COUNTING THE FISSION TREES AND NONABELIAN HODGE GRAPHS

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Abstract. Any algebraic connection on a vector bundle on a smooth complex algebraic curve determines an irregular class and in turn a fission tree at each puncture. The fission trees are the discrete data classifying the admissible deformation classes. Here we explain how to count the fission trees with given slope and number of leaves, in the untwisted case. This also leads to a clearer picture of the "periodic table" of the atoms that play the role of building blocks in 2d gauge theory.

1. Introduction

The fission trees ([17] 3.18, [7] p.29) encode the topology of the possible ways the structure group may be broken/fissioned by a meromorphic connection at a pole. In the untwisted multiplicity 1 case (our focus here), a fission tree looks as follows:

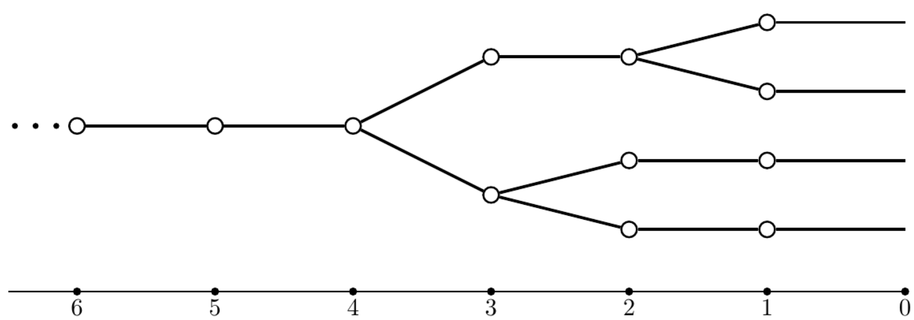
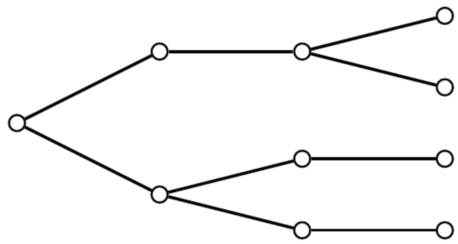


Figure 1. Example fission tree $\mathbb{T}$ (with 4 leaves and slope 3).

Thus it is connected, each node has an integer height, the leaves have height 0, a node of height 1 cannot be a branch node, and there is a highest branch node, the root, above which there is just one node for each integer. The simplest numerical invariants of a fission tree are the number n of leaves and the slope k (= the height of the root minus 1).

The aim of this note is to explain how to compute the number $\phi(k, n)$ of such fission trees of slope k with n leaves, up to isomorphism. We will then adapt this to solve several closely related counting problems.

The above drawing of a fission tree is a bit redundant and it is often more convenient to draw a pruned form of the tree, obtained by truncating the tree at the root, and trimming the leaves off (cutting at height 1). Thus the pruned rendering of the above tree is as follows:



For example it is not so hard to find the other eight fission trees with slope 3 and 4 leaves (= the number of nodes of height 1), as well as those with 2 or 3 leaves:

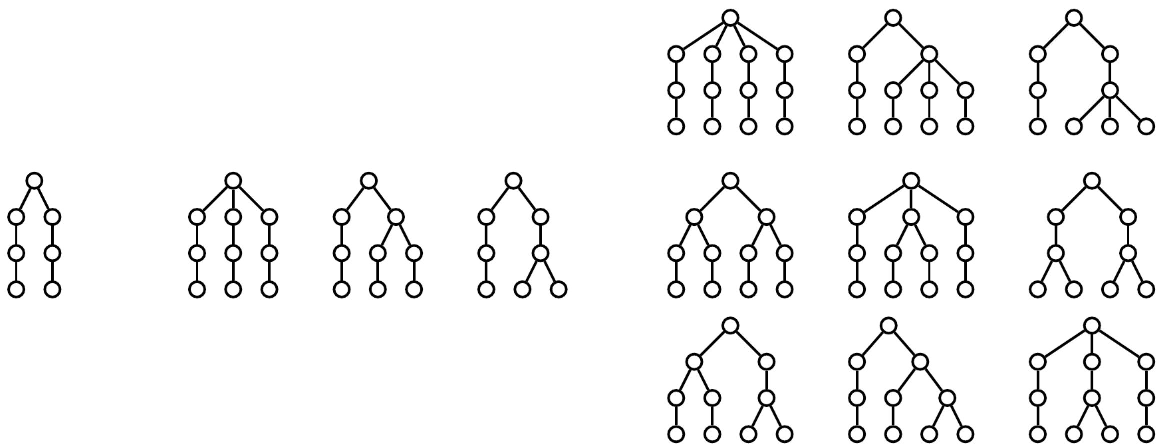


FIGURE 2. $1+3+9$ fission trees with slope 3 and $2, 3$ or 4 leaves.

For other slopes/leaf counts, direct counting quickly becomes laborious. We will explain how a method dating back to Euler may be adapted to give a quick way to count the fission trees, for example leading to the construction of the following table:

$k \setminus n$	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1	1
2	0	1	2	4	6	10	14	21	29	41
3	0	1	3	9	20	47	96	201	394	775
4	0	1	4	16	48	148	407	1121	2933	7612
5	0	1	5	25	95	365	1271	4383	14479	47198
6	0	1	6	36	166	766	3237	13466	53933	212645
7	0	1	7	49	266	1435	7140	34853	164324	761829
8	0	1	8	64	400	2472	14162	79430	431242	2301016
9	0	1	9	81	573	3993	25893	164157	1009029	6094011
10	0	1	10	100	790	6130	44392	314011	2156113	14544961

TABLE 1. Counting (untwisted, mult. 1) slope k fission trees with n leaves

Note for example that row three: $0, 1, 3, 9, 20, 47, 96, \ldots$ is not currently in the online encyclopaedia of integer sequences oeis.org.

Three aspects of the underlying (somewhat elaborate) motivation are as follows. The reader interested mainly in counting fission trees might skip straight to §2.

1.1. **Topological classification of irregular types.** The main motivation for studying the fission trees comes from global Lie theory, or more precisely the classification of wild Riemann surfaces $\mathbf{\Sigma}$, and in turn the deformation classes of their character varieties $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma})$ and nonabelian Hodge moduli spaces $\mathfrak{M}(\mathbf{\Sigma}, \mathcal{C})$ (i.e. we wish to understand the deformation classes of these complete hyperkähler manifolds). In simple terms one chooses a compact Riemann surface Σ, an integer $n \geq 1$ (the rank), a finite subset $\mathbf{a} \subset \Sigma$ and an *irregular type* Q_a at each point $a \in \mathbf{a} \subset \Sigma$, and defines $\mathbf{\Sigma}$ to be the triple $(\Sigma, \mathbf{a}, \mathbf{Q})$ where $\mathbf{Q} = \{Q_a \mid a \in \mathbf{a}\}$. This data determines an algebraic Poisson moduli space $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma})$ as in [11] Cor. 8.3 (and [5, 4] in the generic case) and its symplectic leaves have hyperkähler metrics (they are the wild nonabelian Hodge moduli spaces $\mathfrak{M}(\mathbf{\Sigma}, \mathcal{C})$ constructed in [3]). If we choose a local coordinate z vanishing at $a \in \Sigma$ then an irregular type at a is just an element of the form

$$Q_a = \frac{A_k}{z^k} + \cdots + \frac{A_2}{z^2} + \frac{A_1}{z}$$

where the A_i are all diagonal complex $n \times n$ matrices. The choice of irregular types picks out a finite dimensional space of meromorphic connections/Higgs bundles with good properties (generalising the tame case when $Q = 0$).[1]

Now, each irregular type $Q = \sum_1^k A_i/z^i$ determines a fission tree $\mathbb{T}(Q)$, whose set of nodes of height $h \in \mathbb{N}$ is the set of eigenspaces of the truncation

$$\tau_h(Q) = \sum_{i \geq h} A_i/z^i$$

of Q (we take the truncation to be zero if $h > k$, so there is then just one node), as in [17] Lem. 3.22 (cf. also [7] Apx. C, [11]). For example $Q = A/z^3 + B/z^2 + C/z$ gives the tree in Fig. 1 if $A = \mathrm{diag}(0, 0, 1, 1), B = \mathrm{diag}(0, 0, 0, 1), C = \mathrm{diag}(1, 0, 0, 0)$.

The fission tree $\mathbb{T}(Q)$ encodes the topology of the irregular type Q, and leads to the topological skeleton $\mathrm{Sk}(\mathbf{\Sigma}) := (g, \mathbf{F})$ of $\mathbf{\Sigma}$, where g is the genus of Σ and $\mathbf{F} = \sum_{a \in \mathbf{a}} [\mathbb{T}(Q_a)]$ is the fission forest, made up of the multiset of isomorphism classes of fission trees (as in [17] §3.7). Now the key point here is that the moduli spaces $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma})$ vary smoothly ([11] Thm. 10.2), provided that the wild Riemann surface $\mathbf{\Sigma} = (\Sigma, \mathbf{a}, \mathbf{Q})$ varies in an admissible fashion, i.e. that it varies smoothly and the topological skeleton $\mathrm{Sk}(\mathbf{\Sigma})$ is constant. Thus the topological skeleta *parameterise* the admissible deformation classes. In more detail the notion of admissible deformation of $\mathbf{\Sigma}$ was defined in [11, 17] and in the current setting it says that:

[1]Given an irregular type Q, one considers meromorphic connections that are locally of the form $d - (dQ + M(z)dz/z)$ for some holomorphic matrix M, as in the Hukuhara–Turrittin normal form, so that solutions of such connections involve the essentially singular functions $\exp(Q)$. One can then look at moduli spaces of such connections, and their Betti descriptions [28, 4, 5, 11] leading to the hyperkahler manifolds of [3], classifying wild harmonic bundles on $\Sigma \setminus \mathbf{a}$.

- The underlying Riemann surface Σ varies smoothly, and
- the points $\mathbf{a} \subset \Sigma$ vary smoothly and do not coalesce, and
- each irregular type Q_a varies smoothly and its fission tree $\mathbb{T}(Q_a)$ does not change.

Thus it is important to classify the fission trees in order to understand the deformation classes of the wild character varieties $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma})$ and in turn wild nonabelian Hodge moduli spaces $\mathfrak{M}(\mathbf{\Sigma}, \mathcal{C})$ (which are isomorphic to symplectic leaves of $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma})$)—in general distinct fission trees will lead to distinct deformation classes of moduli spaces.

Note that the underlying deformation spaces themselves, the moduli spaces $\mathcal{M}_{g,\mathbf{F}}$ of wild Riemann surfaces with given skeleton, also form interesting geometric objects, and their fundamental groups are the wild mapping class groups, [12] §8, [11, 17, 26]. By definition a map $\mathbb{B} \to \mathcal{M}_{g,\mathbf{F}}$ is the same thing as an admissible family over $\mathbb{B}$ of wild Riemann surfaces (with skeleton $(g, \mathbf{F})$), as defined in [11] §10. The spaces $\mathcal{M}_{g,\mathbf{F}}$ generalise the usual moduli spaces of curves with marked points (that appear in the tame case, in effect when the fission trees all just have one leaf). Enumerating the fission trees is the key step in enumerating the forests $\mathbf{F}$ and thus the spaces $\mathcal{M}_{g,\mathbf{F}}$.

1.2. **Nonabelian Hodge graphs.** Secondly the fission trees are key to seeing which (non-affine) Kac–Moody root systems appear in 2d gauge theory. For example the nonabelian Hodge spaces of complex dimension two (the H3 surfaces) are related to ADE surface singularities (cf. [31, 6]), and are classified by certain, special, affine Dynkin diagrams. For example in the six Painlevé cases one gets the diagrams:

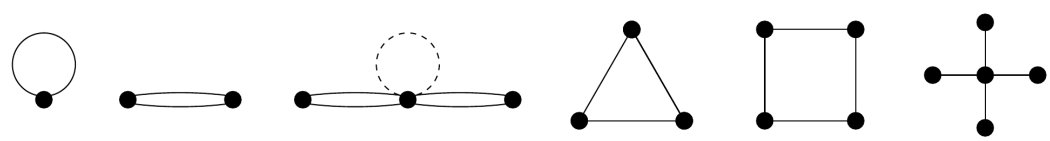


FIGURE 3. The diagrams of the Painlevé moduli spaces 1,2,3,4,5,6 (see [18]).

This story generalises to nonabelian Hodge spaces of arbitrary dimension on the Riemann sphere: any such space has a diagram attached to it [21, 22, 7, 10, 27, 36, 18, 23] (see [15] §1.6 for more on nonabelian Hodge spaces). Some of the diagrams are just graphs (possibly with multiple edges, but no edge loops, or negative/dashed edges) and others, the *twisted cases*, involve loops or negative edges (such as the dashed loop for Painlevé 3 above), that may be embedded in the untwisted case by pullback. In all cases one gets a generalisation of a Cartan matrix whose inner product controls the dimension of the moduli space [18, 23]. It is natural to ask the basic question:

What is the special class of graphs that appear as diagrams of nonabelian Hodge spaces?

This question looks to be open, but we know a large class of examples of such nonabelian Hodge graphs, the *supernova graphs*, that may in fact cover all the cases. The supernova graphs are simple extensions of the fission graphs, and their classification reduces to the classification of fission trees. The two basic statements are as follows:

• There is a special class of graphs, the *fission graphs*, that are in bijection with the (untwisted, multiplicity 1) fission trees of slope > 1. If $\mathbb{T}$ is a fission tree with leaves I then the fission graph $\Gamma(\mathbb{T})$ of $\mathbb{T}$ has vertices I and the number of edges between two vertices i, j in Γ is $h_{ij} - 2$ where h_{ij} is the height of the nearest common ancestor of the leaves i, j. (The slope > 1 condition implies the graph is connected.) They are very special: for example there are only $\phi(2, 10) = 41$ simple fission graphs with 10 nodes, out of the 11.7 million or so simple graphs with 10 nodes oeis/A001349. For example the fission graphs of the nine fission trees of slope 3 with 4 leaves are:

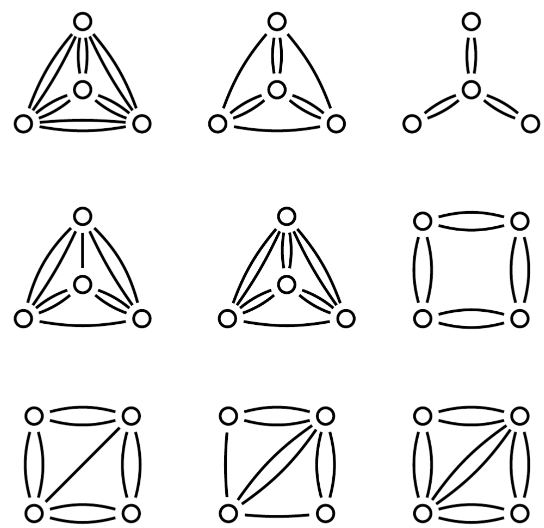


FIGURE 4. Fission graphs with 4 nodes and maximal edge multiplicity 2.

• Suppose Γ is a fission graph with nodes I and we choose an integer ≥ 0 for each node, i.e. we choose $\mathbf{l} = \{l_i \in \mathbb{N} \mid i \in I\} \in \mathbb{N}^I$. Then the *supernova graph* $\widehat{\Gamma}(\mathbf{l})$ determined by $(\Gamma, \mathbf{l})$ is the graph obtained by gluing a leg of length l_i onto the node $i \in I$ of Γ for each $i \in I$, ([10] Def. 9.1, [7] Apx. C). The *core* of $\widehat{\Gamma}(\mathbf{l})$ is Γ.

It is known that any supernova graph $\widehat{\Gamma}(\mathbf{l})$ is a nonabelian Hodge graph. In other words, there is a (nonempty) wild nonabelian Hodge moduli space $\mathfrak{M}(\mathbf{\Sigma}, \mathcal{C})$ with diagram $\widehat{\Gamma}(\mathbf{l})$ (for some choice of marking of the formal monodromy orbits). This follows (as explained in the simply-laced case in [7] p.21) using the Kac–Moody root system of the graph $\widehat{\Gamma}(\mathbf{l})$ and results of Vinberg [34] and Crawley–Boevey [20]—in the general non-simply-laced case, this uses Hiroe–Yamakawa's proof [27, 36] of the quiver modularity conjecture of [7] Apx. C.

This gives further motivation for enumerating the fission trees (and thus the fission graphs). In particular this shows where these special types of Kac–Moody root systems appear in 2d gauge theory. As mentioned above, we don't know any examples of nonabelian Hodge graphs that are not supernova graphs.

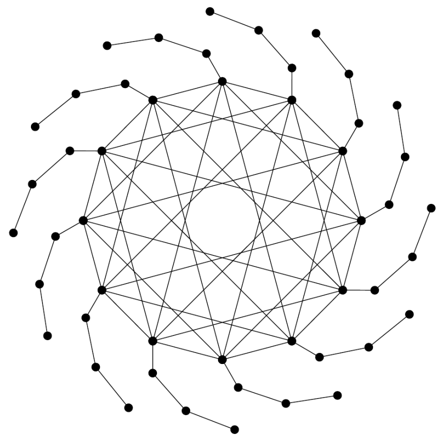


FIGURE 5. Example supernova graph, with $l_i = 3$ for all core nodes i.

The original definition of the fission graph of an irregular type ([7] Apx. C) involved a sequence of fission operations, as illustrated in Fig. 6:

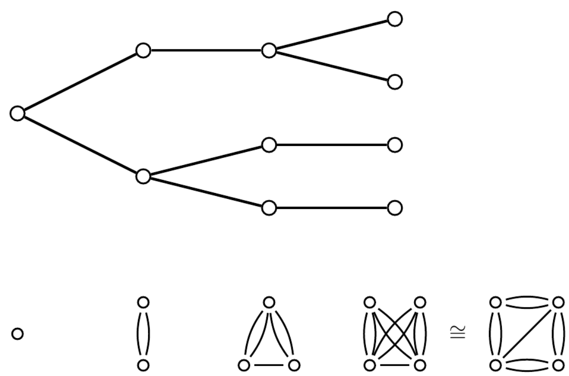


FIGURE 6. Fission construction of the fission graph of the fission tree of Fig. 1 (as drawn in [14] p.79).

Remark 1. The direct meaning of the fission graph/tree in terms of Stokes arrows was pointed out in [18] p.65 (recall that the Stokes arrows [16] §5.5 parameterise all the possible nontrivial entries of the Stokes automorphisms, rephrasing the original definition in [30]): If we increase each edge multiplicity of a fission graph $\Gamma(\mathbb{T})$ by one, then we get the *Stokes quiver* of the fission tree $\mathbb{T}$ (thus the Stokes quivers are also in bijection with the fission trees)—If $\mathbb{T} = \mathbb{T}(Q)$ is the fission tree of an irregular type, then the Stokes quiver contains all the Stokes arrows at the pole (the union of all the Stokes arrows for all the singular directions of Q, as in [16] §5.5).

1.3. **New multiplicative quiver varieties.** The "quasi-Hamiltonian" theory of Lie group valued moment maps [1] gives a precise mathematical language for constructing symplectic moduli spaces via TQFT-type gluing of real surfaces. Although originally set-up for compact Lie groups it may be transposed to the complex algebraic setting, and then leads to many new algebraic symplectic and Poisson moduli spaces [5, 33, 35, 8, 11].

The (new) multiplicative quiver varieties appear as follows: Given a fission graph Γ with nodes I and a dimension vector $\mathbf{d} \in \mathbb{N}^I$, one may define a Zariski open subset:

$$\mathcal{B}(\Gamma, \mathbf{d}) = \mathrm{Rep}^*(\Gamma, \mathbf{d}) \subset \mathrm{Rep}(\Gamma, \mathbf{d})$$

of the space $\mathrm{Rep}(\Gamma, \mathbf{d})$ of representations of the graph Γ on the vector space $\mathbb{C}^{\mathbf{d}}$, the reduced fission space/invertible graph representations (here we identify Γ as a quiver, by viewing each edge as two opposite arrows). The reduced fission space is a quasi-Hamiltonian variety equipped with a moment map

$$\mu : \mathcal{B}(\Gamma, \mathbf{d}) \to \mathrm{GL}_{\mathbf{d}}(\mathbb{C}) := \prod_{i \in I} \mathrm{GL}_{d_i}(\mathbb{C}).$$

In brief if Γ is the fission graph of a fission tree $\mathbb{T}(Q)$ of an irregular type Q at $\infty \in \mathbb{P}^1$ then $\mathcal{B}$ is isomorphic to the wild representation variety $\mathcal{R} = \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ of $\mathbf{\Sigma} = (\mathbb{P}^1, \infty, Q)$, from [11] Thm 1.1 (and independent of Q by [11] Thm 10.4).

Now suppose we have an arbitrary finite graph Γ (possibly with multiple edges) equipped with a colouring map $\gamma : \Gamma \to C$ to some set C of colours, such that each monochromatic subgraph $\Gamma(c) := \gamma^{-1}(c)$ is a fission graph. Then for each dimension vector $\mathbf{d} \in \mathbb{N}^I$ and parameters $q \in (\mathbb{C}^*)^I$ (where I is the set of nodes of Γ), there is a symplectic variety called the (new) multiplicative quiver variety:

$$\mathcal{M}(\Gamma, q, \mathbf{d}) = \mathrm{Rep}^*(\Gamma, \mathbf{d}) /\!\!/_q \mathrm{GL}_{\mathbf{d}}(\mathbb{C}) = \mu^{-1}(q)/\mathrm{GL}_{\mathbf{d}}(\mathbb{C})$$

where $\mathrm{Rep}^*(\Gamma, \mathbf{d})$ is defined by fusing together the spaces $\mathcal{B}(\Gamma(c), \mathbf{d}(c))$ for each colour c. The classical multiplicative quiver varieties [22, 33, 35] are the special case when each fission graph is a single edge $\circ - \circ$, corresponding to the (unique) fission tree of slope 2 and two leaves. The simply-laced case (when each fission graph has no multiple edges, i.e. the corresponding fission trees have slope 2) is described in detail in [13], and the general case follows the same pattern (noting Rmk 5.4 of [13]). In general the classical multiplicative quiver variety $\mathcal{M}_{cl}(\Gamma, q, \mathbf{d})$ looks to be a proper open subset of $\mathcal{M}(\Gamma, q, \mathbf{d})$ (i.e. when we modify the colouring map so that each edge has a distinct colour, one obtains a slightly smaller symplectic variety).

In any case we see the fission graphs $\Gamma(\mathbb{T})$ parameterise the basic building blocks $\mathcal{B}(\Gamma, \mathbf{d})$ here, giving more motivation for the classification of the underlying fission trees $\mathbb{T}$. They form a kind of "periodic table" for the building blocks, or "atoms". (Via [3] they parameterise harmonic bundles on $\mathbb{A}^1$, and it is hoped $\mathcal{M}(\Gamma, q, \mathbf{d})$ still parameterises harmonic bundles; they are special cases of fission varieties (cf. [8])).

Remark 2. The general definition of the fission trees of a meromorphic connection on a vector bundle on a Riemann surface in [17] involves the choice of a dimension vector (i.e. the multiplicity map n in [17] 3.4.1). Here we are just looking at the untwisted case (and so the definition is essentially the same as that in [7] Apx. C), and we are forgetting the choice of dimension vector in the classification (as it is easy to add this afterwards: see §3). The extension to the twisted case may be accomplished by considering automorphisms of fission trees, and will be discussed elsewhere.

2. Counting

2.1. **Encoding via surjective maps.** First observe that the combinatorics of an untwisted fission tree $\mathbb{T}$ (as in Fig. 1) can be encoded as follows. Observe that $\mathbb{T}$ is determined by its sets $J_1, J_2, \ldots$ of nodes of each height ≥ 1, together with the "parent maps", which are the surjective maps:

$$\pi_i : J_i \twoheadrightarrow J_{i+1}$$

taking any node to its parent. Each of the sets J_i is finite, and the maps converge in the sense that $|J_i| = 1$ for all sufficiently large i. The number of leaves of $\mathbb{T}$ equals $|J_1|$ and the slope of the tree is $\min\{i-1 \bigm| |J_i| = 1\}$. Thus we can just as well consider sequences of surjective maps of finite sets:

$$\cdots J_{k+2} \cong J_{k+1} = \{*\} \twoheadleftarrow J_k \twoheadleftarrow \cdots \twoheadleftarrow J_2 \twoheadleftarrow J_1 \tag{1}$$

where all the maps to the left of $*$ are isomorphisms. It follows that the number $\phi(k, n)$ is (also) the number of isomorphism classes of such sequences of finite sets and surjective maps, such that $|J_1| = n$ and $\min\{\, i \bigm| |J_i| = 1\} = k + 1$.

Up to relabelling this is the way the fission trees appeared in [7] Apx. C, where J_i was the set of simultaneous eigenspaces of the leading coefficients $\{A_k, \ldots, A_i\}$ of an irregular type of the form:

$$Q = \frac{A_k}{z^k} + \cdots + \frac{A_2}{z^2} + \frac{A_1}{z}$$

where the A_i are all diagonal matrices. Thus the fission trees encode how the eigenspaces of the truncation $\tau_h(Q) = \sum_{i\geq h} A_i/z^i$ are refined as h decreases (geometrically this encodes the types of growth of the maps $\exp(Q)$ that occur in solutions of the corresponding connections).

2.2. **Counting.** We will say that a fission tree is *generic* if it has just one branch node (so exactly one of the parent maps is not bijective). There is just one fission tree with one leaf, and it has slope 0, and moreover any tree with slope 0 has just one leaf, so:

$$\phi(0, 1) = 1 \quad \text{and} \quad \phi(k, 1) = \phi(0, n) = 0 \text{ if } k > 0, n > 1.$$

For slope 1 there is just one fission tree with any number $n > 1$ of leaves (a generic fission tree), so that:

$$\phi(1, n) = 1 \quad \text{for any } n > 1, \text{ and } \phi(1, 1) = 0 \text{ as above.}$$

For slope 2, we have a surjective map $J_1 \twoheadrightarrow J_2$ of finite sets where $n = |J_1|$ is the number of leaves. Thus the map gives a partition of n, where the parts are labelled by the set J_2. By definition $|J_2| > 1$ so we are counting the partitions of n with at least two parts. Since there is just one partition with just one part it follows that:

$$\phi(2, n) = p(n) - 1$$

where $p(n)$ is the number of integer partitions of n.

For slope 3 and $n = 1, 2, 3, 4, 5$ leaves, we find directly that there are $0, 1, 3, 9, 20$ fission trees, respectively. The first few are as drawn in Fig. 2. Now this sequence $0, 1, 3, 9, 20$ yields only 5 matches at oeis.org, none of which seem relevant. Thus it seems no-one has counted the fission trees before.

It turns out however that the desired numbers appear as the *difference* of some easily generated sequences, using a method going back to Euler. In particular we will easily be able to tabulate the numbers $\phi(n, k)$ far beyond anything we could count by hand.

2.3. **Changing the question.** If we have a fission tree of slope $k > 1$ and delete the root node (at height $k + 1$) to disconnect it, then we get a multiset of fission trees of slope $< k$ (where we may have to prune the tops a little bit, as the new roots may be lower down).

This suggests we should change the question: Let $\Phi(k, n)$ be the number of isomorphism classes of fission trees of slope $\leq k$ and n leaves. This is useful since $\Phi(k, n) = \sum_{r \leq k} \phi(r, n)$ and so it is clear that:

Lemma 3. $\phi(k, n) = \Phi(k, n) - \Phi(k - 1, n)$.

In turn, to compute $\Phi(k, n)$ recall that the *Euler transform* of an integer sequence $a_1, a_2, a_3, \ldots$ is the integer sequence (b_n) given by:

$$1 + \sum_1^\infty b_n u^n = \prod_1^\infty (1 - u^m)^{-a_m}.$$

It converts the counting of connected trees with a given property into the counts of forests of trees with the same property (see [32] p.20). For completeness a self-contained proof is in Appx. B below.

Lemma 4. *1)* $\Phi(2, n) = p(n)$ *is the number of integer partitions of* n*,*

2) for any fixed k *the sequence* $\Phi(k, *)$ *is the Euler transform of the sequence* $\Phi(k - 1, *)$*.*

Proof. For 1), as mentioned above, note that the map $J_1 \twoheadrightarrow J_2$ partitions the set J_1 into $|J_2|$ parts.

For 2) first recall that a forest is a disjoint union of (not necessarily distinct) trees, i.e. it is an (unordered) multiset of isomorphism classes of trees. Then note that 2) means exactly that $\Phi(k, n)$ is the number of fission forests, each with n leaves in total, such that each tree in the forest has slope $\leq k - 1$. Any such forest determines a unique fission tree with n leaves and of slope $\leq k$: View each tree in the forest as a full (nontruncated) tree and then truncate it at height $k + 1$. Then glue all the top nodes (height $k + 1$) of all the trees together, to a single node $\{*\}$. This gives a fission tree of slope $\leq k$ (if the original forest had just one tree then the resulting tree has slope $< k$, and needs to be pruned to get a truncated tree). Conversely any fission tree counted by $\Phi(k,n)$ determines such a forest by deleting the root node (and pruning the tops). $\square$

Note that 2) holds in general and the numbers $\Phi(2,n)$ are the Euler transform of the sequence $[1,1,1,1,1,\ldots]$ essentially counting the fission trees of slope 1. This is the original instance:

$$1+\sum_1^\infty p(n)u^n = \prod_1^\infty (1-u^m)^{-1}$$

of the Euler transform, giving the count $p(n)=\Phi(2,n)$ of partitions.

Note (of course) that the case before this also holds:

$$1+u+u^2+u^3+\cdots=(1-u)^{-1}$$

which says that the all 1's sequence $\Phi(1,n)$ is the Euler transform of the sequence $[1,0,0,0,\ldots]$ counting the fission trees of slope 0.

The next case

$$1+\sum_1^\infty \Phi(3,n)u^n = \prod_1^\infty (1-u^m)^{-p(m)}$$

is discussed by Cayley 1855 [19] p.316, who was counting "partitions of partitions" (i.e. unordered multisets of partitions, whose total sum is n), which we can now identify with fission trees of slope ≤ 3. They are sometimes called double partitions [29] or 2-dimensional partitions[2]. Thus the number $\phi(3,n)=\Phi(3,n)-p(n)$ we are interested in counts the *proper* double partitions of n (i.e. those that do not just consist of a single partition). As Cayley wrote, the list of counts $\Phi(3,n)$ is:

$$1,3,6,14,27,58,111,223,424,817,\ldots$$

which is oeis/A001970, and $p(n)$ is:

$$1,2,3,5,7,11,15,22,30,42,\ldots$$

so that the count $\phi(3,n)=\Phi(3,n)-p(n)$ of slope 3 fission trees with n leaves is:

$$0,1,3,9,20,47,96,201,394,775,\ldots$$

extending the list of counts we constructed above.

For example there are 14 double partitions of 4, of which $5=p(4)$ are just single partitions. The remaining 9, the proper double partitions of 4, are as follows:

[[1][1][1][1]]	[[111][1]]	[[3][1]]
[[11][11]]	[[11][1][1]]	[[2][2]]
[[2][11]]	[[21][1]]	[[2][1][1]]

[2]But beware that a "plane partition" is a special type of $2d$ partition (and they have different counts for $n\ge 4$).

and correspond to the slope 3 fission trees with 4 leaves, drawn on the right of Fig. 2. The corresponding fission graphs are in Fig. 4.

In any case it is now easy to compute the numbers $\phi(k,n)$ and construct Table 1.

$k \setminus n$	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1	1
2	0	1	2	4	6	10	14	21	29	41
3	0	1	3	9	20	47	96	201	394	775
4	0	1	4	16	48	148	407	1121	2933	7612
5	0	1	5	25	95	365	1271	4383	14479	47198
6	0	1	6	36	166	766	3237	13466	53933	212645
7	0	1	7	49	266	1435	7140	34853	164324	761829
8	0	1	8	64	400	2472	14162	79430	431242	2301016
9	0	1	9	81	573	3993	25893	164157	1009029	6094011
10	0	1	10	100	790	6130	44392	314011	2156113	14544961

Table 1. Counting untwisted multiplicity 1 fission trees with slope k and n leaves

By inspection, there appear to be simple formulae for the first few columns:

$$\phi(k,3)=k,\quad \phi(k,4)=k^2,\quad \phi(k,5)=k(5k^2-3k+4)/6,$$

the last of which equals oeis/A203552 (although no links to trees are noted there). It seems one gets a (numerical) polynomial of degree $n-2$ for the nth column in general[3], for example: $\phi(k,6)=k(2k^3-2k^2+4k-1)/3$.

One can compute much larger versions of this table with the following maple code.

```
# Maple code for n terms of kth row of table 1
# Euler transform adapted from https://oeis.org/transforms.txt
k:=4: n:=12:
EULER:=proc(a)  local b,c,i,d: b:=[]:c:=[]:
for i to nops(a) do c:=[op(c), add( d*max(0,1-irem(i,d))*a[d], d=1..i)]: od:
for i to nops(a) do b:=[op(b),(1/i)*(c[i]+add( c[d]*b[i-d], d=1..i-1))]: od:
RETURN(b);  end:
(EULER@@(k-2))([1$n]):EULER(%):convert(<op(%)> - <op(%%)>,list);
```

The simply-laced fission graphs (those with no multiple edges) are exactly the complete multipartite graphs, and they are parameterised by the *nontrivial* integer partitions (i.e. partitions with at least two parts), coming from the slope 2 fission trees. Recall that the complete k partite graph with n nodes determined by the partition $n_1+\cdots+n_k=n$ of n, is the graph $\Gamma(n_1,\dots,n_k)$ with n nodes partitioned into parts of sizes $n_1,\dots,n_k$ such that two nodes are connected by an edge if and only if they are in different parts. For example the core of the supernova graph in Fig. 5 is the complete bipartite graph $\Gamma(6,6)$. Looking at the slope 2 row of the table, we see the number of simply-laced fission graphs with ≤ 6 nodes is $1+2+4+6+10=23$.

[3]We will prove that $\phi(k,4)=k^2$ in the appendix—the others should be viewed as experimental observations that one might want to try to prove directly.

This includes the 5 star-shaped graphs $\Gamma(1,n), n = 1,2,3,4,5$. The remaining $18 = 1+3+5+9$ were drawn in [7] Fig. 1 (and in [10] Fig. 3):

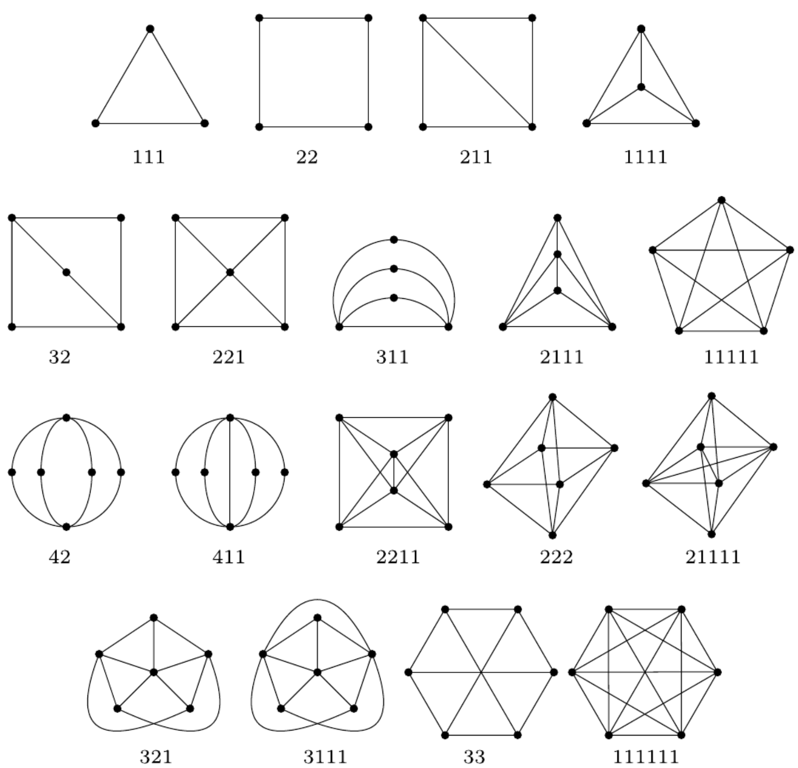


FIGURE 7. The 18 non-starshaped simply-laced fission graphs with ≤ 6 nodes

3. Allowing arbitrary multiplicities

In the above discussion we counted the number of untwisted fission trees that appear when we forget the multiplicity of the leaves (in effect setting the multiplicity of each leaf to be 1). Here we will explain how to add back in the multiplicities, and find essentially the same table but shifted by 1. (Recall that the general definition of fission trees in [17] Defn. 3.18 includes the data of a multiplicity map $n : \mathbb{V}_0 \to \mathbb{Z}_{\geq 1}$ assigning an integer multiplicity to each leaf $\mathbb{V}_0$ of the tree.)

Define the *rank* of a fission tree $\mathbb{T}$ to be the sum of all the multiplicities:

$$\mathrm{rank}(\mathbb{T}) = \sum_{i\in\mathbb{V}_0} n(i)$$

where $\mathbb{V}_0$ is the set of leaves and n is the multiplicity map. (If $\mathbb{T} = \mathbb{T}(Q)$ comes from an $n \times n$ irregular type Q, then $\mathbb{T}$ has rank n and the multplicities come from the multiplicities of the diagonal entries of Q.) Of course in the multiplicity 1 setting above, the rank is the same as the number of leaves. Also, since $\mathbb{V}_1 \cong \mathbb{V}_0$ in the present untwisted setting, and we like to truncate the leaves, we can just as well view n as a map $n : \mathbb{V}_1 \to \mathbb{Z}_{\geq 1}$.

Define $\psi(k,n)$ to be the size of the set of isomorphism classes of all untwisted fission trees of slope k and rank n, with arbitrary multiplicities. Recall that $\phi(k,n)$ was the size of the set of isomorphism classes of untwisted multiplicity 1 fission trees of slope k and rank n. Clearly $\psi(0,n)=1$ for all n as there is just one fission tree with slope zero and rank n (the tree with one leaf of multiplicity n). For higher slopes k one can reduce to the case already studied:

Proposition 5. *If $k \geq 1$ then there is a bijection between the set of isomorphism classes of untwisted fission trees of slope k and rank n, and the set of isomorphism classes of multiplicity 1 untwisted fission trees of slope $k+1$ and rank n, so that $\psi(k,n)=\phi(k+1,n)$ for any n.*

Proof. The bijection arises as follows: Suppose $\mathbb{T}$ is a multiplicity 1 untwisted fission tree of slope $k+1 \geq 2$ and rank n, with nodes $\mathbb{V}$. Define a tree $\mathbb{T}'$ with nodes $\mathbb{V}'_i := \mathbb{V}_{i+1}$ for $i \geq 1$ (and the same edges between these nodes), in effect chopping off all of $\mathbb{T}$ below height 2, and shifting the height by 1. Then define the multiplicity $n(i)$ of any element $i \in \mathbb{V}'_1$ to be the number ≥ 1 of children of the node $i \in \mathbb{V}_2 \subset \mathbb{T}$. Thus $\mathbb{T}'$ is an untwisted fission tree of slope k and rank n, (with possibly nontrivial multiplicities). This process is clearly bijective. □

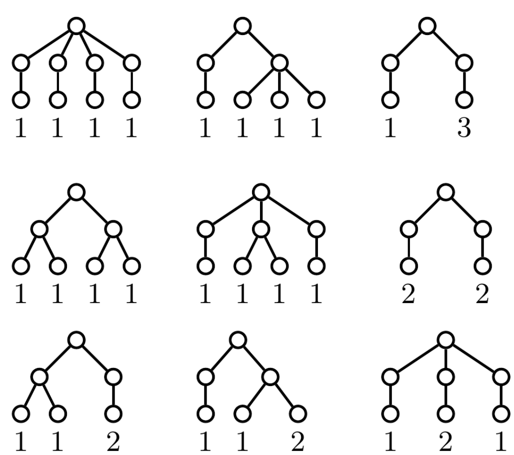


FIGURE 8. The 9 untwisted fission trees with slope 2 and rank 4.

Thus it is easy to modify table 1 to get the counts of all untwisted fission trees.

Remark 6. Whereas a fission tree includes a multiplicity map (assigning an integer to each leaf), by definition a quiver or graph does not include a dimension vector. Thus since $\phi(k,n)$ counts the fission graphs with n nodes and maximal edge multiplicity exactly $k-1$, the number $\psi(k,n)$ counts the "equipped fission graphs" (i.e. equipped with a dimension vector) such that the maximal edge multiplicity is exactly $k-1$ and the total dimension is n (summing the dimension at each node—if a node has dimension zero it is treated as absent, and the graphs are all connected).

$k \setminus n$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	0	1	2	4	6	10	14	21	29	41
2	0	1	3	9	20	47	96	201	394	775
3	0	1	4	16	48	148	407	1121	2933	7612
4	0	1	5	25	95	365	1271	4383	14479	47198
5	0	1	6	36	166	766	3237	13466	53933	212645
6	0	1	7	49	266	1435	7140	34853	164324	761829
7	0	1	8	64	400	2472	14162	79430	431242	2301016
8	0	1	9	81	573	3993	25893	164157	1009029	6094011
9	0	1	10	100	790	6130	44392	314011	2156113	14544961
10	0	1	11	121	1056	9031	72248	564201	4280870	31910879

Table 2. $\psi(k,n)$ counting all untwisted fission trees with slope k and rank n

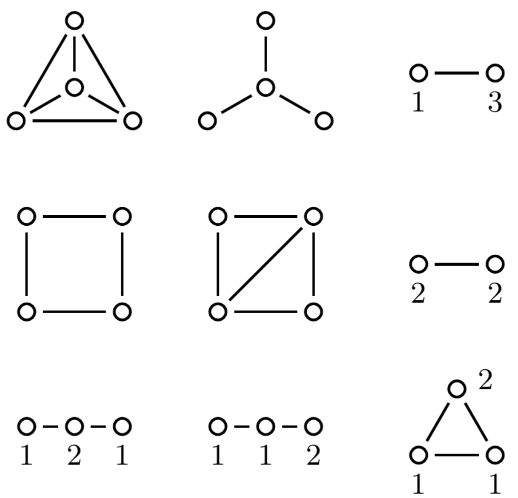


FIGURE 9. The $\psi(2,4) = 9$ equipped fission graphs of rank 4 with maximum edge multiplicity exactly 1 (unmarked nodes have dim. 1).

4. Counting the supernovas

Recall from §1.2 that a supernova graph is a graph of the form $\widehat{\Gamma}(\mathbf{l})$ obtained by gluing a leg of length $l_i \geq 0$ on to each node i of a fission graph Γ (and that they seem to most closely play the role of Dynkin graphs for moduli spaces in 2d gauge theory). Define $\sigma(k,n)$ to be the number of supernova graphs with n nodes and maximal edge multiplicity k. In the last section we counted the "equipped fission graphs", i.e. the fission graphs Γ equipped with a dimension vector $\mathbf{d}$ assigning an integer $d_i \geq 1$ to each node. Thus we have a map from equipped fission graphs to supernova graphs, defined by taking $l_i = d_i - 1$. The number of nodes of the resulting supernova graph equals the rank $\sum d_i$ of the equipped fission graph.

This map is not always bijective: For $k = 1$ (the simply-laced supernova graphs, i.e. those that are simple graphs), the core is not uniquely determined, and so there is overcounting: The issue is clear already in Fig. 9: the 9 equipped fission graphs there only determine 6 distinct supernova graphs (D_4 appears twice and A_4 three times).

However this map gives a bijection most of the time, and thus enables us to count the supernova graphs, provided that the core is not star-shaped.

Proposition 7. *Suppose $k \geq 2$. The map above gives a bijection between the equipped fission graphs of rank n with maximal edge multiplicity exactly k, and the supernova graphs with n nodes and maximal edge multiplicity k. Thus $\sigma(k,n) = \psi(k+1,n) = \phi(k+2,n)$.*

Proof. By definition all supernova graphs occur in this way, so the map is surjective. The core of such a supernova graph is uniquely determined (exercise), and so distinct equipped fission graphs determine distinct supernova graphs. □

For $k = 1$ the issue is just with star-shaped core graphs (a supernova graph that is not star-shaped has a unique core) and we can count them directly to correct the counting:

$$\sigma(1,n) = \psi(2,n) - N_1(n) + N_2(n)$$

where

$$N_1(n) := \#\{\text{equipped, star-shaped, rank } n \text{ fission graphs}\}$$

$$N_2(n) := \#\{\text{star-shaped (supernova) graphs with } n \text{ nodes}\}.$$

For example for $n = 4$ as in Fig. 9, we have $N_1 = 5, N_2 = 2$ so that there are $\sigma(1,4) = 9 - 5 + 2 = 6$ simple supernova graphs with 4 nodes.

Lemma 8. *$N_1(n) = \lfloor n/2 \rfloor + p(2) + \cdots + p(n-1) + 2 - n$ where $p(k)$ is the partition function.*

Proof. This is the same as counting the (untwisted mult. 1) fission trees of slope 2 with n leaves such that: a) exactly two nodes have height 3 and b) at most one of the nodes at height 3 has > 1 child. If both nodes at height 3 have 1 child, then we are counting the partitions of n with 2 parts: there are $\lfloor n/2 \rfloor$ of them. If the first node at height 3 has 1 child, and the second has > 1 child, then they have $n_1 + n_2 = n$ total descendants respectively, then the count is given by the number of partitions of n_2 into ≥ 2 parts. Thus we get

$$(p(n-1)-1) + (p(n-2)-1) + \cdots + (p(2)-1)$$

going through the cases when $n_1 = 1, 2, \ldots, n-2$. Adding these up gives the answer. □

If we define $\theta(n) = \sum_0^n p(k) = 2 + \sum_2^n p(k)$ (as in the sequence oeis/A000070) then the lemma says that $N_1(n) = \theta(n-1) - \lceil n/2 \rceil$.

Next we can count the stars with n nodes. (Here a star-shaped graph is a finite connected simple graph with at most one vertex of degree > 2.)

Lemma 9. *There are $N_2(n) = p(n-1) - \lfloor (n-1)/2 \rfloor$ star-shaped graphs with n nodes, where $p(n)$ is the number of partitions of n.*

Proof. If there are less than 3 legs then it is just a type A_n Dynkin graph. Otherwise we need to specify the length of the ≥ 3 legs, i.e. choose a partition of $n-1$ with at least 3 parts. There is one partition with 1 part and $\lfloor (n-1)/2 \rfloor$ with 2 parts, and so we get $p(n) - 1 - \lfloor (n-1)/2 \rfloor$ partitions of $n-1$ with at least 3 parts. Adding on 1 corresponding to A_n gives the answer. □

Corollary 10. *The number of simple supernova graphs with n nodes is: $\sigma(1, n) = \psi(2, n) + 1 - \theta(n-2)$ where $\theta(k) = \sum_0^k p(i)$.*

Proof. This is immediate now, since $\theta(n-1) - p(n-1) = \theta(n-2)$ and $\lceil n/2 \rceil - \lfloor (n-1)/2 \rfloor = 1$. □

This is now easily computed (recall that $\psi(2, n) = \Phi(3, n) - p(n)$ where $\Phi(3, n)$ is the double partition function oeis/A001970). In summary the supernova counts are as in Table 3 below (which continues as in Table 2, with k shifted by 1, as $\sigma(k, n) = \psi(k+1, n)$ for $k \geq 2$).

$k \setminus n$	1	2	3	4	5	6	7	8	9	10
1	0	1	2	6	14	36	78	172	350	709
2	0	1	4	16	48	148	407	1121	2933	7612
3	0	1	5	25	95	365	1271	4383	14479	47198
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 3. $\sigma(k, n)$: supernova graphs with n nodes & max. edge mult. k.

In fact we are most interested in the non-Dynkin supernova graphs, since the Dynkin cases have no imaginary roots so only lead to rigid irreducible connections (nonabelian Hodge spaces of dimension zero). Thus we can subtract the counts of the ADE Dynkin graphs $A_1, A_2, A_3, (A_4, D_4), (A_5, D_5), (A_6, D_6, E_6), \ldots$, to obtain table 4 below.

5. Extended fission trees

The fission trees really only capture *part* of the breaking of structure group at a pole, namely that due to the irregular/wild part of the connection. In general there is further breaking of the structure group of a slightly different nature (*tame fission*) from the tame/logarithmic part of the connection, passing to the associated graded of the tame (Levelt–Simpson) filtrations and then taking generalised eigenspaces (cf. the tame fission spaces $\mathbb{M}$ in [9] Thm. 9). In terms of deformation classes, this amounts to upgrading the fission trees as follows (adding the "fine hidden structure" of the leaves).

Table 4. Counting the simply-laced supernova graphs.

#nodes	#starshaped	#other	Total	Total non-Dynkin
2	1	0	1	0
3	1	1	2	1
4	2	4	6	4
5	3	11	14	12
6	5	31	36	33
7	8	70	78	75
8	12	160	172	169
9	18	332	350	348
10	26	683	709	707

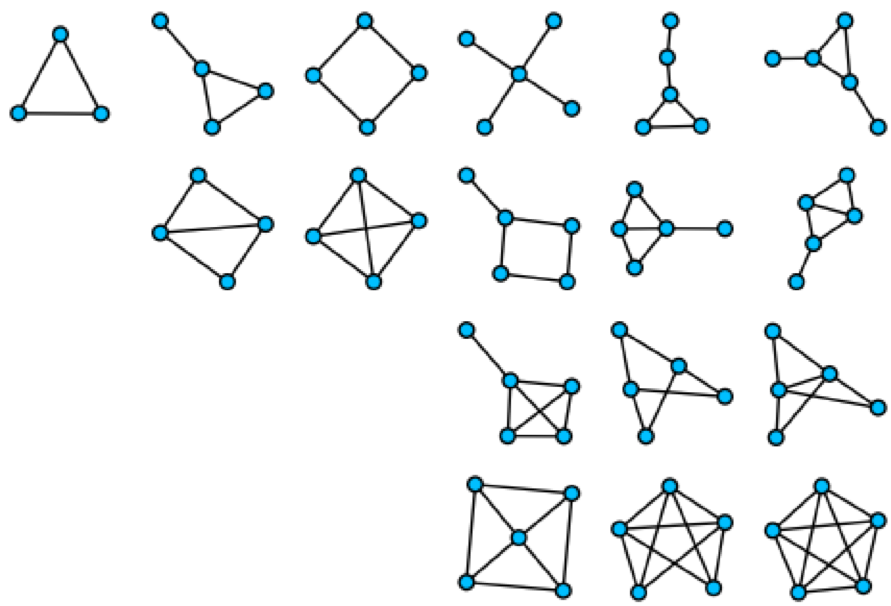


Figure 10. The non-Dynkin simple supernova graphs with $3, 4$ or 5 nodes, including the graphs of the simply-laced H3 surfaces $\widehat{A}_2, \widehat{A}_3, \widehat{D}_4$.

Definition 11. A <u>tame fission tree</u> τ of rank n is a surjective map between two finite sets:

$$J_0 \twoheadleftarrow J_{-1}$$

equipped with a multiplicity map $\mu : J_{-1} \to \mathbb{Z}_{>0}$ such that $\sum_{j\in J_{-1}} \mu(j) = n$.

The points of J_0 will be called *residual eigenspaces*, and the points of J_{-1} will be called *Jordan blocks*. A tame fission tree is *semisimple* if each Jordan block has size

1, i.e. $\mu(J_{-1}) = \{1\}$. Observe that the choice of a rank n tame fission tree is the same as the choice of a double partition of n:

Lemma 12. *The choice of a tame fission tree is the same as the choice of a finite set J_0 and a non-zero integer partition for each $j \in J_0$. Any matrix $A \in \mathrm{GL}(\mathbb{C}^n)$ determines a rank n tame fission tree by taking its generalised eigenspaces, and the sizes of the Jordan blocks in each eigenspace. More generally any automorphism of an $\mathbb{R}$-graded vector space of dimension n determines a rank n tame fission tree (where the "eigenvalues" are now in $\mathbb{R} \times \mathbb{C}^*$).*

Definition 13. An <u>extended fission tree</u> $\widehat{\mathcal{T}}$ is the data of a fission tree $\mathcal{T}$ (as in [17] Defn. 3.18) together with a tame fission tree τ_i of rank n_i for each leaf $i \in \mathcal{T}$ (where $n_i = n(i)$ is the multiplicity of the leaf i). An extended fission tree $\widehat{\mathcal{T}}$ is <u>untwisted</u> if $\mathcal{T}$ is untwisted. It is semisimple if each τ_i is semisimple. The slope of $\widehat{\mathcal{T}}$ is defined to be the slope (i.e. Poincaré–Katz rank ≥ 0) of $\mathcal{T}$.

The untwisted extended fission trees are now easy to count, due to the statement:

Lemma 14. *The operation of shifting the height by 2 gives a bijection between rank n untwisted fission trees, and untwisted extended fission trees of rank n. Under this correspondence the multiplicity one fission trees correspond to the semisimple extended fission trees. If $k \geq 1$, the extended fission trees of slope k correspond to fission trees of slope $k+2$, whereas those of slope 0 correspond to all the fission trees of slope ≤ 2.*

Corollary 15. *For any $n, k \geq 1$, the number of rank n untwisted:*

a) extended fission trees of slope k is $\psi(k+2, n)$, and

b) semisimple extended fission trees of slope k is $\phi(k+2, n)$,

c) extended fission trees of slope 0 is equal to $\Phi(3, n) = \psi(2, n) + \psi(1, n) + \psi(0, n)$, and there are $\phi(2, n) + \phi(1, n) + \phi(0, n) = p(n)$ of them that are semisimple.

Any (very) good meromorphic connection (in the sense of [3, 15]) determines an extended fission tree at each marked point, and we expect the resulting forest of extended fission trees determines the deformation class of the corresponding wild nonabelian Hodge space $\mathfrak{M}(\mathbf{\Sigma}, \mathcal{C})$ (cf. [2]).

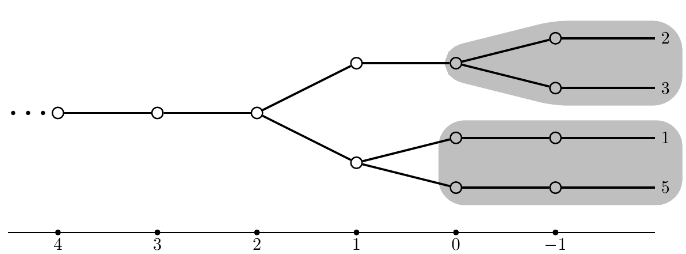


FIGURE 11. Extended fission tree $\widehat{\mathbb{T}}$ (slope 1, rank 11, & $\mathbb{T}$ has 2 leaves).

6. Conclusion

In this paper we explained how to count the multiplicity 1 untwisted fission trees by iterating the Euler transform and looking at the difference of the results. Thus the fission trees (amongst the most basic objects in 2d gauge theory) are counted by iterating a very basic counting method (in effect generalising the counting of nontrivial integer partitions and double partitions to "deeper" cases). This enabled the counting of several closely related objects (fission trees with any multiplicities, fission graphs, equipped fission graphs, supernova graphs, extended fission trees) and will be used in the classification of wild nonabelian Hodge moduli spaces (cf. the Lax project [15]). In the sequel we will consider automorphisms of the untwisted fission trees, in order to obtain the twisted fission trees. One can also consider the analogous question for other structure groups, beyond the $\mathrm{GL}_n(\mathbb{C})$ case considered here[4].

Appendix A. Squares of squares

We saw that $\phi(3,4) = \psi(2,4) = 9$ and will prove here that $\phi(k,4) = \psi(k-1,4) = k^2$ in general, i.e.:

Proposition 16. *There are exactly k^2 untwisted fission trees of rank 4 & slope $k-1$ (i.e. there are k^2 untwisted multiplicity one fission trees with 4 leaves & slope k).*

Said differently this means that:

Proposition 17. *If $k \geq 2$ then there are exactly k^2 fission graphs with 4 nodes such that the maximal edge multiplicity is exactly $k-1$.*

This will be proved by induction. (In the rest of this section a fission tree means a "multiplicity one untwisted fission tree".) First note two simpler facts:

Lemma 18. *If $k \geq 1$ then:*

- *There is exactly 1 fission tree with two leaves and slope k, and*
- *There are exactly k fission trees with three leaves and slope k.*

Proof. The first statement is clear. For the second: there are either one or two branch nodes in the corresponding fission tree. There is just one with one (the generic tree). If there are two branch nodes, then the only choice is the height of the lower branch node: it can be $2, 3, \ldots, k$, so we get exactly $1 + (k-1) = k$ trees. □

Now we can prove Proposition 16:

Proof. The induction we will do reduces to the identity:

$$k^2 = (k-1)^2 + 1 + 2(k-2) + 2.$$

[4] In effect the untwisted fission trees parameterise the possible chains of fission subgroups as in [11] (33) p.35, and this definition is for any complex reductive group G (see also [3] (2.2), the general discussion of fission in [8] and the study of G-fission trees in [25, 24]).

To see this consider the map $J_2 \twoheadleftarrow J_1$ from the bottom of the fission tree. If this map is bijective then the tree is just the extension of one of slope $k-1$ (increasing the length of each branch by 1). By induction there are $(k-1)^2$ of these fission trees.

If the map $J_2 \twoheadleftarrow J_1$ is not bijective then J_2 has either 2 or 3 elements.

If J_2 has 3 elements then the tree comes from a fission tree of slope $k-1$ and 3 leaves, by adding exactly one branch node (and extending the other branches by 1). There are $1+(k-2)$ of these trees. The first is generic, and up to isomorphism there is just one way to extend it to a tree with four nodes. The other $k-2$ can be extended in two ways, so yield $2(k-2)$ trees.

If J_2 has 2 elements then the tree comes from the unique fission tree of slope $k-1$ and 2 leaves. There are two ways to extend it to get 4 leaves $(2+2$ or $1+3)$. This gives 2 fission trees.

This gives exactly $k^2=(k-1)^2+1+2(k-2)+2$ in total. □

Appendix B. Euler transform.

For completeness we will give a proof of the Euler transform. This is well-known (cf. Kaneiwa [29]), but it might be helpful to give a direct proof of the main facts in the (simple) language of multisets.

Let S be a set. Recall that a "multiset with underlying set contained in S" is the same thing as a map

$$\mu : S \to \mathbb{Z}_{\geq 0}$$

(the multiplicity map) assigning an integer multiplicity ≥ 0 to each element of S. A multiset is finite if the underlying set

$$\mathrm{supp}(\mu) := \{ s \in S \mid \mu(s) > 0 \} \subset S$$

is a finite set. The size of a finite multiset with multiplicity map μ is

$$|\mu| = \sum_{s\in S} \mu(s).$$

Two multisets are equal if they have the same underlying set and the same multiplicity map (i.e. we ignore everything in S with multiplicity zero).

Lemma 19. *Choose $k, n \in \mathbb{N}$. The number $\mathrm{MS}(k,n)$ of distinct multisets of size n with underlying set contained in $\{1,\dots,k\}$ is given by*

$$\mathrm{MS}(k,n) = \binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

Proof. Consider ordered sequences of symbols of the form:

$$\circ\circ\circ|\circ||\circ\circ|$$

where there are n circles and $k-1$ vertical bars (in this example $n = 6, k = 5$). Such a sequence determines a multiset with underlying set contained in $\{1, \dots, k\}$. For example here we get $\mu = 3, 1, 0, 2, 0$ for $1, 2, 3, 4, 5$ respectively (counting the circles from left to right). This gives a bijection, so $\mathrm{MS}(k, n)$ counts all such sequences. But there are exactly $n + k - 1$ symbols in such a sequence and the sequence is uniquely determined by choosing which n of the symbols is a circle (or equivalently which $k-1$ of them should be a bar). Thus there are $\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$ such sequences, and thus this number of multisets of the desired type. $\square$

Let $A(1), A(2), \dots$ be a sequence of disjoint finite sets. We will say that the elements in $A(i)$ have weight i, and write $A = \bigsqcup A(i)$. Thus A is an $\mathbb{N}$-graded set. Let $a(i) = \#A(i)$ be the number of elements of weight i.

Note that if M is a finite multiset with underlying set contained in A and multiplicity map $\mu : A \to \mathbb{Z}_{\geq 0}$, then M has a well-defined weight, given by

$$w(M) = \sum_{x \in A} \mu(x) w(x)$$

where $w(x)$ is the weight of $x \in A$.

Let $B(i) = \{M \mid w(M) = i\}$ be the set of finite multisets with underlying set contained in A, that have weight i. Define $B = \bigsqcup B(i) =: \mathrm{Euler}(A)$ to be the Euler transform of the graded set A. Let $b(i) = \#B(i)$ be the number of elements of $B(i)$.

Lemma 20.

$$b(i) = \sum_{\substack{s_1, s_2, \dots \geq 0 \\ 1 \cdot s_1 + 2 \cdot s_2 + \cdots = i}} \prod_{k=1}^{i} \mathrm{MS}(a(k), s_k).$$

Proof. This should be self-explanatory: Any such M leads to a sum of the form

$$1 \cdot s_1 + 2 \cdot s_2 + \cdots = i$$

where $s_k = \#M(k)$ is the number of elements of M of weight k (counted with multiplicity). Thus for each possible such sum we need to count the number of possible M's that lead to that sum. By definition of $\mathrm{MS}(k, n)$, the number of them is $\prod_{k=1}^{i} \mathrm{MS}(a(k), s_k)$. $\square$

In turn the generating functions are related by the Euler transform:

Proposition 21.

$$1 + \sum_{1}^{\infty} b(n) u^n = \prod_{1}^{\infty} (1 - u^m)^{-a(m)} \in \mathbb{Z}[\![u]\!].$$

Proof. This is now straightforward, using the previous two lemmas, since the binomial theorem in this context says that:

$$(1-u^m)^{-a} = \sum_{n=0}^{\infty} \mathrm{MS}(a,n)u^{mn}$$

for integers a, m. □

Thus if $A = \bigsqcup_{i\geq 1} A(i)$ is any graded set such that $A(i)$ is finite, then $\mathrm{Euler}(A)$ is defined to be the set of finite multisets with underlying set contained in A, and there is a natural inclusion $A \hookrightarrow \mathrm{Euler}(A)$ taking $a \in A$ to the multiset given by a with multiplicity 1.

In effect Table 1 arises by iterating this Euler transform, starting with the set $A = \{1\}$ consisting of one element of weight 1, and counting the new elements that arise. Thus $\mathrm{Euler}(A) \cong \mathbb{Z}_{>0}$ and row 1 of Table 1 lists the counts of elements of the set $\mathrm{Euler}(A) \setminus A$, of each weight. In turn $\mathrm{Euler}^2(A) = \mathrm{Euler} \circ \mathrm{Euler}(A)$ is the set $\mathcal{P}^*$ of all partitions of all integers ≥ 1. Row 2 of Table 1 lists the counts of elements of the set $\mathrm{Euler}^2(A) \setminus \mathrm{Euler}(A)$ of each weight, i.e. the partitions with > 1 part ($\sim$ the untwisted fission trees of slope 2, mult. 1, weighted by their leaf counts). Similarly row k of Table 1 lists the counts of elements of the set $\mathrm{Euler}^k(A) \setminus \mathrm{Euler}^{k-1}(A)$, i.e. the (multiplicity 1) untwisted fission trees of slope k.

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