Formulas for Odd Zeta Values and Powers of π

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Abstract

Plouffe conjectured rapidly converging series formulas for π^{2n+1} and $\zeta(2n+1)$ for small values of n. We find the general pattern for all nonnegative integer values of n and offer a proof.

1 Introduction

It took nearly one hundred years for the Basel Problem — finding a closed form solution to $\sum_{k=1}^{\infty} 1/k^2$ — to see a solution. Euler solved this in 1735 and essentially solved the problem where the power of two is replaced with any even power. This formula is now usually written as

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!},$$

where $\zeta(s)$ is the Riemann zeta function and B_k is the k-th Bernoulli number, defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}, \quad |x| < 2\pi,$$

whose first few values are $0, -1/2, 1/6, 0, -1/30, \ldots$. However, finding a closed form for $\zeta(2n+1)$ has remained an open problem. Only in 1979 did Apéry show that $\zeta(3)$ is irrational. His proof involved the snappy acceleration

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$

This tidy formula does not generalize to the other odd zeta values, but other representations, such as nested sums or integrals, have been well studied. The hunt for a clean result like Euler's has largely been abandoned, leaving researchers with the goal of finding formulas which either converge quickly or have an elegant form.

Following his success in discovering a new formula for π, Simon Plouffe [5] conjectured several identities which relate either π^m or $\zeta(m)$ to three infinite series. Letting

$$S_n(r) = \sum_{k=1}^{\infty} \frac{1}{k^n(e^{\pi r k} - 1)},$$

the first few examples are[1]

$$\begin{aligned}
\pi &= 72S_1(1) - 96S_1(2) + 24S_1(4) \\
\pi^3 &= 720S_3(1) - 900S_3(2) + 180S_3(4) \\
\pi^5 &= 7056S_5(1) - 6993S_5(2) - 63S_5(4) \\
\pi^7 &= \frac{907200}{13}S_7(1) - 70875S_7(2) + \frac{14175}{13}S_5(4).
\end{aligned}$$

and

$$\begin{aligned}
\zeta(3) &= 28S_3(1) - 37S_3(2) + 7S_3(4) \\
\zeta(5) &= 24S_5(1) - \frac{259}{10}S_5(2) - \frac{1}{10}S_5(4) \\
\zeta(7) &= \frac{304}{13}S_7(1) - \frac{103}{4}S_7(2). + \frac{19}{52}S_7(4).
\end{aligned}$$

Plouffe conjectured these formulas by first assuming, for example, that there exist constants a, b, and c such that

$$\pi = aS_1(1) + bS_1(2) + cS_1(4).$$

By obtaining accurate approximations of each the three series, he wrote some computer code to postulate rational values for a, b, c. Today, such integer relations algorithms have been used to discover many formulas. The widely used PSLQ algorithm, developed by Ferguson and Bailey [4], is implemented in Maple. The following Maple code solves the above problem:

```
> with(IntegerRelations):
> Digits := 100;
> S := r -> sum( 1/k/( exp(Pi*r*k)-1 ), k=1..infinity );
> PSLQ( [ Pi, S(1), S(2), S(4) ] );
```

The PSLQ command returns the vector $[-1, 72, -96, 24]$, producing the first formula.

While the computer can be used to conjecture the coefficients for a specific power, finding the general sequences of rationals has remained an open problem. This note finds these sequences and offers formal proofs.

[1]There is a typographical error in the sign of the last coefficient in the formula for π^5 in [5], which is corrected here.

2 Exact Formulas

While it does not seem that $\zeta(2n+1)$ is a rational multiple of π^{2n+1}, a result in Ramanujan's notebooks gives a relationship with rapidly convergent infinite series. See Entry 21(i) in Chapter 14 of [2], and see [3] for additional commentary.

Theorem 1 (Ramanujan)**.** *If $\alpha > 0$, $\beta > 0$, and $\alpha\beta = \pi^2$, then*

$$\alpha^{-n}\left\{\frac{1}{2}\zeta(2n+1) + S_{2n+1}(2\alpha/\pi)\right\} =$$
$$(-\beta)^{-n}\left\{\frac{1}{2}\zeta(2n+1) + S_{2n+1}(2\beta/\pi)\right\} - 4^n\sum_{k=0}^{n+1}(-1)^k\frac{B_{2k}B_{2n+2-2k}}{(2k)!(2n+2-2k)!}\alpha^{n+1-k}\beta^k.$$

Using $\alpha = \beta = \pi$ in Proposition 1 and defining

$$F_n = \sum_{k=0}^{n+1}(-1)^k\frac{B_{2k}B_{2n+2-2k}}{(2k)!(2n+2-2k)!},$$

we have

$$\left(\pi^{-n} - (-\pi)^{-n}\right)\left(\frac{1}{2}\zeta(2n+1) + S_{2n+1}(2)\right) = -4^n\pi^{n+1}F_n.$$

To find formulas for the odd zeta values and powers of π, we will divide these into two classes: $\zeta(4m-1)$ and $\zeta(4m+1)$. Such distinctions can be seen in other studies; see [1, pp. 137–139].

First, we find the formulas for π^{4m-1} and $\zeta(4m-1)$. If n is odd, then

$$\frac{1}{2}\zeta(2n+1) + S_{2n+1}(2) = \frac{-4^n}{2}\pi^{2n+1}F_n. \tag{1}$$

Using $\alpha = \pi/2$ and $\beta = 2\pi$ in Theorem 1 and defining

$$G_n = \sum_{k=0}^{n+1}(-4)^k\frac{B_{2k}B_{2n+2-2k}}{(2k)!(2n+2-2k)!},$$

one has

$$\zeta(2n+1) = -\frac{2\cdot 4^n S_{2n+1}(1) + 2S_{2n+1}(4) + 4^n\pi^{2n+1}G_n}{4^n+1}.$$

Combining this with equation (1) yields

$$\frac{4^n S_{2n+1}(1) - (4^n+1)S_{2n+1}(2) + S_{2n+1}(4)}{\frac{4^n}{2}(4^n+1)F_n - \frac{4^n}{2}G_n} = \pi^{2n+1}.$$

Substituting $n = 2m-1$ and defining

$$D_m = \frac{4^{2m-1}\left[(4^{2m-1}+1)F_{2m-1} - G_{2m-1}\right]}{2}$$

produces

$$\pi^{4m-1} = \frac{4^{2m-1}}{D_m} S_{4m-1}(1) - \frac{4^{2m-1}+1}{D_m} S_{4m-1}(2) + \frac{1}{D_m} S_{4m-1}(4).$$

This identity may be combined with equation (1) to give

$$\zeta(4m-1) = -\frac{F_{2m-1}4^{4m-2}}{D_m} S_{4m-1}(1) + \frac{G_{2m-1}4^{2m-1}}{D_m} S_{4m-1}(2) - \frac{F_{2m-1}4^{2m-1}}{D_m} S_{4m-1}(4).$$

To obtain formulas for the $4m+1$ cases, set $\alpha = 2\pi$, $\beta = \pi/2$, and $n = 2m$ in Theorem 1 to obtain

$$\zeta(4m+1) = \frac{-2 \cdot 16^m S_{4m+1}(1) + 2S_{4m+1}(4) - 16^m \pi^{4m+1} G_{2m}}{16^m - 1}. \tag{2}$$

Define $T_n(r)$ by

$$T_n(r) = \sum_{k=1}^{\infty} \frac{1}{k^n (e^{\pi r k} + 1)},$$

and another finite sum of Bernoulli numbers by

$$H_n = \sum_{k=0}^{n} (-4)^{n+k} \frac{B_{4k} B_{4n+2-4k}}{(4k)!(4n+2-4k)!}.$$

We begin with the case $m \geq 1$. Under this hypothesis, Vepstas established the following expression (refer to the calculation following [6, Theorem 7], and the statement in the introduction of [6]):

$$(1 + (-4)^m - 2^{4m+1})\zeta(4m+1) =$$
$$2T_{4m+1}(2) + 2(2^{4m+1} - (-4)^m) S_{4m+1}(2) + 2^{4m+1}\pi^{4m+1} H_m + 2^{4m}\pi^{4m+1} G_{2m}.$$

Vepstas also gave a formula relating T_k and S_k:

$$T_k(x) = S_k(x) - 2S_k(2x).$$

Combining the last two equations produces

$$\frac{1 + (-4)^m - 2^{4m+1}}{\frac{1}{2}(1 - 4^{2m})} \left(\frac{4^{2m}}{2} \pi^{4m+1} G_{2m} - S_{4m+1}(4) + 4^{2m} S_{4m+1}(1) \right) =$$
$$2[2^{4m+1} - (-4)^m + 1] S_{4m+1}(2) - 4S_{4m+1}(4) + 2^{4m+1}\pi^{4m+1} H_m + 2^{4m}\pi^{4m+1} G_{2m}.$$

Letting

$$K_m = \frac{\frac{1}{2}(1 - 4^{2m})}{1 + (-4)^m - 2^{4m+1}}$$

and

$$E_m = \frac{4^{2m}}{2} G_{2m} - 2^{4m+1} K_m H_m - 2^{4m} K_m G_{2m},$$

one finds

$$\pi^{4m+1} = -\frac{4^{2m}}{E_m} S_{4m+1}(1) + \frac{2K_m[2^{4m+1} - (-4)^m + 1]}{E_m} S_{4m+1}(2) + \frac{(1-4K_m)}{E_m} S_{4m+1}(4).$$

Substituting this into equation (2) produces

$$\begin{aligned} \zeta(4m+1) \quad = \quad & -\frac{16^m(2E_m - 16^m G_{2m})}{(16^m-1)E_m} S_{4m+1}(1) \\ & -\frac{2\cdot 16^m G_{2m} K_m (2\cdot 16^m - (-4)^m + 1)}{(16^m-1)E_m} S_{4m+1}(2) \\ & -\frac{16^m G_{2m}(1-4K_m) - 2E_m}{(16^m-1)E_m} S_{4m+1}(4). \end{aligned}$$

It remains to consider the case $m = 0$ and establish the formula for π stated in the introduction. This formula is a consequence of classical results about q-series, which permit the direct evaluation of $S_1(1)$, $S_1(2)$, and $S_1(4)$. First, note that [2, Equation (22.11)] gives

$$S_1(2) = \frac{1}{4}\log\left(\frac{4}{\pi}\right) - \frac{\pi}{12} + \log\Gamma\left(\frac{3}{4}\right),$$

where log denotes the natural logarithm. A straightforward adaptation of the proof of this formula in [2] yields, after using the tables of Zucker cited therein with the choice $c^2 = 4$,

$$S_1(4) = -\frac{1}{6}\log(2^{-33/4}) + \frac{1}{6}\log(\pi^{9/2}) - \log\Gamma\left(\frac{1}{4}\right) - \frac{\pi}{6}.$$

Further, [3, Equation (2.4)] gives

$$S_1(1) = S_1(4) + \frac{1}{4}\log\left(\frac{1}{4}\right) + \frac{\pi}{8}.$$

Substituting these expressions into the claimed formula for π and simplifying completes the proof.

References

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