A BBP-style computation for π in base 5

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Abstract. We joke about how to compute (promptly) the digits of π, in base 5, from a given place without computing preceding ones.

1. The BBP algorithm

Given a base $b \in \mathbb{Z}_{\ge 2}$ and a 'related to the base' series expansion of constant $\xi \in \mathbb{R}$ (a BBP-type formula), the Bailey–Borwein–Plouffe algorithm [1] computes r base b digits of ξ beginning at the position $d+1$ after the floating point. A traditional illustrative example is the constant

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n}$$

and $b = 2$. To access what begins at the binary position $d+1$, we look for the fractional part of

$$2^d \log 2 = \sum_{n=1}^{d} \frac{2^{d-n}}{n} + \sum_{n=d+1}^{\infty} \frac{2^{d-n}}{n}.$$

In the first (finite) sum fractional parts of individual terms are computed using

$$\frac{2^{d-n}}{n} \bmod \mathbb{Z} = \frac{2^{d-n} \bmod n}{n},$$

with $2^{d-n} \bmod n$ calculated with the help of fast modular exponentiation; the second sum — the tail — converges quickly at rate $1/2$, so that its first few digits can be easily computed.

On the other hand, one can vary this recipe slightly by writing

$$2^d \log 2 = \sum_{n=1}^{d} \frac{2^{d-n}}{n} + \sum_{n=d+1}^{d+2r} \frac{2^{d-n}}{n} + \sum_{n=d+2r+1}^{\infty} \frac{2^{d-n}}{n}.$$

For the first sum here we do exactly the same as before — compute individual fractional parts with the help of fast exponentiation. For the second sum we just sum up the corresponding $2r$ fractions, each being the reciprocal of an integer, while for the third sum we use the estimate

$$\Bigg| \sum_{n=d+2r+1}^{\infty} \frac{2^{d-n}}{n} \Bigg| < \frac{1}{d+2r+1} \sum_{n=d+2r+1}^{\infty} 2^{d-n} = \frac{2^{-2r}}{d+2r+1}$$

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showing that this tail will not in general affect the first r binary digits of $\{2^d \log 2\}$.

2. A BBP-type formula for π

Several BBP-type formulae for π are known for bases which are powers of 2. This gives access, for example, to computing hexadecimal digits of π starting at a particular far-away position without computing predecessors. It is an open question whether this type of computation is possible for π related to other bases.

In what follows i stands for $\sqrt{-1}$.

Rational approximations to π constructed by Salikhov in [2, 3] and later refined by Zeilberger and this author in [4] are based on the representation

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{1}{1-2i}\right)^{2n+1} - \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{1}{1+2i}\right)^{2n+1} = \frac{\pi i}{4}. \tag{1}$$

Because the norm of the quadratic irrationalities $1 \pm 2i$ is 5, one can think of the expansion as of 'base $\sqrt{5}$' BBP-type formula for π — in fact 'base 5' as we encounter the powers of $(1 \pm 2i)^2$ rather than of $1 \pm 2i$. Can it be used for computing base 5 digits of π?

Before proceeding with this, we make some related comments about computation of powers $(1 \pm 2i)^n$. Their real and imaginary parts are read off from the 2×2 matrix $\left(\begin{smallmatrix} 1 & -2 \\ 2 & 1 \end{smallmatrix}\right)^n$; they are also generated via the recurrence equation $a_n = 2a_{n-1} - 5a_{n-2}$. The latter circumstance leads to the following interpretation of identity (1): Define the sequence b_n through the recursion $b_n = -6b_{n-1} - 25b_{n-2}$ for $n \ge 2$ and initial data $b_0 = 1$, $b_1 = -1$. Then

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{b_n}{5^{2n}} = \frac{5\pi}{16}$$

and $|b_n| < 2 \cdot 5^n$ for all n.

Formula (1) allows one to cast π as the imaginary part of

$$\xi = 8 \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{1}{1-2i}\right)^{2n+1},$$

and we can use the BBP strategy to compute the base 5 expansion for both its real and imaginary parts beginning at the position $d+1$ till the position $d+r$, say, under a mild condition we give below (see condition (2)). This demands for computing first r base 5 digits after the floating point of the number $5^d \xi$ (again we do this for both real and imaginary parts!). Because $|1-2i| = \sqrt{5}$, the tail

$$8 \cdot 5^d \sum_{n=d+2r}^{\infty} \frac{1}{2n+1} \left(\frac{1}{1-2i}\right)^{2n+1}$$

is bounded above by

$$\frac{8 \cdot 5^d}{2d+4r+1} \sum_{n=d+2r}^{\infty} \frac{1}{5^{n+1/2}} = \frac{2 \cdot 5^{1/2-2r}}{2d+4r+1},$$

which means that the positions before the position $r+1$ in the base 5 expansion of $5^d\xi$ are hardly affected, and we look for the first r digits of

$$\sum_{n=0}^{d+2r-1} \frac{8\cdot 5^d}{2n+1}\left(\frac{1}{1-2i}\right)^{2n+1}.$$

We are only interested in this expression in $\mathbb{Q}[i]$ modulo 1 for real and imaginary parts. Assuming that

$$\frac{\log(2d+4r-1)}{\log 5} \le d \tag{2}$$

and writing $2n+1 = 5^k\cdot m$ with $(m,5)=1$ for the nth term in the sum, we look for

$$\frac{8\cdot 5^d}{2n+1}\left(\frac{1}{1-2i}\right)^{2n+1} \bmod \mathbb{Z}[i] = \frac{8\cdot 5^{d-k}(1-2i)^{-(2n+1)} \bmod m}{m}; \tag{3}$$

thus, we only need executing fast exponentiation for $5^{d-k} \bmod m$ and $(1-2i)^{2n+1} \bmod m$ in $\mathbb{Z}[i]$, and then computing the reciprocal of the latter modulo m. The latter inversion is possible because m is coprime with the norm of any power of $1-2i$; alternatively, one can take an integer a such that $a \equiv \frac{1}{5} \bmod m$, with the motive that $(1-2i)^{-1} = \frac{1}{5}+\frac{2}{5}i \equiv a(1+2i) \bmod m\mathbb{Z}[i]$, and compute $a^{2n+1}(1+2i)^{2n+1} \bmod m$.

3. An obvious flaw

The equality in (3) is incorrect when $2n+1 > d-k$, since the left-hand side in the latter case has the denominator $5^{2n+1-d+k}m$, while the denominator of the right-hand side is m. Using a shorter finite sum over n to meet this constraint is not an option either, because there is no bound for (the fractional part of) the tail in such cases. Are there more suitable formulae for π or other interesting constants, defined over quadratic or more general algebraic extensions, that could be of use?

References

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