ON THE BOUNDARY OF THE HARTER-HEIGHWAY DRAGON CURVE

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Abstract. In this article we apply an L-system to prove a recurrence formula for the length of the boundary of iterands $\mathcal{C}_n$ of the well known Harter-Heighway dragon curve, a space filling curve with fractal boundary. This leads to finding formulas for related sequences of certain binary strings and ternary matrices. This proves some long standing conjectures for the recurrence relation for the number of terms in the boundary of the dragon curve, first stated in unpublished work Daykin and Tucker from 1975 [3].

1. Introduction

This article proves some results about two sequences, which count the number of components on the left and right sides of the iterands of the Harter-Heighway dragon. The sequences appear in [7] as sequences A227036 and A203175, but formulas given there are conjectural. We provide proofs of the formulas, which give the sequences in terms of generating function, appearing in [8].

The Harter-Heighway dragon curve, also known as the Heighway dragon, or the dragon curve was discovered in 1967 by John Heighway and William Harter [9], [5, 1.5]. It is a fascinating curve, since it is plane-filling and has a fractal boundary. There are many variations on this curve. See e.g., [4], [2]. This curve, which we refer to as $\mathcal{C}_\infty$, is given as the limit of a sequence of curves, $\mathcal{C}_n$ for non-negative integers n. The curve $\mathcal{C}_n$ is formed of 2^n equal length line segments, which we also will refer to as the *edges*, of $\mathcal{C}_n$, with a 90° angle between each edge. The curves can be described in various ways, as follows.

1.1. **Construction (A): paper folding.** The Harter-Heighway curve was first obtained by repeatedly folding a strip of paper in half n times, and then opening out all the folds to have angle 90°. The first few cases are shown in Figure 1.

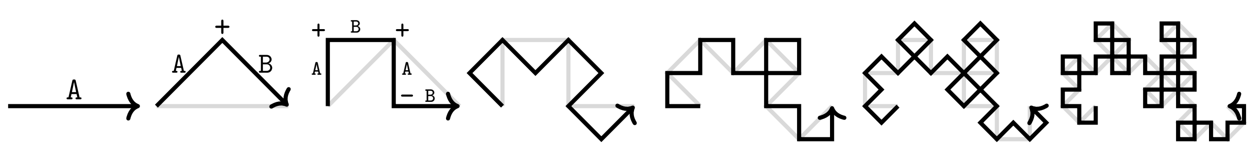

Figure 1. Iterates $\mathcal{C}_0$ to $\mathcal{C}_6$ of Heighway's dragon curve.

1.2. **Construction (B): alternating triangles.** The dragon curve may be described by setting $\mathcal{C}_0$ to be a unit line. Then $\mathcal{C}_n$ is obtained from $\mathcal{C}_{n-1}$ as follows. Place right angle isosceles triangles on each edge of $\mathcal{C}_{n-1}$, with hypotenuse along the edge, with the triangles on alternating sides of the curve. Then remove the edges of the original curve, as shown in Figure 1, where $\mathcal{C}_{n-1}$ is shown in gray, under $\mathcal{C}_n$ in black, so the triangles are formed with two black sides, and a gray hypotenuse.

1.3. **Construction (C): L-systems.** Another method of describing the curves $\mathcal{C}_n$ is by the use of turtle geometry [1] and L-systems [6]. This is the method used in this article.

An L-system $\mathcal{L} = (\Omega, A, P)$ is a rewriting system, defined to be a triple, consisting of an alphabet of symbols, Ω; a starting symbol $A \in \Omega^*$, called an axiom; and a function $P : \Omega \to \Omega^*$, from Ω

to the set Ω^* of of finite length words with letters in Ω. This induces a function $P : \Omega^* \to \Omega^*$; each letter x of a word in Ω^* is replaced by $P(x)$. In the case of the Heighway dragon, we have $\Omega = \{A, B, +, -\}$, the axiom is A, and the function is given by

$$P(A) = A + B, P(B) = A - B, P(+) = +, P(-) = -.$$

The curve $\mathcal{C}_n$ is defined to be the path corresponding to the word $P^n(A)$, by following the letters of $P^n(A)$ as instructions for building a curve according to a turtle geometry construction [1], with the symbols of Ω interpreted as

A	make a horizontal unit move
B	make a vertical unit move
$+$	turn 90° clockwise (no move)
$-$	turn 90° counterclockwise (no move)

For example, labels in Ω are drawn on $\mathcal{C}_n$ for $n = 0, 1, 2$ in Figure 1.

1.4. **The L-system for the boundary of $\mathcal{C}_n$.** In [10], it is shown that the boundary of the curve $\mathcal{C}_n$, or more accurately, the boundary of a polyomino $\mathcal{S}_n$ containing $\mathcal{C}_n$, can be described by an L-system. The polyomino $\mathcal{S}_n$ is defined to be the union of a collection of squares, each having diagonal one edge of $\mathcal{C}_n$, as shown in Figure 2.

The L-system $\mathcal{L}_1 = (\Omega_1, Rr, P_1)$ for the boundary of $\mathcal{S}_n$ is given by taking $\Omega = \{\mathtt{R}, \mathtt{r}, \mathtt{L}, \mathtt{l}, \mathtt{S}, \mathtt{s}\}$, with P_1 given by

$$\mathtt{R} \mapsto \mathtt{Rr}, \quad \mathtt{r} \mapsto \mathtt{S}, \quad \mathtt{L} \mapsto \mathtt{S}, \quad \mathtt{l} \mapsto \mathtt{Ll}, \quad \mathtt{S} \mapsto \mathtt{Rl}, \quad \mathtt{s} \mapsto \mathtt{Lr}. \tag{1}$$

We consider an element (x, y) of $\mathbb{Z}^2$ to be even or odd according to whether the parity of $x + y$ is even or odd. Each symbol of Ω_1 corresponds to a path element in the turtle geometry sense, which consists of (1) starting from a point $(x, y) \in \mathbb{Z}^2$, a diagonal movement of length $1/\sqrt{2}$, continuing in an already specified direction, (2) no turn, or a turn of 90° left or right (3) another diagonal movement of length $1/\sqrt{2}$, resulting in the path which ends at one of the eight neighbouring points of $\mathbf{Z}^2$, i.e., $(x + \epsilon, y + \delta)$ for $\epsilon, \delta \in \{-1, 0, 1\}$, not both zero. Additional positional information gives the parity of the starting vertex. The symbols of Ω_1 correspond to two halves of an edge of $\mathcal{S}_n$, rather than straight line edge segments in the case of $\mathcal{C}_n$ and letters in Ω. The direction of turn, or lack of turn, for each symbol, and the starting parities, are as follows:

R	right turn in the middle of the path, starting point even
r	right turn in the middle of the path, starting point odd
L	left turn in the middle of the path, starting point even
l	left turn in the middle of the path, starting point odd
S	no turn in the middle of the path, starting point even
s	no turn in the middle of the path, starting point odd

Figure 2 illustrates the relationship between $\mathcal{C}_n$ and $\mathcal{S}_n$. In the figure, in passing from $\mathcal{C}_n$ to $\mathcal{S}_{n+1}$, the unit lengths are drawn scaled down by a factor of $\sqrt{2}$, and the unit grid rotates by 45° counterclockwise. The starting point is always $(0, 0)$, which is even. The parity of the vertices of the grid through which $\mathcal{C}_n$ and the boundary of $\mathcal{S}_n$ passes are marked by black and white dots for even and odd vertices respectively. For example, the left boundary of $\mathcal{S}_4$ is described by the word RrSRlRrLl.

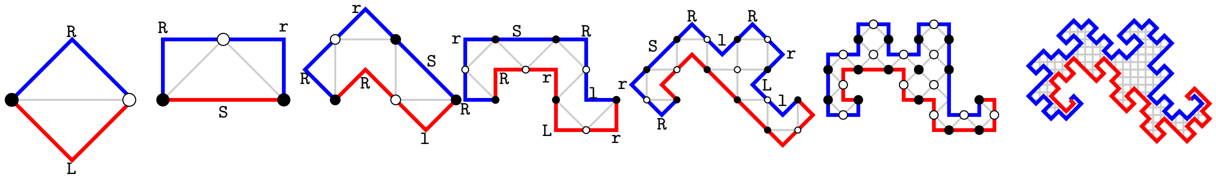

FIGURE 2. Iterates of the boundary of $\mathcal{S}_n$, for $n = 0$ to 5 and 8. The left boundary is in blue, the right boundary is in red, and the curve $\mathcal{C}_n$ is in gray. The red and blue curves together form the boundary of the polyomino $\mathcal{S}_n$.

In [10] it is shown that this construction produces a self-avoiding description of the boundary of $\mathcal{S}_n$. Thus the number of edges of $\mathcal{S}_n$ is equal to the number of letters in $P^n(\mathtt{Rr})$. We also write $||w||$ to mean the number of letters in w for w a word with letters in some alphabet.

The L-system $\mathcal{L}_1$ gives a sequence of words $P^n(\mathtt{Rr})$ describing the complete boundary of $\mathcal{S}_n$. However, since $\mathcal{C}_n$ is a curve with different start and end points, it is considered to have two sides, a left and right side. So we also consider the curves defined by the L-systems $\mathcal{L}_L = (\Omega_1, \mathtt{R}, P_1)$ and $\mathcal{L}_R = (\Omega_1, \mathtt{L}, P_1)$, which describe the left and right boundaries of $\mathcal{S}_n$ respectively.

In this paper we consider the boundary of the curves to be the boundary of the polyomino, $\mathcal{S}_n$. In the limit as n tends to infinity, the shapes $\mathcal{S}_n$ and $\mathcal{C}_n$ are the same. However, the actual unit length of the boundary is different for these two curves. The difference between the two boundaries can be observed in Figure 2, where the gray curve is $\mathcal{C}_n$, and the red curve the boundary of $\mathcal{S}_n$.

2. THE LEFT SIDE OF THE HEIGHWAY DRAGON $\mathcal{C}_n$

Given the L-system description of the boundary (1), we can prove the following result, which has been a long standing conjecture on the length of the boundary of the Heighway dragon, a sequence starting $2, 4, 8, 16, 28, \ldots$, [7, A227036].

Theorem 1. *The length of the boundary of the Heighway dragon curve $\mathcal{C}_n$, that is, the number of horizontal and vertical line segments on the left side of the curve $\mathcal{C}_n$, is equal to the coefficients of the expansion of the Taylor series about $x = 0$ of*

$$\frac{2(1+x^2)}{(1-x)(1-x-2x^3)} = \sum_{n=0}^{\infty} a_n x^n = 2 + 4x + 8x^2 + 16x^3 + 28x^4 + 48x^5 + 84x^6 + \cdots. \tag{2}$$

Proof. For $\mathtt{w} \in \Omega_1^*$ and $\mathtt{X} \in \Omega_1$ let $a_{\mathtt{X}}$ denote the number of times $\mathtt{X}$ occurs in $\mathtt{w}$. Set

$$v(\mathtt{w}) := (a_{\mathtt{R}}, a_{\mathtt{r}}, a_{\mathtt{L}}, a_l, a_{\mathtt{S}}),$$

which gives a count of the number of times each letter of Ω_1 occurs in $\mathtt{w}$. Note that we do not include $a_{\mathtt{s}}$ since it is not in the image of any element of Ω_1 under the action of P_1, where P_1 is the map corresponding to the L-system for the boundary of the polyomino containing the Harter-Heighway dragon curve, as in 1. Define a matrix M, with columns $v(P(\mathtt{X}))$ for $\mathtt{X} = \mathtt{R}, \mathtt{r}, \mathtt{L}, l, \mathtt{S}, \mathtt{s}$, so that

$$v(P(\mathtt{w})) = Mv(\mathtt{w}).$$

We have

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}. \tag{3}$$

Note that the left side of $\mathcal{S}_0$ corresponds to the word $\mathtt{R}$, as illustrated in Figure 2. Since $M(v(\mathtt{w}))$ counts the occurrences of each letter in $P(\mathtt{w})$, we have that the components of $M^n(v(\mathtt{R}))$ count the number of occurrences of each letter of Ω_1. The total number of edges of the left boundary of $\mathcal{P}_n$ is thus given by the sum of these components, i.e., since $v(\mathtt{R}) = (1,0,0,0,0)$, the value

$$(1,1,1,1,1)M^n(1,0,0,0,0)^T.$$

Examining the relationship between the curve $\mathcal{C}_n$ and the polyomino $\mathcal{S}_n$, shown in Figure 2, we see that we have a correspondence:

segment in left side of $\mathcal{S}_n$		A or B segment in left side boundary of $\mathcal{C}_n$
R or r	$\to$	1
S or s	$\to$	2
L or l	$\to$	3.

Therefore, given that our L-system gives us the matrix $M := M_{P_1}$ in (3), we must have that the number of A and B segments comprising the left boundary of $\mathcal{C}_n$ is given by

$$b_n := (1,1,3,3,2)M_{P_1}^n(1,0,0,0,0)^T.$$

We compute that the first few terms of this sequence are $1, 2, 4, 8, 16, 28$. The characteristic polynomial of M is

$$x^5 - 2x^4 + x^3 - 2x^2 + 2x = x(x-1)(x^3 - x^2 - 2).$$

So we have

$$\begin{aligned} 0 &= (1,1,3,3,2)(M^{5+n} - 2M^{4+n} + M^{3+n} - 2M^{2+n} + 2M^{n+1})(1,0,0,0,0)^T \\ &= b_{n+5} - 2b_{n+4} + b_{n+3} - 2b_{n+2} + 2b_{n+1} \end{aligned} \tag{4}$$

from which we obtain, for $n \geq 5$ the relationship

$$b_n = 2b_{n-1} - b_{n-2} + 2b_{n-3} - 2b_{n-4}. \tag{5}$$

Now we turn to the Taylor series. With the a_n as in (14), and $a_n = 0$ for $n < 0$, we have

$$\begin{aligned} 2(1+x^2) &= \sum_{n=0}^{\infty} a_n(1-x)(1-x-2x^3)x^n \\ &= \sum_{n=0}^{\infty} a_n(2x^4 - 2x^3 + x^2 - 2x + 1)x^n \\ &= \sum_{n=4}^{\infty} (2a_{n-4} - 2a_{n-3} + a_{n-2} - 2a_{n-1} + a_n)x^n. \end{aligned}$$

Therefore, equating coefficients of x^n, we have

$$\begin{aligned} 2 &= a_0 (n = 0) \\ 0 &= -2a_0 + a_1 (n = 1) \\ 2 &= a_0 - 2a_1 + a_2 (n = 2) \\ 0 &= 2a_{n-4} - 2a_{n-3} + a_{n-2} - 2a_{n-1} + a_n (n \geq 3). \end{aligned}$$

So, we have that the a_n satisfy a recurrence relation,

$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3} - 2a_{n-4} \tag{6}$$

with the initial terms given by

$$\begin{aligned} a_n &= 0 \text{ for } n < 0 \\ a_0 &= 2 \\ a_1 &= 4 \\ a_2 &= 8 \\ a_3 &= 16 \end{aligned}$$

Comparing equations (5) and (6) we see that the the two sequences satisfy the same recurrence relation. Also the first few terms are the same, up to an offset, with $a_n = b_{n+1}$. Therefore, the two sequences are equal (up to the offset). □

3. Right side boundary of $\mathcal{S}_n$ and A203175.

In this section we discuss the number of edges of the right side boundary of $\mathcal{S}_n$, and find that this is the same as the sequence [7, A203175]. In [7, A203175], this sequence has several conjectured descriptions, which with our L-system $(\Omega_1, \mathtt{R}, P_1)$ from Section 1.4 can now be proved. We will consider each description in turn, and show that in each case we have the same recurrence relation satisfied, and the same first few terms, so the sequences are the same.

3.1. **Right side boundary of $\mathcal{S}_n$.** In the notation of Section 1.3, the right side of the boundary of the polyomino $\mathcal{S}_n$ is described by the word $P_1^n(\mathtt{L})$, for $n \geq 0$. We define a sequence

$$a_n = |P_1^n(\mathtt{L})|, \tag{7}$$

which by [10] counts the length of the right side of the boundary of $\mathcal{S}_n$. The first few words $P_1^n(\mathtt{L})$ are $\mathtt{L}, \mathtt{S}, \mathtt{Rl}, \mathtt{RrLr}, \mathtt{RrSSLl}, \mathtt{RrSRlR}l\mathtt{SLl}$, corresponding to the red (lower) sides of the polyominos in Figure 2. So the first few terms of the sequence a_n are

$$1, 1, 2, 4, 6, 10. \tag{8}$$

Theorem 2. *The length of the right side boundary of the polyomino $\mathcal{S}_n$, containing the Harter-Heighway dragon curve $\mathcal{C}_n$ is given by the sequence a_n with*

$$\begin{aligned} &a_n = a_{n-1} + 2a_{n-3} \text{ for } n \geq 4 \\ &a_0 = 1, \ \ a_1 = 1, \ \ a_2 = 2. \end{aligned} \tag{9}$$

Proof. As in the proof of Theorem 1, there is a matrix M, given by (3) such that $M^n(v(\mathtt{L}))$ counts the number of each kind of right boundary unit of $\mathcal{S}_n$. Since $v(\mathtt{L}) = (0, 0, 1, 0, 0)$, the total number of right edges is

$$a_n = (1, 1, 1, 1, 1)M^n(0, 0, 1, 0, 0)^T. \tag{10}$$

Note that P_1 in (1) is invariant under switching $\mathtt{L} \leftrightarrow \mathtt{r}$, $\mathtt{R} \leftrightarrow \mathtt{l}$ (in domain and image), which corresponds to conjugating M by the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

i.e., $PMP = M$, and so we also have

$$\begin{aligned} |P_1^n(\mathtt{r})| &= (1, 1, 1, 1, 1)M^n(0, 1, 0, 0, 0)^T = (1, 1, 1, 1, 1)PM^nP(0, 1, 0, 0, 0)^T \\ &= (1, 1, 1, 1, 1)M^n(0, 0, 1, 0, 0)^T = a_n = |P_1^n(\mathtt{L})|. \end{aligned}$$

So we can write

$$a_n = |P_1^n(\mathtt{L})| + |P_1^n(\mathtt{r})| = (1, 1, 1, 1, 1)M^n(0, 1, 1, 0, 0)^T.$$

Now notice that

$$M^3 - M^2 - 2I = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and that $(0,1,1,0,0) = v(\mathtt{r}) + v(\mathtt{L})$ is in the kernel of this matrix. Thus

$$(1,1,1,1,1)M^n(M^3 - M^2 - 2I)(0,1,1,0,0)^T = 0, \tag{12}$$

from which we obtain

$$a_n - a_{n-1} - 2a_{n-3} = 0. \tag{13}$$

Together with the first few terms, given in (8), we obtain the recurrence relation (9). □

3.2. **Generating function.** For integers $n \geq 1$, let b_n be the coefficients of the Taylor series expansion of $x(1+x^2)/(1-x-2x^3)$, which is the generating function for these coefficients, and is taken from [8] and [7, A203175]. By definition we have,

$$\frac{x(1+x^2)}{1-x-2x^3} = \sum_{n=1}^{\infty} b_n x^n = x + x^2 + 2x^3 + 4x^4 + 6x^5 + 10x^6 + 18x^7 + \cdots. \tag{14}$$

Theorem 3. *The coefficients b_n of the Taylor series expansion of $x(1+x^2)/(1-x-2x^3)$ satisfy*

$$\begin{array}{l} b_n = b_{n-1} + 2b_{n-3} \text{ for } n \geq 4 \\ b_1 = 1,\ \ b_2 = 1,\ \ b_3 = 2. \end{array} \tag{15}$$

Proof. Multiplying both sides of (14) by $(1-x-2x^3)$, we have

$$\begin{aligned} x + x^3 &= \sum_{n=0}^{\infty} b_n(1-x-2x^3)x^n \\ &= \sum_{n=0}^{\infty} b_n(1-x-2x^3)x^n \\ &= \sum_{n=4}^{\infty} (b_n - b_{n-1} - 2b_{n-3})x^n. \end{aligned}$$

Therefore, by comparing coefficients of x^n, we see that the b_n satisfy the given recurrence relation. The initial terms are taken from the expansion in (14). □

3.3. **Binary sequences without certain runs of zeros.** We now consider binary sequences without runs of zeros of length 1 mod 3, following the definition of Milan Janjic in the entry [7, A203175].

Definition 1. *S_n is the set of binary sequences of length n with no run of zeros of length 1 mod 3. Let*

$$c_n := |S_n|.$$

Note that here the length of a run of zeros means the maximum length of any substring of zeros, not the possible lengths of substrings of zeros, for for example $00000 \in S_5$. For example, S_n and c_n for 1 to 5 are shown in the Table 1.

n	1	2	3	4	5
c_n	1	2	4	6	10
S_n	$1(C)$	$11(E)$	$111(E)$	$1111(E)$	$11111(E)$
					$11100(B)$
				$1100(B)$	$11001(D)$
					$11000(A)$
			$100(B)$	$1001(D)$	$10011(E)$
				$1000(A)$	$10001(C)$
		$00(B)$	$001(D)$	$0011(E)$	$00111(E)$
					$00100(B)$
			$000(A)$	$0001(C)$	$00011(E)$
					$00000(B)$

TABLE 1. Elements of S_n for $n = 1, \ldots, 5$, arranged according to the appending rules in Table 3. Each sequence is followed by its type in brackets, according to Table 2.

Definition 2. *We define six types of sequences, as in Table 2.*

name	*description*	*example*
A	*sequences ending in a string of* $0 < m \equiv 0 \mod 3$ *zeros*	1011000
B	*sequences ending in a string of* $0 < m \equiv 2 \mod 3$ *zeros*	10100000
C	*sequences ending in a string of* $0 < m \equiv 0 \mod 3$ *zeros, followed by 1*	10001
D	*sequences ending in a string of* $0 < m \equiv 2 \mod 3$ *zeros, followed by 1*	001
E	*sequences ending in 11*	100111

TABLE 2. Types of elements of S_n.

Now we define rules for building sequences of length $n + 1$ from sequences of length n, as in Table 3.

A	add 1 to get a sequence of type C
B	add 0 to get a sequence of type A
	or add 1 to get a sequence of type D
C	add 1 to get a sequence of type E
	or remove the last 1 and add 00 to get a sequence of type B
D	add 1 to get a sequence of type E
E	add 1 to get a sequence of type E
	or remove the last 1, and add 00 to get a sequence of type B

TABLE 3. how to transform elements of S_n to elements of S_{n+1}.

We denote the power set of S_{n+1} by $\mathcal{P}(S_{n+1})$. The rules in Table 3. can be used to define a function as follows, where $\mathbf{v}$ denotes a binary sequence of length n, and $\mathbf{w}$ denotes a binary sequence of length $n - 2$.

$$
(16)\qquad
\begin{aligned}
f_n : S_n &\to \mathcal{P}(S_{n+1})\\
\mathbf{v} &\mapsto \begin{cases}
\{\mathbf{v}1\} & \text{if } \mathbf{v} \text{ has type A (image type C)}\\
\{\mathbf{v}0, \mathbf{v}1\} & \text{if } \mathbf{v} \text{ has type B (image types A, D)}\\
\{\mathbf{w}011, \mathbf{w}000\} & \text{if if } \mathbf{v} = \mathbf{w}01 \text{ has type C (image types E, B)}\\
\{\mathbf{v}1\} & \text{if } \mathbf{v} \text{ has type D (image type E)}\\
\{\mathbf{w}111, \mathbf{w}100\} & \text{if } \mathbf{v} = \mathbf{w}11 \text{ has type E (image types E, B)}
\end{cases}
\end{aligned}
$$

Lemma 1. *We have a disjoint union*

$$S_{n+1} = \bigcup_{\mathbf{v}\in S_n} f_n(\mathbf{v})$$

and

$$|S_{n+1}| = \sum_{\mathbf{v}\in S_n} |f_n(\mathbf{v})|.$$

Proof. We must show that each element $\mathbf{x} \in S_{n+1}$ is contained in exactly one of the sets of the form $f_n(\mathbf{v})$ for some $\mathbf{v} \in S_n$. First we prove existence of some $\mathbf{v}$ with $\mathbf{x} \in f_n(\mathbf{v})$.

Suppose that $\mathbf{x} = \mathbf{v}1$. Then $\mathbf{v}$ must be in S_n, and can have any type, and we have $\mathbf{x} \in f_n(\mathbf{v})$, since we can always add 1 to any element of S_n to obtain an element of S_{n+1}. In other words, if $\mathbf{x}$ ends in a 1, we can always remove it to obtain an element of S_n.

Suppose that $\mathbf{x} = \mathbf{v}0$. Then either $\mathbf{v}$ ends in a string of zeros of length 1 or 2 mod 3.

(1) In the case that $\mathbf{v}$ ends in a string of $3k+2$ zeros (for some $k \in \mathbb{Z}$), $\mathbf{v} \in S_n$ has type B, and $\mathbf{x} \in f_n(\mathbf{v})$. I.e., $\mathbf{v}$ is obtained from $\mathbf{x}$ precisely by removing the last 0 from $\mathbf{x}$.
(2) If $\mathbf{v}$ ends in a string of $3k+1$ zeros, for example if $\mathbf{x} = 100$, then $\mathbf{v} \notin S_n$. In this case, we have that $\mathbf{x}$ ends in at least two zeros. Either these are immediately preceded by a 1 or a 0. Suppose we have $\mathbf{x} = \mathbf{y}100$ for some word $\mathbf{y}$. Then $\mathbf{x} \in f_n(\mathbf{y}11)$, where $\mathbf{y}11$ has type E. E.g., $00100 \in f_n(0011)$.
(3) If we are the the case where $\mathbf{x} = \mathbf{y}000$, then $\mathbf{x} \in f_n(\mathbf{y}01)$, where $\mathbf{y}01$ has type C. E.g., $001100000 \in f_n(00110001)$.

Now we must show that $\mathbf{x}$ is in $f_n(\mathbf{v})$ for some unique $\mathbf{v} \in S_n$.

(1) For the case that $\mathbf{x}$ ends in 1, this is because we only obtain elements of S_{n+1} in $f_n(\mathrm{v})$ by adding 1 to the end of elements of S_n.
(2) In the case that $\mathbf{x}$ ends in a zero, considering the definition, and (16), $\mathbf{x}$ must be in $f_n(\mathbf{v})$ for some $\mathbf{v}$ of type B, C, or E, corresponding to $\mathbf{x}$ having type A, B, B respectively.
(3) In the case where $\mathbf{x}$ has type A, ending in $3k$ zeros, we can only obtain $\mathbf{x}$ from an element of S_n by removing the last 0, to obtain an element of type B. So this gives a unique element $\mathbf{v}$ with $\mathbf{x} \in f_n(\mathbf{v})$.
(4) In the case where $\mathbf{x}$ has type B, and ends in at least two zeros, in which case $\mathbf{v}$ is obtained uniquely from $\mathbf{x}$ by removing the last two zeros and replacing with 1, resulting in a uniquely defined element either of type C or E.

Thus we obtain the stated equalities. □

Theorem 4. *The sequence $c_n = |S_n|$ satisfies a recurrence relation*

$$
(17)\qquad
\begin{aligned}
&c_n = c_{n-1} + 2c_{n-3} \text{ for } n \geq 4\\
&c_1 = 1,\ c_2 = 2,\ c_3 = 4
\end{aligned}
$$

Proof. We can rewrite the map in (16) as follows:

$$\begin{aligned} A &\mapsto C \\ B &\mapsto AD \\ C &\mapsto EB \\ D &\mapsto E \\ E &\mapsto EB \end{aligned}$$

By Lemma 1, any element of S_n is obtained uniquely from some element of S_{n-1}. Define a transition matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \tag{18}$$

with columns corresponding to A, B, C, D, E. This tells us how to pass from elements of S_n to elements of S_{n+1} by type. So we have that the size of S_n is given by

$$|S_n| = (1,1,1,1,1)M^{n-1}(0,0,1,0,0)^T \tag{19}$$

where the vector $(0,0,1,0,0)$ corresponds to the initial string 1 of length one. We compute that the characteristic polynomial of M is $x^2(x^3 - x^2 - 2)$, and then, as in the computation in (4), we find that

$$c_n = c_{c-1} + 2c_{n-3}. \tag{20}$$

The initial terms are as given in Table 1. □

3.4. **Certain arrays with elements** $0, 1, 2$**.** Sequence [7, A203175] has description as follows.

Definition 3. *Let A_n be the set of $n \times 2$ arrays, containing only elements of the set $\{0,1,2\}$, such that*

- *every 1 is immediately preceded by 0 to the left or above,*
- *no 0 is immediately preceded by a 0, either above or to the left,*
- *every 2 is immediately preceded by 0 1, in the two rows above.*

I.e., if $m \in A_n$ has elements $m_{i,j}$, with $m_{0,0} = 0$, then $m_{i,j} = 1 \Rightarrow m_{i-1,j} = 0$ (and $i > 0$) or $m_{i,j-1} = 0$ (and $j > 0$); $m_{i,j} = 0 \Rightarrow m_{i-1,j} \neq 0$ (if $i > 0$) and $m_{i,j-1} \neq 0$ (if $j > 0$); and $m_{i,j} = 2 \Rightarrow m_{i,j-1} = 1$ and $m_{i,j-2} = 0$ (and $j > 1$). Let

$$d_n = |A_n|. \tag{21}$$

For example, $A_1 = \{(0,1)\}$, $A_2 = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ so $d_1 = d_2 = 1$. The elements of A_5 are shown in Figure 3, and $d_5 = 6$.

Just as we constructed S_{n+1} from S_n in the previous section, we can construct A_{n+1} from A_n as follows. We define ten different types of arrays, depending on the last two rows:

Definition 4. *An $n \times 2$ array of elements $0, 1, 2$ is said to have type A, B, C, D, E, F, G, H depending on the last row, according to the following table*

type	A	B	C	D	E	F	G	H
row	$(0,1^0)$	$(0,1^x)$	$(0,2)$	$(1,0)$	$(1,2)$	$(2,0)$	$(2,1^0)$	$(2,1^x)$

Here, in the second entry, 1^0 means 1 with a 0 above it, and 1^x means a 1 with a 1 or 2 above it. In the first column, 1 always has a 0 above it. We can never have a row of the form $(1,1)$, since this would have to be preceded by a row $(0,0)$, and 0 is not allowed to be next to 0, so this is impossible. Similarly, $(2,2)$ would have to be preceded by $(1,1)$ so is not allowed. So the above list contains all the possible last rows of elements of A_n for all n (not all of which will be achieved for all n). We also refer to the last row as having the given type.

In the definition of A_n, we see that the elements are built up in terms of the previous rows, so all elements of A_{n+1} can be obtained from an element of A_n by adding one more row which satisfies the rules in Definition 3.

Each type of matrix in A_n can be extended to a matrix in A_{n+1} by adding a row, with type E.g., suppose a matrix m in A_n has type A, and so ends with a row $m_n = (m_{n,1}, m_{n,2}) = (0, 1^0)$. Then for the next row, with new elements $(m_{n+1,1}, m_{n+1,2})$, we must have $m_{n+1,1} = 1$, since 0 in the first column can only be followed by a 1 below it. Since $m_{n,2}$ has a 0 above it, we could have $m_{n+1,2} = 2$. Since $m_{n+1,1} \neq 0$, and $m_{n,2} \neq 0$, we could have $m_{n+1,2} = 0$, but we can't obtain $m_{n+1,2} = 1$. So type A can be followed by type D or E. By similar considerations, we obtain the following table, which shows all the possible ways of extending a matrix of a given type in A_n to a matrix of a given type in A_{n+1}. We define a corresponding function f on $\{A, B, C, D, E, F, G, H\}$ as in the column on the right in Table 4.

type of m_n	possible type of m_{n+1}	f
A	D, E	$f(A) = \{D, E\}$
B	D	$f(B) = \{D\}$
C	D	$f(C) = \{D\}$
D	A, G	$f(D) = \{A, G\}$
E	B, F	$f(E) = \{B, F\}$
F	A	$f(F) = \{A\}$
G	B, C	$f(G) = \{B, C\}$
H	B	$f(H) = \{B\}$

TABLE 4. Rules for elements of A_{n+1} following from elements of A_n.

Theorem 5. *The number of elements of A_n in Definition 3 is given by*

$$\begin{aligned} &d_n = d_{n-1} + 2d_{n-3} \text{ for } n \geq 4 \\ &d_1 = 1,\ d_2 = 1,\ d_3 = 2. \end{aligned} \tag{22}$$

Proof. We have discussed above how rows A, B, C, D, E, F, G, H of an element of A_n transition to the next possible row of an element of A_{n+1}, as shown in Table 4. Since row type H never occurs in the image of f, we will leave this out from now on. The transition function f in Table 4 can be represented by the matrix

$$N = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{23}$$

with rows and columns corresponding to A to G in alphabetical order. Since $(0, 1, 0, 0, 0, 0, 0)$ corresponds to B, in the initial set $A_1 = \{B\}$. We have that the number of elements of each type

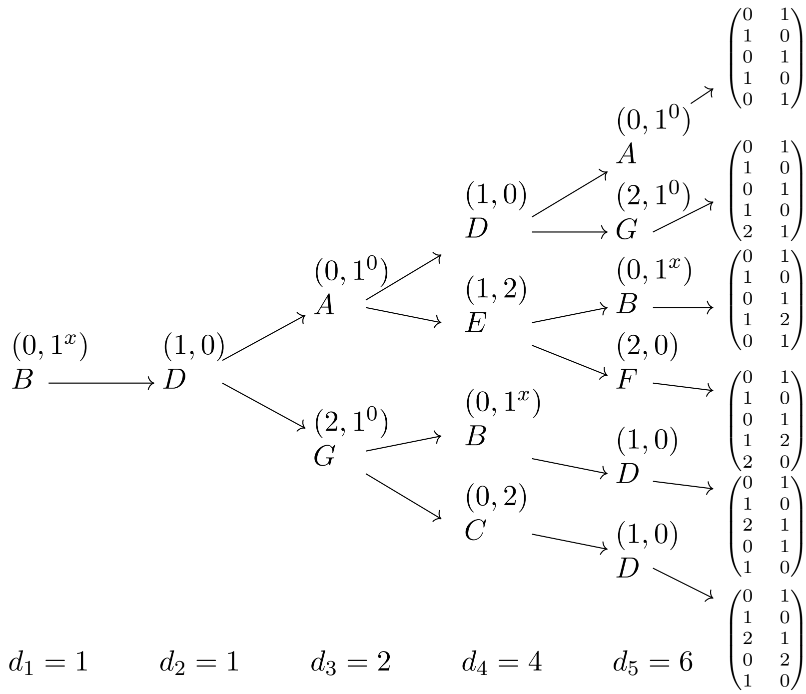

FIGURE 3. Construction of elements of A_5, row by row, using the rules in Table 4

in A_n is given by the corresponding component of the vector $N^{n-1}(0,1,0,0,0,0,0)^T$, and the count of all of these is

$$d_n = (1,1,1,1,1,1,1)N^{n-1}(0,1,0,0,0,0,0)^T. \tag{24}$$

The matrix N has characteristic polynomial $x(x^3-x^2-2)(x^3+x^2-1)$. The factor (x^3-x^2-2) corresponds to our expected recurrence relation. However, $(0,1,0,0,0,0,0,0)$, does not belong to the kernel of this matrix, so we can't immediately conclude our proof.

The kernel of $N(N^3+N^2-I)$ is spanned by $(1,0,0,-1,0,0,0)$, $(0,0,0,0,1,0,-1)$, $(0,1,-1,0,0,0,0)$, $(0,1,0,0,0,-1,0)$, which corresponds to a partition of $\{A,B,\ldots,G\}$ into the sets

$$X=\{A,D\}, Y=\{E,G\}, Z=\{B,C,F\}.$$

We can rewrite Table 4 in terms of X,Y,Z:

type of m_n	possible type of m_{n+1}
X	X,Y
Y	Z,Z
Z	X

TABLE 5. Rules for elements of A_{n+1} following from elements of A_n, in terms of types X,Y,Z.

This table shows how each type of matrix in A_n can be extended to a matrix of some type in A_{n+1}, in terms of the types X,Y,Z. For example, the initial element $(0,1)\in A_1$ has type Z, and can only be followed by an element in A_2 of type X, which can be followed in A_3 by elements of type X and Y. For example, to obtain the elements of A_5, we have sequences corresponding to elements of A_5 as in Table 6. We can rewrite Table 5 as a function

applications of f					resulting sequence	sequence in terms of A to G.
Z	X	X	X	X	$ZXXXX$	$BDADA$
				Y	$ZXXXY$	$BDADG$
			Y	Z	$ZXXYZ$	$BDAEB$
				Z	$ZXXYZ$	$BDAEF$
		Y	Z	X	$ZXYZX$	$BDGBD$
			Z	X	$ZXYZX$	$BDGCD$

Table 6. Example of words in X, Y, Z, and the corresponding words in A, B, C, D, E, F, G.

$$f(X) = \{X, Y\},\ f(Y) = \{Z, Z\},\ f(Z) = \{X\}, \tag{25}$$

where the images are not sets, but ordered lists of elements of the set $\{X, Y, Z\}$. The map f (25) can now be written in matrix format as

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix},$$

where the first, second and third rows and columns correspond to the sets X, Y, Z respectively. Since P tells us how we can continue sequences of the rows of elements of A_n to A_{n+1}, and the starting element B is contained in Z, which corresponds to the vector $(0, 0, 1)$, heuristically, we have that

$$d_n = (1, 1, 1)P^{n-1}(0, 0, 1)^T. \tag{26}$$

To prove more formally that (26) holds, we view $\mathbb{R}^3$ as a quotient of $\mathbb{R}^7$ by the kernel of $N(N^3 + N^2 - I)$. Corresponding to this description, we find a quotient map, $V : \mathbb{R}^7 \to \mathbb{R}^3$, and a right inverse inclusion map $U : \mathbb{R}^3 \hookrightarrow \mathbb{R}^7$. For simplicity of notation, we denote the corresponding matrices by the same symbols. The maps U and V are given by the matrices

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{27}$$

We have $P = VNU$, and can also verify that $VU = I_3$. (the identity in $\mathrm{GL}(n)$ is denoted by I for all n, or I_n for clarity.) Define linear maps $Q : \mathbb{R} \to \mathbb{R}^3$, $R : \mathbb{R} \to \mathbb{R}^7$, $S : \mathbb{R}^3 \to \mathbb{R}$ and $T : \mathbb{R}^7 \to \mathbb{R}$ by

$$R = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad T = (1, 1, 1, 1, 1, 1, 1), \qquad S = (1, 1, 1). \tag{28}$$

Then the RHS of (24) is the value of $TN^{n-1}R(1)$, and the RHS of (26) is the value of $SP^{n-1}Q(1)$. So to prove that (26) holds, we must check that $TN^{n-1}R = SP^{n-1}Q$ for all $n \geq 1$. This is equivalent to showing that the following diagram commutes, that is, the top row is the same map as the bottom row:

$$\begin{array}{ccccccccccccc} \mathbb{R} & \xrightarrow{R} & \mathbb{R}^7 & \xrightarrow{N} & \mathbb{R}^7 & \xrightarrow{N} & \mathbb{R}^7 & \xrightarrow{N^{n-4}} & \mathbb{R}^7 & \xrightarrow{N} & \mathbb{R}^7 & \xrightarrow{T} & \mathbb{R} \\ \| & & \downarrow V & & \downarrow V & & \downarrow V & & \downarrow V & & \downarrow V & & \| \\ \mathbb{R} & \xrightarrow[Q]{} & \mathbb{R}^3 & \xrightarrow[P]{} & \mathbb{R}^3 & \xrightarrow[P]{} & \mathbb{R}^3 & \xrightarrow[P^{n-4}]{} & \mathbb{R}^3 & \xrightarrow[P]{} & \mathbb{R}^3 & \xrightarrow[S]{} & \mathbb{R} \end{array} \tag{29}$$

To see this, first note that we have equalities

$$Q = VR, \qquad T = SV, \qquad PV = VN, \tag{30}$$

which can be checked computationally. Then by an inductive argument, we have for $i = 0, \ldots, n-1$ that $TN^{n-1}R = SVN^{n-1}R = SP^iVN^{n-1-i}R = SP^{n-1}VR = SP^{n-1}Q$, which just corresponds to following through the diagram, and we see that (29) does commute. So (26) does indeed hold. We find that the matrix P has characteristic polynomial $x^3 - x^2 - 2$, so $P^3 - P^2 - 2I = 0$. Thus from (26), using the same method as in (12), we obtain the recurrence

$$d_n - d_{n-1} - 2d_{n-3} = 0, \text{ for } n \geq 3$$

with initial terms $d_1 = d_2 = 1$ and $d_3 = 2$ as in Figure 3. □

Corollary 1. *We have $a_n = b_{n-1} = c_n = d_{n-1}$, with a_n, b_n, c_n, d_n as defined in (7), (14), Definition 1, and (21) respectively.*

Proof. The first few terms, $1, 1, 2, 4, 6$, of these sequences are given in (8), (15), Table 1, and Figure 3 respectively. The only difference is that a_n starts from $n = 0$, b_n starts from $n = 1$, c_n starts from $n = 0$, (inserting an extra $c_0 = 1$ term) and d_n starts from $n = 1$. We showed that they all satisfy the same recurrence relation, in Theorems 2, 3, 4, and 5 respectively. Thus the result follows. □

4. Conclusions

By using the L-system for the boundary of the Harter-Heighway dragon curve, we have been able to prove results not only about the dragon curve, but also related sequences, found in [7], by also viewing them in terms of transition matrices inspired by L-systems. Given that the sequences in Section 3 count sizes of various sets, which all turn out to have the same size, we have actually constucted bijections between these sets. The L-system for the Heighway dragon results in a word, with letters in an order which is lost by just using the matrix M, which only counts the total number of letters. The elements of the sets in S_n and A_n in Sections 3.3 and 3.4 do not a proiri have a natural ordering, but we can use the L-system for the right side of the dragon curve to impose an ordering on the elements of the sets S_n and A_n (Definitions 3 and 1), though we must make a choice, e.g., lexicographical on the elements $A, \ldots, G$ in Definition 4. This gives orderings for example as in Table 1 and Figure 3. Given such a choice, this results in a corresponding bijection between the elements of the sets A_n and S_n and the edges of the right side of the dragon curve $\mathcal{S}_n$. This may well just be numerology, but perhaps there is an interesing geometrical or number theoretical meaning waiting to be discovered.

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