A NUMBER FIELD ANALOGUE OF RAMANUJAN'S IDENTITY FOR $\zeta(2m+1)$

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Dedicated to Professor Bruce Berndt on the occasion of his 85th birthday

ABSTRACT. Ramanujan's famous formula for $\zeta(2m+1)$ has captivated the attention of numerous mathematicians over the years. Grosswald, in 1972, found a simple extension of Ramanujan's formula which in turn gives transformation formula for Eisenstein series over the full modular group. Recently, Banerjee, Gupta and Kumar found a number field analogue of Ramanujan's formula. In this paper, we present a new number field analogue of the Ramanujan-Grosswald formula for $\zeta(2m+1)$ by obtaining a formula for Dedekind zeta function at odd arguments. We also obtain a number field analogue of an identity of Chandrasekharan and Narasimhan, which played a crucial role in proving our main identity. As an application, we generalize transformation formula for Eisenstein series $G_{2k}(z)$ and Dedekind eta function $\eta(z)$. A new formula for the class number of a totally real number field is also obtained, which provides a connection with the Kronceker's limit formula for the Dedekind zeta function.

1. Introduction

The theory of the Riemann zeta function $\zeta(s)$ is a central object of study in number theory that holds a substantial place in the mathematical landscape. The nature of special values of $\zeta(s)$ has a rich history. Euler gave an exact evaluation for $\zeta(2k)$ in 1734, which establishes a relation between Bernoulli numbers and even zeta values. More precisely, for every positive integer k,

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2(2k)!},\tag{1.1}$$

where B_{2k} denotes 2kth Bernoulli number. The above formula instantly implies that even zeta values are transcendental. However, we have very little information about algebraic nature of positive odd zeta values. In 1979, Apery [1, 2] achieved a breakthrough by proving the irrationality of $\zeta(3)$. In 2001, Rivoal [31], Ball and Rivoal [3]

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proved that there exist infinitely many odd zeta values which are irrational. A result due to Zudilin [34], states that at least one of $\zeta(5), \zeta(7), \zeta(9)$ and $\zeta(11)$ are irrational, which is the most notable achievement in this area as of now. Prior to all these results, Ramanujan in his Notebook [28, p. 173, Ch. 14, Entry 21(i)] as well as in Lost Notebook [29, p. 319, Entry (28)] gave an intriguing formula involving odd zeta values, that has drawn the attention of many mathematicians. For any $\alpha, \beta > 0$ with $\alpha\beta = \pi^2, k \in \mathbb{Z} \setminus \{0\}$, we have

$$H_{2k+1}(\alpha) + (-1)^{k+1} H_{2k+1}(\beta) = \sum_{j=0}^{k+1} (-1)^{j-1} \frac{B_{2j}}{(2j)!} \frac{B_{2k+2-2j}}{(2k+2-2j)!} \alpha^{k+1-j} \beta^j,$$
 (1.2)

where

$$H_{2k+1}(x) = (4x)^{-k} \left(\frac{1}{2} \zeta(2k+1) + \sum_{n=1}^{\infty} \sigma_{-(2k+1)}(n) e^{-2xn} \right),$$

and the generalized divisor function $\sigma_z(n) = \sum_{d|n} d^z, z \in \mathbb{C}$.

Over the years, numerous mathematicians have generalized Ramanujan's formula in a variety of ways. Ramanujan [28, Ch. 14, Entry 8(iii)] himself gave a huge generalization of (1.2). An analogue of Ramanujan's formula (1.2) for L-functions associated to modular forms was discussed by Razar [30, Theorem 2] and Weil [35] independently. An extension for Dirichlet L-function, Lerch zeta function, and more generally for any Dirichlet series with periodic coefficients was given by Bradley [10] in 2002. Quite surprisingly, in 1977, Berndt [8] showed that Euler's formula (1.1) and Ramanujan's formula (1.2) are branches of a bigger tree, that is, they can be obtained from a single transformation formula for a generalized Eisenstien series. Recently, Dixit and the second author [16] found an interesting one variable generalization of (1.2), and later Dixit et. al. [18] and Chavan [12] established different generalizations of (1.2) for the Hurwitz zeta function. To know more detailed information about the Ramanujan's formula (1.2), we refer to [5, p. 276] and a survey article by Berndt and Straub [9], and an expository paper by Dixit [15] where one can find recent developments. Readers are also encouraged to see [6, 7, 13, 14, 20, 21, 22].

Ramanujan's formula (1.2) exhibits a profound correlation with Eisenstein series. Let \mathbb{H} be the upper half plane. For $z \in \mathbb{H}$ and an integer $k \geq 2$, we define the holomorphic Eisenstein series $G_{2k}(z)$ of weight 2k for the full modular group $SL_2(\mathbb{Z})$,

$$G_{2k}(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+nz)^{2k}}.$$

For $a, b, c, d \in \mathbb{Z}$ with ad - bc = 1, it satisfies the following modular transformation:

$$G_{2k}\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k}G_{2k}(z).$$
 (1.3)

Then the Fourier series expansion of $G_{2k}(z)$ is given by

$$G_{2k}(z) = 2\zeta(2k) \left(1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i n z} \right).$$

In particular, $G_{2k}(z)$ satisfies the following two main transformation formulae,

$$G_{2k}(z+1) = G_{2k}(z), (1.4)$$

$$G_{2k}\left(-\frac{1}{z}\right) = z^{2k}G_{2k}(z),$$
 (1.5)

which yield (1.3). We now introduce a simple extension of (1.2) given by Grosswald [19], in 1972, which shows how the above transformation formula (1.5) of $G_{2k}(z)$ is connected to Ramanujan's identity. It states that for $z \in \mathbb{H}$, $k \in \mathbb{Z} \setminus \{0\}$,

$$F_{2k+1}(z) - z^{2k} F_{2k+1}\left(-\frac{1}{z}\right) = \frac{1}{2}\zeta(2k+1)(z^{2k}-1) + \frac{(2\pi i)^{2k+1}}{2z} R_{2k+1}(z), \tag{1.6}$$

where

$$F_k(z) = \sum_{n=1}^{\infty} \sigma_{-k}(n)e^{2\pi i n z}$$

$$\tag{1.7}$$

and $R_{2k+1}(z)$ is the Ramanujan polynomial, introduced by Gun, Murty and Rath [20], defined as

$$R_{2k+1}(z) = \sum_{j=0}^{k+1} z^{2k+2-2j} \frac{B_{2j}}{(2j)!} \frac{B_{2k+2-2j}}{(2k+2-2j)!}.$$
 (1.8)

More about the above polynomial (1.8) can be seen in the paper by Murty, Smyth and Wang [26]. Setting $z = i\beta/\pi$, $\alpha\beta = \pi^2$, with $\alpha, \beta > 0$, Grosswald's identity immediately gives Ramanujan's formula (1.2). If k < -1 in (1.6), then

$$F_{2k+1}(z) - z^{2k} F_{2k+1}\left(-\frac{1}{z}\right) = \frac{1}{2}\zeta(2k+1)(z^{2k}-1). \tag{1.9}$$

One can easily check that the above formula is nothing but the transformation formula (1.5) of the Eisenstien series $G_{2k}(z)$. We further note that the above formula is equivalent to an identity of Ramanujan [5, p. 261, Entry 13].

In the literature, there are several generalizations of $\zeta(s)$, however, the Dedekind zeta function is considered as one of the most noteworthy generalizations of $\zeta(s)$ to the number fields. In this paper, our main aim is to establish a new generalization of

Ramanujan's formula (1.2) for Dedekind zeta function. Before delving deeper, we first define the Dedekind zeta function.

1.0.1. Dedekind zeta function. Let \mathbb{F} be a number field with the degree $[\mathbb{F} : \mathbb{Q}] = d = r_1 + 2r_2$, where r_1 and r_2 denote the number of real and complex embeddings (upto conjugates) of \mathbb{F} . Let D be the absolute value of the discriminant $D_{\mathbb{F}}$ of \mathbb{F} . Let $\mathcal{O}_{\mathbb{F}}$ be the ring of integers and \mathfrak{N} be the norm map of \mathbb{F} over \mathbb{Q} . Let $\mathbf{a}_{\mathbb{F}}(n)$ be the number of ideals in $\mathcal{O}_{\mathbb{F}}$ with norm n. Then the Dedekind zeta function associated to \mathbb{F} is defined by the following Dirichlet series:

$$\zeta_{\mathbb{F}}(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_{\mathbb{F}}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{\mathsf{a}_{\mathbb{F}}(n)}{n^s}, \quad \text{Re}(s) > 1, \tag{1.10}$$

where \mathfrak{a} runs over the non-zero integral ideals of $\mathcal{O}_{\mathbb{F}}$. It is well known that $\zeta_{\mathbb{F}}(s)$ has a simple pole at s=1 and the residue is given by the class number formula:

$$\lim_{s \to 1} (s - 1)\zeta_{\mathbb{F}}(s) = \frac{2^{r_1} (2\pi)^{r_2}}{\sqrt{D}} \frac{R_{\mathbb{F}} h_{\mathbb{F}}}{w_{\mathbb{F}}} := H_{\mathbb{F}}, \tag{1.11}$$

where $w_{\mathbb{F}}$ denotes the number of roots of unity contained in $\mathcal{O}_{\mathbb{F}}$, $h_{\mathbb{F}}$ is the class number, and $R_{\mathbb{F}}$ denotes the regulator of \mathbb{F} . Throughout the paper, for simplicity, we denote the above residue as $H_{\mathbb{F}}$. Further, we know that $r = r_1 + r_2 - 1$ is the rank of the unit group of \mathbb{F} and $\zeta_{\mathbb{F}}(s)$ has a zero of order r at s = 0 with

$$\lim_{s \to 0} \frac{\zeta_{\mathbb{F}}(s)}{s^r} = -\frac{R_{\mathbb{F}}h_{\mathbb{F}}}{w_{\mathbb{F}}} := C_{\mathbb{F}}.$$
(1.12)

Note that the constants $H_{\mathbb{F}}$ and $C_{\mathbb{F}}$ are related by the relation $\sqrt{D}H_{\mathbb{F}} = -2^{r_1}(2\pi)^{r_2}C_{\mathbb{F}}$. Now we define a generalized divisor function attached to a given number field \mathbb{F} , which was recently studied by Gupta and Pandit [23, Equation (1.5)]:

$$\sigma_{\mathbb{F},\ell}(n) = \sum_{d|n} \mathbf{a}_{\mathbb{F}}(d) \mathbf{a}_{\mathbb{F}}\left(\frac{n}{d}\right) d^{\ell}. \tag{1.13}$$

In their paper, they investigated Riesz sum associated to $\sigma_{\mathbb{F},\ell}(n)$. One can check that the Dirichlet series associated to $\sigma_{\mathbb{F},\ell}(n)$ is given by

$$\sum_{n=1}^{\infty} \frac{\sigma_{\mathbb{F},\ell}(n)}{n^s} = \zeta_{\mathbb{F}}(s)\zeta_{\mathbb{F}}(s-\ell), \quad \text{Re}(s) > \max\{1, 1 + \text{Re}(\ell)\}. \tag{1.14}$$

A number field analogue of Euler's identity (1.1) has been given by Klingen [25] and Siegel [32]. It states that for any totally real number field \mathbb{F} of degree n, we have

$$\zeta_{\mathbb{F}}(2m) = \frac{q_m \pi^{2mn}}{\sqrt{D}}, \quad m \in \mathbb{N},$$
(1.15)

where q_m is some fixed non-zero rational number. From the above identity, we notice that like $\zeta(2m)$, even zeta values over totally real number fields are also transcendental. Recently, Murty and Pathak [27] have also studied the arithmetic nature of Dedekind zeta function at odd positive integers. Their result states that for a given integer $n \geq 1$, at most one of $\zeta_{\mathbb{F}}(2n+1)$ is rational when \mathbb{F} runs over all imaginary quadratic fields. Recently, Banerjee, Gupta and Kumar [4] have studied transformation formula for Dedekind zeta function at odd integers which gives a generalization of (1.2). However, our generalization is different from their result.

Now we discuss the Steen function before mentioning the main result. The Steen function $V(z|a_1, a_2, \ldots, a_n)$ is defined by

$$V(z|a_1, a_2, \dots, a_n) = \frac{1}{2\pi i} \int_{(c)} \prod_{j=1}^n \Gamma(s + a_j) z^{-s} ds.$$
 (1.16)

Here and throughout the article the symbol (c) denotes the vertical line from $c - i\infty$ to $c + i\infty$. We assume that all the poles of $\Gamma(s + a_j)$ lie on one side of the vertical line (c). One can check that this function is a particular case of Meijer G-function. Special cases of Steen function are associated to many other well-known functions such as,

$$V(z|0) = e^{-z}$$
, if $c > 0$, (1.17)

$$V(z|a,b) = 2z^{\frac{1}{2}(a+b)} K_{a-b}(2z^{\frac{1}{2}}), \text{ if } c > \max\{-a, -b\},$$
(1.18)

where K_{ν} denotes the modified Bessel function of the second kind. Further information about Steen function can be found in [33], [24, p. 63].

Recall that Ramanujan's formula (1.2) as well as Grosswald's identity (1.6) has an infinite series (1.7) containing the divisor function $\sigma_z(n)$ and the exponential function, that is, for $k \in \mathbb{Z}, z \in \mathbb{H}$,

$$F_k(z) = \sum_{n=1}^{\infty} \sigma_{-k}(n)e^{2\pi i n z}.$$
 (1.19)

In the current paper, we are interested to study transformation formula for the following infinite series:

$$\mathfrak{F}_{\mathbb{F},k}(z) := \sum_{n=1}^{\infty} \sigma_{\mathbb{F},-k}(n) V\left(-\frac{(2\pi)^d niz}{D} \middle| \bar{0}_d\right). \tag{1.20}$$

One can easily check that the above series (1.20) reduces to (1.19) when $\mathbb{F} = \mathbb{Q}$. Now we are ready to state the main results of this paper in the next section.

2. Main Results

Theorem 2.1. Let \mathbb{F} be a number field of degree $d = r_1 + 2r_2$ and $\zeta_{\mathbb{F}}(s)$ be the Dedekind zeta function defined in (1.10). We consider $r = r_1 + r_2 - 1$. For any non-zero integer k, we define

$$\Lambda_{\mathbb{F},k}(s) := \Gamma(s)^d \zeta_{\mathbb{F}}(s) \zeta_{\mathbb{F}}(s+2k+1) \left(\frac{(2\pi)^d}{D}\right)^{-s}.$$
 (2.1)

Let $\mathfrak{F}_{\mathbb{F},k}(z)$ be the infinite series defined as in (1.20). Then for any $z \in \mathbb{H}$ and k > 0, we have

$$\mathfrak{S}_{\mathbb{F},2k+1}(z) = (-1)^{k(r_1+1)+r_2} z^{2k} \mathfrak{S}_{\mathbb{F},2k+1} \left(-\frac{1}{z} \right) + \sum_{j=1}^{k} \mathfrak{R}_{-(2j-1)}(z) + \sum_{j=1}^{k-1} \mathfrak{R}_{-2j}(z), \quad (2.2)$$

where

$$\mathfrak{S}_{\mathbb{F},2k+1}(z) := \mathfrak{F}_{\mathbb{F},2k+1}(z) - \mathfrak{R}_0(z) - \mathfrak{R}_1(z),$$

and the residual terms are defined as

$$\mathfrak{R}_{0}(z) = \frac{1}{(r_{2})!} \lim_{s \to 0} \frac{\mathrm{d}^{r_{2}}}{\mathrm{d}s^{r_{2}}} \left(s^{r_{2}+1} \Lambda_{\mathbb{F},k}(s) (-iz)^{-s} \right),$$

$$\mathfrak{R}_{1}(z) = H_{\mathbb{F}} \zeta_{\mathbb{F}}(2k+2) \frac{iD}{(2\pi)^{d}z},$$

$$\mathfrak{R}_{-(2j-1)}(z) = \frac{1}{(r)!} \lim_{s \to -(2j-1)} \frac{\mathrm{d}^{r}}{\mathrm{d}s^{r}} \left((s+2j-1)^{r+1} \Lambda_{\mathbb{F},k}(s) (-iz)^{-s} \right),$$

$$\mathfrak{R}_{-2j}(z) = \frac{1}{(r_{2}-1)!} \lim_{s \to -2j} \frac{\mathrm{d}^{r_{2}-1}}{\mathrm{d}s^{r_{2}-1}} \left((s+2j)^{r_{2}} \Lambda_{\mathbb{F},k}(s) (-iz)^{-s} \right).$$

Again, for k < 0, we have

$$\mathfrak{U}_{\mathbb{F},2k+1}(z) = (-1)^{k(r_1+1)+r_2} z^{2k} \mathfrak{U}_{\mathbb{F},2k+1} \left(-\frac{1}{z}\right) + \mathfrak{R}(z), \tag{2.3}$$

where

$$\mathfrak{U}_{\mathbb{F},2k+1}(z) := \mathfrak{F}_{\mathbb{F},2k+1}(z) - \frac{C_{\mathbb{F}}\zeta_{\mathbb{F}}^{(r_2)}(2k+1)}{(r_2)!},$$

and

$$\mathfrak{R}(z) = \begin{cases} -\frac{i}{4\pi z}, & \text{if } (k, r_1, r_2) = (-1, 1, 0), \\ \frac{H_{\mathbb{F}}\zeta_{\mathbb{F}}(0)Di}{(2\pi)^2 z}, & \text{if } (k, r_1, r_2) = (-1, 0, 1), \\ 0, & \text{otherwise.} \end{cases}$$
(2.4)

Remark 1. We note that an explicit evaluation of the terms $\mathfrak{R}_0(z)$, $\mathfrak{R}_{-(2j-1)}(z)$ and $\mathfrak{R}_{-2j}(z)$ is not easy as it involves higher derivatives. However, we can say that $\mathfrak{R}_0(z)$ is a polynomial in $\mathbb{C}[\log(z)]$ of degree r_2 , whereas $\mathfrak{R}_{-(2j-1)}(z)$ and $\mathfrak{R}_{-2j}(z)$ are polynomials of the form $z^{2j-1}g(\log(z))$ and $z^{2j}h(\log(z))$, where g(z) and h(z) are some polynomials of degree $r = r_1 + r_2 - 1$ and $r_2 - 1$, respectively.

Corollary 2.2. For $\mathbb{F} = \mathbb{Q}$, Theorem 2.1 reduces to the Ramanujan-Grosswald formula (1.6).

In the next subsection, we highlight some of the intriguing identities that are by product of our main result in the case of totally real number field.

2.1. Transformation formulae for totally real number fields. The next implication of Theorem 2.1 is stated as a separate theorem since it can be regarded as a formula for $\zeta_{\mathbb{F}}(2k+1)$ over a totally real number field.

Theorem 2.3. Let \mathbb{F} be a totally real number field of degree r_1 and k be a positive integer. Then we have

$$\mathfrak{S}_{\mathbb{F},2k+1}(z) = (-1)^{k(r_1+1)} z^{2k} \mathfrak{S}_{\mathbb{F},2k+1} \left(-\frac{1}{z} \right) + \sum_{j=1}^{k} \mathfrak{R}_{-(2j-1)}(z), \tag{2.5}$$

where

$$\mathfrak{S}_{\mathbb{F},2k+1}(z) = \mathfrak{F}_{\mathbb{F},2k+1}(z) - C_{\mathbb{F}}\zeta_{\mathbb{F}}(2k+1) - H_{\mathbb{F}}\zeta_{\mathbb{F}}(2k+2) \frac{iD}{(2\pi)^{r_1}z},$$

and the residual term is given by

$$\mathfrak{R}_{-(2j-1)}(z) = \frac{1}{(r_1 - 1)!} \lim_{s \to -(2j-1)} \frac{\mathrm{d}^{r_1 - 1}}{\mathrm{d}s^{r_1 - 1}} \left((s + 2j - 1)^{r_1} \Lambda_{\mathbb{F}, k}(s) (-iz)^{-s} \right).$$

In particular, for the real quadratic fields we obtain the following identity.

Corollary 2.4. Let k > 0, and \mathbb{F} be a real quadratic field i.e. $\mathbb{F} = \mathbb{Q}(\sqrt{m})$ where m is a positive square free integer. Then we have

$$\mathfrak{S}_{\mathbb{F},2k+1}(z) = (-1)^k z^{2k} \mathfrak{S}_{\mathbb{F},2k+1} \left(-\frac{1}{z} \right) + \sum_{j=1}^k \mathfrak{R}_{-(2j-1)}(z), \tag{2.6}$$

where

$$\mathfrak{S}_{\mathbb{F},2k+1}(z) = 2\sum_{n=1}^{\infty} \sigma_{\mathbb{F},-2k-1}(n) K_0 \left(2\pi \sqrt{\frac{nz}{m}} e^{-\frac{i\pi}{4}} \right) - \zeta_{\mathbb{F}}'(0) \zeta_{\mathbb{F}}(2k+1) - \frac{H_{\mathbb{F}} \zeta_{\mathbb{F}}(2k+2) im}{\pi^2 z}.$$

The term $\mathfrak{R}_{-(2j-1)}(z)$ in (2.6) is given by

$$\mathfrak{R}_{-(2j-1)}(z) = \lim_{s \to -(2j-1)} \frac{\mathrm{d}}{\mathrm{d}s} \left((s+2j-1)^2 \Lambda_{\mathbb{F},k}(s) (-iz)^{-s} \right).$$

Next, we present identities for Dedekind zeta function over totally real number fields at negative odd integers.

Theorem 2.5. Let k be a positive integer and \mathbb{F} be a totally real number field of degree r_1 . Then, we have

$$z^{2k} \left\{ \mathfrak{F}_{\mathbb{F},-2k+1}(z) - C_{\mathbb{F}} \zeta_{\mathbb{F}}(1-2k) \right\}$$

$$= (-1)^{k(r_1+1)} \left\{ \mathfrak{F}_{\mathbb{F},-2k+1} \left(-\frac{1}{z} \right) - C_{\mathbb{F}} \zeta_{\mathbb{F}} (1-2k) \right\} + \begin{cases} \frac{z^{2k}}{4\pi z i}, & if (k, r_1, r_2) = (1, 1, 0), \\ 0, & otherwise. \end{cases}$$

Remark 2. When $(r_1, r_2) = (1, 0)$ and k > 1, then the above formula immediately transforms into the Grosswald's identity (1.9). In particular, when $(k, r_1, r_2) = (1, 1, 0)$, the above formula yields the well known transformation formula for the Eisenstein series of weight 2, that is,

$$E_2\left(-\frac{1}{z}\right) = z^2\left(E_2(z) + \frac{6}{\pi i z}\right).$$

The next result gives an explicit evaluation of the infinite series $\mathfrak{F}_{\mathbb{F},-2k+1}(z)$ at z=i that involves the class number and value of the Dedekind zeta function at negative odd integers.

Corollary 2.6. Let $k \geq 3$ and $r_1 \geq 1$ be odd integers. Then for a totally real field \mathbb{F} of degree r_1 , we have

$$\sum_{n=1}^{\infty} \sigma_{\mathbb{F},2k-1}(n) V\left(\frac{(2\pi)^{r_1} n}{D} \middle| \bar{0}_{r_1}\right) = -\frac{h_{\mathbb{F}} R_{\mathbb{F}} \zeta_{\mathbb{F}} (1-2k)}{w_{\mathbb{F}}}.$$
 (2.7)

This gives a new identity for the class number of a totally real field. Using functional equation (3.4) and Siegel's identity (1.15), one can check that $\zeta_{\mathbb{F}}(1-2k)$ is a non-zero rational number.

Remark 3. In particular, when $\mathbb{F} = \mathbb{Q}$, Corollary 2.6 gives an exact evaluation of the following well-known Lambert series

$$\sum_{n=1}^{\infty} \sigma_{2k-1}(n)e^{-2\pi n} = -\frac{\zeta(1-2k)}{2} = \frac{B_{2k}}{4k}.$$

This identity was first obtained by Glaisher and later rediscovered by Ramanujan [5, p. 262, Equation (13.1)].

Next, we provide an interesting analogue of the transformation formula (1.5) for Eisenstien series $G_{2k}(z)$, namely, a transformation formula that sends $z \to -\frac{1}{z}$ for real quadratic fields.

Corollary 2.7. Let m be a positive square free integer and $\mathbb{F} = \mathbb{Q}(\sqrt{m})$. Then for $z \in \mathbb{H}$ and $k \in \mathbb{N}$, we have

$$(iz)^{2k}G_{\mathbb{F},2k}(z) = G_{\mathbb{F},2k}\left(-\frac{1}{z}\right),\,$$

where

$$G_{\mathbb{F},2k}(z) := 1 - \frac{2}{\zeta'_{\mathbb{F}}(0)\zeta_{\mathbb{F}}(1-2k)} \sum_{n=1}^{\infty} \sigma_{\mathbb{F},2k-1}(n) K_0 \left(2\pi \sqrt{\frac{nz}{m}} e^{-\frac{i\pi}{4}}\right).$$

Further, as an application of Theorem 2.1, we give number field analogue of Ramanujan-Grosswald identity for imaginary number fields.

2.2. Transformation formulae for imaginary number fields.

Theorem 2.8. Let \mathbb{F} be a purely imaginary number field with degree of extension $2r_2$ over \mathbb{Q} . Then for $k > 0, z \in \mathbb{H}$, we have

$$\mathfrak{S}_{\mathbb{F},2k+1}(z) = (-1)^{k+r_2} z^{2k} \mathfrak{S}_{\mathbb{F},2k+1} \left(-\frac{1}{z} \right) + \sum_{j=1}^{k} \mathfrak{R}_{-(2j-1)}(z) + \sum_{j=1}^{k-1} \mathfrak{R}_{-2j}(z), \qquad (2.8)$$

where

$$\mathfrak{S}_{\mathbb{F},2k+1}(z) = \mathfrak{F}_{\mathbb{F},2k+1}(z) - \mathfrak{R}_0(z) - \mathfrak{R}_1(z),$$

and the residual terms are defined as

$$\mathfrak{R}_{0}(z) = \frac{1}{(r_{2})!} \lim_{s \to 0} \frac{\mathrm{d}^{r_{2}}}{\mathrm{d}s^{r_{2}}} \left(s^{r_{2}+1} \Lambda_{\mathbb{F},k}(s) (-iz)^{-s} \right),$$

$$\mathfrak{R}_{1}(z) = H_{\mathbb{F}} \zeta_{\mathbb{F}}(2k+2) \frac{iD}{(2\pi)^{2r_{2}} z},$$

$$\mathfrak{R}_{-(2j-1)}(z) = \frac{1}{(r_{2}-1)!} \lim_{s \to -(2j-1)} \frac{\mathrm{d}^{r_{2}-1}}{\mathrm{d}s^{r_{2}-1}} \left((s+2j-1)^{r_{2}} \Lambda_{\mathbb{F},k}(s) (-iz)^{-s} \right),$$

$$\mathfrak{R}_{-2j}(z) = \frac{1}{(r_{2}-1)!} \lim_{s \to -2j} \frac{\mathrm{d}^{r_{2}-1}}{\mathrm{d}s^{r_{2}-1}} \left((s+2j)^{r_{2}} \Lambda_{\mathbb{F},k}(s) (-iz)^{-s} \right).$$

Again, for k < 0, we have

$$\mathfrak{U}_{\mathbb{F},2k+1}(z) = (-1)^{k+r_2} z^{2k} \mathfrak{U}_{\mathbb{F},2k+1} \left(-\frac{1}{z} \right) + \mathfrak{R}(z), \tag{2.9}$$

where

$$\mathfrak{U}_{\mathbb{F},2k+1}(z) := \mathfrak{F}_{\mathbb{F},2k+1}(z) - \frac{C_{\mathbb{F}}\zeta_{\mathbb{F}}^{(r_2)}(2k+1)}{(r_2)!},\tag{2.10}$$

and

$$\mathfrak{R}(z) = \begin{cases} \frac{H_{\mathbb{F}}\zeta_{\mathbb{F}}(0)Di}{(2\pi)^2 z}, & \text{if } (k, r_1, r_2) = (-1, 0, 1), \\ 0, & \text{otherwise.} \end{cases}$$
(2.11)

In particular, for quadratic imaginary fields we obtain the following transformation formula.

Corollary 2.9. Let m be a positive square free integer and $\mathbb{F} = \mathbb{Q}(\sqrt{-m})$ be a quadratic imaginary number field. Then for k > 0, we have

$$\mathfrak{S}_{\mathbb{F},2k+1}(z) = (-1)^{k+1} z^{2k} \mathfrak{S}_{\mathbb{F},2k+1} \left(-\frac{1}{z} \right) + \sum_{j=1}^{k} \mathfrak{R}_{-(2j-1)}(z) + \sum_{j=1}^{k-1} \mathfrak{R}_{-2j}(z), \qquad (2.12)$$

where

$$\mathfrak{S}_{\mathbb{F},2k+1}(z) = 2\sum_{n=1}^{\infty} \sigma_{\mathbb{F},-2k-1}(n) K_0 \left(2\pi \sqrt{\frac{nz}{m}} e^{-\frac{i\pi}{4}} \right) - \mathfrak{R}_0(z) - \mathfrak{R}_1(z),$$

and the residual terms are defined as

$$\mathfrak{R}_{0}(z) = \lim_{s \to 0} \frac{\mathrm{d}^{2}}{\mathrm{d}s^{2}} \left(s^{2} \Lambda_{\mathbb{F},k}(s) (-iz)^{-s} \right),$$

$$\mathfrak{R}_{1}(z) = H_{\mathbb{F}} \zeta_{\mathbb{F}}(2k+2) \frac{im}{\pi^{2}z},$$

$$\mathfrak{R}_{-(2j-1)}(z) = -\frac{\zeta_{\mathbb{F}}'(1-2j)}{[(2j-1)!]^{2}} \zeta_{\mathbb{F}}(2k-2j+2) \left(\frac{\pi^{2}iz}{m} \right)^{2j-1},$$

$$\mathfrak{R}_{-2j}(z) = \frac{\zeta_{\mathbb{F}}'(-2j)}{[(2j)!]^{2}} \zeta_{\mathbb{F}}(2k-2j+1) \left(\frac{\pi^{2}iz}{m} \right)^{2j}.$$

The next result also gives another modular transformation property associated to quadratic imaginary fields.

Corollary 2.10. Let m be a square free positive integer and $\mathbb{F} = \mathbb{Q}(\sqrt{-m})$. Then for $z \in \mathbb{H}$, we have

$$z^{2}\mathfrak{U}_{\mathbb{F},-1}(z) = \mathfrak{U}_{\mathbb{F},-1}\left(-\frac{1}{z}\right) + \frac{H_{\mathbb{F}}\zeta_{\mathbb{F}}(0)izm}{\pi^{2}},\tag{2.13}$$

where

$$\mathfrak{U}_{\mathbb{F},-1}(z) = \mathfrak{F}_{\mathbb{F},-1}(z) - \zeta_{\mathbb{F}}(0)\zeta_{\mathbb{F}}'(-1).$$

Moreover, letting z = i in (2.13), we obtain the following exact evaluation:

$$\sum_{n=1}^{\infty} \sigma_{\mathbb{F},1}(n) K_0 \left(2\pi \sqrt{\frac{n}{m}} \right) = \frac{1}{2} \zeta_{\mathbb{F}}(0) \zeta_{\mathbb{F}}'(-1) + \frac{H_{\mathbb{F}} \zeta_{\mathbb{F}}(0) m}{4\pi^2}.$$
 (2.14)

The below identity holds for any purely imaginary number field.

Corollary 2.11. Let \mathbb{F} be a purely imaginary field of degree $2r_2$ and k be a positive integer. If $(k, r_2) \neq (1, 1)$, then we have

$$z^{2k}\mathfrak{U}_{\mathbb{F},1-2k}(z) = (-1)^{k+r_2}\mathfrak{U}_{\mathbb{F},1-2k}\left(-\frac{1}{z}\right),\tag{2.15}$$

where

$$\mathfrak{U}_{\mathbb{F},1-2k}(z) = \mathfrak{F}_{\mathbb{F},1-2k}(z) - \frac{C_{\mathbb{F}}\zeta_{\mathbb{F}}^{(r_2)}(1-2k)}{(r_2)!}.$$

Remark 4. Moreover, substituting z = i in (2.15) and considering r_2 as an odd positive integer, we obtain the following interesting exact evaluation:

$$\sum_{n=1}^{\infty} \sigma_{\mathbb{F},2k-1}(n) V\left(\frac{(2\pi)^{2r_2}n}{D} \middle| \bar{0}_{2r_2}\right) = -\frac{h_{\mathbb{F}} R_{\mathbb{F}} \zeta_{\mathbb{F}}^{(r_2)}(1-2k)}{w_{\mathbb{F}}(r_2)!}.$$
 (2.16)

This result is true for purely imaginary number fields and it can be considered as an analogue of Corollary 2.6, which is true only for real number fields. In particular, when $\mathbb{F} = \mathbb{Q}(\sqrt{-m})$, then for $k \geq 3$, we have

$$\sum_{n=1}^{\infty} \sigma_{\mathbb{F},2k-1}(n) K_0\left(2\pi\sqrt{\frac{n}{m}}\right) = \frac{1}{2} \zeta_{\mathbb{F}}(0) \zeta_{\mathbb{F}}'(1-2k). \tag{2.17}$$

2.3. A number field analogue of transformation formula for Dedekind eta function. One of the key observations is that our main result, i.e., Theorem 2.1 loses its validity for k = 0 as $\zeta_{\mathbb{F}}(2k+1)$ exhibits a simple pole at 1. Therefore, corresponding k = 0, we must handle it separately. Quite interestingly, in this case, we will derive a number field analogue of transformation formula for Dedekind eta function $\eta(z)$, which is a half integral weight modular form.

Theorem 2.12. Let \mathbb{F} be a number field of degree d and $\mathfrak{F}_{\mathbb{F},1}(z)$ be the infinite series defined as in (1.20). Then we have

$$\mathfrak{T}_{\mathbb{F}}(z) = (-1)^{r_2} \mathfrak{T}_{\mathbb{F}} \left(-\frac{1}{z} \right) + \mathfrak{R}_0(z),$$

where

$$\mathfrak{T}_{\mathbb{F}}(z) := \mathfrak{F}_{\mathbb{F},1}(z) - \mathfrak{R}_1(z),$$

with

$$\mathfrak{R}_{0}(z) = \frac{1}{(r_{2}+1)!} \lim_{s \to 0} \frac{\mathrm{d}^{r_{2}+1}}{\mathrm{d}s^{r_{2}+1}} \left(s^{r_{2}+2} \Lambda_{\mathbb{F},0}(s) (-iz)^{-s} \right), \quad \mathfrak{R}_{1}(z) = H_{\mathbb{F}} \zeta_{\mathbb{F}}(2) \frac{iD}{(2\pi)^{d}z}.$$

In particular, we have the following modular transformation property for totally real number fields.

Corollary 2.13. Let \mathbb{F} be a totally real number field of degree r_1 and $\mathfrak{F}_{\mathbb{F},1}(z)$ be the infinite series defined in (1.20). Let $H_{\mathbb{F}}, \gamma_{\mathbb{F}}$ be the constants defined in (3.5). Then we have

$$\mathfrak{F}_{\mathbb{F},1}(z) - \mathfrak{F}_{\mathbb{F},1}\left(-\frac{1}{z}\right) = a_0 \gamma_{\mathbb{F}} - a_0 H_{\mathbb{F}}\left\{r_1 \gamma + \log\left(-\frac{(2\pi)^{r_1} iz}{D}\right)\right\} + a_1 H_{\mathbb{F}} + \mathfrak{R}_1(z) - \mathfrak{R}_1\left(-\frac{1}{z}\right), \tag{2.18}$$

where

$$\mathfrak{R}_1(z) = \frac{iDH_{\mathbb{F}}\zeta_{\mathbb{F}}(2)}{(2\pi)^{r_1}z}, \quad a_0 = \frac{\zeta_{\mathbb{F}}^{(r_1-1)}(0)}{(r_1-1)!} = C_{\mathbb{F}}, \quad a_1 = \frac{\zeta_{\mathbb{F}}^{(r_1)}(0)}{(r_1)!}.$$

Remark 5. Quite surprisingly, substituting z = i in (2.18) and using the relation between $H_{\mathbb{F}}$ and $C_{\mathbb{F}}$, we obtain a connection between class number of a totally real number field of degree r_1 and the constant term $\gamma_{\mathbb{F}}$ of the Laurent series expansion (3.5) of the Dedekind zeta function at s = 1. Mainly, we obtain

$$C_{\mathbb{F}} = \frac{a_1 - A\gamma_{\mathbb{F}}}{r_1\gamma + \log\left(\frac{(2\pi)^{r_1}}{D}\right)},\tag{2.19}$$

where $A = \frac{\sqrt{D}}{2^{r_1}(2\pi)^{r_2}}$. This suggests that finding a formula for class number is also connected with the Kronceker's limit formula for the Dedekind zeta function.

Further, letting $\mathbb{F} = \mathbb{Q}$ in Corollary 2.13, we have the following result.

Corollary 2.14. For any complex number $z \in \mathbb{H}$, we have

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n} e^{2\pi i n z} - \sum_{n=1}^{\infty} \frac{\sigma(n)}{n} e^{-\frac{2\pi i n}{z}} = \frac{i\pi z^2 + i\pi}{12z} + \frac{1}{2} \log(-iz), \tag{2.20}$$

which is equivalent to the logarithm of the transformation formula for the Dedekind eta function $\eta(z)$, namely,

$$\eta\left(-\frac{1}{z}\right) = \sqrt{-iz}\,\eta(z).$$

3. Required Tools

In this section, we state a few essential results which will be frequently used in the proof of the main results. The gamma function $\Gamma(z)$ satisfies the following functional

equation and reflection formula:

$$\Gamma(z+1) = z\Gamma(z), \quad z \in \mathbb{C},$$
 (3.1)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad z \in \mathbb{C}\backslash\mathbb{Z}$$
 (3.2)

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z). \tag{3.3}$$

We now discuss analytic properties of the Dedekind zeta function $\zeta_{\mathbb{F}}(s)$.

Lemma 3.1. The Dedekind zeta function $\zeta_{\mathbb{F}}(s)$ has an analytic continuation in the whole complex plane except for a simple pole at s = 1. It also satisfies the following functional equation relating its values at s and 1 - s,

$$\Omega_{\mathbb{F}}(s) = \Omega_{\mathbb{F}}(1-s),\tag{3.4}$$

where $\Omega_{\mathbb{F}}(s) = \left(\frac{D}{\pi^{d}4^{r_2}}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_{\mathbb{F}}(s)$.

The Dedekind zeta function $\zeta_{\mathbb{F}}(s)$ satisfies the following Laurent series expansion at s=1:

$$\zeta_{\mathbb{F}}(s) = \frac{H_{\mathbb{F}}}{s-1} + \gamma_{\mathbb{F}} + O(s-1). \tag{3.5}$$

Utilizing functional equation (3.4) of $\zeta_{\mathbb{F}}(s)$, one can verify that it has a zero at s=0 of order $r=r_1+r_2-1$. Therefore, the following Laurent series expansion at s=0 holds:

$$\zeta_{\mathbb{F}}(s) = a_0 s^{r_1 + r_2 - 1} + a_1 s^{r_1 + r_2} + O\left(s^{r_1 + r_2 + 1}\right), \tag{3.6}$$

where $a_0 = \frac{\zeta^{(r_1+r_2-1)}(0)}{(r_1+r_2-1)!}$ and $a_1 = \frac{\zeta^{(r_1+r_2)}(0)}{(r_1+r_2)!}$. From (1.12), one can observe that a_0 is nothing but the constant $C_{\mathbb{F}}$.

Now we mention an interesting identity due to Chandrasekharan and Narasimhan [11, Equation (57)]. For any integer k, one can prove that

$$\Lambda_k(s) = (-1)^k \Lambda_k(-s - 2k),$$

where $\Lambda_k(s) = (2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s+2k+1)$. We generalize this identity for Dedekind zeta function $\zeta_{\mathbb{F}}(s)$. Mainly, we prove the below result, which will play a crucial role in proving our main identity.

Lemma 3.2. Let \mathbb{F} be a number field of degree d and $\zeta_{\mathbb{F}}(s)$ be the Dededkind zeta function defined in (1.10). For any $k \in \mathbb{Z}$, we have

$$\Lambda_{\mathbb{F},k}(s) = (-1)^{kr_1 + r_2} \Lambda_{\mathbb{F},k}(-s - 2k),$$

where $\Lambda_{\mathbb{F},k}(s) = D^s(2\pi)^{-ds}\Gamma(s)^d\zeta_{\mathbb{F}}(s)\zeta_{\mathbb{F}}(s+2k+1).$

Proof. We first prove this identity for any non-negative integer k. From the functional equation (3.4) of $\zeta_{\mathbb{F}}(s)$, we have

$$\left(\frac{D}{\pi^d 4^{r_2}}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_{\mathbb{F}}(s) = \left(\frac{D}{\pi^d 4^{r_2}}\right)^{\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right)^{r_1} \Gamma(1-s)^{r_2} \zeta_{\mathbb{F}}(1-s). \quad (3.7)$$

Replace s by s + 2k + 1 in (3.7) to see that

$$\left(\frac{D}{\pi^{d}4^{r_2}}\right)^{\frac{s+2k+1}{2}} \Gamma\left(\frac{s+2k+1}{2}\right)^{r_1} \Gamma(s+2k+1)^{r_2} \zeta_{\mathbb{F}}(s+2k+1)
= \left(\frac{D}{\pi^{d}4^{r_2}}\right)^{\frac{-s-2k}{2}} \Gamma\left(-\frac{s}{2}-k\right)^{r_1} \Gamma(-s-2k)^{r_2} \zeta_{\mathbb{F}}(-s-2k).$$
(3.8)

Now we multiply (3.7) and (3.8) to get

$$\left(\frac{D}{\pi^{d}4^{r_2}}\right)^{2s+2k} \left\{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}+k\right)\right\}^{r_1} \left\{\Gamma(s)\Gamma(s+2k+1)\right\}^{r_2} \zeta_{\mathbb{F}}(s)\zeta_{\mathbb{F}}(s+2k+1)$$

$$= \left\{\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\Gamma\left(-\frac{s}{2}-k\right)\right\}^{r_1} \left\{\Gamma(1-s)\Gamma(-s-2k)\right\}^{r_2} \zeta_{\mathbb{F}}(1-s)\zeta_{\mathbb{F}}(-s-2k). \quad (3.9)$$

To simplify the above equation further, we use the functional equation (3.1) of $\Gamma(s)$ repeatedly and find that, for any non-negative integer k,

$$\Gamma\left(\frac{s+1}{2} + k\right) = \frac{1}{2^k}(s+2k-1)(s+2k-3)\cdots(s+1)\Gamma\left(\frac{s}{2} + \frac{1}{2}\right). \tag{3.10}$$

Again, using (3.1) repeatedly, one can show that

$$\Gamma\left(-\frac{s}{2} - k\right) = \frac{(-2)^k \Gamma\left(-\frac{s}{2}\right)}{(s+2k)(s+2k-2)\cdots(s+2)}.$$
(3.11)

Utilize (3.10), (3.11) and duplication formula (3.3) for $\Gamma(s)$ to derive

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}+k\right) = \frac{2}{2^{s+k}}\sqrt{\pi}(s+2k-1)(s+2k-3)\cdots(s+1)\Gamma(s), \quad (3.12)$$

$$\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)\Gamma\left(-\frac{s}{2} - k\right) = \frac{(-1)^k 2^{1+s+k} \sqrt{\pi}\Gamma(-s)}{(s+2k)(s+2k-2)\cdots(s+2)}.$$
(3.13)

We again employ (3.1) to see that, for any non-negative integer k,

$$\Gamma(s)\Gamma(s+2k+1) = s(s+1)\cdots(s+2k)\Gamma(s)^2,$$
 (3.14)

$$\Gamma(-s - 2k)\Gamma(1 - s) = \frac{-s(\Gamma(-s))^2}{(s + 2k)(s + 2k - 1)\cdots(s + 1)}.$$
(3.15)

Now substituting expressions from (3.12)-(3.15) in (3.9) and upon simplification, we have

$$\left(\frac{D}{(2\pi)^d}\right)^{2s+2k} \left\{ (s+1)(s+2)\cdots(s+2k) \right\}^d \Gamma(s)^d \zeta_{\mathbb{F}}(s) \zeta_{\mathbb{F}}(s+2k+1)
= (-1)^{kr_1+r_2} (\Gamma(-s))^d \zeta_{\mathbb{F}}(1-s) \zeta_{\mathbb{F}}(-s-2k).$$
(3.16)

Here we have used the fact that $r_1 + 2r_2 = d$. Applying (3.1) again, one can check that

$$\frac{\Gamma(-s)}{(s+2k)(s+2k-1)\dots(s+2)(s+1)} = \Gamma(-s-2k). \tag{3.17}$$

Using (3.17) in (3.16), we obtain

$$\left(\frac{D}{(2\pi)^d}\right)^{2s+2k} \Gamma(s)^d \zeta_{\mathbb{F}}(s) \zeta_{\mathbb{F}}(s+2k+1)$$
$$= (-1)^{kr_1+r_2} \Gamma(-s-2k)^d \zeta_{\mathbb{F}}(1-s) \zeta_{\mathbb{F}}(-s-2k).$$

Now letting $\Lambda_{\mathbb{F},k}(s) = D^s(2\pi)^{-ds}\Gamma(s)^d\zeta_{\mathbb{F}}(s)\zeta_{\mathbb{F}}(s+2k+1)$, the above identity reduces to $\Lambda_{\mathbb{F},k}(s) = (-1)^{kr_1+r_2}\Lambda_{\mathbb{F},k}(-s-2k).$

This completes the proof of this lemma for any non-negative integer k. In a similar fashion, one can prove that this identity also holds for any negative integer as well. \square

In the next section, we present proofs of our main results.

4. Proof of Main Results

Proof of Theorem 2.1. Using the definition (1.16) of the Steen function, for y > 0, we write

$$\sum_{n=1}^{\infty} \sigma_{\mathbb{F},-2k-1}(n) V\left(\frac{(2\pi)^d ny}{D}\middle| \bar{0}_d\right) = \sum_{n=1}^{\infty} \sigma_{\mathbb{F},-2k-1}(n) \frac{1}{2\pi i} \int_{(c)} \Gamma(s)^d \left(\frac{(2\pi)^d ny}{D}\right)^{-s} ds$$

$$= \frac{1}{2\pi i} \int_{(c)} \Gamma(s)^d \sum_{n=1}^{\infty} \frac{\sigma_{\mathbb{F},-2k-1}(n)}{n^s} \left(\frac{(2\pi)^d y}{D}\right)^{-s} ds,$$

$$(4.1)$$

where we choose the line of integration as $\max\{1, -2k\} < \text{Re}(s) = c < \max\{1, -2k\} + \epsilon$, with $0 < \epsilon < 1$, so that the above Dirichlet series converges absolutely and uniformly. Hence, the interchange of summation and integration is justifiable. Now utilize the definition of the Dirichlet series (1.14) in (4.1), to see that

$$\sum_{n=1}^{\infty} \sigma_{\mathbb{F},-2k-1}(n) V\left(\frac{(2\pi)^d ny}{D} \middle| \bar{0}_d\right) = \frac{1}{2\pi i} \int_{(c)} \Gamma(s)^d \zeta_{\mathbb{F}}(s) \zeta_{\mathbb{F}}(s+2k+1) \left(\frac{(2\pi)^d y}{D}\right)^{-s} \mathrm{d}s. \tag{4.2}$$

Now our aim is to study the following vertical line integral,

$$J_{\mathbb{F},k}(y) = \frac{1}{2\pi i} \int_{(c)} \Gamma(s)^d \zeta_{\mathbb{F}}(s) \zeta_{\mathbb{F}}(s+2k+1) \left(\frac{(2\pi)^d y}{D}\right)^{-s} ds.$$

From the definition (2.1) of $\Lambda_{\mathbb{F},k}(s)$, it is clear that the above integral can be rewritten as

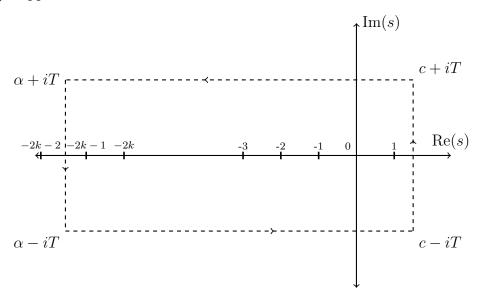
$$J_{\mathbb{F},k}(y) = \frac{1}{2\pi i} \int_{(c)} \Lambda_{\mathbb{F},k}(s) y^{-s} ds.$$

We now set up a rectangular contour C made up of the vertices c - iT, c + iT, $\alpha + iT$ and $\alpha - iT$ taken in counter-clockwise direction. We already have $c > \max\{1, -2k\}$ with T being some large positive quantity and we wisely choose

$$\min\{-1, -2k - 2\} < \alpha < \min\{0, -2k - 1\}.$$

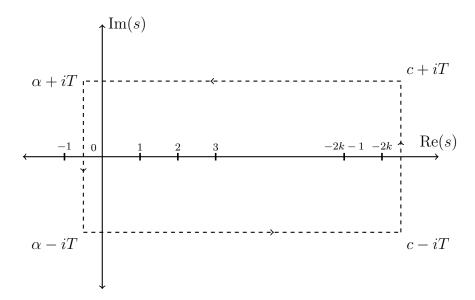
Before applying Cauchy's residue theorem, we need to examine all the poles of $\Lambda_{\mathbb{F},k}(s)$ inside the contour \mathcal{C} . We know that $\Gamma(s)^d$ has poles of order d at non-positive integers. Other poles will depend upon the value of k which will affect location of vertices of \mathcal{C} . Here we divide in two cases.

Case I) Suppose k > 0.



Now inside the contour C, for $\zeta_{\mathbb{F}}(s)$, s=0 is a zero of order r_1+r_2-1 , $s\in\{-2,-4,\ldots,-2k\}$ are zeros of order r_1+r_2 , $s\in\{-1,-3,\ldots,-2k-1\}$ are zeros of order r_2 and s=1 is a simple pole. For $\zeta_{\mathbb{F}}(s+2k+1)$, s=-2k-1 is a zero of order r_1+r_2-1 and s=-2k is a simple pole. Hence final poles of $\Lambda_{\mathbb{F},k}(s)$ are at s=0,-2k of order r_2+1 , s=1,-2k-1 are simple poles, $s\in\{-2,-4,\ldots,-2k+2\}$ are poles of order r_2 and $s\in\{-1,-3,\ldots,-2k+1\}$ are poles of order r_1+r_2 .

Case II) When k < 0.



We see that s=0 is a zero of order r_1+r_2-1 for $\zeta_{\mathbb{F}}(s)$ and a zero of order r_2 for $\zeta_{\mathbb{F}}(s+2k+1)$. Note that s=1 is a simple pole for $\zeta_{\mathbb{F}}(s)$ and s=-2k is a simple pole for $\zeta_{\mathbb{F}}(s+2k+1)$. Hence final poles for $\Lambda_{\mathbb{F},k}(s)$, when k<0, are at s=0,1,-2k, which are all simple poles.

Now, considering the above two cases and applying Cauchy's residue theorem, we have

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \Lambda_{\mathbb{F},k}(s) y^{-s} ds = \mathcal{R}, \tag{4.3}$$

where $\Lambda_{\mathbb{F},k}(s) = \Gamma(s)^d \zeta_{\mathbb{F}}(s) \zeta_{\mathbb{F}}(s+2k+1)(2\pi)^{-s} D^s$ and the residual term is given by

where
$$\Lambda_{\mathbb{F},k}(s) = \Gamma(s)^d \zeta_{\mathbb{F}}(s) \zeta_{\mathbb{F}}(s+2k+1) (2\pi)^{-s} D^s$$
 and the residual term is given by
$$\mathcal{R} = \begin{cases}
R_0^{(r_2+1)}(y) + R_1^{(1)}(y) + R_{-2k}^{(r_2+1)}(y) + R_{-2k-1}^{(1)}(y) + \sum_{j=1}^k R_{-(2j-1)}^{(r_1+r_2)}(y) \\
+ \sum_{j=1}^k R_{-2j}^{(r_2)}(y), & \text{for } k > 0, \\
R_0^{(1)}(y) + R_1^{(1)}(y) + R_{-2k}^{(1)}(y), & \text{for } k < 0,
\end{cases}$$
(4.4)

where $R_{\gamma}^{(l)}(y)$ denotes the residue of $\Lambda_{\mathbb{F},k}(s)y^{-s}$ of order l at $s=\gamma$. Letting $T\to\infty$ and making use of Stirling's formula for $\Gamma(s)$, we can show that both the horizontal integrals vanish. Now, after taking the left vertical integral to the right side and using (4.2) in (4.3), we are left with

$$\sum_{n=1}^{\infty} \sigma_{\mathbb{F},-2k-1}(n) V\left(\frac{(2\pi)^d y}{D} \middle| \bar{0}_d\right) = \mathcal{R} + \frac{1}{2\pi i} \int_{(\alpha)} \Lambda_{\mathbb{F},k}(s) y^{-s} ds. \tag{4.5}$$

Now our main aim is to simplify the above vertical integral, which we defined as

$$I_{\mathbb{F},k}(y) := \frac{1}{2\pi i} \int_{(\alpha)} \Lambda_{\mathbb{F},k}(s) y^{-s} ds.$$
 (4.6)

Replace s by -t - 2k, so that $Re(t) = -Re(s) - 2k = -\alpha - 2k > \max\{1, -2k\}$. Let $\beta = Re(t) = -\alpha - 2k$. After this change of variable, the above integral becomes

$$I_{\mathbb{F},k}(y) = \frac{1}{2\pi i} \int_{(\beta)} \Lambda_{\mathbb{F},k}(-t - 2k) y^{t+2k} dt.$$

Now we bring in use of Lemma 3.2 to write as

$$I_{\mathbb{F},k}(y) = (-1)^{kr_1 + r_2} y^{2k} \frac{1}{2\pi i} \int_{(\beta)} \Lambda_{\mathbb{F},k}(t) y^t dt$$
$$= (-1)^{(kr_1 + r_2)} y^{2k} \sum_{n=1}^{\infty} \sigma_{\mathbb{F},-2k-1}(n) V\left(\frac{(2\pi)^d n}{Dy} \middle| \bar{0}_d\right), \tag{4.7}$$

where we have used (4.2) in the last step. Now substituting (4.7) in (4.5), we obtain

$$\sum_{n=1}^{\infty} \sigma_{\mathbb{F},-2k-1}(n) V\left(\frac{(2\pi)^d ny}{D} \middle| \bar{0}_d\right) = (-1)^{kr_1+r_2} y^{2k} \sum_{n=1}^{\infty} \sigma_{\mathbb{F},-2k-1}(n) V\left(\frac{(2\pi)^d n}{Dy} \middle| \bar{0}_d\right) + \mathcal{R}.$$
(4.8)

Now we are left with calculating the term \mathcal{R} containing all the residual terms.

Case 1: Let k > 0. In this case we need to calculate the following terms:

$$\mathcal{R} = R_0^{(r_2+1)}(y) + R_1^{(1)}(y) + R_{-2k}^{(r_2+1)}(y) + R_{-2k-1}^{(1)}(y) + \sum_{j=1}^k R_{-(2j-1)}^{(r_1+r_2)}(y) + \sum_{j=1}^{k-1} R_{-2j}^{(r_2)}(y).$$

Using the definition of residue calculation, one can check that

$$R_0^{(r_2+1)}(y) = \frac{1}{(r_2)!} \lim_{s \to 0} \frac{\mathrm{d}^{r_2}}{\mathrm{d}s^{r_2}} \left(s^{r_2+1} \Lambda_{\mathbb{F},k}(s) y^{-s} \right), \tag{4.9}$$

$$R_1^{(1)}(y) = \lim_{s \to 1} (s-1) \Lambda_{\mathbb{F},k}(s) y^{-s}$$

$$= \lim_{s \to 1} (s-1) \Gamma(s)^d \zeta_{\mathbb{F}}(s) \zeta_{\mathbb{F}}(s+2k+1) \left(\frac{(2\pi)^d y}{D} \right)^{-s}$$

$$= H_{\mathbb{F}} \zeta_{\mathbb{F}}(2k+2) \frac{D}{(2\pi)^d y}, \tag{4.10}$$

in the last step we have used the class number formula (1.11). Further, we have

$$R_{-2k}^{(r_2+1)}(y) = \frac{1}{(r_2)!} \lim_{s \to -2k} \frac{\mathrm{d}^{r_2}}{\mathrm{d}s^{r_2}} \left((s+2k)^{r_2+1} \Lambda_{\mathbb{F},k}(s) y^{-s} \right). \tag{4.11}$$

Now using Lemma 3.2, the above residual term can be rewritten as follows

$$R_{-2k}^{(r_2+1)}(y) = \frac{(-1)^{kr_1+r_2}}{(r_2)!} \lim_{s \to -2k} \frac{\mathrm{d}^{r_2}}{\mathrm{d}s^{r_2}} \left((s+2k)^{r_2+1} \Lambda_{\mathbb{F},k} (-s-2k) y^{-s} \right),$$

$$= \frac{(-1)^{kr_1+r_2+1}y^{2k}}{(r_2)!} \lim_{s\to 0} \frac{\mathrm{d}^{r_2}}{\mathrm{d}s^{r_2}} \left(s^{r_2+1}\Lambda_{\mathbb{F},k}(s)y^s\right),$$

$$= (-1)^{kr_1+r_2+1}y^{2k}R_0^{(r_2+1)}\left(\frac{1}{y}\right). \tag{4.12}$$

This shows that the residue at s=0 and s=-2k are linked by the above relation. Next, the residue at s=-2k-1 is given by

$$R_{-2k-1}^{(1)}(y) = \lim_{s \to -2k-1} (s+2k+1) \Lambda_{\mathbb{F},k}(s) y^{-s}.$$

Further, use Lemma 3.2 to see that

$$R_{-2k-1}^{(1)}(y) = (-1)^{kr_1+r_2} \lim_{s \to -2k-1} (s+2k+1) \Lambda_{\mathbb{F},k}(-s-2k) y^{-s}$$

$$= (-1)^{kr_1+r_2+1} y^{2k} \lim_{s \to 1} (s-1) \Lambda_{\mathbb{F},k}(s) y^{s}$$

$$= (-1)^{kr_1+r_2+1} y^{2k} R_1^{(1)} \left(\frac{1}{y}\right)$$

$$= (-1)^{kr_1+r_2+1} y^{2k} H_{\mathbb{F}} \zeta_{\mathbb{F}}(2k+2) \frac{Dy}{(2\pi)^d}.$$
(4.14)

The above relation indicates the connection between the residues at s = 1 and s = -2k - 1. The remaining residues at s = -(2j - 1) and s = -2j can be evaluated as follows:

$$R_{-(2j-1)}^{(r_1+r_2)}(y) = \frac{1}{(r_1+r_2-1)!} \lim_{s \to -(2j-1)} \frac{\mathrm{d}^{r_1+r_2-1}}{\mathrm{d}s^{r_1+r_2-1}} \left((s+2j-1)^{r_1+r_2} \Lambda_{\mathbb{F},k}(s) y^{-s} \right),$$

$$= \frac{1}{(r)!} \lim_{s \to -(2j-1)} \frac{\mathrm{d}^r}{\mathrm{d}s^r} \left((s+2j-1)^{r+1} \Lambda_{\mathbb{F},k}(s) y^{-s} \right), \tag{4.15}$$

$$R_{-(2j)}^{(r_2)}(y) = \frac{1}{(r_2 - 1)!} \lim_{s \to -2j} \frac{\mathrm{d}^{r_2 - 1}}{\mathrm{d}s^{r_2 - 1}} \left((s + 2j)^{r_2} \Lambda_{\mathbb{F},k}(s) y^{-s} \right). \tag{4.16}$$

As we know the poles of $\Lambda_{\mathbb{F},k}(s)y^{-s}$ at s=-(2j-1) and s=-2j are of higher order, so it is a difficult task to simplify the above residual terms. However, one can say that these residual terms will be polynomials in $\mathbb{C}[y,\log(y)]$ with degree being $2j+r_1+r_2-2$ and $2j+r_2-1$, respectively. This is because for the residual term (4.15) the highest degree of y will be 2j-1 and the highest power of $\log(y)$ will be $r=r_1+r_2-1$. Similarly, for the residual term (4.16) the highest degree of y will be 2j and the highest degree of $\log(y)$ will be r_2-1 . In a similar fashion, one can verify that the residue at s=0, i.e., the residual term (4.9) is a polynomial in $\mathbb{C}[\log(y)]$ of degree r_2 . This completes the calculations of all residues for k>0. Now we shall compute residues for k<0.

Case 2: When k < 0, from (4.4), we know that the residual term \mathcal{R} is given by

$$\mathcal{R} = R_0^{(1)}(y) + R_1^{(1)}(y) + R_{-2k}^{(1)}(y).$$

Recall that, in this case, poles of $\Lambda_{\mathbb{F},k}(s)$ are at 0,1,-2k and all are simple. Therefore, we have

$$R_0^{(1)}(y) = \lim_{s \to 0} s \Lambda_{\mathbb{F},k}(s) y^{-s} = \lim_{s \to 0} s \Gamma(s)^d \zeta_{\mathbb{F}}(s) \zeta_{\mathbb{F}}(s+2k+1) \left(\frac{(2\pi)^d y}{D}\right)^{-s}$$

$$= \lim_{s \to 0} (s\Gamma(s))^d \frac{\zeta_{\mathbb{F}}(s)}{s^{r_1+r_2-1}} \frac{\zeta_{\mathbb{F}}(s+2k+1)}{s^{r_2}} \left(\frac{(2\pi)^d y}{D}\right)^{-s}$$

$$= \lim_{s \to 0} \Gamma(s+1)^d \frac{\zeta_{\mathbb{F}}(s)}{s^{r_1+r_2-1}} \frac{\zeta_{\mathbb{F}}(s+2k+1)}{s^{r_2}} \left(\frac{(2\pi)^d y}{D}\right)^{-s}$$

$$= \frac{C_{\mathbb{F}} \zeta_{\mathbb{F}}^{(r_2)}(2k+1)}{(r_2)!}.$$
(4.17)

To obtain the final step we used (1.12) and s = 0 is a zero of order r_2 for $\zeta_{\mathbb{F}}(s + 2k + 1)$ as k < 0 and negative odd integers are zeros of $\zeta_{\mathbb{F}}(s)$ of order r_2 . Now we calculate the residual term, at s = -2k, which is given by

$$R_{-2k}^{(1)}(y) = \lim_{s \to -2k} (s+2k) \Lambda_{\mathbb{F},k}(s) y^{-s}.$$

To obtain a simplified form, we use Lemma 3.2. Thus, we get

$$R_{-2k}^{(1)}(y) = (-1)^{kr_1+r_2} \lim_{s \to -2k} (s+2k) \Lambda_{\mathbb{F},k}(-s-2k) y^{-s}$$

$$= (-1)^{kr_1+r_2+1} y^{2k} \lim_{s \to 0} s \Lambda_{\mathbb{F},k}(s) y^{s}$$

$$= (-1)^{kr_1+r_2+1} y^{2k} R_0^{(1)} \left(\frac{1}{y}\right)$$

$$= (-1)^{kr_1+r_2+1} y^{2k} \frac{C_{\mathbb{F}} \zeta_{\mathbb{F}}^{(r_2)}(2k+1)}{(r_2)!}, \tag{4.18}$$

here in the penultimate step and final step, we have used the definition of $R_0^1(y)$ and its final expression (4.17). This indicates that the residues at s = 0 and s = -2k are associated with each other. Thus, from (4.17) and (4.18), we have

$$R_0^{(1)}(y) + R_{-2k}^{(1)}(y) = \left\{1 + (-1)^{kr_1 + r_2 + 1} y^{2k}\right\} \frac{C_{\mathbb{F}} \zeta_{\mathbb{F}}^{(r_2)}(2k+1)}{(r_2)!}.$$
 (4.19)

Finally, the residue at s=1, a simple pole of $\Lambda_{\mathbb{F},k}(s)y^{-s}$, can be evaluated as follows:

$$R_1^{(1)}(y) = \lim_{s \to 1} (s-1)\Lambda_{\mathbb{F},k}(s)y^{-s} = \lim_{s \to 1} (s-1)\Gamma(s)^d \zeta_{\mathbb{F}}(s)\zeta_{\mathbb{F}}(s+2k+1) \left(\frac{(2\pi)^d y}{D}\right)^{-s}$$
$$= H_{\mathbb{F}}\zeta_{\mathbb{F}}(2k+2)\frac{D}{(2\pi)^d y},$$

$$= \begin{cases} -\frac{1}{4\pi y}, & \text{if } (k, r_1, r_2) = (-1, 1, 0), \\ \frac{H_{\mathbb{F}}\zeta_{\mathbb{F}}(0)D}{(2\pi)^2 y}, & \text{if } (k, r_1, r_2) = (-1, 0, 1), \\ 0, & \text{otherwise.} \end{cases}$$
(4.20)

Till now, we assumed that y > 0. However, by analytic continuation, one can show that the identity can be extended for Re(y) > 0. Thus, we substitute y = -iz with $z \in \mathbb{H}$ in (4.8) to get the following form, for $z \in \mathbb{H}$,

$$\sum_{n=1}^{\infty} \sigma_{\mathbb{F},-2k-1}(n) V\left(\frac{-(2\pi)^d niz}{D} \middle| \bar{0}_d\right) = (-1)^{k(r_1+1)+r_2} z^{2k} \sum_{n=1}^{\infty} \sigma_{\mathbb{F},-2k-1}(n) V\left(\frac{(2\pi)^d ni}{Dz} \middle| \bar{0}_d\right) + \mathcal{R},$$

where, for k > 0, the term \mathcal{R} is given by

$$\mathcal{R} = \left\{ R_0^{(r_2+1)}(-iz) + (-1)^{k(r_1+1)+r_2+1} z^{2k} R_0^{(r_2+1)} \left(\frac{i}{z}\right) \right\}$$

$$+ \left\{ R_1^{(1)}(-iz) + (-1)^{k(r_1+1)+r_2+1} z^{2k} R_1^{(1)} \left(\frac{i}{z}\right) \right\}$$

$$+ \sum_{j=1}^k R_{-(2j-1)}^{(r_1+r_2)}(-iz) + \sum_{j=1}^{k-1} R_{-2j}^{(r_2)}(-iz).$$

Here we note that to get the above form we have used the relations (4.12) and (4.13) among residues. To simplicity further, we define

$$\begin{split} \mathfrak{R}_0(z) &:= R_0^{(r_2+1)}(-iz), \mathfrak{R}_1(z) := R_1^{(1)}(-iz), \\ \mathfrak{R}_{-(2j-1)}(z) &:= R_{-(2i-1)}^{(r_1+r_2)}(-iz), \mathfrak{R}_{-(2j)}(z) := R_{-2j}^{(r_2)}(-iz). \end{split}$$

Then the above the term \mathcal{R} becomes, for k > 0,

$$\mathcal{R} = \left\{ \mathfrak{R}_{0}(z) + (-1)^{k(r_{1}+1)+r_{2}+1} z^{2k} \mathfrak{R}_{0} \left(-\frac{1}{z} \right) \right\}$$

$$+ \left\{ \mathfrak{R}_{1}(z) + (-1)^{k(r_{1}+1)+r_{2}+1} z^{2k} \mathfrak{R}_{1} \left(-\frac{1}{z} \right) \right\}$$

$$\sum_{j=1}^{k} \mathfrak{R}_{-(2j-1)}(z) + \sum_{j=1}^{k-1} \mathfrak{R}_{-(2j)}(z).$$

In a similar way, after substituting y = -iz, for k < 0, from (4.17)-(4.19), the residual term \mathcal{R} can be written as

$$\mathcal{R} = \left\{ 1 + (-1)^{k(r_1+1)+r_2+1} z^{2k} \right\} \frac{C_{\mathbb{F}} \zeta_{\mathbb{F}}^{(r_2)}(2k+1)}{(r_2)!} + \mathfrak{R}(z),$$

where the term $\Re(z)$ is given by

$$\mathfrak{R}(z) = \begin{cases} -\frac{i}{4\pi z}, & \text{if } (k, r_1, r_2) = (-1, 1, 0), \\ \frac{H_{\mathbb{F}}\zeta_{\mathbb{F}}(0)Di}{(2\pi)^2 z}, & \text{if } (k, r_1, r_2) = (-1, 0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Finally, combining all these residual terms and using the definition (1.20) of $\mathfrak{F}_{\mathbb{F},k}(z)$, we obtain the final identity.

Proof of Corollary 2.2. If we consider $\mathbb{F} = \mathbb{Q}$, then one can easily check that

$$(r_1, r_2, d, D, H_{\mathbb{F}}, C_{\mathbb{F}}) = (1, 0, 1, 1, 1, -1/2).$$

Moreover, the number field analogous generalized divisor function $\sigma_{\mathbb{F},-k}(n)$ reduces to the usual generalized divisor function $\sigma_{-k}(n)$. Therefore, for k > 0, we have

$$\mathfrak{F}_{\mathbb{F},2k+1}(z) = \sum_{n=1}^{\infty} \sigma_{-(2k+1)}(n) e^{2\pi i n z},$$

$$\mathfrak{S}_{\mathbb{F},2k+1}(z) = \sum_{n=1}^{\infty} \sigma_{-(2k+1)}(n) e^{2\pi i n z} - \mathfrak{R}_{0}(z) - \mathfrak{R}_{1}(z),$$

where $\mathfrak{R}_0(z) = -\frac{\zeta(2k+1)}{2}$ and $\mathfrak{R}_1(z) = \frac{i\zeta(2k+2)}{2\pi z} = \frac{(2\pi i)^{2k+1}}{2z} \frac{B_{2k+2}}{(2k+2)!}$. Again, one can check that

$$\mathfrak{R}_{-(2j-1)}(z) = \lim_{s \to -(2j-1)} (s+2j-1)\Gamma(s)\zeta(s)\zeta(s+2k+1)(-2\pi i z)^{-s}$$
$$= \frac{(2\pi i)^{2k+1}}{2z} \frac{B_{2j}}{(2j)!} \frac{B_{2k-2j+2}}{(2k-2j+2)!} z^{2j}.$$

Here we have used the fact that $\zeta(1-2j) = -\frac{B_{2j}}{(2j)!}$ and Euler's formula (1.1). Substituting these values in Theorem 2.1, we get

$$\sum_{n=1}^{\infty} \sigma_{-(2k+1)}(n)e^{2\pi i n z} + \frac{\zeta(2k+1)}{2} - \frac{(2\pi i)^{2k+1}}{2z} \frac{B_{2k+2}}{(2k+2)!}$$

$$= z^{2k} \left\{ \sum_{n=1}^{\infty} \sigma_{-(2k+1)}(n)e^{-2\pi i n/z} + \frac{\zeta(2k+1)}{2} + \frac{(2\pi i)^{2k+1}}{2z} \frac{z^2 B_{2k+2}}{(2k+2)!} \right\}$$

$$+ \frac{(2\pi i)^{2k+1}}{2z} \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!} \frac{B_{2k-2j+2}}{(2k-2j+2)!} z^{2j}.$$

This implies that

$$\sum_{n=1}^{\infty} \sigma_{-(2k+1)}(n)e^{2\pi i n z} + \frac{\zeta(2k+1)}{2} = z^{2k} \left\{ \sum_{n=1}^{\infty} \sigma_{-(2k+1)}(n)e^{-2\pi i n/z} + \frac{\zeta(2k+1)}{2} \right\} + \frac{(2\pi i)^{2k+1}}{2z} \sum_{j=0}^{k+1} \frac{B_{2j}}{(2j)!} \frac{B_{2k-2j+2}}{(2k-2j+2)!} z^{2j},$$

which is exactly same as the identity (1.6). This completes the proof.

Proof of Theorem 2.3. Given that \mathbb{F} is a totally real number field of degree r_1 , so we have $r_2 = 0$ and $d = r_1$. Thus, in Theorem 2.1, we have

$$\mathfrak{R}_{0}(z) = \lim_{s \to 0} s \Gamma^{r_{1}}(s) \zeta_{\mathbb{F}}(s) \zeta_{\mathbb{F}}(s+2k+1) \left(-\frac{(2\pi)^{r_{1}}iz}{D}\right)^{-s}$$

$$= \lim_{s \to 0} \Gamma^{r_{1}}(s+1) \frac{\zeta_{\mathbb{F}}(s)}{s^{r_{1}-1}} \zeta_{\mathbb{F}}(s+2k+1) \left(-\frac{(2\pi)^{r_{1}}iz}{D}\right)^{-s}$$

$$= C_{\mathbb{F}} \zeta_{\mathbb{F}}(2k+1),$$

$$\mathfrak{R}_{1}(z) = H_{\mathbb{F}} \zeta_{\mathbb{F}}(2k+2) \frac{iD}{(2\pi)^{r_{1}}z},$$

$$\mathfrak{R}_{-(2j-1)}(z) = \frac{1}{(r_{1}-1)!} \lim_{s \to (1-2j)} \frac{\mathrm{d}^{r_{1}-1}}{\mathrm{d}s^{r_{1}-1}} \left((s+2j-1)^{r_{1}} \Lambda_{\mathbb{F},k}(s)(-iz)^{-s}\right).$$

Note that we have used (1.12) to evaluate $\mathfrak{R}_0(z)$. Moreover, we point out that the terms $\mathfrak{R}_{-2j}(z)$ would not appear in this case since we are dealing with totally real fields. Substituting the above values in Theorem 2.1, we finish the proof of Theorem 2.3.

Proof of Corollary 2.4. Given that m is a square-free positive integer and $\mathbb{F} = \mathbb{Q}(\sqrt{m})$. In this case, we have $r_1 = 2, r_2 = 0, d = 2$ and D = 4m. In this case, we use (1.18) to see that

$$V\left(-\frac{(2\pi)^2 niz}{4m}\middle|\bar{0}_2\right) = 2K_0\left(2\pi\sqrt{\frac{nz}{m}}e^{-\frac{i\pi}{4}}\right). \tag{4.21}$$

Moreover, the residual terms become

$$\mathfrak{R}_0(z) = \zeta_{\mathbb{F}}'(0)\zeta_{\mathbb{F}}(2k+1),$$

$$\mathfrak{R}_1(z) = H_{\mathbb{F}}\zeta_{\mathbb{F}}(2k+2)\frac{im}{\pi^2 z},$$

$$\mathfrak{R}_{-(2j-1)}(z) = \lim_{s \to (1-2j)} \frac{\mathrm{d}}{\mathrm{d}s} \left((s+2j-1)^2 \Lambda_{\mathbb{F},k}(s)(-iz)^{-s} \right).$$

To calculate $\mathfrak{R}_0(z)$ we have used the fact $C_{\mathbb{F}} = \zeta_{\mathbb{F}}'(0)$ as we are working on real quadratic field. Putting these terms in Theorem 2.3, one can complete the proof of this result. \square

Proof of Theorem 2.5. This is result is an implication of our main Theorem 2.1 for negative integers k and totally real fields. Mainly, from (2.3) with $r_2 = 0$, we have

$$\mathfrak{F}_{\mathbb{F},2k+1}(z) - C_{\mathbb{F}}\zeta_{\mathbb{F}}(2k+1) = (-1)^{k(r_1+1)}z^{2k} \left\{ \mathfrak{F}_{\mathbb{F},2k+1}\left(-\frac{1}{z}\right) - C_{\mathbb{F}}\zeta_{\mathbb{F}}(2k+1) \right\} + \begin{cases} -\frac{i}{4\pi z}, & \text{if } (k,r_1,r_2) = (-1,1,0), \\ 0, & \text{otherwise.} \end{cases}$$

Now replacing k by -k and simplifying, we complete the proof.

Proof of Corollary 2.6. Given that $k \geq 3$ and r_1 both are positive odd integers. Substituting z = i in Theorem 2.5, we obtain

$$\mathfrak{F}_{\mathbb{F},-2k+1}(i) = C_{\mathbb{F}}\zeta_{\mathbb{F}}(1-2k).$$

Now the proof of this corollary follows by using the value (1.12) of $C_{\mathbb{F}}$ and the definition (1.20) of $\mathfrak{F}_{\mathbb{F},-2k+1}(i)$.

Proof of Corollary 2.7. Substituting $\mathbb{F} = \mathbb{Q}(\sqrt{m})$, i.e., $r_1 = 2, r_2 = 0, D = 4m$ in Theorem 2.5, we can see that

$$(iz)^{2k} \left\{ \mathfrak{F}_{\mathbb{F},-2k+1}(z) - \zeta_{\mathbb{F}}'(0)\zeta_{\mathbb{F}}(1-2k) \right\} = \left\{ \mathfrak{F}_{\mathbb{F},-2k+1}\left(-\frac{1}{z}\right) - \zeta_{\mathbb{F}}'(0)\zeta_{\mathbb{F}}(1-2k) \right\}, \tag{4.22}$$

where

$$\mathfrak{F}_{\mathbb{F},-2k+1}(z) = \sum_{n=1}^{\infty} \sigma_{\mathbb{F},2k-1}(n) V\left(-\frac{(2\pi)^2 niz}{4m} \middle| \bar{0}_2\right).$$

Now employing the expression (4.21) for $V\left(-\frac{(2\pi)^2 niz}{4m}\middle|\bar{0}_2\right)$ in the above series and then putting it in (4.22), the proof of Corollary 2.7 follows.

Proof of Theorem 2.8. In this case, we have taken \mathbb{F} to be a purely imaginary number field with degree $d=2r_2$. Thus, the proof of this theorem immediately follows by substituting $r_1=0$ in Theorem 2.1.

Proof of Corollary 2.9. Given that $\mathbb{F} = \mathbb{Q}(\sqrt{-m})$ is a quadratic imaginary field and k > 0. Therefore, substituting $(r_1, r_2, d, D) = (0, 1, 2, 4m)$ in Theorem 2.8, it yields that

$$\mathfrak{S}_{\mathbb{F},2k+1}(z) = (-1)^{k+1} z^{2k} \mathfrak{S}_{\mathbb{F},2k+1} \left(-\frac{1}{z} \right) + \sum_{j=1}^{k} \mathfrak{R}_{-(2j-1)}(z) + \sum_{j=1}^{k-1} \mathfrak{R}_{-2j}(z),$$

where

$$\mathfrak{S}_{\mathbb{F},2k+1}(z) = \sum_{n=1}^{\infty} \sigma_{\mathbb{F},-2k-1}(n) V\left(-\frac{\pi^2 niz}{m} \middle| \bar{0}_2\right) - \mathfrak{R}_0(z) - \mathfrak{R}_1(z),$$

and the residual terms are defined as

$$\begin{split} \mathfrak{R}_{0}(z) &= \lim_{s \to 0} \frac{\mathrm{d}^{2}}{\mathrm{d}s^{2}} \left(s^{2} \Lambda_{\mathbb{F},k}(s) (-iz)^{-s} \right), \\ \mathfrak{R}_{1}(z) &= H_{\mathbb{F}} \zeta_{\mathbb{F}}(2k+2) \frac{im}{\pi^{2}z}, \\ \mathfrak{R}_{-(2j-1)}(z) &= \lim_{s \to -(2j-1)} \left((s+2j-1) \Lambda_{\mathbb{F},k}(s) (-iz)^{-s} \right), \\ &= \lim_{s \to -(2j-1)} \left((s+2j-1)^{2} \Gamma(s)^{2} \frac{\zeta_{\mathbb{F}}(s)}{s+2j-1} \zeta_{\mathbb{F}}(s+2k+1) \left(\frac{-i\pi^{2}z}{m} \right)^{-s} \right) \\ &= -\frac{\zeta_{\mathbb{F}}'(1-2j)}{((2j-1)!)^{2}} \zeta_{\mathbb{F}}(2k-2j+2) \left(\frac{\pi^{2}iz}{m} \right)^{2j-1}. \end{split}$$

In a similar way, one can show that

$$\mathfrak{R}_{-2j}(z) = \frac{\zeta_{\mathbb{F}}'(-2j)}{((2j)!)^2} \zeta_{\mathbb{F}}(2k - 2j + 1) \left(\frac{\pi^2 iz}{m}\right)^{2j}.$$

Now combining all these residual terms and using the expression (4.21) for Steen function, we complete the proof of (2.12).

Proof of Corollary 2.10. Substitute k = -1 and $\mathbb{F} = \mathbb{Q}(\sqrt{-m})$, that is, $r_1 = 0, r_2 = 1$ in Theorem 2.8. Thus, from (2.9)-(2.11), we arrive at

$$\mathfrak{U}_{\mathbb{F},-1}(z) = z^{-2}\mathfrak{U}_{\mathbb{F},-1}\left(-\frac{1}{z}\right) + \frac{H_{\mathbb{F}}\zeta_{\mathbb{F}}(0)im}{\pi^2 z},\tag{4.23}$$

where

$$\mathfrak{U}_{\mathbb{F},-1}(z) = \mathfrak{F}_{\mathbb{F},-1}(z) - \zeta_{\mathbb{F}}(0)\zeta_{\mathbb{F}}'(-1).$$

Here we used (3.6) with the fact that $C_{\mathbb{F}} = \zeta_{\mathbb{F}}(0)$ as \mathbb{F} is quadratic imaginary field. Now multiplying by z^2 on both sides of (4.23), we finish the proof of (2.13). Further, to obtain (2.14), substituting z = i in (4.23) it yields that

$$\mathfrak{F}_{\mathbb{F},-1}(i) = \zeta_{\mathbb{F}}(0)\zeta_{\mathbb{F}}'(-1) + \frac{H_{\mathbb{F}}\zeta_{\mathbb{F}}(0)m}{2\pi^2}.$$

Now using the definition (1.20) of $\mathfrak{F}_{\mathbb{F},-1}(i)$ we derive that

$$\sum_{n=1}^{\infty} \sigma_{\mathbb{F},1}(n) V\left(\frac{n\pi^{2}}{m} | \bar{0}_{2}\right) = \zeta_{\mathbb{F}}(0) \zeta_{\mathbb{F}}'(-1) + \frac{H_{\mathbb{F}}\zeta_{\mathbb{F}}(0)m}{2\pi^{2}}.$$

Finally, using the fact that $V\left(\frac{n\pi^2}{m}|\bar{0}_2\right)=2K_0\left(2\pi\sqrt{\frac{n}{m}}\right)$, we complete the proof of (2.14).

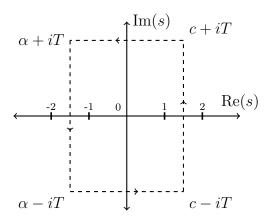
Proof of Corollary 2.11. The proof of this corollary immediately follows from (2.9)-(2.11) by replacing k by -k. As it is given that $(k, r_2) \neq (1, 1)$, so we have to use the fact that $\Re(z) = 0$ and multiply by z^{2k} on both sides of (2.9) after replacing k by -k.

Proof of Theorem 2.12. The proof of this theorem goes along the same line as in Theorem 2.1, however, we give brief outline. This identity is due to the case k = 0. In the way as we proceeded in (4.1) and (4.2), one can show that

$$\sum_{n=1}^{\infty} \sigma_{\mathbb{F},-1}(n) V\left(\frac{(2\pi)^d ny}{D} \middle| \bar{0}_d\right) = \frac{1}{2\pi i} \int_{(c)} \Gamma(s)^d \zeta_{\mathbb{F}}(s) \zeta_{\mathbb{F}}(s+1) \left(\frac{(2\pi)^d y}{D}\right)^{-s} ds$$

$$= \frac{1}{2\pi i} \int_{(c)} \Lambda_{\mathbb{F},0}(s) y^{-s} ds, \tag{4.24}$$

where $\Lambda_{\mathbb{F},0}(s) = \Gamma(s)^d \zeta_{\mathbb{F}}(s) \zeta_{\mathbb{F}}(s+1) (2\pi)^{-ds} D^s$, which is exactly same as we defined in (2.1), and $1 < \text{Re}(s) = c < 1 + \epsilon$ with $0 < \epsilon < 1$. Further proceeding in a similar way as in Theorem 2.1, here also we set up a rectangular contour \mathcal{C} made up of the vertices c - iT, c + iT, $\alpha + iT$ and $\alpha - iT$, with $-2 < \alpha < -1$ and T is some large positive quantity.



We now examine the poles of our integrand function $\Lambda_{\mathbb{F},0}(s)$. At s=0, $\Gamma(s)^d$ has a pole order d, $\zeta_{\mathbb{F}}(s)$ has a zero of order r_1+r_2-1 , and $\zeta_{\mathbb{F}}(s+1)$ has a simple pole. Therefore, the order of the pole of the integrand function $\Lambda_{\mathbb{F},0}(s)$ at s=0 is $d-(r_1+r_2-1)+1=r_2+2$ since $d=r_1+2r_2$. It is easy to see that the integrand function has a simple pole at s=1. From Lemma 3.2, we can see that $\Lambda_{\mathbb{F},0}(s)=(-1)^{r_2}\Lambda_{\mathbb{F},0}(-s)$. This indicates that s=-1 is also a simple pole of the

integrand function. Now utilizing Cauchy residue theorem, we arrive

$$\sum_{n=1}^{\infty} \sigma_{\mathbb{F},-1}(n) V\left(\frac{(2\pi)^d ny}{D} \middle| \bar{0}_d\right) = (-1)^{r_2} \sum_{n=1}^{\infty} \sigma_{\mathbb{F},-1}(n) V\left(\frac{(2\pi)^d n}{Dy} \middle| \bar{0}_d\right) + \mathcal{R}, \qquad (4.25)$$

where the residual term \mathcal{R} is given by

$$\mathcal{R} = R_0^{(r_2+2)}(y) + R_1^{(1)}(y) + R_{-1}^{(1)}(y). \tag{4.26}$$

The above term \mathcal{R} makes the main difference with Theorem 2.1, see (4.4). The terms of \mathcal{R} can be calculated as follows:

$$R_0^{(r_2+2)}(y) = \frac{1}{(r_2+1)!} \lim_{s \to 0} \frac{\mathrm{d}^{r_2+1}}{\mathrm{d}s^{r_2+1}} \left(s^{r_2+2} \Lambda_{\mathbb{F},0}(s) y^{-s} \right), \tag{4.27}$$

$$R_1^{(1)}(y) = \lim_{s \to 1} (s-1) \Lambda_{\mathbb{F},0}(s) y^{-s}$$

$$= \lim_{s \to 1} (s-1) \Gamma(s)^d \zeta_{\mathbb{F}}(s) \zeta_{\mathbb{F}}(s+1) (2\pi)^{-ds} D^s y^{-s}$$

$$= \frac{H_{\mathbb{F}} \zeta_{\mathbb{F}}(2) D}{(2\pi)^d y}, \tag{4.28}$$

$$R_{-1}^{(1)}(y) = \lim_{s \to -1} (s+1) \Lambda_{\mathbb{F},0}(s) y^{-s}$$

$$= \lim_{s \to -1} (s+1) (-1)^{r_2} \Lambda_{\mathbb{F},0}(-s) y^{-s}$$

$$= (-1)^{r_2+1} \lim_{s \to 1} (s-1) \Lambda_{\mathbb{F},0}(s) y^s$$

$$= (-1)^{r_2+1} \frac{H_{\mathbb{F}} \zeta_{\mathbb{F}}(2) D y}{(2\pi)^d}. \tag{4.29}$$

Here we used class number formula (1.11) and Lemma 3.2 to simplify the above residual terms $R_1^{(1)}(y)$ and $R_{-1}^{(1)}(y)$. Substituting the above residual terms in (4.26) and together with (4.25), we get

$$\begin{split} & \sum_{n=1}^{\infty} \sigma_{\mathbb{F},-1}(n) V \left(\frac{(2\pi)^d n y}{D} \middle| \bar{0}_d \right) - \frac{H_{\mathbb{F}} \zeta_{\mathbb{F}}(2) D}{(2\pi)^d y} \\ &= (-1)^{r_2} \left\{ \sum_{n=1}^{\infty} \sigma_{\mathbb{F},-1}(n) V \left(\frac{(2\pi)^d n}{D y} \middle| \bar{0}_d \right) - \frac{H_{\mathbb{F}} \zeta_{\mathbb{F}}(2) D y}{(2\pi)^d} \right\} \\ &+ \frac{1}{(r_2+1)!} \lim_{s \to 0} \frac{\mathrm{d}^{r_2+1}}{\mathrm{d} s^{r_2+1}} \left(s^{r_2+2} \Lambda_{\mathbb{F},0}(s) y^{-s} \right). \end{split}$$

The above identity holds for y > 0, however, by analytic continuation, one can show that this identity is also true for Re(y) > 0. We substitute y = -iz to get the following form, for $z \in \mathbb{H}$,

$$\sum_{n=1}^{\infty} \sigma_{\mathbb{F},-1}(n) V\left(-\frac{(2\pi)^d niz}{D} \middle| \bar{0}_d\right) - \frac{H_{\mathbb{F}} \zeta_{\mathbb{F}}(2) iD}{(2\pi)^d z}$$

$$= (-1)^{r_2} \left\{ \sum_{n=1}^{\infty} \sigma_{\mathbb{F},-1}(n) V\left(\frac{(2\pi)^d ni}{Dz} \middle| \bar{0}_d\right) + \frac{H_{\mathbb{F}} \zeta_{\mathbb{F}}(2) Diz}{(2\pi)^d} \right\}$$

$$+ \frac{1}{(r_2+1)!} \lim_{s \to 0} \frac{\mathrm{d}^{r_2+1}}{\mathrm{d}s^{r_2+1}} \left(s^{r_2+2} \Lambda_{\mathbb{F},0}(s) (-iz)^{-s} \right).$$

Now if we use the definition (1.20) of $\mathfrak{F}_{\mathbb{F},1}(z)$, then the above expression can be rewritten as

$$\mathfrak{F}_{\mathbb{F},1}(z)-\mathfrak{R}_1(z)=(-1)^{r_2}\left\{\mathfrak{F}_{\mathbb{F},1}\left(-\frac{1}{z}\right)-\mathfrak{R}_1\left(-\frac{1}{z}\right)\right\}+\mathfrak{R}_0(z),$$

where

$$\mathfrak{R}_1(z) = \frac{H_{\mathbb{F}}(\zeta_{\mathbb{F}}(2)iD}{(2\pi)^d z}, \quad \text{and} \quad \mathfrak{R}_0(z) = \frac{1}{(r_2+1)!} \lim_{s \to 0} \frac{\mathrm{d}^{r_2+1}}{\mathrm{d}s^{r_2+1}} \left(s^{r_2+2} \Lambda_{\mathbb{F},0}(s)(-iz)^{-s}\right).$$

Letting $\mathfrak{T}_{\mathbb{F}}(z) := \mathfrak{F}_{\mathbb{F},1}(z) - \mathfrak{R}_1(z)$, one can complete the proof of Theorem 2.12.

Proof of Corollary 2.13. Letting \mathbb{F} to be a totally real number field of degree $d=r_1$ in Theorem 2.12, we get

$$\mathfrak{F}_{\mathbb{F},1}(z) - \mathfrak{F}_{\mathbb{F},1}\left(-\frac{1}{z}\right) = \mathfrak{R}_1(z) - \mathfrak{R}_1\left(-\frac{1}{z}\right) + \mathfrak{R}_0(z),\tag{4.30}$$

where

$$\mathfrak{R}_1(z) = \frac{H_{\mathbb{F}}\zeta_{\mathbb{F}}(2)iD}{(2\pi)^{r_1}z}, \quad \mathfrak{R}_0(z) = \lim_{s \to 0} \frac{\mathrm{d}}{\mathrm{d}s} \left(s^2 \Gamma^{r_1}(s) \zeta_{\mathbb{F}}(s) \zeta_{\mathbb{F}}(s) + 1 \right) \left(-\frac{(2\pi)^{r_1}iz}{D} \right)^{-s} \right).$$

Now we shall try to simplify the term $\mathfrak{R}_0(z)$. Utilizing the fact that

$$\Gamma(s) = \frac{1}{s} - \gamma + O(s),$$

one can see that

$$(s\Gamma(s))^{r_1} = 1 - r_1\gamma s + O(s^2). \tag{4.31}$$

Since \mathbb{F} is a totally real number field of degree r_1 , so from (3.6), the Laurent series expansion of $\zeta_{\mathbb{F}}(s)$ at s=0, one has

$$\frac{\zeta_{\mathbb{F}}(s)}{s^{r_1-1}} = a_0 + a_1 s + O(s^2), \tag{4.32}$$

where

$$a_0 = \frac{\zeta_{\mathbb{F}}^{(r_1-1)}(0)}{(r_1-1)!} = C_{\mathbb{F}}, \quad a_1 = \frac{\zeta_{\mathbb{F}}^{(r_1)}(0)}{(r_1)!}.$$

Further, utilizing (3.5), the Laurent series expansion of $\zeta_{\mathbb{F}}(s)$, it yields that

$$s\zeta_{\mathbb{F}}(s+1) = H_{\mathbb{F}} + \gamma_{\mathbb{F}}s + O(s^2). \tag{4.33}$$

Moreover, we have

$$\left(-\frac{(2\pi)^{r_1}iz}{D}\right)^{-s} = 1 - s\log\left(-\frac{(2\pi)^{r_1}iz}{D}\right) + O(s^2). \tag{4.34}$$

Now combining all the above Laurent series expansions (4.31)-(4.34), one can check that the coefficient of s in the Laurent series expansion of $s^2\Gamma^{r_1}(s)\zeta_{\mathbb{F}}(s)\zeta_{\mathbb{F}}(s+1)(-2\pi iz)^{-s}$ is $a_0\gamma_{\mathbb{F}} - a_0H_{\mathbb{F}}\left\{r_1\gamma + \log\left(-\frac{(2\pi)^{r_1}iz}{D}\right)\right\} + a_1H_{\mathbb{F}}$, which shows that

$$\mathfrak{R}_0(z) = a_0 \gamma_{\mathbb{F}} - a_0 H_{\mathbb{F}} \left\{ r_1 \gamma + \log \left(-\frac{(2\pi)^{r_1} iz}{D} \right) \right\} + a_1 H_{\mathbb{F}}.$$

Finally, substituting the above value of $\mathfrak{R}_0(z)$ in (4.30), we complete the proof of (2.18).

Proof of Corollary 2.14. We know that $\zeta_{\mathbb{Q}}(s) = \zeta(s)$. Therefore, considering $\mathbb{F} = \mathbb{Q}$, Corollary 2.13 reduces to

$$\mathfrak{F}_{\mathbb{Q},1}(z) - \mathfrak{F}_{\mathbb{Q},1}\left(-\frac{1}{z}\right) = \mathfrak{R}_1(z) - \mathfrak{R}_1\left(-\frac{1}{z}\right) + \mathfrak{R}_0(z),\tag{4.35}$$

where

$$\mathfrak{F}_{\mathbb{Q},1}(z) = \sum_{n=1}^{\infty} \sigma_{\mathbb{Q},-1}(n) V\left(-2\pi niz \middle| \bar{0}\right) = \sum_{n=1}^{\infty} \sigma_{-1}(n) \exp(2\pi inz),$$

$$\mathfrak{R}_{1}(z) = \frac{H_{\mathbb{Q}}\zeta_{\mathbb{Q}}(2)iD}{2\pi z} = \frac{i\pi}{12z},$$

$$\mathfrak{R}_{0}(z) = a_{0}\gamma_{\mathbb{Q}} - a_{0}H_{\mathbb{Q}}\left\{r_{1}\gamma + \log(-2\pi iz)\right\} + a_{1}H_{\mathbb{Q}} = \frac{1}{2}\log(-iz).$$

The above simplified forms have been obtained by using the following well-known values:

$$H_{\mathbb{Q}} = 1, \gamma_{\mathbb{Q}} = \gamma, \zeta_{\mathbb{Q}}(2) = \frac{\pi^2}{6}, a_0 = \zeta(0) = -\frac{1}{2}, a_1 = \zeta'(0) = -\frac{1}{2}\log(2\pi).$$

Eventually, putting the above values of $\mathfrak{F}_{\mathbb{Q},1}(z), \mathfrak{R}_1(z), \mathfrak{R}_0(z)$ in (4.35) and using the fact that $n\sigma_{-1}(n) = \sigma(n)$, one can finish the proof of (2.20).

5. Concluding Remarks

Ramanujan's formula (1.2) involves the following Lambert series

$$\sum_{n=1}^{\infty} \frac{n^{-k}}{e^{ny} - 1} = \sum_{n=1}^{\infty} \sigma_{-k}(n)e^{-ny}, \quad \text{Re}(y) > 0,$$
 (5.1)

associated with the generalized divisor function $\sigma_k(n) = \sum_{d|n} d^k$, $k \in \mathbb{C}$. Upon suitable change of variable, one can easily observe that the above series is exactly same as the following series:

$$F_k(z) = \sum_{n=1}^{\infty} \sigma_{-k}(n)e^{2\pi i n z}, \quad z \in \mathbb{H},$$
(5.2)

which is also present in Grosswald's identity (1.6). Recently, Banerjee, Gupta and Kumar [4, Equation (1.7)] have generalized the above Lambert series (5.1), while obtaining a number field analogue of Ramanujan's identity, in the following way:

$$\sum_{\mathfrak{a}\subset\mathcal{O}_{\mathbb{K}}}\mathfrak{N}(\mathfrak{a})^{k}\Omega_{\mathbb{F}}\left(\frac{\mathfrak{N}(\mathfrak{a})y}{D}\right)=\sum_{n=1}^{\infty}\mathsf{a}_{\mathbb{F}}(n)n^{k}\Omega_{\mathbb{F}}\left(\frac{ny}{D}\right),$$

where the function $\Omega_{\mathbb{F}}(x)$, which involves the Meijer G-function, is defined as

$$\Omega_{\mathbb{F}}(x) := \frac{2^{1-r_1-r_2}}{\pi^{1-\frac{r_1}{2}}} \sum_{j=1}^{\infty} \mathsf{a}_{\mathbb{F}}(j) G_{0,2d}^{d+1,0} \left(\begin{array}{c} - \\ \left(0\right)_{r_1+r_2}, \left(\frac{1}{2}\right)_{r_2+1}; \left(\frac{1}{2}\right)_{r_1+r_2-1}, \left(0\right)_{r_2} \end{array} \right| \frac{x^2 j^2}{4^d} \right),$$

with $d = r_1 + 2r_2$ (the degree of the field \mathbb{F} over \mathbb{Q}), and D is the absolute value of the discriminant of \mathbb{F} . In this paper, we obtained a new number field analogue of Ramanujan's identity by generalizing the Lambert series $F_k(z)$, defined in (5.2), in a different way than Banerjee et. al. Mainly, we studied modular transformation formula for the following infinite series:

$$\mathfrak{F}_{\mathbb{F},k}(z) = \sum_{n=1}^{\infty} \sigma_{\mathbb{F},-k}(n) V\left(-\frac{(2\pi)^d niz}{D} \middle| \bar{0}_d\right),\,$$

where $\sigma_{\mathbb{F},k}(n)$ is the number field analogue (1.13) of the generalized divisor function $\sigma_k(n)$ and $V(z|\bar{0}_d)$ is the Steen function (1.16). By obtaining a number field analogue of Ramanujan-Grosswald identity for odd zeta values, we are able to find a formula for Dedekind zeta function at odd arguments. As an application our main identity i.e., Theorem 2.1, we obtain many interesting modular transformation formulae that are true for totally real number fields and purely imaginary fields, see Theorem 2.3, Theorem 2.8. In particular, for real quadratic fields and imaginary quadratic fields, we obtained modular transformation formulae, i.e., Corollary 2.7, Corollary 2.11 that are perfect analogue of the transformation formula (1.5) for the Eisenstien series. Further, we found an exact evaluation of $\mathfrak{F}_{\mathbb{F},2k-1}(i)$ in terms of class number and the values of the Dedekind zeta function at negative odd integers, see (2.7), (2.16) and (2.17). We also obtain a modular transformation formula which generalizes transformation formula for the Dedekind eta function $\eta(z)$, see Theorem 2.12. As one of the interesting applications of Theorem 2.12, we derived a formula (2.19) for the class number of a totally real field which also indicates a relation with the Kronceker's limit formula for the Dedekind

zeta function. This observation might be of an independent interest to the readers of this article.

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