1) = \{k_1 \cdots 1 k_2 \cdots (k_1+1) k_3 \cdots (k_{\ell-1}+1) n \cdots (k_\ell + 1)\}.$$

That is the involutions can be considered as consisting of segments, such that

(a) all letters in segment i are smaller than all letters in segment $(i+1)$,
(b) the elements in a segment are in decreasing order.

Let $\pi = a_1 a_2 \cdots a_n$ be such an involution. We want to investigate what happens when we add the restriction to avoid p, where p is one of the patterns in $\{(1\text{-}2\text{-}3), (1\text{-}3\text{-}2), (2\text{-}1\text{-}3)\}$.

The avoidance of (1-2-3) limits the number of segments of π to two. Indeed, because of property (a) above, if π has more than two segments, a (1-2-3)-subword will be formed as $a_{i_1} a_{i_2} a_{i_3}$, where a_{i_1}, a_{i_2} and a_{i_3} can be arbitrarily chosen from the first, second and third segment respectively. Thus it follows that π in $\mathcal{I}_n(1\text{-}2\text{-}3, 2\text{-}3\text{-}1)$ is of the form

$$\pi = k \cdots 1 n \cdots (k+1),$$

that is, the involution π is uniquely determined by the choice of k, hence

$$|\mathcal{I}_n(1\text{-}2\text{-}3, 2\text{-}3\text{-}1)| = |\mathcal{I}_n(1\text{-}2\text{-}3, 3\text{-}1\text{-}2)| = n.$$

When the patterns (1-3-2) and (2-3-1) are to be simultaneously avoided, π can not have any "peaks". No letter a_i can be both preceded and succeeded by smaller letters. Consequently, if the letter 1 is in position k, then π must consist of the k smallest letters in decreasing order, followed by the letters that are larger than k in increasing order. To use the above notation, all segments except for the first one contain only one letter. Accordingly $\mathcal{I}_n(1\text{-}3\text{-}2, 2\text{-}3\text{-}1)$ consists of all permutations π of the form $\pi = k(k-1)\cdots 1(k+1)\cdots n$. Again the choice of k fixes the remaining involution, so it follows that

$$|\mathcal{I}_n(1\text{-}3\text{-}2, 2\text{-}3\text{-}1)| = |\mathcal{I}_n(1\text{-}3\text{-}2, 3\text{-}1\text{-}2)| = n.$$

Likewise, avoiding the patterns (2-1-3) and (3-1-2) implies that there can not be any "valleys", so, if the letter n is in position $(k+1)$, it must be preceded by the k smallest letters in increasing order and followed by the larger letters in decreasing order. This time each segment but the first one consists of a single letter, thus an involution π in $\mathcal{I}_n(2\text{-}1\text{-}3, 3\text{-}1\text{-}2)$

can be written $\pi = 12\cdots kn(n-1)\cdots(k+1)$, from which we conclude that

$$|\mathcal{I}_n(2\text{-}1\text{-}3, 2\text{-}3\text{-}1)| = |\mathcal{I}_n(2\text{-}1\text{-}3, 3\text{-}1\text{-}2)| = n,$$

since each involution is fully determined by k. □

Proposition 29. *We have that*

$$|\mathcal{I}_n(3\text{-}2\text{-}1, 2\text{-}3\text{-}1)| = |\mathcal{I}_n(3\text{-}2\text{-}1, 3\text{-}1\text{-}2)| = F_{n+1},$$

where F_n denotes the nth Fibonacci number.

Note that, as in the proof of Proposition 28, it suffices to study either $\mathcal{I}_n(3\text{-}2\text{-}1, 2\text{-}3\text{-}1)$ or $\mathcal{I}_n(3\text{-}2\text{-}1, 3\text{-}1\text{-}2)$ since the two sets are indeed the same.

Consider π in $\mathcal{I}_n(3\text{-}2\text{-}1, 2\text{-}3\text{-}1)$. Being a (2-3-1)-avoiding involution, π can be described as consisting of segments, within which the letters are decreasingly ordered, according to the above characterization of $\mathcal{I}_n(2\text{-}3\text{-}1)$. Furthermore, the avoidance of (3-2-1) implies that the decreasing sequences must be of length at most two, so π consists of fixed points and 2-cycles of consecutive letters. Hence

$$\pi = \cdots(k_i)\cdots(k_j, k_j+1)\cdots$$

gives a description of π in cycle form.

We prove that the involutions of the above form are counted by the Fibonacci numbers, first by combining two of the proofs of Proposition 6 with well known properties of the Fibonacci numbers and then by recursively constructing $\mathcal{I}_n(3\text{-}2\text{-}1, 2\text{-}3\text{-}1)$ from $\mathcal{I}_{n-1}(3\text{-}2\text{-}1, 2\text{-}3\text{-}1)$ and $\mathcal{I}_{n-2}(3\text{-}2\text{-}1, 2\text{-}3\text{-}1)$.

First proof. We begin with a proof that refers to the first proof of Proposition 6, in which a bijection Φ_n from the binary strings of length $(n-1)$, to $\mathcal{I}_n(2\text{-}3\text{-}1)$ is constructed. Given a binary string $x = x_1\cdots x_{n-1}$ in B_{n-1}, the corresponding involution is recursively built up from $[n]$ by considering the letters x_i, one at at time. We recall that $x_i = 1$ causes an inversion to be formed as the letter i is placed before $(i-1)$, whereas $x_i = 0$ implies that i is placed as the last element so far. From the construction it is easily seen that π contains a decreasing subsequence of length larger than 3 whenever x has two consecutive 1's and conversely that an x with no two consecutive 1's maps to an involution of the form

$$\cdots(k_i)\cdots(k_j, k_j+1)\cdots.$$

Hence there is a one-to-one correspondence between $\mathcal{I}_n(3\text{-}2\text{-}1, 2\text{-}3\text{-}1)$ and the binary strings of length $n-1$ with no consecutive 1's, which are known to be counted by F_{n+1}. For a reference, see for example [11]. Thus it follows that

$$|\mathcal{I}_n(3\text{-}2\text{-}1, 2\text{-}3\text{-}1)| = |\mathcal{I}_n(3\text{-}2\text{-}1, 3\text{-}1\text{-}2)| = F_{n+1}.$$

□

Second proof. Next we relate to the third proof of Proposition 6, in which a bijection between $\mathcal{P}_{n-1}$, the subsets of $[n-1]$, and $\mathcal{I}_n(2\text{-}3\text{-}1)$ is defined. Let A be in $\mathcal{P}_{n-1}$. The corresponding involution is constructed from A by letting i be preceded by a larger letter if and only if i belongs to A. From the appearance of $\mathcal{I}_n(2\text{-}3\text{-}1)$ we see that there is only one choice of the larger letter preceding i, namely $i+1$. Thus, π has an occurrence of (3-2-1) if and only if A contains two or more consecutive integers. It is well known that the number of subsets of $[n]$ with no consecutive integers is the nth Fibonacci number, see for instance [11]. Thus the result follows. □

Third proof. Finally we give a proof by induction. Recall that the Fibonacci numbers are defined by

$$F_n = F_{n-1} + F_{n-2}, \text{ where } F_0 = 0, F_1 = 1.$$

We will now show that the number of (3-2-1, 2-3-1)-avoiding involutions of $[n]$ satisfies the same recursion. Let π be such an involution. From the above description of $\mathcal{I}_n(3\text{-}2\text{-}1, 2\text{-}3\text{-}1)$ as consisting only of fixed points and 2-cycles of consecutive letters we see that the letter n will be either a fixed point or contained in the cycle $(n-1, n)$. This gives us two ways of recursively constructing $\mathcal{I}_n(p,q)$ from $\mathcal{I}_{n-1}(p,q)$ and $\mathcal{I}_{n-2}(p,q)$ (for convenience we let the patterns (3-2-1) and (2-3-1) be denoted by p and q). Either n is added to a (p,q)-avoiding involution of $[n-1]$ or the cycle $((n-1)n)$ is added to a (p,q)-avoiding involution of $[n-2]$. Hence

$$\begin{aligned}\mathcal{I}_n(p,q) &= \{b_1 \cdots b_{n-1} n,\ b_1 \cdots b_{n-1} \in \mathcal{I}_{n-1}(p,q)\} \cup \\ &\quad \{c_1 \cdots c_{n-2} n(n-1),\ c_1 \cdots c_{n-2} \in \mathcal{I}_{n-2}(p,q)\},\end{aligned}$$

so

$$|\mathcal{I}_n(p,q)| = |\mathcal{I}_{n-1}(p,q)| + |\mathcal{I}_{n-2}(p,q)|.$$

Since $\mathcal{I}_0(p,q) = 0$ and $\mathcal{I}_1(p,q) = 1$, we conclude that

$$|\mathcal{I}_n(3\text{-}2\text{-}1, 2\text{-}3\text{-}1)| = |\mathcal{I}_n(3\text{-}2\text{-}1, 3\text{-}1\text{-}2)| = F_{n+1}.$$

□

Proposition 30. *We have that*

$$|\mathcal{I}_n(1\text{-}3\text{-}2, 3\text{-}2\text{-}1)| = |\mathcal{I}_n(2\text{-}1\text{-}3, 3\text{-}2\text{-}1)| = \lfloor n/2 \rfloor + 1.$$

Proof. Consider $\mathcal{I}_n(2\text{-}1\text{-}3, 3\text{-}2\text{-}1)$. We recall from the proof of Proposition 18 that a permutation π, with 1 in position k, avoids (2-1-3) if and only if it can be written as $\sigma 1 \tau$, where $\sigma = a_1 a_2 \cdots a_{k-1}$ is a (2-1-3)-avoiding permutation of $\{n, (n-1), \ldots, (n-k+2)\}$ and $\tau = a_{k+1} a_{k+2} \cdots a_n$ is a (2-1-3)-avoiding permutation of $\{2, 3, \ldots, (n-k+1)\}$. Furthermore we recall that, when π is an involution, the letter 1 can be either a fixed point or in position k, where $k \geq n/2$. Let us investigate the latter case. Clearly, the letter k is in position 1. In order to avoid (3-2-1), the remaining σ must consist of letters larger than k in increasing order. We realize that this leads to absurdity whenever $k > n/2$ (since there are not enough larger letters). Thus, k must simultaneously be larger than or equal to $n/2$ and less then or equal to $n/2$, which is possible for integer k only when n is even. Then $k = n/2$, which determines π to be equal to $n/2 \cdots n1 \cdots (n/2-1)$. When 1 is a fixed point we can recursively apply the above reasoning to τ, so that an involution in $\mathcal{I}_n(2\text{-}1\text{-}3, 3\text{-}2\text{-}1)$ can be written as $\pi = 12\cdots(n-2k-1)(n-2k)\rho$. Here ρ is the (2-1-3, 3-2-1)-avoiding involution of $\{(n-2k), (n-2k+1), \ldots, n\}$ in which the smallest letter is in the middle position. Thus, these involutions are fully characterized by the choice of k, where k has to be less than or equal to $n/2$, hence

$$|\mathcal{I}_n(2\text{-}1\text{-}3, 3\text{-}2\text{-}1)| = \lfloor n/2 \rfloor + 1.$$

For the (1-3-2)- and (3-2-1)-avoiding involutions the proposition can be proved in a similar way, for which we omit the details. Let $\pi = a_1 a_2 \cdots a_n$ be such an involution. The letter n can either be a fixed point or, if n is even, in position $n/2$, in which case it determines the rest of π. By recursively repeating the arguments to the segment $a_1 a_2 \cdots a_{n-1}$ when n is a fixed point, we see that an (1-3-2, 3-2-1)-avoiding involution π can be written as $\rho(2k)(2k+1)\cdots n$, where ρ is the (1-3-2, 3-2-1)-avoiding involution of $[2k-1]$, in which the largest letter is in the middle position. Again the involutions are uniquely determined by the choice of k, hence the result follows. □

Proposition 31. *We have that*

$$\begin{aligned} |\mathcal{I}_n(1\text{-}2\text{-}3, 1\text{-}3\text{-}2)| &= |\mathcal{I}_n(1\text{-}2\text{-}3, 2\text{-}1\text{-}3)| = \\ |\mathcal{I}_n(1\text{-}3\text{-}2, 2\text{-}1\text{-}3)| &= 2^{\lfloor n/2 \rfloor}. \end{aligned}$$

Proof. We start with the case of (1-2-3)- and (2-1-3)-avoiding involutions of $[n]$. Note that the largest letter, n, has to be in position 1 or 2, because otherwise n will be preceded by two smaller letters that are either ordered as (1-2) or (2-1), causing occurrences of (1-2-3) and (2-1-3) respectively. On the other hand if n is the first (or second) letter, there can not be any (1-2-3)- or (2-1-3)-subwords containing 1 (or 2) or n, since n can not act as a 1 or a 2, as well as 1 (or 2) in position n will not do as a 3. Therefore, letting (1-2-3) and (2-1-3) be denoted by p and q, we can recursively construct $\mathcal{I}_n(p,q)$ from $\mathcal{I}_{n-2}(p,q)$, according to

$$\begin{aligned} \mathcal{I}_n(p,q) &= \{na_2 \cdots a_{n-1}1, \ \ \mathrm{proj}(a_2 \cdots a_{n-1}) \in \mathcal{I}_{n-2}(p,q)\} \cup \\ &\quad \{a_1 n a_3 \cdots a_{n-1}2, \ \ \mathrm{proj}(a_1 a_3 \cdots a_{n-1}) \in \mathcal{I}_{n-2}(p,q)\}. \end{aligned}$$

Hence we get the recursion formula

$$|\mathcal{I}_n(p,q)| = 2 \cdot |\mathcal{I}_{n-2}(p,q)|, \ \text{ where } \mathcal{I}_1(p,q) = 1 \text{ and } \mathcal{I}_2(p,q) = 2,$$

from which it follows that

$$|\mathcal{I}_n(1\text{-}2\text{-}3, 1\text{-}3\text{-}2)| = 2^{\lfloor n/2 \rfloor}.$$

Next we consider (1-2-3)- and (2-1-3)-avoidance. This is similar to the above case, but now with the letter 1 playing the role of n. For an involution π to be in $\mathcal{I}_n(1\text{-}2\text{-}3, 2\text{-}1\text{-}3)$, the 1 can be placed either as the last or the penultimate letter of π. As above, none of the two corresponding cycles $(1, n)$ or $(1, n-1)$ can possibly contribute to the formation of (1-2-3)- or (2-1-3)-subwords. So, letting p and q denote the patterns (1-2-3) and (2-1-3) respectively, we see that $\mathcal{I}_n(p, q)$ can be recursively constructed from $\mathcal{I}_{n-2}(p, q)$ as

$$\begin{aligned}\mathcal{I}_n(p,q) \quad = \quad & \{na_2 \cdots a_{n-1}1, \ \mathrm{proj}(a_2 \cdots a_{n-1}) \in \mathcal{I}_{n-2}(p,q)\} \cup \\ & \{(n-1)a_2 \cdots a_{n-2}1a_n, \ \mathrm{proj}(a_2 \cdots a_{n-2}a_n) \in \mathcal{I}_{n-2}(p,q)\}.\end{aligned}$$

This will once again result in the recursion

$$|\mathcal{I}_n(p,q)| = 2 \cdot |\mathcal{I}_{n-2}(p,q)|,$$

with initial conditions $\mathcal{I}_1(p, q) = 1$ and $\mathcal{I}_2(p, q) = 2$. Thus the result follows.

Finally we turn to the (1-3-2)- and (2-1-3)-avoiding involutions of $[n]$. Let π be such an involution. From the proof of Proposition 18 we recall that, in order to avoid (2-1-3), the letter 1 must be in position $k \geq n/2+1$, or it is a fixed point. However, the simultaneous avoidance of (1-3-2) precludes the latter alternative in all cases except the identity permutation $\pi = 12 \cdots n$. Assume therefore that the letter 1 is in position k. According to the proof of Proposition 18, π can be written as $\sigma 1 \tau$ where τ is a (2-1-3)-avoiding permutation of $\{2, \ldots, (n-k+1)\}$. We realize that the only choice of τ that makes π (1-3-2)-avoiding is in fact $\tau = 23 \cdots (n-k+1)$, which corresponds to the initial segment $a_2 \cdots a_k$ of π. We can then write $\pi = k(k+1) \cdots n\rho 12 \cdots (n-k+1)$, where ρ is a (1-3-2)-avoiding involution of $\{n-k+2, n-k+1, \ldots k-1\}$, that is $\mathrm{proj}(\rho) \in \mathcal{I}_{n-2k}(1\text{-}3\text{-}2, 2\text{-}1\text{-}3)$. Accordingly, with p and q denoting (1-3-2) and (2-1-3) respectively, we can construct $\mathcal{I}_n(p, q)$ from $\{\mathcal{I}_{n-2k}(p, q)\}$, where

$k \leq n/2$. We have that

$$\begin{aligned}\mathcal{I}_n(p,q) &= \{na_2\cdots a_{n-1}1,\ \ \mathrm{proj}(a_2\cdots a_{n-1}) \in \mathcal{I}_{n-2}(p,q)\} \cup \\ &\quad \{(n-1)na_3\cdots a_{n-2}12,\ \ \mathrm{proj}(a_3\cdots a_{n-2}) \in \mathcal{I}_{n-4}(p,q)\} \cup \\ &\quad \vdots \\ &\quad \{k(k+1)\cdots na_{n-k+2}\cdots a_{k-1}12\cdots(n-k+1), \\ &\quad\ \ \mathrm{proj}(a_{n-k+2}\cdots a_{k-1}) \in \mathcal{I}_{n-2(n-k+1)}(p,q)\} \cup \\ &\quad \vdots \\ &\quad 12\cdots n.\end{aligned}$$

Thus $|\mathcal{I}_n(p,q)|$ satisfies the recursion

$$|\mathcal{I}_n(p,q)| = \sum_{k=1}^{\lfloor n/2 \rfloor} |\mathcal{I}_{n-2k}(p,q)|$$

and, since $|\mathcal{I}_1(p,q)| = 1$ and $|\mathcal{I}_2(p,q)| = 2$, we conclude that

$$|\mathcal{I}_n(1\text{-}3\text{-}2, 2\text{-}1\text{-}3)| = 2^{\lfloor n/2 \rfloor}.$$

□

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