

Counting 1324-avoiding Permutations

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Abstract

We consider permutations that avoid the pattern 1324. By studying the generating tree for such permutations, we obtain a recurrence formula for their number. A computer program provides data for the number of 1324-avoiding permutations of length up to 20.

1 Introduction

Let S_n denote the set of all permutations of length n . A permutation $\pi = (p_1, p_2, \dots, p_n) \in S_n$ contains a pattern $\tau = (t_1, t_2, \dots, t_k) \in S_k$ if there is a sequence $1 \leq i_{t_1} < i_{t_2} < \dots < i_{t_k} \leq n$ such that $p_{i_{t_1}} < p_{i_{t_2}} < \dots < p_{i_{t_k}}$. A permutation π avoids a pattern τ , in other words π is τ -avoiding, if π does not contain τ . We write $S_n(\tau)$ for the set of all τ -avoiding permutations of length n , and $s_n(\tau)$ for the cardinality of $S_n(\tau)$. Patterns τ_1 and τ_2 are *Wilf-equivalent* if $s_n(\tau_1) = s_n(\tau_2)$ [Wil02]. A permutation π is $\{\tau_1, \tau_2, \dots, \tau_n\}$ -avoiding if π does not contain any of the patterns from the set.

It is a natural and easy-looking question to ask for the exact formula for $s_n(\tau)$. However, this problem turns out to be very difficult. Although a lot of results on this and related problems have been discovered in the last thirty years, exact answers are only known in a few cases. For all patterns τ of length 3, $s_n(\tau) = C_n$ [Knu73], where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n -th Catalan number, a classical sequence [Sta99]. When τ is of length 4, it has been shown that the only essentially different patterns are 1234, 1342 and 1324; all other patterns of length 4 are Wilf-equivalent to one of these three [Sta94, Sta96, BW00]. Regev [Reg81] showed that $s_n(1234)$ asymptotically equals $c \frac{9^n}{n^4}$, where c is a constant given by a multiple integral. Gessel [Ges90] later used theory of symmetric functions to give a generating function for 1234-avoiding permutations. Bóna [Bón97a] enumerated

1342-avoiding permutations, giving their ordinary generating function:

$$\sum_n s_n(1342)x^n = \frac{32x}{-8x^2 + 20x + 1 - (1 - 8x)^{3/2}}.$$

However, the exact enumeration of 1324-avoiding permutations is still an outstanding open problem that we address in this paper.

The problem of avoiding more than one pattern was first studied by Simion and Schmidt [SS85], who determined the number of permutations avoiding two or three patterns of length 3. The numbers of permutations avoiding certain pairs of patterns of length 4 give the Schröder numbers [Wes95]. West [Wes96] also used *generating trees* [CGHK78] to enumerate permutations avoiding all pairs of a pattern of length 3 and a pattern of length 4. Recently, Albert et al. [AAA⁺03] enumerated {1324, 31524}-avoiding permutations, while finding connections with queue jumping.

We provide a full characterization for the generating tree of 1324-avoiding permutations. This result, combined with a simple computer program, provides data for $s_n(1324)$ for n up to 20. In particular, we show the following:

Theorem 1. *The number $s_n(1324)$ of 1324-avoiding permutations of length n is $g(\langle 1 \rangle, n)$, where g is determined by the following recursive formula:*

$$g(\langle a_1 \dots a_m \rangle, n) = \begin{cases} \sum_{i=1}^m a_i & \text{if } n = 1, \\ \sum_{i=1}^m g(f(\langle a_1 \dots a_m \rangle, i), n - 1) & \text{if } n > 1 \end{cases} \quad (1)$$

and $f(\langle a_1 \dots a_m \rangle, i) = \langle b_1 \dots b_{a_i} \rangle$, where:

$$b_j = \begin{cases} a_i + 1 & \text{if } j = 1, \\ \min(i + 1, a_j) & \text{if } 2 \leq j \leq i, \\ a_{j-1} + 1 & \text{if } i < j \leq a_i. \end{cases} \quad (2)$$

We conclude by enumerating 1324-avoiding permutations in a specific *strong* class, which is conjectured to be the largest.

2 Proof of Theorem 1

We apply generating trees to count 1324-avoiding permutations. First, we briefly describe succession rules and generating trees. They were introduced in [CGHK78] for the study of Baxter permutations and further applied to the study of pattern-avoiding permutations by Stankova and West [Sta94, Sta96, Wes95, Wes96]. Recently, Barcucci et al. developed ECO [BDLPP99], a methodology for the enumeration of combinatorial objects, which is based on the technique of generating trees.

Definition 2. A generating tree is a rooted, labelled tree such that the labels of the set of children of each node v can be determined from the label of v itself. In other words, a generating tree can be specified by a recursive definition consisting of:

1. **basis:** the label of the root
2. **inductive step:** a set of succession rules that yields a multiset of labelled children depending solely on the label of the parent.

Given $\pi = (p_1, p_2, \dots, p_n) \in S_n$, we call the position to the left of p_1 position 0, the position between p_i and p_{i+1} , where $1 \leq i \leq n - 1$, position i , and the position to the right of p_n position n . We will refer to any of these positions as a *site* of π .

Definition 3. Let τ be a forbidden pattern. The position i , $0 \leq i \leq n$, of a permutation $\pi \in S_n(\tau)$ is an *active site* if inserting $n + 1$ into position i gives a permutation belonging to the set $S_{n+1}(\tau)$; otherwise it is said to be an *inactive site*.

Following the methodology developed in [Wes96, Wes95], the generating tree for τ -avoiding permutations is a rooted tree whose nodes on level n are exactly the elements of $S_n(\tau)$. The children of a permutation π of length $n - 1$ are all the τ -avoiding permutations obtained by inserting n into π . Each node in the tree is assigned a label; in the simplest case, the label is the number of active sites of π . Typical applications of generating trees analyze changes in the number of active sites after inserting n in a permutation of length $n - 1$. These changes determine the labels in the tree and the list of succession rules. Our application considers one more step: to keep the label of every node completely determined from the label of its parent, we consider the changes after inserting n and also $n + 1$.

Given a node π at level $n - 1$ in the generating tree for 1324-avoiding permutations, let π_n^i be π 's children obtained by inserting n into the i -th active site of π . The label assigned to π_n^i is the pair $(s(\pi), i)$, where the sequence $s(\pi) = \langle l(\pi_n^1) \dots l(\pi_n^{l(\pi)}) \rangle$ contains the number of active sites $l(\pi_n^j)$ for all children π_n^j of π , i.e., for π_n^i and all its siblings. The following completely characterizes this generating tree.

Lemma 4. All 1324-avoiding permutations of length n lie on the n -th level of the generating tree (Figure 1) defined by the following succession rules:

$$\begin{cases} \text{basis:} & ((2), 1) \\ \text{inductive step:} & (\langle a_1 \dots a_m \rangle, i) \rightarrow (\langle b_1 \dots b_{a_i} \rangle, a_i) (\langle b_1 \dots b_{a_i} \rangle, a_i - 1) \dots (\langle b_1 \dots b_{a_i} \rangle, 1) \end{cases}$$

where $\langle b_1 \dots b_{a_i} \rangle = f(\langle a_1 \dots a_m \rangle, i)$ as in (2).

Proof. First, we make the following observation. Given a 1324-avoiding permutation $\pi = (p_1, p_2, \dots, p_{n-1})$ of length $n - 1$, the active sites of π are actually the first $l(\pi)$ sites; we can order 132 patterns in π by the occurrence of their 2 and n can be inserted anywhere to the left of the first 2, but nowhere to the right of it.

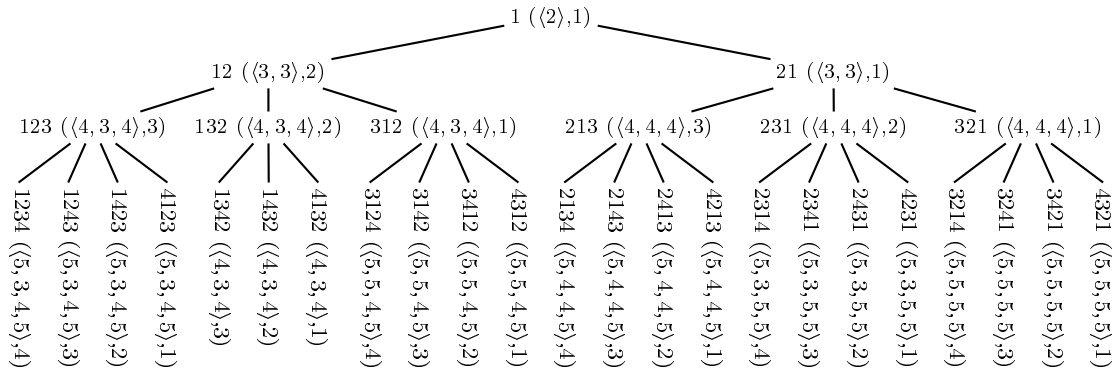


Figure 1: The generating tree for 1324-avoiding permutations

Inserting n into the i -th active site of π certainly creates one new active site in π_n^i , since $n + 1$ can be inserted into π_n^i right in front and right behind n . However, inserting n into π may deactivate some active sites in π , because n can play a role of 3 for some 132 pattern in π_n^i that was not in π . In other words, if we order 132 patterns in π and π_n^i by the occurrence of their 2, the first 2 in π_n^i may be to the left of the first 2 in π . The index of the first 2 that n introduces in π_n^i is $\min_{k > i-1, p_k > \min(p_1, p_2, \dots, p_{i-1})} k$. Since the active sites of π_n^i are exactly the sites to the left of the first 2, the number of active sites in π_n^i is:

$$l(\pi_n^i) = 1 + \min\{l(\pi), \min_{k > i-1, p_k > \min(p_1 \dots p_{i-1})} k\} \quad (3)$$

Notice that $l(\pi_n^i) > i$, since $l(\pi) \geq i$ and $k \geq i$.

In the special case when $i = 1$, i.e., when π_n^i starts with n , we have $l(\pi_n^1) = 1 + l(\pi)$, since n cannot play the role of 3 for any 132 pattern. In general, however, the equation (3) does not express $l(\pi_n^i)$ solely in terms of $l(\pi)$. This is why we consider the next step, inserting $n + 1$ into π_n^i .

Let $\pi_{n,n+1}^{i,j}$ be the permutation obtained by inserting $n + 1$ into the j -th active site of π_n^i (which is not necessarily the j -th active site of π). We do a case analysis based on j ; in each of three cases, the position of the first 2 is the key of our analysis:

- $j = 1$

Then $\pi_{n,n+1}^{i,j}$ starts with $n + 1$ and $l(\pi_{n,n+1}^{i,j}) = 1 + l(\pi_n^i)$.

- $2 \leq j \leq i$

Then $n + 1$ is inserted to the left of n and we have

$$\pi_{n,n+1}^{i,j} = (p_1, \dots, p_{j-1}, n + 1, p_j, \dots, p_{i-1}, n, p_i, \dots, p_{n-1})$$

Hence, $\pi_{n,n+1}^{i,j}$ has a 132 pattern where any element to the left of $n + 1$ serves as 1, $n + 1$ serves as 3, and n serves as 2. Thus, n may be the first 2 in $\pi_{n,n+1}^{i,j}$. Further, the number of active sites in $\pi_{n,n+1}^{i,j}$ equals the number of active sites in

$\pi_n^j = (p_1, \dots, p_{j-1}, n, p_j, \dots, p_{n-1})$, unless n is the first 2 in $\pi_{n,n+1}^{i,j}$, which reduces the number of active sites in $\pi_{n,n+1}^{i,j}$ to the index of entry n . Therefore, $l(\pi_{n,n+1}^{i,j}) = \min(i+1, l(\pi_n^j))$.

- $i < j \leq l(\pi_n^i)$

Then $n+1$ is inserted to the right of n giving

$$\pi_{n,n+1}^{i,j} = (p_1, \dots, p_{i-1}, n, p_i, \dots, p_{j-2}, n+1, p_{j-1}, \dots, p_{n-1})$$

Note that $n+1$ is inserted right behind p_{j-2} , and not p_{j-1} , because the position to the right of p_{j-2} is the j -th active site in π_n^i . The number of active sites in $\pi_{n,n+1}^{i,j}$ equals the number of active sites in $\pi_n^{j-1} = (p_1, \dots, p_{j-2}, n, p_{j-1}, \dots, p_{n-1})$ plus the additional active site next to entry n : $l(\pi_{n,n+1}^{i,j}) = l(\pi_n^{j-1}) + 1$.

In summary, we have obtained the number of active sites in a 1324-avoiding permutation of length $n+1$ in terms of the number of active sites in 1324-avoiding permutations of length n :

$$l(\pi_{n,n+1}^{i,j}) = \begin{cases} l(\pi_n^i) + 1 & \text{if } j = 1, \\ \min(i+1, l(\pi_n^j)) & \text{if } 2 \leq j \leq i, \\ l(\pi_n^{j-1}) + 1 & \text{if } i < j \leq l(\pi_n^i). \end{cases}$$

Clearly, the values $l(\pi_{n,n+1}^{i,j})$, $1 \leq j \leq l(\pi_n^i)$, depend on i and the values $l(\pi_n^j)$, $1 \leq j \leq l(\pi_n^i)$. Hence, if we assign label $(s(\pi), i)$, where $s(\pi) = \langle l(\pi_n^1) \dots l(\pi_n^{l(\pi)}) \rangle$, to each π_n^i , for $1 \leq i \leq l(\pi)$, then the label of $\pi_{n,n+1}^{i,j}$ is completely determined by the label of its parent, π_n^i . More precisely, the label of $\pi_{n,n+1}^{i,j}$ is $(s(\pi_n^i), j)$; the sequence $s(\pi_n^i) = \langle l(\pi_{n,n+1}^{i,1}) \dots l(\pi_{n,n+1}^{i,l(\pi_n^i)}) \rangle$ is given by the succession rule $s(\pi_n^i) = f(\langle l(\pi_n^1) \dots l(\pi_n^{l(\pi)}) \rangle, i)$, where f is the function defined in (2). The root of the tree has the label $(\langle 2 \rangle, 1)$, which represents the unique permutation of length 1. This completes the proof of the lemma. \square

We next prove Theorem 1. Let T be the generating tree for 1324-avoiding permutations.

Proof. Let $d[\langle a_1 \dots a_m \rangle, i, n]$ be the number of 1324-avoiding permutations on the n -th level of the subtree of T , rooted at (the node with label) $(\langle a_1 \dots a_m \rangle, i)$. Then,

$$d[\langle a_1 \dots a_m \rangle, i, n] = \begin{cases} 1 & \text{if } n = 0, \\ \sum_{j=1}^{a_i} d[\langle b_1 \dots b_{a_i} \rangle, j, n-1] & \text{if } n > 0. \end{cases}$$

Note that $d[\langle a_1 \dots a_m \rangle, i, 1] = \sum_{j=1}^{a_i} d[\langle b_1 \dots b_{a_i} \rangle, j, 0] = a_i$, since $d[\langle b_1 \dots b_{a_i} \rangle, j, 0] = 1$.

Let $g(\langle a_1 \dots a_m \rangle, n)$ be the number of 1324-avoiding permutations on the n -th level of the subforest of T , which consists of trees whose roots are $(\langle a_1 \dots a_m \rangle, i)$, $1 \leq i \leq m$. Then,

$$\begin{aligned}
g(\langle a_1 \dots a_m \rangle, n) &= \sum_{i=1}^m d[\langle a_1 \dots a_m \rangle, i, n] = \sum_{i=1}^m \sum_{j=1}^{a_i} d[(f(\langle a_1 \dots a_m \rangle, i), j), n-1] \\
&= \sum_{i=1}^m g(f(\langle a_1 \dots a_m \rangle, i), n-1).
\end{aligned}$$

□

3 Concluding remarks

Theorem 1 provides a recurrence formula for the number of 1324-avoiding permutations, which, with the help of a computer, gives values of $s_n(1324)$ up to $n = 20$ [SPBC96]. Figure 2 shows a simple Maple code that directly corresponds to Theorem 1; the procedure `count1324` counts the number of all 1324-avoiding permutations of length n , and the procedure `g` corresponds to g , with inlined f .

Note that `g` has `option remember` modifier. It instructs Maple to use memoization [Bel57, Mic68] for `g`. Namely, Maple maintains a table of the input pairs \mathbf{s} and \mathbf{n} and corresponding values for `g`. Before computing the value for some pair, Maple first checks if that pair is already in the table. If so, Maple immediately returns the value; otherwise, it computes the value and stores the pair and the value in the table. The use of memoization significantly reduces time for computing the values of `g` for larger n . However, the memoization table requires space. On machines on which we used Maple, it ran out of memory when n was 15. We rewrote the code from Figure 2 in Java to speed up the computation and to reduce the memory consumption. The Java code uses a more compact representation of sequences of small numbers. It also has a selective memoization that stores in the table only the input pairs (and their corresponding values) for which `g` is likely to be invoked several times. We ran the Java code on the Sun JVM version 1.3.0 running under Linux on a 2GHz Pentium IV machine with 2GB of memory. Computing the number of 1324-avoiding permutations of length 20 took about 5 hours.

Although we have obtained a recurrence formula for the number of all 1324-avoiding permutations, we do not have a closed form for $s_n(1324)$. The occurrence of the `min` function in the definition of f , together with the fact that the length of the sequences assigned to nodes of the generating tree increase with the node level in the tree, complicate any attempt to obtain a closed formula. But, the formula may help finding the asymptotic growth of $s_n(1324)$.

In 1990, Stanley and Wilf conjectured that $s_n(\tau) < (c(\tau))^n$, where $c(\tau)$ is a constant. This conjecture clearly holds for patterns of length 3. Results of Bóna and Regev [Bón97a, Reg81] imply that $s_n(1342) < 8^n$ and $s_n(1234) < 9^n$, these bounds being asymptotically tight. Moreover, Bóna [Bón97b] proves that $s_n(1324)$ is asymptotically larger than $s_n(1234)$, and sketches an argument to prove that $s_n(1324) < 36^n$, this bound almost certainly not being tight. His techniques use the idea of dividing permutations into strong classes.

```

count1324 := proc(n)
  return g([1], n);
end:

g := proc(s, n) option remember;
  local i, j, sum, sNext;
  if (n = 1) then
    return convert(s, '+');
  fi;

  sum := 0;
  for i from 1 to nops(s) do
    sNext := s[i] + 1;
    for j from 2 to i do
      sNext := sNext, 'min'(i + 1, s[j]);
    od;
    for j from i + 1 to s[i] do
      sNext := sNext, s[j - 1] + 1;
    od;
    sum := sum + g([sNext], n - 1);
  od;
  return sum;
end:

```

Figure 2: The Maple code for counting 1324-avoiding permutations

n	$s_n(1324)$
0	1
1	1
2	2
3	6
4	23
5	103
6	513
7	2,762
8	15,793
9	94,776
10	591,950
11	3,824,112
12	25,431,452
13	173,453,058
14	1,209,639,642
15	8,604,450,011
16	62,300,851,632
17	458,374,397,312
18	3,421,888,118,907
19	25,887,131,596,018
20	198,244,731,603,623

Figure 3: The number of 1324-avoiding permutations for length up to 20

Definition 5. Two permutations π and σ are said to be in the same *strong* class if the left-to-right minima of π are the same as those of σ and they occur in the same position; and the same is true also of the right-to-left maxima.

Strong classes are denoted by specifying the positions of their minima and maxima and writing a '*' in the other positions. For example, $7*5*3*1*13*11*9$ denotes the strong class whose left-to-right minima are 7,5,3,1 (at positions 1,3,5,7) and right-to-left maxima are 13,11,9 (at positions 9,11,13). This particular strong class is, in fact, the class $S_{4,3}$ where, in general, $S_{l,r}$ is the strong class whose left-to-right minima $2l + 1, 2l - 1, \dots$ occur at the odd numbered positions followed by the right-to-left maxima $2(l + r) - 1, 2(l + r) - 3, \dots$ occurring at the remaining odd numbered positions.

Bóna showed that there are at most 9^n non-empty strong classes and sketched a proof that each one contains at most 4^n 1324-avoiding permutations. From our experiments with the Java applet [Str03] provided by Atkinson and his group we conjecture with some confidence that

Conjecture 6. *If $n = 2(l + r) - 1$, the strong class $S_{l,r}$ contains more 1324-avoiding permutations than any other strong class with l left-to-right minima and r right-to-left maxima. Furthermore, the strong class $S_{r,r}$ contains more 1324-avoiding permutations than any other strong class of that length.*

We actually know the exact formula for $|S_{l,r}|$.

Proposition 7. $|S_{l,r}| = \binom{l+r-1}{l-1}$.

Proof. Let $n = 2k + 1$. Let a_1, \dots, a_l be the left-to-right minima, and b_r, \dots, b_1 be the right-to-left maxima. Here, the sequence $a_1, \dots, a_l, b_1, \dots, b_r$ is actually the sequence $1, 3, \dots, n$. Let $\sigma \in S_{l,r}$. It is easy to see that: 1) if $k + 1$ occurs to the left of $b_r = n$, then $k + 1$ has to be the second entry of σ ; and 2) if $k + 1$ occurs to the right of $a_1 = 1$, then $k + 1$ has to be the next-to-last entry of σ . Hence, 1324-avoiding permutations in $S_{l,r}$ fall into two categories: the ones with $\sigma(2) = k + 1$ and the ones with $\sigma(n - 1) = k + 1$. We map each $\sigma = (k, k + 1, k - 1, \gamma) \in S_{l,r}$ to $\sigma' = (k - 1, \gamma') \in S_{l-1,r}$, and vice versa, where γ' is obtained from γ by reducing all the entries of γ that are greater than $k + 1$ by 2. Therefore, 1324-avoiding permutations in $S_{l,r}$ with $k + 1$ as the second entry are in one-to-one correspondence with 1324-avoiding permutations in $S_{l-1,r}$. Similarly, 1324-avoiding permutations in $S_{l,r}$ with $k + 1$ as the next-to-last entry are in one-to-one correspondence with 1324-avoiding permutations in $S_{l,r-1}$. Thus, $|S_{l,r}| = |S_{l-1,r}| + |S_{l,r-1}|$, completing the proof by induction. \square

Since $\binom{2r-1}{r-1} < 2^{n/2}$, the conjecture would prove that $s_n(1324) < (9\sqrt{2})^n$, which would be a considerable improvement on Bóna's bound. It remains plausible that $s_n(1324) < 9^n$.

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