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# Generalizing Ramaujan Summation

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**Abstract:**

Ever since Cesaro introduced the idea of assigning particular value to divergent series of real numbers, various mathematicians had provided novel techniques to assign fixed values for divergent series of real numbers. One such noticeable method is Ramanujan Summation Method. In this paper, we have generalized the notion of Ramanujan Summation method by considering arithmetic progressions. The final result surprisingly provides an alternate form of the original Ramanujan summation method. Several new results regarding Ramanujan summation methods were derived in detail along with suitable figures to enhance the understanding of ideas provided.

**Keywords**: Ramanujan Summation, Arithmetic Progression, Sum of Powers, Binomial coefficients, Definite Integral.

## 1. Introduction

The most prominent Indian of twentieth century, Srinivasa Ramanujan was well known for his wonderful jottings of extra-ordinary formulas in his notebooks. In one of such jottings, he claimed $1+2+3+4+\cdots=-\frac{1}{12}, 1^2+2^2+3^2+4^2+\cdots=0$ and several weird looking equations. Obviously on surface level these results don't seem to make any sense. But at the same time, how a genius like Ramanujan could have scribbled such meaningless equations? In fact only a genius like Littlewood could correctly identify what Ramanujan was thinking in his mind. Ramanujan notoriously presented the summation jottings in connection with Riemann Zeta Function existing in extended complex plane. Ever since, this connection was realized, several new results emerged expanding the literature to vast extent. In this paper, we present one such new viewpoint in generalizing Ramanujan Summation method by considering Arithmetic Progressions.

## 2. Definitions and Preliminary Results

**2.1** Let $a$, $d$ be two real numbers where $d$ is non–zero. Then the sequence $a, a+d, a+2d. a+3d, a+4d, a+5d, \ldots$ is defined as Arithmetic Progression. The number $a$ is called the first term and $d$ is called the common difference of the corresponding Arithmetic Progression.

**2.2** Let $\sum_{n=1}^{\infty} a_n$ represent any divergent series of real numbers. The Ramanujan Summation (*RS*) of $\sum_{n=1}^{\infty} a_n$ (see **[1]**) is defined by $(RS)\left(\sum_{n=1}^{\infty} a_n\right) = \int_{n=-1}^{0}\left(\sum_{k=1}^{n} a_k\right)dn$ (2.1)

**2.3** The sum of *m*th powers of first ($n$ – 1) natural numbers (see **[2], [3]**) for $m$ = 1 to 8 are given by

$$1^1 + 2^1 + \cdots + (n-1)^1 = \frac{1}{2}n^2 - \frac{1}{2}n$$
$$1^2 + 2^2 + \cdots + (n-1)^2 = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$$
$$1^3 + 2^3 + \cdots + (n-1)^3 = \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2$$
$$1^4 + 2^4 + \cdots + (n-1)^4 = \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$
$$1^5 + 2^5 + \cdots + (n-1)^5 = \frac{1}{6}n^6 - \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$
$$1^6 + 2^6 + \cdots + (n-1)^6 = \frac{1}{7}n^7 - \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$$
$$1^7 + 2^7 + \cdots + (n-1)^7 = \frac{1}{8}n^8 - \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2$$
$$1^8 + 2^8 + \cdots + (n-1)^8 = \frac{1}{9}n^9 - \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$$

Figure 1: Sum of powers of first ($n$ – 1) natural numbers

**2.4** Let $S_{n,m} = \sum_{k=1}^{n} \left(a+(k-1)d\right)^m = a^m + (a+d)^m + \cdots + \left(a+(n-1)d\right)^m$ (2.2) denote the sum of *m*th powers of first $n$ terms of the Arithmetic Progression given by $a$, $a+d$, $a+2d$. $a+3d$, . . . , $a+(n-1)d$.

**3. Ramanujan Summation for first few powers of Arithmetic Progression**

**3.1 First Power**

In this section, we will determine the Ramanujan summation of first power of first $n$ terms of Arithmetic progression. From first equation of Figure 1 and (2.2), we have

$$S_{n,1} = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d)$$
$$= na + (1+2+\cdots+(n-1))d = na + \left(\frac{n^2}{2} - \frac{n}{2}\right)d \quad (3.1)$$

Now, from (2.1), the Ramanujan summation of first power of arithmetic progression is given by

$$(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)\right)=\int_{n=-1}^{0}\left(\sum_{k=1}^{n}(a+(k-1)d)\right)dn=\int_{n=-1}^{0}S_{n,1}\,dn$$

$$=\int_{n=-1}^{0}\left[na+\left(\frac{n^2}{2}-\frac{n}{2}\right)d\right]dn$$

$$=\left(a-\frac{d}{2}\right)\left(\frac{n^2}{2}\right)_{n=-1}^{0}+\frac{d}{2}\left(\frac{n^3}{3}\right)_{n=-1}^{0}$$

$$=\frac{5d-6a}{12}$$

$$(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)\right)=(RS)(a+(a+d)+(a+2d)+\cdots)=\frac{5d-6a}{12} \quad (3.2)$$

In particular, if $a$ =1, $d$ = 1 then from (3.2) we get $(RS)(1+2+3+\cdots)=-\frac{1}{12}$ (3.3)

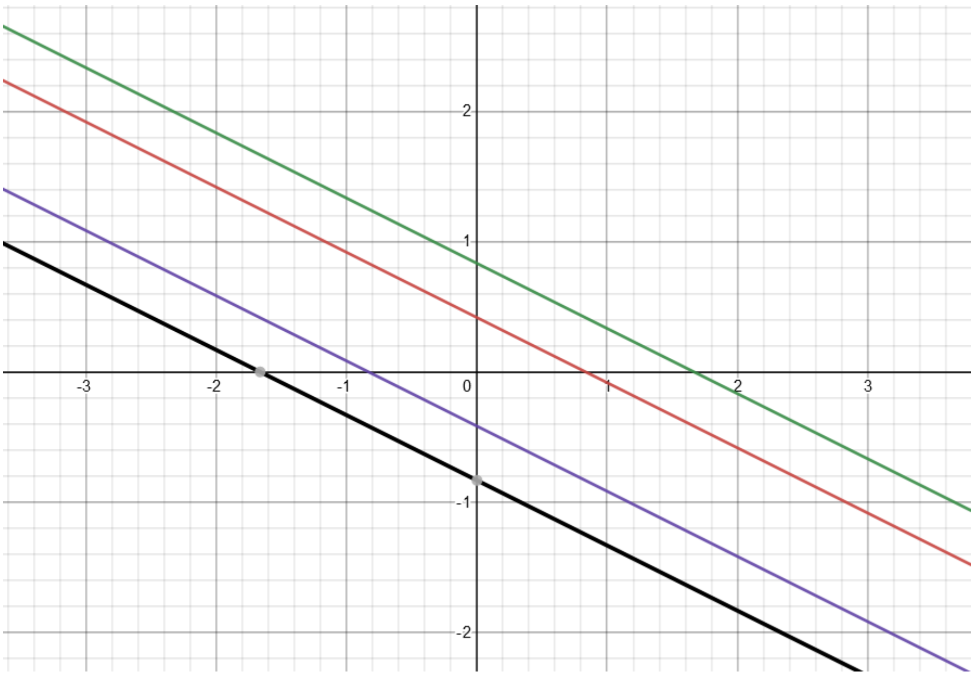


Figure 1: Graphs of $\frac{5d-6x}{12}$ for $d=-2,-1,1,2$

### 3.2 Second Power

Now, we will determine the Ramanujan summation of second powers of first $n$ terms of Arithmetic progression. From first and second equations of figure 1 and (2.2), we have

$$S_{n,2} = a^2 + (a+d)^2 + (a+2d)^2 + \cdots + (a+(n-1)d)^2$$
$$= na^2 + 2ad(1+2+3+\cdots+(n-1)) + d^2(1^2+2^2+3^2+\cdots+(n-1)^2)$$
$$= na^2 + ad\left(n^2-n\right) + \frac{d^2}{6}\left(2n^3-3n^2+n\right) \quad (3.4)$$

From (2.1), the Ramanujan summation of second powers of arithmetic progression is given by

$$(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)^2\right) = \int_{n=-1}^{0}\left(\sum_{k=1}^{n}(a+(k-1)d)^2\right)dn = \int_{n=-1}^{0} S_{n,2}\,dn$$
$$= \int_{n=-1}^{0}\left[na^2 + ad\left(n^2-n\right) + \frac{d^2}{6}\left(2n^3-3n^2+n\right)\right]dn$$
$$= a^2\left(\frac{n^2}{2}\right)_{n=-1}^{0} + ad\left(\frac{n^3}{3}-\frac{n^2}{2}\right)_{n=-1}^{0} + \frac{d^2}{6}\left(\frac{n^4}{2}-n^3+\frac{n^2}{2}\right)_{n=-1}^{0}$$
$$= \frac{5ad-3a^2-2d^2}{6}$$

$$(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)^2\right) = (RS)(a^2+(a+d)^2+(a+2d)^2+\cdots) = \frac{5ad-3a^2-2d^2}{6} \quad (3.5)$$

In particular, if $a$ =1, $d$ = 1 then from (3.5) we get $(RS)(1^2+2^2+3^2+\cdots) = 0 \quad (3.6)$

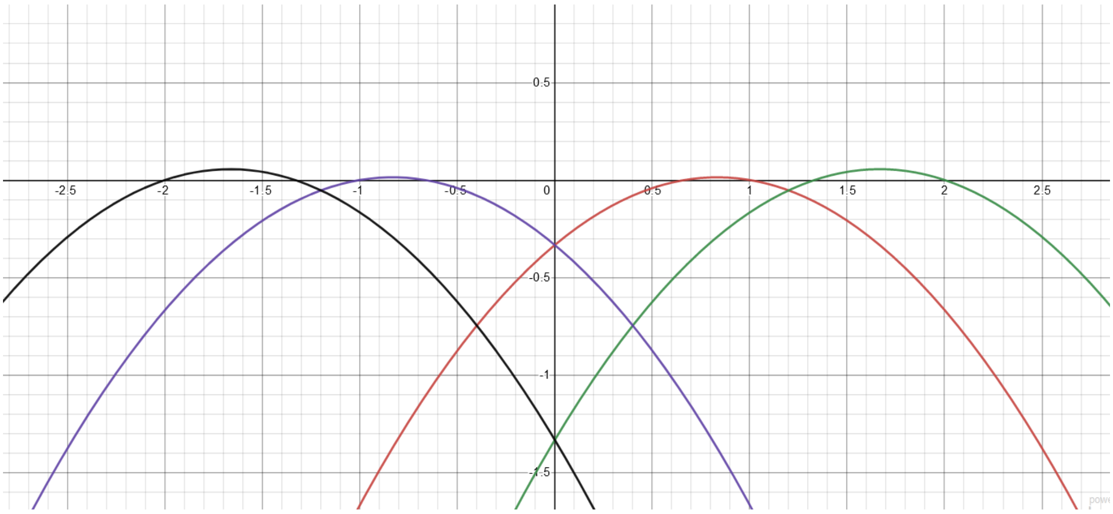


Figure 2: Graphs of $\frac{5xd-3x^2-2d^2}{6}$ for $d = -2,-1,1,2$

**3.3 Third Power**

We will determine the Ramanujan summation of third powers of first $n$ terms of Arithmetic progression. From first three equations of figure 1 and (2.2), we have

$$S_{n,3} = a^3 + (a+d)^3 + (a+2d)^3 + \cdots + (a+(n-1)d)^3$$
$$= a^3\left(\sum_{k=0}^{n-1} 1\right) + 3a^2 d\left(\sum_{k=1}^{n-1} k\right) + 3ad^2\left(\sum_{k=1}^{n-1} k^2\right) + d^3\left(\sum_{k=1}^{n-1} k^3\right)$$
$$= na^3 + 3a^2 d\left(\frac{n^2-n}{2}\right) + 3ad^2\left(\frac{2n^3-3n^2+n}{6}\right) + d^3\left(\frac{n^4-2n^3+n^2}{4}\right) \quad (3.7)$$

From (2.1), the Ramanujan summation of third powers of arithmetic progression is given by

$$(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)^3\right) = \int_{n=-1}^{0}\left(\sum_{k=1}^{n}(a+(k-1)d)^3\right)dn = \int_{n=-1}^{0} S_{n,3}\, dn$$
$$= \int_{n=-1}^{0}\left[na^3 + 3a^2 d\left(\frac{n^2-n}{2}\right) + 3ad^2\left(\frac{2n^3-3n^2+n}{6}\right) + d^3\left(\frac{n^4-2n^3+n^2}{4}\right)\right] dn$$
$$= a^2\left(-\frac{1}{2}\right) + 3a^2 d\left(\frac{5}{12}\right) + 3ad^2\left(-\frac{1}{3}\right) + d^3\left(\frac{31}{120}\right)$$
$$= \frac{-60a^3 + 150a^2 d - 120ad^2 + 31d^3}{120}$$

$$(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)^3\right) = (RS)(a^3 + (a+d)^3 + (a+2d)^3 + \cdots) = \frac{-60a^3 + 150a^2 d - 120ad^2 + 31d^3}{120} \quad (3.8)$$

In particular, if $a$ =1, $d$ = 1 then from (3.5) we get $(RS)(1^3 + 2^3 + 3^3 + \cdots) = \frac{1}{120}$ (3.9)

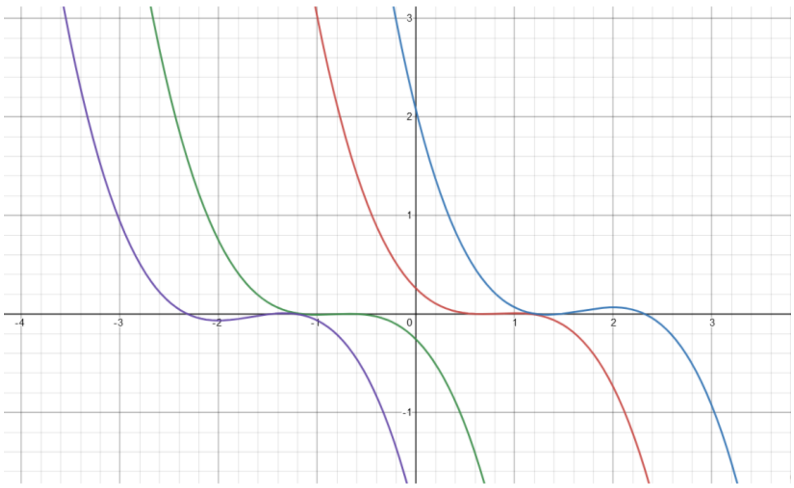


Figure 3: Graphs of $\frac{-60x^3 + 150x^2 d - 120xd^2 + 31d^3}{120}$ for $d = -2, -1, 1, 2$

### 3.4 Fourth Power

We will determine the Ramanujan summation of fourth powers of first $n$ terms of Arithmetic progression. From first four equations of figure 1 and (2.2), we have

$$S_{n,4}=a^4+(a+d)^4+(a+2d)^4+\cdots+(a+(n-1)d)^4=a^4\left(\sum_{k=0}^{n-1}1\right)+4a^3d\left(\sum_{k=1}^{n-1}k\right)+6a^2d^2\left(\sum_{k=1}^{n-1}k^2\right)$$

$$+4ad^3\left(\sum_{k=1}^{n-1}k^3\right)+d^4\left(\sum_{k=1}^{n-1}k^4\right)=na^4+4a^3d\left(\frac{n^2-n}{2}\right)+6a^2d^2\left(\frac{2n^3-3n^2+n}{6}\right)+$$

$$4ad^3\left(\frac{n^4-2n^3+n^2}{4}\right)+d^4\left(\frac{6n^5-15n^4+10n^3-n}{4}\right) \quad (3.10)$$

From (2.1), the Ramanujan summation of fourth powers of arithmetic progression is given by

$$(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)^4\right)=\int_{n=-1}^{0}\left(\sum_{k=1}^{n}(a+(k-1)d)^4\right)dn=\int_{n=-1}^{0}S_{n,4}\,dn$$

$$=a^4\left(-\frac{1}{2}\right)+4a^3d\left(\frac{5}{12}\right)+6a^2d^2\left(-\frac{1}{3}\right)+4ad^3\left(\frac{31}{120}\right)+d^4\left(-\frac{1}{5}\right)$$

$$=\frac{-15a^4+50a^3d-60a^2d^2+31ad^3-6d^4}{30}$$

$$(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)^4\right)=(RS)\left(a^4+(a+d)^4+\cdots\right)=\frac{-15a^4+50a^3d-60a^2d^2+31ad^3-6d^4}{30} \quad (3.11)$$

In particular, if $a$ =1, $d$ = 1 then from (3.11) we get $(RS)(1^4+2^4+3^4+\cdots)=0$ (3.12)

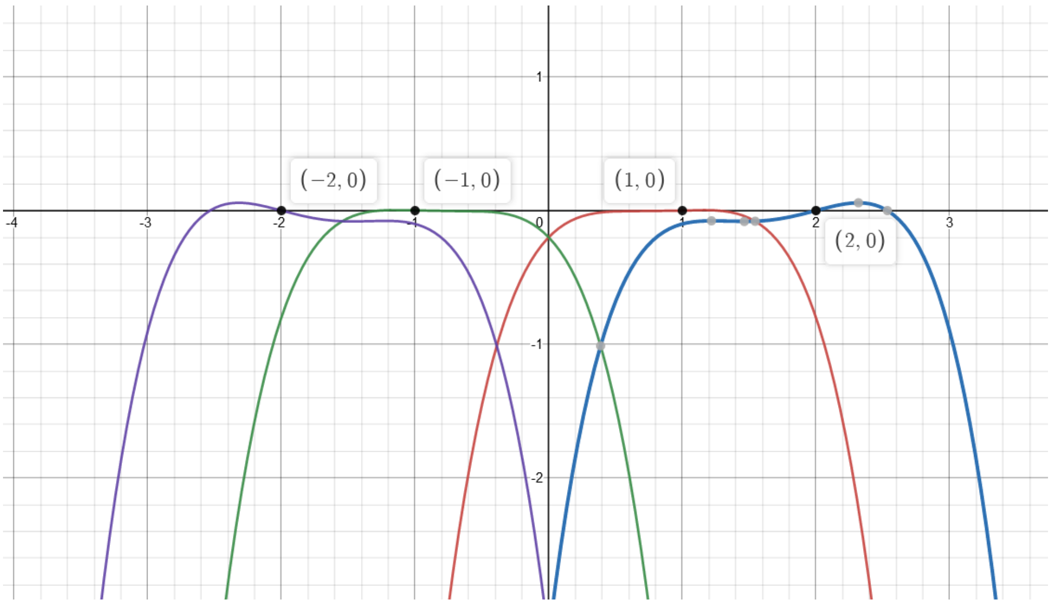


Figure 4: Graphs of $\dfrac{-15x^4+50x^3d-60x^2d^2+31xd^3-6d^4}{30}$ for $d=-2,-1,1,2$

## 4. Bernoulli Numbers and Definite Integrals

In this section, we will introduce Bernoulli numbers and certain definite integrals.

**4.1 Bernoulli Numbers**

The Bernoulli numbers occur as coefficient of $\frac{x^n}{n!}$ in the Maclaurin's series expansion of $\frac{x}{e^x-1}$. We denote the *n*th Bernoulli number by $B_n$.

Thus from the definition we observe that $\frac{x}{e^x-1}=\sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$ (4.1)

The list of Bernoulli numbers (see **[4]** – **[15]**) are given by

$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_3=0, B_4=-\frac{1}{30}, B_5=0, B_6=\frac{1}{42}, B_7=0, B_8=-\frac{1}{30},$$
$$B_9=0, B_{10}=\frac{5}{66}, B_{11}=0, B_{12}=-\frac{691}{2730}, B_{13}=0, B_{14}=\frac{7}{6}, B_{15}=0, B_{16}=-\frac{3617}{510}, \ldots \quad (4.2)$$

We notice that except $B_1$, $B_n=0$ for all odd positive integers *n*.

**4.2 Definite Integrals**

For any positive integer *r*, let $I_r=\int_{n=-1}^{0}\left(\sum_{k=1}^{n-1} k^r\right) dn$ (4.3) denote the definite integral of sum of *r*th powers of first *n* – 1 natural numbers. For *r* = 0, $I_0$ is defined by $I_0=\int_{n=-1}^{0}\left(\sum_{k=0}^{n-1} k^0\right) dn$

Since $\sum_{k=1}^{n-1} k^r$ is a polynomial of degree *n*, it is continuous at all points in $[-1,0]$. Hence the definite integral defined in (4.3) is Riemann Integrable over $[-1,0]$. Using the expressions given in Figure 1, we find that

$$I_0=\int_{n=-1}^{0}\left(\sum_{k=0}^{n-1} k^0\right) dn=\int_{n=-1}^{0} n\, dn=-\frac{1}{2}, I_1=\int_{n=-1}^{0}\left(\sum_{k=1}^{n-1} k^1\right) dn=\int_{n=-1}^{0}\left(\frac{n^2}{2}-\frac{n}{2}\right) dn=\frac{5}{12}$$

$$I_2=\int_{n=-1}^{0}\left(\sum_{k=1}^{n-1} k^2\right) dn=\int_{n=-1}^{0}\left(\frac{n^3}{3}-\frac{n^2}{6}+\frac{n}{6}\right) dn=-\frac{1}{3}, I_3=\int_{n=-1}^{0}\left(\sum_{k=1}^{n-1} k^3\right) dn=\int_{n=-1}^{0}\left(\frac{n^4}{2}-\frac{n^3}{4}+\frac{n^2}{2}\right) dn=\frac{31}{120}$$

Similarly, we find that $I_4=-\frac{1}{5}, I_5=\frac{41}{252}, I_6=-\frac{1}{7}, I_7=\frac{31}{240}, I_8=-\frac{1}{9}, I_9=\frac{61}{660}, I_{10}=-\frac{1}{11}$

Thus, the first eleven values of the definite integrals defined above are given by

$$I_0=-\frac{1}{2}, I_1=\frac{5}{12}, I_2=-\frac{1}{3}, I_3=\frac{31}{120}, I_4=-\frac{1}{5}, I_5=\frac{41}{252},$$
$$I_6=-\frac{1}{7}, I_7=\frac{31}{240}, I_8=-\frac{1}{9}, I_9=\frac{61}{660}, I_{10}=-\frac{1}{11} \qquad (4.4)$$

From the list of Bernoulli numbers in (4.2) and list of definite integrals in (4.4), we have the following equations

$I_0 = -\frac{1}{2}$, $I_r = -\frac{1}{r+1}$ if $r$ is even and $I_r = \frac{1-B_{r+1}}{r+1}$ if $r$ is odd (4.5)

**5. General Case**

In this section, we will provide methods for determining Ramanujan summation of $m$th powers of Arithmetic progression, where $m$ is any positive integer.

**5.1 Theorem 1**

If $m$ is any positive integer, $a$, $d$ are first term and common difference of any Arithmetic progression, then $(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)^m\right) = \sum_{r=0}^{m}\binom{m}{r}a^{m-r}d^r I_r$ (5.1) where $I_r$ is the definite integral given by

$$I_r = \int_{n=-1}^{0}\left(\sum_{k=1}^{n-1}k^r\right)dn$$

**Proof**: Let $S_{n,m}$ denote sum of $m$th powers of first $n$ terms of the arithmetic progression $a$, $a + d$, $a + 2d$, . . . Then $S_{n,m} = \sum_{k=1}^{n}(a+(k-1)d)^m$. Using Binomial expansion, we have

$$S_{n,m} = \sum_{k=1}^{n}(a+(k-1)d)^m = \sum_{r=0}^{m}\binom{m}{r}a^{m-r}d^r\left(\sum_{k=1}^{n-1}k^r\right)$$

Now using (2.1), we get

$$(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)^m\right) = \int_{n=-1}^{0}\left(\sum_{k=1}^{n}(a+(k-1)d)^m\right)dn = \int_{n=-1}^{0}S_{n,m}\,dn$$

Now using the expression for $S_{n,m}$ and using additive property of Riemann Integrals, we obtain

$$(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)^m\right) = \int_{n=-1}^{0}\left[\sum_{r=0}^{m}\binom{m}{r}a^{m-r}d^r\left(\sum_{k=1}^{n-1}k^r\right)\right]dn = \sum_{r=0}^{m}\binom{m}{r}a^{m-r}d^r\int_{n=-1}^{0}\left(\sum_{k=1}^{n-1}k^r\right)dn$$

Now using the definition of the definite integral $I_r$ we have

$$(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)^m\right) = \sum_{r=0}^{m}\binom{m}{r}a^{m-r}d^r I_r$$

This proves (5.1) and hence completes the proof.

**5.2 Corollary 1**

The Ramanujan summation of first four powers of terms of Arithmetic progression $a$, $a+d$, $a+2d$. $a+3d$, . . . are given by $(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)^m\right) = \sum_{r=0}^{m}\binom{m}{r}a^{m-r}d^r I_r$ for $m$ = 1, 2, 3, 4 respectively.

**Proof**: For $m$ = 1, we obtain

$$(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)^{1}\right)=\sum_{r=0}^{1}\binom{1}{r}a^{1-r}d^{r}I_{r}=aI_{0}+dI_{1}=a\left(-\frac{1}{2}\right)+d\left(\frac{5}{12}\right)=\frac{5d-6a}{12}$$ which is same as that of obtained in (3.6)

Similarly, for $m$ = 2, 3 and 4 we get the following expressions

$$(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)^{2}\right)=\sum_{r=0}^{2}\binom{2}{r}a^{2-r}d^{r}I_{r}=a^{2}I_{0}+2adI_{1}+d^{2}I_{2}$$

$$=a^{2}\left(-\frac{1}{2}\right)+2ad\left(\frac{5}{12}\right)+d^{2}\left(-\frac{1}{3}\right)=\frac{-3a^{2}+5ad-d^{2}}{6}$$

$$(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)^{3}\right)=\sum_{r=0}^{3}\binom{3}{r}a^{3-r}d^{r}I_{r}=a^{3}I_{0}+3a^{2}dI_{1}+3ad^{2}I_{2}+d^{3}I_{3}$$

$$=a^{3}\left(-\frac{1}{2}\right)+3a^{2}d\left(\frac{5}{12}\right)+3ad^{2}\left(-\frac{1}{3}\right)+d^{3}\left(\frac{31}{120}\right)$$

$$=\frac{-60a^{3}+150a^{2}d-120ad^{2}+31d^{3}}{120}$$

$$(RS)\left(\sum_{k=1}^{\infty}(a+(k-1)d)^{4}\right)=\sum_{r=0}^{4}\binom{4}{r}a^{4-r}d^{r}I_{r}=a^{4}I_{0}+4a^{3}dI_{1}+6a^{2}d^{2}I_{2}+4ad^{3}I_{3}+d^{4}I_{4}$$

$$=a^{4}\left(-\frac{1}{2}\right)+4a^{3}d\left(\frac{5}{12}\right)+6a^{2}d^{2}\left(-\frac{1}{3}\right)+4ad^{3}\left(\frac{31}{120}\right)+d^{4}\left(-\frac{1}{5}\right)$$

$$=\frac{-15a^{4}+50a^{3}d-60a^{2}d^{2}+31ad^{3}-6d^{4}}{30}$$

We notice that the above three expressions are precisely the values given by (3.5), (3.8) and (3.11) respectively. This completes the proof.

**5.3 Corollary 2**

If $m$ is any positive integer, then the Ramanujan summation of $m$th powers of positive integers is given by

$$(RS)\left(1^{m}+2^{m}+3^{m}+\cdots\right)=\sum_{r=0}^{m}\binom{m}{r}I_{r}\quad(5.2)$$ where $I_r$ is the definite integral given by

$$I_{r}=\int_{n=-1}^{0}\left(\sum_{k=1}^{n-1}k^{r}\right)dn$$

**Proof**: Taking $a$ = 1, $d$ = 1 in (5.1), we get

$$(RS)\left(\sum_{k=1}^{\infty}(1+(k-1))^{m}\right)=(RS)(1^{m}+2^{m}+\cdots)=\sum_{r=0}^{m}\binom{m}{r}I_{r}$$

This proves (5.2) and hence completes the proof.

**5.4 Corollary 3**

The Ramanujan summation of first few powers of sum of natural numbers are given by

$$(RS)(1+2+3+4+\cdots)=-\frac{1}{12} \qquad (5.3)$$

$$(RS)(1^2+2^2+3^2+4^2+\cdots)=0 \qquad (5.4)$$

$$(RS)(1^3+2^3+3^3+4^3+\cdots)=\frac{1}{120} \qquad (5.5)$$

$$(RS)(1^4+2^4+3^4+4^4+\cdots)=0 \qquad (5.6)$$

$$(RS)(1^5+2^5+3^5+4^5+\cdots)=-\frac{1}{252} \qquad (5.7)$$

$$(RS)(1^6+2^6+3^6+4^6+\cdots)=0 \qquad (5.8)$$

$$(RS)(1^7+2^7+3^7+4^7+\cdots)=\frac{1}{240} \qquad (5.9)$$

**Proof**: Using the list of integral values from (4.4), and using (5.2) for each value of *m*, we obtain

For *m* = 1, $(RS)(1+2+3+\cdots)=\sum_{r=0}^{1}\binom{1}{r}I_r=I_0+I_1=-\frac{1}{2}+\frac{5}{12}=-\frac{1}{12}$

For *m* = 2, $(RS)(1^2+2^2+3^2+\cdots)=\sum_{r=0}^{2}\binom{2}{r}I_r=I_0+2I_1+I_2=-\frac{1}{2}+\frac{10}{12}-\frac{1}{3}=0$

For *m* = 3, $(RS)(1^3+2^3+3^3+\cdots)=\sum_{r=0}^{3}\binom{3}{r}I_r=I_0+3I_1+3I_2+I_3=-\frac{1}{2}+\frac{5}{4}-1+\frac{31}{120}=\frac{1}{120}$

For *m* = 4, $(RS)(1^4+2^4+3^4+\cdots)=\sum_{r=0}^{4}\binom{4}{r}I_r=I_0+4I_1+6I_2+4I_3+I_4=-\frac{1}{2}+\frac{5}{3}-2+\frac{31}{30}-\frac{1}{5}=0$

For *m* = 5,

$$(RS)(1^5+2^5+3^5+\cdots)=\sum_{r=0}^{5}\binom{5}{r}I_r=I_0+5I_1+10I_2+10I_3+5I_4+I_5$$

$$=-\frac{1}{2}+\frac{25}{12}-\frac{10}{3}+\frac{31}{12}-1+\frac{41}{252}=-\frac{1}{252}$$

For *m* = 6,

$$(RS)(1^6+2^6+3^6+\cdots)=\sum_{r=0}^{6}\binom{6}{r}I_r=I_0+6I_1+15I_2+20I_3+15I_4+6I_5+I_6$$

$$=-\frac{1}{2}+\frac{5}{2}-5+\frac{31}{6}-3+\frac{41}{42}-\frac{1}{7}=0$$

For *m* = 7,

$$(RS)(1^7+2^7+3^7+\cdots)=\sum_{r=0}^{7}\binom{7}{r}I_r=I_0+7I_1+21I_2+35I_3+35I_4+21I_5+7I_6+I_7$$

$$=-\frac{1}{2}+\frac{35}{12}-7+\frac{217}{24}-7+\frac{41}{12}-1+\frac{31}{240}=\frac{1}{240}$$

We have thus obtained all the results provided from equations (5.3) to (5.9)

This completes the proof.

**6. Conclusion**

The main objective of this paper is to generalize the concept of Ramanujan Summation and derive the same results that Ramanujan provided in different form. While Ramanujan could have obtained the summation method for sum of *m*th powers of positive integers, we had generalized it to obtain a summation method for sum of *m*th powers of general arithmetic progressions which include positive integers as special case by choosing the first term and common difference each to be 1.

In section 3, through four sub-sections, we had obtained the Ramanujan summation formulas for first, second, third and fourth powers of arithmetic progression through equations (3.2), (3.5), (3.8), (3.11) respectively. As special case of these results by considering $a = 1$, $d = 1$, the equations (3.3), (3.6), (3.9) and (3.12) reveal the secret of how Ramanujan could have obtained those mystic results in his work. Moreover, four figures were provided at respective places to understand the meaning of the answers obtained. In particular, the figures were plotted by considering the value of the common difference $d$ to take four values – 2, – 1, 1, 2. The range of the function plotted for different values of $d$ will provide not only the results given by Ramanujan but does more than that, since by choosing convenient values of $a$ and $d$, we can determine Ramanujan summation of several divergent series of real numbers. In this sense, the results obtained in equations (3.2), (3.5), (3.8), (3.11) were generalized versions of original Ramanujan summation methods.

The usual Ramanujan Summation formula depends on Bernoulli numbers. For this sake, we had introduced such numbers in section 4 through (4.2). We had also defined set of definite integrals which are Riemann integrable over $[-1, 0]$ to obtain a more general formula. The connection between the Bernoulli numbers and definite integrals were provided in (4.5).

In section 5, through equation (5.1) of theorem 1, we had proved a general compact formula for Ramanujan summation method. In particular, we notice that the Ramanujan Summation for sum of *m*th powers of arithmetic progressions is the sum of products of binomial coefficients and definite integral values obtained in (4.4). Note that the formula (5.1) contain only $m + 1$ terms along with binomial coefficients and certain definite integral values. Three useful corollaries were obtained using (5.1). Thus the ideas in this paper not only generalize the concept of Ramanujan summation but it provides a completely new formula for obtaining the same. In particular, equation (5.2) provides the known Ramanujan summation method in a completely new perspective, which is the main purpose of this paper.

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