

Sequential Rank and Preference Learning with the Bayesian Mallows Model

Øystein Sørensen* Anja Stein† Waldir Leoncio Netto‡
David S. Leslie†

Abstract

The Bayesian Mallows model is a flexible tool for analyzing data in the form of complete or partial rankings, and transitive or intransitive pairwise preferences. In many potential applications of preference learning, data arrive sequentially and it is of practical interest to update posterior beliefs and predictions efficiently, based on the currently available data. Despite this, most algorithms proposed so far have focused on batch inference. In this paper we present an algorithm for sequentially estimating the posterior distributions of the Bayesian Mallows model using nested sequential Monte Carlo. As it requires minimum user input in form of tuning parameters, is straightforward to parallelize, and returns the marginal likelihood as a direct byproduct of estimation, the algorithm is an alternative to Markov chain Monte Carlo techniques also in batch estimation settings.

Keywords: Mallows mixtures, partial rankings, particle filter, preference learning, SMC2.

1 Introduction

Data in the form of rankings and preferences arise naturally across a variety of domains. Examples include content recommendation based on click data (Liu et al., 2019b), algorithm comparison (Rojas-Delgado et al., 2022), consumer preferences (Courcoux and Semenou, 1997; Kamishima, 2003; Krivulin et al., 2022; Manuel et al., 2015), grant panel reviews (Pearce and Erosheva, 2022), genome-wide transcriptomic analyses (Eliseussen et al., 2022; Vitelli et al., 2023), analysis of mutual funds’ preferences for governance structures (Yi, 2021), social hierarchies (Nicholls et al., 2022), and reinforcement learning from human feedback (Hwang et al., 2023). An early model for analyzing rankings (Thurstone, 1927) assumed a judge ranks m items by assigning a score to each item, and then ordering them according to the score. Further developments of this approach include the Plackett-Luce model (Luce, 1959; Plackett, 1975), the Babington Smith model (Babington Smith, 1950), and the Bradley-Terry model (Bradley and Terry, 1952), all of which are based on assigning real-valued utilities to each item, yielding a large number of parameters to be estimated.

Now consider a collection of items $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$, and let ρ be a permutation of the integers $[m] := \{1, 2, \dots, m\}$ denoting the items’ modal ranking in the population of

*Department of Psychology, University of Oslo. oystein.sorensen@psykologi.uio.no

†School of Mathematical Sciences, Lancaster University. a.k.stein@outlook.com

‡Oslo Centre for Biostatistics and Epidemiology, University of Oslo. w.l.netto@medisin.uio.no

interest, such that ρ_i denotes the modal ranking of item A_i . For a particular individual, let $A_i \succ A_j$ imply that the individual prefers A_i to A_j , and let \mathbf{r} be the permutation of $[m]$ encoding the individual’s rankings. Mallows (1957) showed that if the probability that an individual ranks a pair of items in agreement with their relative position in $\boldsymbol{\rho}$ is given by

$$P(A_i \succ A_j | \rho_i < \rho_j) = 0.5 + 0.5 \tanh\{(\rho_j - \rho_i) \log \theta + \log \phi\}$$

we obtain an exponential model $P(\mathbf{r} | \boldsymbol{\rho}) \propto \exp\{-\alpha d(\boldsymbol{\rho}, \mathbf{r})\}$ for the observed ranking \mathbf{r} . When $\phi = 1$, $d(\cdot, \cdot)$ is Spearman’s rank correlation (Spearman, 1904) and when $\theta = 1$, $d(\cdot, \cdot)$ is the Kendall distance (Kendall, 1938). The precision parameter α quantifies how far observed rankings typically are from the modal ranking. Advantages of the Mallows model over utility-based models include a lower number of parameters (α and $\boldsymbol{\rho}$) and the fact that its support is defined on the space of rankings. The model has later been extended to incorporate additional distance functions (Diaconis, 1988) and to item-dependent precision parameters (Fligner and Verducci, 1986). We refer to the reviews by Liu et al. (2019a) and Yu et al. (2019) and the monograph by Marden (1995) for further details.

Vitelli et al. (2017) proposed a Bayesian Mallows model and a Markov chain Monte Carlo (MCMC) algorithm for its estimation. Compared to other approaches focusing on the Kendall or Cayley distances (Irurozki et al., 2018; Lu and Boutilier, 2014; Meila and Bao, 2010), Vitelli et al. (2017)’s algorithm works naturally with any of the distance functions proposed by Diaconis (1988) for the Mallows model, and it incorporates data in the form of partial rankings or pairwise preferences. Its fully Bayesian approach allows predicting users’ preferences of items they have not yet seen, allowing the model to be used as a probabilistic recommender system (Liu et al., 2019b).

The MCMC algorithm of Vitelli et al. (2017) has some drawbacks, however. The user has to set tuning parameters for the proposal distributions for precision parameters, modal rankings, and latent rankings. It hence may require several pilot runs for obtaining sufficient acceptance probabilities. Second, in settings where data arrive sequentially, updating the posteriors requires running the full algorithm from scratch. The goal of this paper is to alleviate these issues. To this end, we propose a nested sequential Monte Carlo (SMC²) algorithm (Chopin et al., 2013; Fulop and Li, 2013), which requires minimum user input and efficiently updates posterior distributions when new data arrive. The algorithm is straightforward to parallelize and is an extension of the work by Stein (2023), who proposed a sequential Monte Carlo (SMC) algorithm using a resample-move scheme (Berzuini and Gilks, 2001; Chopin, 2002; Gilks and Berzuini, 2001).

The paper proceeds as follows. In Section 2 we provide necessary background on the Bayesian Mallows model. In Section 3 we propose an SMC² algorithm for the Bayesian Mallows model with partial rankings or pairwise preference data. In Section 4 we report the results of simulation experiments demonstrating the algorithm. We conclude and discuss further developments in Section 5. An R (R Core Team, 2024) package providing an API to our C++ implementation is available from GitHub¹ and R code for reproducing all the results in the paper is available from our OSF repository.²

¹<https://github.com/osorensen/BayesMallowsSMC2>

²<https://osf.io/pquk4/>

2 Background and Model Setup

We now introduce the Bayesian Mallows model as it was defined in [Vitelli et al. \(2017\)](#) and [Crispino et al. \(2019\)](#). As the goal of this paper is to develop generic algorithms we present the model in full generality, and do not discuss modeling choices. Any particular application will typically use special cases of the presented framework.

2.1 Mallows' Model for Partial Rankings and Pairwise Preferences

Let \mathcal{P}_m denote the space of all permutations of $[m]$ and consider rankings $\mathbf{r} \in \mathcal{P}_m$ of a set of items \mathcal{A} distributed according to a mixture of Mallows models ([Diaconis, 1988](#); [Mallows, 1957](#); [Vitelli et al., 2017](#)) with C components

$$p(\mathbf{r}|\theta) = \sum_{c=1}^C \tau_c Z(\alpha_c)^{-1} \exp\{-\alpha_c d(\mathbf{r}, \boldsymbol{\rho}_c)\} 1\{\mathbf{r} \in \mathcal{P}_m\}, \quad (1)$$

where $\theta = \{\alpha_c, \boldsymbol{\rho}_c, \tau_c\}_{c=1}^C$ and $1\{A\}$ is an indicator function for the event A . For each cluster c , $\alpha_c \in \mathbb{R}_{\geq 0}$ is a precision parameter and $\boldsymbol{\rho}_c \in \mathcal{P}_m$ is the modal ranking. $d(\cdot, \cdot)$ is a right-invariant distance function, and

$$Z(\alpha) = \sum_{\mathbf{r} \in \mathcal{P}_m} \exp\{-\alpha d(\mathbf{r}, \mathbf{e})\} \quad (2)$$

is the normalizing constant where $\mathbf{e} = (1, 2, \dots, m)'$. Subject to the constraint $\sum_{c=1}^C \tau_c = 1$, $\tau_c \in [0, 1]$ is the proportion of the population belonging to the c 'th cluster.

A ranking \mathbf{r} is a latent variable, and observations \mathbf{y} are distributed according to $p(\mathbf{y}|\mathbf{r}, \theta)$. The marginal likelihood of N observations $\mathbf{y}_{1:N} = \{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_N\}$ is

$$p(\mathbf{y}_{1:N}|\theta) = \prod_{n=1}^N \sum_{c=1}^C \tau_c Z(\alpha_c)^{-1} \sum_{\mathbf{r}_n \in \mathcal{P}_m} \exp\{-\alpha_c d(\mathbf{r}_n, \boldsymbol{\rho}_c)\} p_\epsilon(\mathbf{y}_n|\mathbf{r}_n) \quad (3)$$

where $p_\epsilon(\mathbf{y}_n|\mathbf{r}_n)$ is the sampling distribution of the observed rankings given the latent rankings where ϵ is an error parameter to be introduced later. Here and in the sequel, Greek letter subscripts imply conditioning.

Complete rankings correspond to $p_\epsilon(\mathbf{y}_n|\mathbf{r}_n) = 1\{\mathbf{y}_n = \mathbf{r}_n\}$. For top- k_n rankings with $k_n \in [m]$ we define the set of items ranked by user n as $\mathcal{A}_n = \{A_i \in \mathcal{A} : r_{ni} \leq k_n\}$ whereas for ranks missing completely at random we let \mathcal{A}_n define the set of ranked items. In both cases we have $p_\epsilon(y_{ni}|\mathbf{r}_n) = 1\{y_{ni} = r_{ni}\}$ for $i : A_i \in \mathcal{A}_n$. Defining the set of latent rankings consistent with the observations

$$\mathcal{S}_n = \{\mathbf{r} \in \mathcal{P}_m : (r_i = y_{ni} \ \forall i : A_i \in \mathcal{A}_n)\} \quad (4)$$

with the complete data case given by $\mathcal{S}_n = \{\mathbf{y}_n\}$, the marginal likelihood (3) reduces to

$$p(\mathbf{y}_{1:N}|\theta) = \prod_{n=1}^N \sum_{c=1}^C \tau_c Z(\alpha_c)^{-1} \sum_{\mathbf{r}_n \in \mathcal{S}_n} \exp\{-\alpha_c d(\mathbf{r}_n, \boldsymbol{\rho}_c)\}. \quad (5)$$

When the data contain pairwise preferences for $p_n \leq \binom{m}{2}$ pairs of items, let y_{ni} denote the i 'th pairwise preference of user n . We define the function

$$g(y_{ni}, \mathbf{r}_n) = \begin{cases} 0 & \text{if } y_{ni} = (A_s \succ A_t) \text{ and } (r_{ns} < r_{nt}) \\ 1 & \text{if } y_{ni} = (A_s \succ A_t) \text{ and } (r_{ns} > r_{nt}) \end{cases}$$

indicating whether the pairwise preferences contradict the latent rankings. Assuming a true latent ranking exists, inconsistencies arise due to errors made by the users.³ If errors occur independently at rate $\epsilon \in [0, 1]$, the direction of the preference relation for the i 'th pair in \mathbf{y}_n has distribution

$$p_\epsilon(y_{ni} | \mathbf{r}_n) = \begin{cases} 1 - \epsilon & \text{if } g(y_{ni}, \mathbf{r}_n) = 0 \\ \epsilon & \text{if } g(y_{ni}, \mathbf{r}_n) = 1. \end{cases}$$

Setting $\epsilon > 0$ allows mutually incompatible preferences (Crispino et al., 2019), and the marginal likelihood (3) becomes

$$p(\mathbf{y}_{1:N} | \theta) = \prod_{n=1}^N \sum_{c=1}^C \tau_c Z(\alpha_c)^{-1} \sum_{\mathbf{r}_n \in \mathcal{P}_m} \exp\{-\alpha_c d(\mathbf{r}_n, \boldsymbol{\rho}_c)\} \left(\frac{\epsilon}{1-\epsilon}\right)^{\sum_{i=1}^{p_n} g(y_{ni}, \mathbf{r}_n)} (1-\epsilon)^{p_n}$$

where θ now also contains ϵ . To only allow mutually compatible pairwise preferences we set $\epsilon = 0$ and define the set of latent rankings consistent with \mathbf{y}_n as

$$\mathcal{S}_n = \{\mathbf{r} \in \mathcal{P}_m : (A_s \succ A_t) \in \text{tc}(\mathbf{y}_n) \Leftrightarrow r_s < r_t\}, \quad (6)$$

where $\text{tc}(\mathbf{y}_n)$ is the transitive closure of the directed acyclic graph induced by the pairwise preferences. In this case we can compute the marginal likelihood using (5), replacing \mathcal{S}_n from (4) with \mathcal{S}_n from (6).

2.2 Prior Distributions

For the precision parameters α_c we follow Crispino et al. (2019) and use independent gamma priors with shape $\gamma > 0$ and rate $\lambda > 0$,

$$\pi(\alpha_c) = \lambda^\gamma \Gamma(\gamma)^{-1} \alpha_c^{\gamma-1} e^{-\lambda \alpha_c}, \quad c = 1, 2, \dots, C,$$

where $\Gamma(\gamma) = \int_0^\infty t^{\gamma-1} e^{-t} dt$. Similar to Vitelli et al. (2017), for the modal ranking we use a uniform prior on \mathcal{P}_m ,

$$\pi(\boldsymbol{\rho}_c) = (m!)^{-1} \mathbf{1}\{\boldsymbol{\rho}_c \in \mathcal{P}_m\}, \quad c = 1, 2, \dots, C.$$

For the cluster probabilities we use a symmetric Dirichlet prior,

$$\pi(\tau_1, \tau_2, \dots, \tau_C) = \Gamma(\psi C) \Gamma(\psi)^{-C} \prod_{c=1}^C \tau_c^{\psi-1}.$$

³This assumption is indeed a mathematical idealization, as empirical evidence suggest that human preferences are inherently non-transitive (Kahneman and Tversky, 1979).

With non-transitive pairwise preferences, we use the Bernoulli model of [Crispino et al. \(2019\)](#) with a truncated Beta prior on $[0, 0.5)$,

$$\pi(\epsilon) \propto \epsilon^{\kappa_1 - 1} (1 - \epsilon)^{\kappa_2 - 1} 1\{\epsilon \in [0, 0.5)\}.$$

A model without non-transitive pairwise preferences corresponds to the limit $\kappa_1 \rightarrow \infty$ and $\kappa_2 \rightarrow 0$, which means that ϵ is fixed to 0 and can be ignored in the analyses. [Crispino et al. \(2019\)](#) also considered a logistic model, in which the logit of the error probability depends on the distance between the items in the latent ranking. While sequential inference with this logistic model is in principle straightforward, we do not consider it further in this paper for ease of presentation.

2.3 Distance Functions and Normalizing Constants

Consider two rankings $\mathbf{a}, \mathbf{b} \in \mathcal{P}_m$. Cayley distance measures the minimum number of pairwise swaps needed for converting \mathbf{a} into \mathbf{b} ([Cayley, 1849](#)), two algorithms for which are given in [Marden \(1995, pp. 25-26\)](#). Ulam distance can be defined as m minus the length of the longest common subsequence of the item orderings corresponding to \mathbf{a} and \mathbf{b} ([Gordon, 1979](#)). We further have Spearman distance $d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|_2^2$ ([Spearman, 1904](#)), Kendall distance measuring the number of discordant pairs $d(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^m \sum_{j=i+1}^m 1\{(a_i - a_j)(b_i - b_j) < 0\}$ ([Kendall, 1938](#)), the footrule $d(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^m |a_i - b_i|$ ([Spearman, 1906](#)), and Hamming distance $d(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^m 1\{a_i \neq b_i\}$ ([Hamming, 1950](#)).

The choice of distance in the Mallows model (1) is ultimately linked to the application at hand. For example, Hamming distance is not likely to work well under the ranking and preference applications considered in this paper but arises naturally when the Mallows model is used for matchings, e.g., when tracking a number of known objects using noisy sensors ([Irurozki et al., 2019](#)). Similarly, Cayley distance is suitable when total disorder is of interest ([Crispino et al., 2019](#)), whereas footrule, Kendall, and Spearman distance are most appropriate for preference data. [Crispino et al. \(2019\)](#) gives an example of two rankings $\mathbf{a} = (1, 2, 3, 4, 5)$ and $\mathbf{b} = (5, 2, 3, 4, 1)$. Their Cayley distance (normalized to be in $[0, 1]$) is 0.25, whereas the normalized footrule, Kendall, and Spearman distances are $2/3$, 0.7, and 0.8, respectively. If \mathbf{a} and \mathbf{b} represent positions on a genome they can be seen as close, and the Cayley distance may be most appropriate. On the other hand, if \mathbf{a} and \mathbf{b} represent preferences for five items, they are far apart and one of the latter three distances are better. An in-depth comparison of Cayley, Kendall and Ulam distances can be found in [Ceberio et al. \(2015\)](#) and further discussion in [Diaconis \(1988, Ch. 6\)](#).

Computing the normalizing constant $Z(\alpha)$ as in (2) requires summing over $|\mathcal{P}_m| = m!$ terms, but tractable exact expressions exist for Cayley, Kendall, and Hamming distances ([Fligner and Verducci, 1986](#); [Irurozki et al., 2018](#)). Furthermore, since $d(\mathbf{r}, \mathbf{e})$ takes on a set of $l < m!$ values $\mathcal{D} = \{d_1, d_2, \dots, d_l\}$, we can define $L_i = \{\mathbf{r} \in \mathcal{P}_m : d(\mathbf{r}, \mathbf{e}) = d_i\}$ and write $Z(\alpha) = \sum_{i=1}^l |L_i| e^{-\alpha d_i}$. For the footrule, $l = \mathcal{O}(m^2)$, for Spearman distance $l = \mathcal{O}(m^3)$, and for Ulam distance $l = \mathcal{O}(m)$ ([Crispino, 2018](#); [Crispino et al., 2023](#); [Irurozki, 2014](#); [Irurozki et al., 2016](#)). Unfortunately, while the set of distances \mathcal{D} is well known, finding the cardinalities $|L_i|$ is hard. The Online Encyclopedia of Integer Sequences ([Sloane, 2023](#)) contains $|L_i|$ up to $m = 50$ for the footrule, up to $m = 20$ for Spearman distance, and $m = 60$ for Ulam distance. Beyond these upper limits, asymptotic approximations exist for the footrule and Spearman distances ([Crispino et al., 2023](#); [Mukherjee, 2016](#)), and an importance sampling scheme has been developed by [Vitelli et al. \(2017\)](#). The latter can in

principle be run to arbitrary precision, and importantly, estimates of the partition function can be precomputed for a given number of items m over a grid of α values (Sørensen et al., 2020). Thus, in the rest of this paper we assume that $Z(\alpha)$ is available and that its Monte Carlo error (if any) is negligible compared to the Monte Carlo error of the proposed SMC² algorithm.

3 Sequential Inference in the Bayesian Mallows Model

Now assume data become available sequentially at timepoints $t = 1, 2, \dots, T$, and let $\mathbf{y}_{\mathcal{I}_t}$ contain partial rankings or pairwise preferences for new users entering the pool at time t , where $\mathcal{I}_t \subset \mathbb{N}$ is the set of user indices. Let $\mathcal{I}_{1:t} = \cup_{t=1}^T \mathcal{I}_t$ contain the indices of all users in the pool at time t . We assume throughout that a given user enters the pool only once, i.e., that $\mathcal{I}_s \cap \mathcal{I}_t = \emptyset$ if $s \neq t$. The target distribution is $\pi(\theta, \mathbf{x}_{\mathcal{I}_{1:t}} | \mathbf{y}_{\mathcal{I}_{1:t}})$, with static parameters $\theta = \{[\alpha_c, \boldsymbol{\rho}_c, \tau_c]_{c=1}^C, \epsilon\}$ and latent variables $\mathbf{x}_{\mathcal{I}_{1:t}} = \{\mathbf{r}_{\mathcal{I}_{1:t}}, \mathbf{z}_{\mathcal{I}_{1:t}}\}$. The goal is to estimate the target distribution at all timepoints, in order to continuously perform inference based on the currently available evidence.

Sequential Monte Carlo (SMC) (Dai et al., 2022; Del Moral et al., 2006; Fearnhead and Künsch, 2018; Naesseth et al., 2019) typically scales better than MCMC for these types of problems, as the latter needs to be completely rerun at each new timepoint. Stein (2023) considered SMC for sequential inference in Bayesian Mallows models using a resample-move framework (Berzuini and Gilks, 2001; Chopin, 2002; Gilks and Berzuini, 2001). Unfortunately, such methods were designed for cases either with only static parameters or latent variables which can be easily integrated out. Integrating over the latent variables in a Bayesian Mallows model, in particular the latent rankings, is computationally demanding and we thus instead base our methodology on SMC² (Chopin et al., 2013; Fulop and Li, 2013) which uses particle marginal Metropolis-Hastings (Andrieu et al., 2010) in the rejuvenation step and was developed specifically for settings with challenging latent variable distributions. We extend the SMC² framework by incorporating the hybrid particle MCMC sampler proposed by Mendes et al. (2020) to allow a combination of Gibbs sampling and Metropolis-Hastings steps.

To set the notation, assume we have R particles each containing static parameters $\theta^r = \{\alpha_c^r, \boldsymbol{\rho}_c^r, \tau_c^r, \epsilon^r\}$ and to each of these we attach S additional particles containing the latent variables for the users entered up to timepoint t , $\mathbf{x}_{\mathcal{I}_{1:t}}^{s,r} = \{\mathbf{r}_{\mathcal{I}_{1:t}}^{s,r}, \mathbf{z}_{\mathcal{I}_{1:t}}^{s,r}\}$. Here and in what follows, superscripts r and s are assumed repeated for all $r = 1, 2, \dots, R$ and $s = 1, 2, \dots, S$ and subscripts c are assumed repeated for $c = 1, 2, \dots, C$. If the data consist of either partial rankings or consistent pairwise preferences we have $\epsilon^r = 0$ and this parameter can be ignored. Similarly, in the absence of mixtures $\tau_c^r = 1$ and $z_i^{s,r} = 1$ are fixed and can be ignored.

The building blocks of the algorithm are particle filters for latent rankings and cluster labels (Section 3.1), iterated batch importance sampling for the static parameters (Section 3.2), and a rejuvenation algorithm (Section 3.3). Proposal distributions for latent rankings are discussed in Section 3.4.

3.1 Particle Filters for Latent Rankings

Define ancestor indices a_{t-1}^s indicating which particle at time $t-1$ is the ancestor of particle s at time t , and set the initial value $a_0^s = s$. At timepoint t , $\mathbf{x}_{n,t-1}^{a_{t-1}^s}$ denotes the latent

variables of user $n \in \mathcal{I}_{1:t-1}$ in particle s . The r superscripts linking latent variables to static parameter particles are omitted for ease of notation. Also let $\mathcal{M}_C(\mathbf{p})$ denote a multinomial distribution over $C \in \mathbb{N}$ categories with probabilities \mathbf{p} . Algorithm 1 approximates $\pi_\theta(\mathbf{x}_{I_{1:T}}) = p(\mathbf{x}_{I_{1:T}} | \mathbf{y}_{I_{1:T}}, \theta)$ for fixed θ .

Algorithm 1 Particle Filter

```

1: for  $t = 1$  to  $T$  do
2:   if  $t > 1$  then
3:     Sample  $a_{t-1}^s \in [S]$  with probabilities  $W_{t-1}^{1:S, \theta}$ .
4:     for  $n \in \mathcal{I}_{1:t-1}$  do
5:        $\mathbf{x}_{n,t}^s \leftarrow \mathbf{x}_{n,t-1}^{a_{t-1}^s}$ .
6:     end for
7:   end if
8:   for  $n \in \mathcal{I}_t$  do
9:     Sample  $\mathbf{r}_{n,t}^s \sim q_\theta(\cdot | \mathcal{S}_n)$ .
10:    Sample  $z_{n,t}^s \sim \mathcal{M}_C(\mathbf{p}_n)$  with probabilities

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$$p_{n,c} = \frac{\tau_c Z(\alpha_c)^{-1} \exp\{-\alpha_c d(\mathbf{r}_{n,t}^s, \boldsymbol{\rho}_c)\}}{\sum_{c=1}^C \tau_c Z(\alpha_c)^{-1} \exp\{-\alpha_c d(\mathbf{r}_{n,t}^s, \boldsymbol{\rho}_c)\}}. \quad (7)$$

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11:  end for
12:  Compute weights

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$$w_{t,\theta}^s = \prod_{n \in \mathcal{I}_t} \frac{\sum_{c=1}^C \tau_c Z(\alpha_c)^{-1} \exp\{-\alpha_c d(\mathbf{r}_{n,t}^s, \boldsymbol{\rho}_c)\}}{q_\theta(\mathbf{r}_{n,t}^s | \mathcal{S}_n)} \left(\frac{\epsilon}{1 - \epsilon} \right)^{\sum_{i=1}^{p_{n,t}} g(y_{ni}, \mathbf{r}_{n,t}^s)} (1 - \epsilon)^{p_{n,t}}. \quad (8)$$

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13:  Normalize weights

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$$W_{t,\theta}^s = \frac{w_{t,\theta}^s}{\sum_{s=1}^S w_{t,\theta}^s}. \quad (9)$$

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14: end for

```

In Algorithm 1 the loop on lines 4-6 ensures that estimated latent rankings for all users are available at each timepoint, but can be omitted to reduce the memory cost. On line 9, \mathcal{S}_n is given by (4) in the case of partial rankings, (6) in the case of consistent pairwise preferences, and \mathcal{P}_m in the case of non-transitive pairwise preferences. We postpone the details of these proposal distributions to Section 3.4.

The expression for the weights (8) is based on Step 2(c) of Chopin et al. (2013, Sec. 2.1) which in our notation becomes

$$w_{t,\theta}^s = \frac{f_\theta(\mathbf{x}_{\mathcal{I}_t}^s | \mathbf{x}_{\mathcal{I}_{1:t-1}}^{a_{t-1}^s}) p_\epsilon(\mathbf{y}_{\mathcal{I}_t} | \mathbf{x}_{\mathcal{I}_t}^s)}{q_{t,\theta}(\mathbf{x}_{\mathcal{I}_t}^s | \mathbf{x}_{\mathcal{I}_{t-1}}^{a_{t-1}^s})}.$$

Because independent users arrive at each timepoint, $\mathbf{x}_{\mathcal{I}_t}^s$ and $\mathbf{x}_{\mathcal{I}_{1:t-1}}^{a_{t-1}^s}$ are independent given

θ and we get

$$\begin{aligned} f_\theta \left(\mathbf{x}_{\mathcal{I}_t}^s | \mathbf{x}_{\mathcal{I}_{1:t-1}}^{a_{t-1}^s} \right) &= f_\theta \left(\mathbf{x}_{\mathcal{I}_t}^s \right) = f_\theta \left(z_{\mathcal{I}_t}^s \right) f_\theta \left(\mathbf{r}_{\mathcal{I}_t}^s | z_{\mathcal{I}_t}^s \right) \\ &= \prod_{n \in \mathcal{I}_t} \frac{\tau_{z_{n,t}^s}}{Z \left(\alpha_{z_{n,t}^s} \right)} \exp \left\{ -\alpha_{z_{n,t}^s} d \left(\mathbf{r}_{n,t}^s, \boldsymbol{\rho}_{z_{n,t}^s} \right) \right\}. \end{aligned}$$

Next, it follows from Section 2.1 that

$$p_\epsilon \left(\mathbf{y}_{\mathcal{I}_t} | \mathbf{x}_{\mathcal{I}_t}^s \right) = \prod_{n \in \mathcal{I}_t} \left(\frac{\epsilon}{1 - \epsilon} \right)^{\sum_{i=1}^{p_{n,t}} g(y_{ni}, \mathbf{r}_{n,t}^s)} (1 - \epsilon)^{p_{n,t}}$$

which simplifies to $p_\epsilon(\mathbf{y}_{\mathcal{I}_t} | \mathbf{x}_{\mathcal{I}_t}^s) = 1$ when $\epsilon = 0$ and where $p_{n,t}$ denotes the number of pairwise preferences in \mathbf{y}_n for some $n \in \mathcal{I}_t$. The proposal distribution is

$$\begin{aligned} q_{t,\theta} \left(\mathbf{x}_{\mathcal{I}_t}^s | \mathbf{x}_{\mathcal{I}_{t-1}}^{a_{t-1}^s} \right) &= \prod_{n \in \mathcal{I}_t} q_\theta \left(\mathbf{r}_{n,t}^s | \mathcal{S}_n \right) q_\theta \left(z_{n,t}^s | \mathbf{r}_{n,t}^s \right) \\ &= \prod_{n \in \mathcal{I}_t} q_\theta \left(\mathbf{r}_{n,t}^s | \mathcal{S}_n \right) \frac{\tau_{z_{n,t}^s} Z \left(\alpha_{z_{n,t}^s} \right)^{-1} \exp \left\{ -\alpha_{z_{n,t}^s} d \left(\mathbf{r}_{n,t}^s, \boldsymbol{\rho}_{z_{n,t}^s} \right) \right\}}{\sum_{c=1}^C \tau_c Z \left(\alpha_c \right)^{-1} \exp \left\{ -\alpha_c d \left(\mathbf{r}_{n,t}^s, \boldsymbol{\rho}_c \right) \right\}}. \end{aligned}$$

These three expressions combine to yield the fraction in (8). For the special case $C = 1$ and $\epsilon = 0$ we recover the weight update formula in Algorithm 12 of Stein (2023) and with complete rankings we recover Algorithm 14 of Stein (2023).

In the resampling step on line 3 in Algorithm 1, as well as in the resampling steps of all subsequent algorithms, both multinomial resampling (Gordon et al., 1993) and the lower variance alternatives residual resampling (Liu and Chen, 1998), stratified resampling (Kitagawa, 1996), and systematic resampling (Kitagawa, 1996) can be used and are part of our implementation. We refer to Douc and Cappe (2005) and Hol et al. (2006) for details.

We also note the important fact that the quantity

$$\hat{Z}_t \left(\theta, \mathbf{x}_{\mathcal{I}_{1:t}}^{1:S}, a_{1:t-1}^{1:S} \right) = \frac{1}{S^t} \prod_{t'=1}^t \left\{ \sum_{s=1}^S w_{t',\theta}^s \right\} \quad (10)$$

is an unbiased estimator of the marginal likelihood $p(\mathbf{y}_{\mathcal{I}_{1:t}} | \theta)$ (Del Moral, 2004, Sec. 7.4.1).

3.1.1 Conditional Particle Filter

To allow particle Gibbs sampling in the rejuvenation step, we need a conditional particle filter (Andrieu et al., 2010) for which the full ancestral history of a given particle $\mathbf{x}_{\mathcal{I}_{1:T}}^k$ is fixed. This particle filter is shown in Algorithm 2 and yields samples approximately distributed according to $p(\mathbf{x}_{\mathcal{I}_{1:T}}^{-k} | \mathbf{y}_{\mathcal{I}_{1:T}}, \theta, \mathbf{x}_{\mathcal{I}_{1:T}}^k)$, where $k \in \{1, 2, \dots, S\}$ and $\mathbf{x}_{\mathcal{I}_{1:T}}^{-k}$ denotes the set of all particles except particle k .

3.2 SMC² Algorithm

The top-level algorithm for sampling the static parameters is an extension of iterated batch importance sampling (Chopin, 2002) which uses the particle filters of the previous section

Algorithm 2 Conditional Particle Filter

- 1: Condition on a trajectory $\mathbf{x}_{\mathcal{I}_{1:T}}^k$ with ancestral lineage $b_T^k = k$ and $b_t^k = a_t^{b_{t+1}^k}$, $t = T - 1, \dots, 1$.
 - 2: **for** $t = 1$ to T **do**
 - 3: **if** $t > 1$ **then**
 - 4: For $s \neq b_t^k$ sample $a_{t-1}^s \in [S]$ with probabilities $W_{t-1, \theta}^{1:S}$.
 - 5: **for** $n \in \mathcal{I}_{1:t-1}$ **do**
 - 6: Set $\mathbf{x}_{n,t}^s \leftarrow \mathbf{x}_{n,t-1}^{a_{t-1}^s}$.
 - 7: **end for**
 - 8: **end if**
 - 9: **for** $n \in \mathcal{I}_t$ **do**
 - 10: For $s \neq b_t^k$ sample $\mathbf{r}_{n,t}^s \sim q_\theta(\cdot | \mathcal{S}_{n,t})$.
 - 11: For $s \neq b_t^k$ sample $z_{n,t}^s \sim \mathcal{M}(C)$ with probabilities (7).
 - 12: **end for**
 - 13: Compute weights using (8) and normalize them using (9).
 - 14: **end for**
-

to integrate out the latent variables, and is stated in Algorithm 3. Since the particle filters yield unbiased estimates of the marginal likelihood, Algorithm 3 targets the correct posterior distribution $\pi(\theta | \mathbf{y}_{\mathcal{I}_{1:t}})$ at each $t = 1, 2, \dots, T$ (Chopin et al., 2013).

Considering Algorithm 3, first note that equation (11) is an unbiased estimator of

$$p(\mathbf{y}_{\mathcal{I}_t} | \mathbf{y}_{\mathcal{I}_{1:t-1}}, \theta^r) = \prod_{n \in \mathcal{I}_t} \sum_{c=1}^C \tau_c^r Z(\alpha_c^r)^{-1} \sum_{\mathbf{r}_n \in \mathcal{S}_n} \exp\{-\alpha_c^r d(\mathbf{r}_n, \boldsymbol{\rho}_c^r)\}. \quad (14)$$

The marginal likelihood increments are given by

$$\hat{p}(\mathbf{y}_{\mathcal{I}_t} | \mathbf{y}_{\mathcal{I}_{1:t-1}}) = \sum_{r=1}^R \Omega^r \times \hat{p}(\mathbf{y}_{\mathcal{I}_t} | \mathbf{y}_{\mathcal{I}_{1:t-1}}, \theta^r) \quad (15)$$

and can be used to estimate the unconditional marginal likelihood

$$\hat{p}(\mathbf{y}_{\mathcal{I}_{1:t}}) = \prod_{t'=1}^t \hat{p}(\mathbf{y}_{\mathcal{I}_{t'}} | \mathbf{y}_{\mathcal{I}_{1:t'-1}}). \quad (16)$$

The rejuvenation threshold A can be set to $R/2$. As in Fulop and Li (2013), we iterate the rejuvenation algorithm at least once, and stop when the number of unique particles exceeds A or when some upper limit on the number of iterations is reached. If the acceptance rate in the rejuvenation step is below some threshold B , which we set to $B = 0.2$ here, the number of particle filters is doubled. The doubling on lines 10-12 implements the exchange importance sampling step of Chopin et al. (2013, Sec. 3.6.1). The components in (13) are readily available from the call to Algorithm 4 in the rejuvenation step.

3.2.1 Parallelization

To reduce the amount of communication between nodes, it seems most sensible to parallelize the top-level Algorithm 3 rather than the particle filters. There are two main approaches

Algorithm 3 SMC² Algorithm

- 1: Sample $\theta^r = \{\alpha_c^r, \rho_c^r, \tau_c^r, \epsilon^r\}$ from their priors and set $\omega^r \leftarrow 1$.
- 2: **for** $t = 1$ to T **do**
- 3: Perform iteration t of the particle filter in Algorithm 1 with $\theta = \theta^r$ and compute

$$\hat{p}(\mathbf{y}_{\mathcal{I}_t} | \mathbf{y}_{\mathcal{I}_{1:t-1}}, \theta^r) = \frac{1}{S} \sum_{s=1}^S w_{t,\theta}^s. \quad (11)$$

- 4: Update and normalize importance weights

$$\omega^r \leftarrow \omega^r \times \hat{p}(\mathbf{y}_{\mathcal{I}_t} | \mathbf{y}_{\mathcal{I}_{1:t-1}}, \theta^r), \quad \Omega^r = \frac{\omega^r}{\sum_{r=1}^R \omega^r}. \quad (12)$$

- 5: Compute the effective sample size $\text{ESS} = \{\sum_{r=1}^R (\Omega^r)^2\}^{-1}$.
- 6: **if** $\text{ESS} < A$ **then**
- 7: Sample $a_t^r \in [R]$ with probabilities Ω^r and set $(\theta^r, \omega^r) \leftarrow (\theta^{a_t^r}, 1)$.
- 8: Rejuvenate with Algorithm 4, letting ζ denote the acceptance rate of (18).
- 9: **if** $\zeta < B$ **then**
- 10: Set $\tilde{S} = 2S$ and sample $i^{\tilde{s}} \in [S]$ for $\tilde{s} = 1, \dots, \tilde{S}$ with probabilities W_{t,θ^r}^s .
- 11: Set $\{\tilde{\mathbf{x}}_{\mathcal{I}_{1:t}}^{1:\tilde{S}}, \tilde{\mathbf{a}}_{1:t-1}^{1:\tilde{S}}\} \leftarrow \{\mathbf{x}_{\mathcal{I}_{1:t}}^{i^{\tilde{s}}}, \mathbf{a}_{1:t-1}^{i^{\tilde{s}}}\}$ and $\tilde{w}_{1:t,\theta^r}^{1:\tilde{S}} \leftarrow w_{1:t,\theta^r}^{i^{\tilde{s}}}$.
- 12: Update $S \leftarrow \tilde{S}$ and the particle weight

$$\omega^r \leftarrow \omega^r \times \left(S / \tilde{S} \right)^t \frac{\prod_{t'=1}^t \left\{ \sum_{s=1}^{\tilde{S}} \tilde{w}_{t',\theta^r}^s \right\}}{\prod_{t'=1}^t \left\{ \sum_{s=1}^S w_{t',\theta^r}^s \right\}}. \quad (13)$$

- 13: **end if**
 - 14: **end if**
 - 15: **end for**
-

to this in the literature (Dai et al., 2022; Naesseth et al., 2019). Jun et al. (2012) and Murray et al. (2016) use the fact that the weight updates in equations (11)-(12) can be done independently for each of the R particles. However, computing effective sample size and subsequently resampling requires communication between the nodes, and hence makes its implementation complicated.

A more straightforward approach, which we use in this paper, is what Naesseth et al. (2019) call importance weighted SMC samplers. In this case the full algorithm with R particles is run independently on P different nodes. Let $\theta^{r:p}$ denote the r th particle of the SMC² algorithm run on the p th compute node and $\Omega^{r:p}$ its weight, for $r = 1, \dots, R$ and $p = 1, \dots, P$. Also let $\hat{p}(\mathbf{y}_{\mathcal{I}_{1:T}})^p$ denote the marginal likelihood estimate (16) from the p 'th node. The combined set of particles $\{\theta^{r:p}\}$ with weights

$$\Omega^{r:p} \times \frac{\hat{p}(\mathbf{y}_{\mathcal{I}_{1:T}})^p}{\sum_{p'=1}^P \hat{p}(\mathbf{y}_{\mathcal{I}_{1:T}})^{p'}} \quad (17)$$

now yield a consistent estimate of the target distribution as $P \rightarrow \infty$ for any R (Naesseth et al., 2019, Sec. 4.4.1). The combined estimate of the marginal likelihood itself can be

obtained by direct averaging, $\hat{p}(\mathbf{y}_{\mathcal{I}_{1:T}}) = \sum_{p'=1}^P \hat{p}(\mathbf{y}_{\mathcal{I}_{1:T}})^{p'}/P$.

3.2.2 Latent Variable Prediction

Latent variable prediction corresponds to state inference in the SMC context. Predicting the latent variables $\mathbf{x}_{\mathcal{I}_t} = \{r_{\mathcal{I}_t}, z_{\mathcal{I}_t}\}$ of the users entering at time t is a filtering problem, and we can obtain R samples, weighted by Ω^r , from $p(\mathbf{x}_{\mathcal{I}_t}|\theta, \mathbf{y}_{\mathcal{I}_{1:t}})$ by drawing an index $s \sim \mathcal{M}(W_{t,\theta}^{r,s})$ for each particle r (Chopin et al., 2013, Sec. 3.3).

Sampling from the posterior $P(\mathbf{x}_{\mathcal{I}_{1:t}}|\theta, \mathbf{y}_{\mathcal{I}_{1:t}})$ of the latent variables of all users entered until time t can be done identically, but requires storing the full path for each particle. That is, if we draw a particle with index s we need to trace its latent variables according to its genealogy back until time 1. As noted by Chopin et al. (2013), this storage requirement can be avoided by triggering the particle doubling step in Algorithm 3 whenever a complete trajectory of the latent variables are needed, and then sampling an index $s \sim \mathcal{M}(W_{t,\theta}^{r,s})$ for each $r \in [R]$.

3.3 Rejuvenation

The rejuvenation step prevents degeneracy by moving each particle independently with an MCMC kernel. The original SMC² rejuvenation algorithms of Chopin et al. (2013) and Fulop and Li (2013) used particle marginal Metropolis-Hastings, but we instead use the algorithm proposed in Mendes et al. (2020) which combines particle marginal Metropolis-Hastings with particle Gibbs. This is useful in the present case because cluster probabilities τ_c and the error probability ϵ can be sampled conditionally, whereas the dispersion parameters α_c and the modal rankings ρ_c require a Metropolis-Hastings algorithm. For ease of notation, now let T denote the current value t of SMC² at the moment the rejuvenation algorithm is called. Also define $a \wedge b = \min\{a, b\}$.

Algorithm 4 starts by computing the variance of each α_c , using unweighted formulas because we always resample before rejuvenating. These variances are used for tuning the random walk proposal on line 4. Next, on line 2, we sample a complete particle history $\mathbf{x}_{\mathcal{I}_{1:T}}^k$ from the particle filters previously run with the parameter value θ . On line 4, $LS(\cdot)$ denotes the leap-and-shift proposal defined in Algorithm 5. To avoid introducing another tuning parameter, and to keep the proposal symmetric, we set leap size to 1. The extension to larger leap sizes is straightforward, and we refer to Vitelli et al. (2017, Sec. 2.4) for details.

On line 5 a new particle filter is run in order to compute the marginal likelihood of the proposed parameters. On line 6 we sample a proposal k' for a new particle history to condition on, using the weights from the particle filter run on line 5. The proposals θ' and k' are accepted with probability given by the Metropolis-Hastings ratio (18) in which the product term follows directly from the priors.

On line 9 we first compute the cluster frequencies \hat{N}_c in the particle filter k that we condition on, after which we sample the cluster probabilities from their conditional posterior (Vitelli et al., 2017, Sec. 4.3). On line 10 we sample the error probability from its conditional posterior (Crispino et al., 2019, p. 504). Lines 12 and 13, which consist of running a conditional particle filter and sampling a new $k \in [S]$ are necessary for computing the denominator in (18) in the next iteration.

A reasonable stopping criterion which is computationally easy to check is that the number of unique parameters exceeds some threshold, say $R/2$. However, the Gibbs sampler is guaranteed to produce new values of all τ_c^r and hence when $C > 1$ we are guaranteed to

Algorithm 4 Rejuvenation Algorithm

- 1: Compute $\hat{\sigma}_{\alpha,c}^2 = \frac{1}{R} \sum_{r=1}^R (\alpha_c^r - \hat{\alpha}_c)^2$ where $\hat{\alpha}_c = \frac{1}{R} \sum_{r=1}^R \alpha_c^r$.
- 2: Sample $k \in [S]$ with probabilities $W_{T,\theta^r}^{1:S}$.
- 3: **while** stopping criterion not met **do**
- 4: Sample proposals $\alpha'_c \sim \log \mathcal{N}(\log \alpha_c^r, \hat{\sigma}_{\alpha,c}^2)$ and $\rho'_c \sim LS(\rho_c^r)$ and set $\theta' \leftarrow \{\alpha'_c, \rho'_c, \tau_c, \epsilon\}_{c=1}^C$.
- 5: Run a particle filter (Algorithm 1) for $t = 1, 2, \dots, T$ and compute

$$\hat{Z}_T(\theta', \mathbf{x}_{\mathcal{I}_{1:T}}^{1:S}, a_{1:T-1}^{1:S}) = \prod_{t=1}^T \left\{ \frac{1}{S} \sum_{s=1}^S w_{t,\theta'}^s \right\}.$$

- 6: Sample $k' \in [S]$ with probabilities $W_{T,\theta'}^{1:S}$ (from particle filter).
- 7: Set $(\theta^r, k) \leftarrow (\theta', k')$ with probability

$$1 \wedge \frac{\hat{Z}_T(\theta', \mathbf{x}_{\mathcal{I}_{1:T}}^{1:S}, a_{1:T-1}^{1:S})}{\hat{Z}_T(\theta^r, \mathbf{x}_{\mathcal{I}_{1:T}}^{1:S}, a_{1:T-1}^{1:S})} \prod_{c=1}^C \left(\frac{\alpha'_c}{\alpha_c^r} \right)^\gamma \exp\{-\lambda(\alpha'_c - \alpha_c^r)\}. \quad (18)$$

- 8: Define $\mathbf{x}_{\mathcal{I}_{1:t}}^k = \{\mathbf{r}_n^k, z_n^k\}_{n \in \mathcal{I}_{1:t}}$ as the latent variables in particle filter k .
- 9: Compute $\hat{N}_c = \sum_{n \in \mathcal{I}_{1:T}} 1\{z_n^k = c\}$ and $\hat{\psi}_c = \psi + \hat{N}_c$, and sample

$$\tau' \sim \text{Dirichlet}(\hat{\psi}_1, \hat{\psi}_2, \dots, \hat{\psi}_C) = \Gamma\left(\sum_{c=1}^C \hat{\psi}_c\right) \left\{ \prod_{c=1}^C \Gamma(\hat{\psi}_c) \right\}^{-1} \prod_{c=1}^C \tau_c^{\hat{\psi}_c - 1}.$$

- 10: Compute $a = \sum_{n \in \mathcal{I}_{1:t}} \sum_{i=1}^{p_n} g(y_{ni}, \mathbf{r}_n^k)$, $b = \sum_{n \in \mathcal{I}_{1:t}} \sum_{i=1}^{p_n} [1 - g(y_{ni}, \mathbf{r}_n^k)]$, and sample

$$\epsilon' \sim f(\epsilon) \propto \epsilon^{\kappa_1 - 1 + a} (1 - \epsilon)^{\kappa_2 - 1 + b} 1\{\epsilon \in [0, 0.5]\}.$$

- 11: Set $\theta^r \leftarrow \{\alpha_c^r, \rho_c^r, \tau_c, \epsilon'\}_{c=1}^C$.
 - 12: Run Algorithm 2, conditional on $\mathbf{x}_{\mathcal{I}_{1:T}}^k$, for $t = 1, 2, \dots, T$.
 - 13: Sample $k \in [S]$ with probabilities $W_{T,\theta^r}^{1:S}$ (from conditional particle filter).
 - 14: **end while**
-

have R unique particles after a single iteration of the algorithm. The same applies to ϵ^r when we have non-transitive pairwise preferences. If this is sufficient, lines 12-13 can be skipped and no conditional particle filter needs to be run. On the other hand, this may lead to degeneracy in α_c and ρ_c , so in this case it might be more useful to monitor to the number of unique values of α_c^r , and stop the rejuvenation when this number exceeds $R/2$.

3.4 Proposals for Latent Rankings

In the particle filters of Section 3.1 we use a proposal for the latent rankings on the form $q_\theta(\cdot | \mathcal{S}_n)$, where \mathcal{S}_n is the set of rankings $\mathbf{r} \in \mathcal{P}_m$ compatible with the preferences given by user n . With partial rankings, \mathcal{S}_n is given by (4) and with consistent pairwise preferences it is given by (6). We now consider these cases in turn. With non-transitive pairwise preferences we have $\mathcal{S}_n = \mathcal{P}_m$ and sampling proposals amounts to simply permuting the

Algorithm 5 Leap-and-Shift Proposal for Modal Ranking (Vitelli et al., 2017)

Input: The current value ρ .

Output: A proposal ρ' separated from ρ by an Ulam distance of 1.

- 1: Sample uniformly $u \sim \mathcal{U}\{1, \dots, m\}$.
- 2: Define $\mathcal{S} = \{\max(1, \rho_u - 1), \min(m, \rho_u + 1)\} \setminus \{\rho_u\}$.
- 3: Sample uniformly $r \sim \mathcal{U}\{\mathcal{S}\}$.
- 4: Define $\rho^* \in \{1, \dots, m\}^m$ with elements $\rho_u^* = r$ and $\rho_i^* = \rho_i$ for $i \in \{1, \dots, m\} \setminus \{u\}$.
- 5: Define $\Delta = \rho_u^* - \rho_u$ and the proposal $\rho' \in \mathcal{P}_m$ with elements

$$\rho'_i = \begin{cases} \rho_u^* & \text{if } \rho_i = \rho_u \\ \rho_i - 1 & \text{if } \rho_u < \rho_i \leq \rho_u^* \text{ and } \Delta > 0 \\ \rho_i + 1 & \text{if } \rho_u > \rho_i \geq \rho_u^* \text{ and } \Delta < 0 \\ \rho_i & \text{otherwise,} \end{cases}$$

for $i = 1, \dots, m$.

integers $[m]$, all of which have probability $1/m!$, and hence no further consideration needs to be given to this case.

3.4.1 Partial Rankings

The simplest approach, used by Vitelli et al. (2017), is to randomly permute the elements of \mathcal{S}_i which are not fixed to a given rank. In this case the proposal distribution takes the form $q_\theta(\mathbf{r}_n | \mathcal{S}_n) = |\mathcal{S}_n|^{-1} \mathbf{1}\{\mathbf{r}_n \in \mathcal{S}_i\}$ and is independent of θ . Note that while this uniform distribution cancels out from the normalized weight formula (9), the probability $1/|\mathcal{S}_n|$ needs to be explicitly added to the unnormalized weights in (8) for the marginal likelihood computation in (15) and subsequently (16) to be correct.

Stein (2023) developed an alternative pseudolikelihood proposal which uses information in θ when proposing a new partial ranking for the user. We present it in Algorithm 6. For a given user n , it first fixes the observed items \mathcal{A}_n to their given value and then iterates through the unranked elements $\mathcal{A} \setminus \mathcal{A}_n$ in random order, sampling conditionally on the hitherto realized ranks. The distance $d(\cdot, \cdot)$ used in (19) needs to be either footrule or Spearman, since only these have a natural definition between single elements of ranking vectors, but note that the Mallows model can use any of the distance functions discussed in Section 2.3. The key difference from a uniform proposal is that the distribution for a latent rank r_{ni} in (19) is designed such that values close to the current estimate of the modal ranking ρ_i for item A_i are more likely to be obtained than values far from the modal ranking.

Pseudolikelihood proposal does not work for mixture models, as this would require knowledge of the cluster label $z_{n,t}^s$ for the given user in order to pick the right parameters $\alpha_{z_{n,t}^s}$ and $\rho_{z_{n,t}^s}$. Since $z_{n,t}^s$ needs to be sampled after $\mathbf{r}_{n,t}^s$ in the particle filters, it is not directly clear how to achieve this.

3.4.2 Consistent Pairwise Preferences

When \mathbf{y}_n contains consistent pairwise preferences, \mathcal{S}_n is given by (6) and contains all topological orderings of the directed acyclic graph given by \mathbf{y}_n , or equivalently all linear extensions

Algorithm 6 Pseudolikelihood Proposal for Latent Rankings (Stein, 2023)

Input: Parameters $\theta = \{\alpha, \rho\}$ and data \mathbf{y}_n .

Output: A proposal \mathbf{r}_n and its probability $q_\theta(\mathbf{r}_n|\mathcal{S}_n)$.

- 1: Define $\mathcal{B}_n = \emptyset$, and $q_\theta(\mathbf{r}_n|\mathcal{S}_n) = 1$.
- 2: **for** $i : A_i \in \mathcal{A}_n$ **do**
- 3: Set $r_{ni} = y_{ni}$ and $\mathcal{B}_n \leftarrow \mathcal{B}_n \cup r_{ni}$.
- 4: **end for**
- 5: Randomize the order of unranked items, $\mathbf{o}_n = \text{Permutation}(\mathcal{A} \setminus \mathcal{A}_n)$.
- 6: **for** $i : A_i \in \mathbf{o}_n$ **do**
- 7: Sample $r_{ni} \in [m] \setminus \mathcal{B}_n$ with probability

$$p(r_{ni}) = \frac{\exp\{-\alpha d(r_{ni}, \rho_i)\}}{\sum_{r_i \in [m] \setminus \mathcal{B}_n} \exp\{-\alpha d(r_i, \rho_i)\}}. \quad (19)$$

- 8: Set $\mathcal{B}_n \leftarrow \mathcal{B}_n \cup r_{ni}$.
 - 9: Set $q_\theta(\mathbf{r}_n|\mathcal{S}_n) \leftarrow q_\theta(\mathbf{r}_n|\mathcal{S}_n) \times p(r_{ni})$.
 - 10: **end for**
 - 11: Define $\mathbf{r}_n \in \mathcal{P}_m$ whose j th element is r_{ni} .
-

of the partially ordered set (poset) \mathbf{y}_n . Vitelli et al. (2017) initiated their MCMC algorithm with a single ordering computed deterministically, and then used a modified leap-and-shift algorithm to propose new latent rankings as local perturbations of the current value. This is not sufficient in our case, as we need both the support set of $q_\theta(\cdot|\mathcal{S}_n)$ and its cardinality.

We will sample latent rankings uniformly on $\text{TO}_n = \text{TO}(\mathbf{y}_n)$, the set of topological orderings of \mathbf{y}_n , and hence need to both count and generate linear extensions. The counting problem itself is known to be #P complete (Brightwell and Winkler, 1991), although faster algorithms exist for special cases, e.g., sparse posets (Kangas et al., 2016). Generation of the linear extensions can be obtained in constant additional time (Pruesse and Ruskey, 1994), and very compact storage of the extensions can be obtained using Gray codes (Ono and Nakano, 2005; Pruesse and Ruskey, 1994) or permutation decision diagrams (Inoue and Minato, 2014). In our implementation we used depth-first search (Cormen et al., 2022, Ch. 20.3-20.4), and generated all orderings by looping over all child nodes at each recursive step of the algorithm, keeping track of the solutions via backtracking. To keep the implementation relatively simple, we stored each element of TO_n to disk and randomly sampled among the files when proposing a new latent ranking.

Our procedure for proposing latent rankings from preference data is summarized in Algorithm 7. Note that for a given \mathbf{y}_n we only need to generate the topological orderings for the items involved in any of the stated pairwise preferences. When all items have been compared ($\bar{\mathcal{A}}_n = \emptyset$), the proposal probability is simply one over the number of orderings. When some set of items have not been involved in the comparisons, we consider two settings. First, if all the compared items are preferred to the non-compared items, denoted $\mathcal{A}_n \succ \bar{\mathcal{A}}_n$ in Algorithm 7, we permute the non-compared elements and place them after the compared items in the resulting order. This setting is relevant in ranked voting systems. The proposal probability now needs to account for the number of ways of ordering the non-compared items. Finally, if there is no preference relation between the compared and non-compared items, we can insert them in any position in the complete ordering vector, and we need to both account for the number of ways of permuting the un-compared items ($|\bar{\mathcal{A}}_n|!$) and the

number of ways of inserting them into the complete ordering $\binom{|\mathcal{A}|}{|\bar{\mathcal{A}}_n|}$.

Algorithm 7 Proposing Latent Rankings from Preference Data

Input: All topological orderings TO_n for items \mathcal{A}_n , unconsidered items $\bar{\mathcal{A}}_n = \mathcal{A} \setminus \mathcal{A}_n$.

Output: A proposal \mathbf{r}_n and its probability $q_\theta(\mathbf{r}_n | \mathcal{S}_n)$.

- 1: Sample an ordering $\mathbf{o}_{\mathcal{A}_n}$ uniformly from TO_n .
 - 2: **if** $\bar{\mathcal{A}}_n = \emptyset$ **then**
 - 3: Convert $\mathbf{o}_{\mathcal{A}_n}$ to a ranking \mathbf{r}_n and set $q_\theta(\mathbf{r}_n | \theta) = |\text{TO}_n|^{-1}$.
 - 4: **else if** $\mathcal{A}_n \succ \bar{\mathcal{A}}_n$ **then**
 - 5: Create ordering $\mathbf{o}_{\bar{\mathcal{A}}_n}$ by permuting the items in $\bar{\mathcal{A}}_n$ and define $\mathbf{o}_n = (\mathbf{o}_{\mathcal{A}_n}, \mathbf{o}_{\bar{\mathcal{A}}_n})$.
 - 6: Convert $\mathbf{o}_{\bar{\mathcal{A}}_n}$ to ranking \mathbf{r}_n and set $q(\mathbf{r}_n | \theta) = \{|\text{TO}_n| \times |\bar{\mathcal{A}}_n|!\}^{-1}$.
 - 7: **else**
 - 8: Sample a vector $\boldsymbol{\iota}$ of $|\bar{\mathcal{A}}_n|$ integers from $\{1, 2, \dots, |\mathcal{A}|\}$.
 - 9: Create ordering \mathbf{o}_n with items $\bar{\mathcal{A}}_n$ in positions $\boldsymbol{\iota}$ and items \mathcal{A}_n in the remaining positions.
 - 10: Convert \mathbf{o}_n to ranking \mathbf{r}_n and set $q(\mathbf{r}_n | \theta) = \{|\text{TO}_n| \times |\bar{\mathcal{A}}_n|! \times \binom{|\mathcal{A}|}{|\bar{\mathcal{A}}_n|}\}^{-1}$.
 - 11: **end if**
-

4 Simulation Experiments

We here report results of simulation experiments aimed at testing the proposed algorithms. Simulations were performed on a MacBook Pro with a 32 GB Apple M1 Max chip, and on the high performance computing cluster Fox, provided by the University of Oslo. The R packages Rcpp (Eddelbuettel and François, 2013) and RcppArmadillo (Eddelbuettel and Sanderson, 2014) were used as interfaces to the C++ implementation of our algorithms, which made heavy use of the Armadillo library (Sanderson and Curtin, 2016). Our parallelization of SMC² used the futures framework (Bengtsson, 2021), through the furr package (Vaughan and Dancho, 2021). Pre- and post-processing of data, as well as visualization, was done mainly with the set of R packages provided by the tidyverse (Wickham et al., 2019).

4.1 Generation of Topological Orderings

The generation of all topological orderings when proposing latent rankings in the pairwise preference case is a potential bottleneck. We here report two numerical experiments investigating the extent of this issue in real data. In both experiments we defined TO_n as the number of orderings of the items compared by user n , since permuting the non-compared items is a computationally easy task.

4.1.1 Topological Orderings for PrefLib Data

We downloaded all datasets containing orders with ties at PrefLib.org (Mattei and Walsh, 2013, 2017). This included 30 election datasets with the number of votes ranging from 2,477 to 298,788 and the number of candidates between 4 and 23, all donated by O’Neill (2013). In addition there was a dataset with 5,000 individuals’ ratings of subsets of a total of 100 sushi items (Kamishima, 2003), and results from an education survey conducted at Instituto Superior Politecnico Jose Antonio Echeverria (Havana, Cuba).

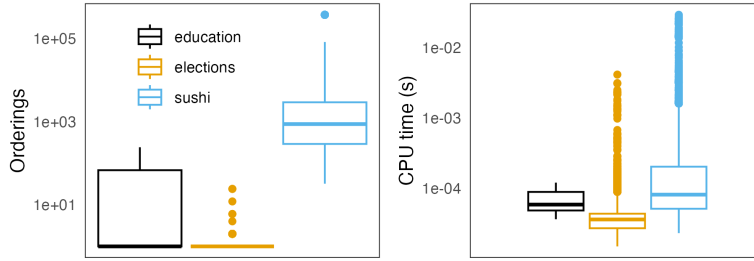


Figure 1: Distributions of the number of topological orderings for each user’s preferences and the required CPU time to compute the orderings.

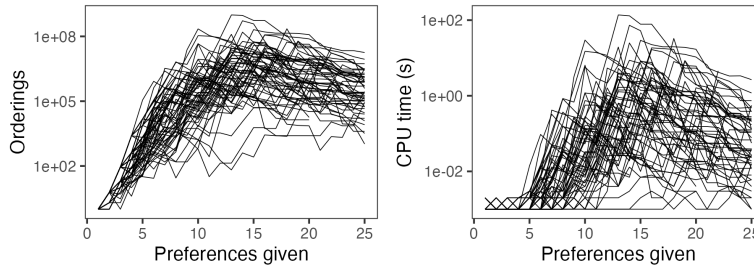


Figure 2: Total number of orderings and associated CPU time as the number of preferences for each user increases, for the beach preference dataset. Each trajectory represents a single user.

For all datasets we computed the number of topological orderings $|TO_n|$. The results are shown in Figure 1, in which the counts for all 30 election datasets have been combined. The largest number of topological orderings occurred for the sushi data, for which the average was 1.2×10^4 and the maximum was 3.6×10^5 . The average central processing unit (CPU) time was 0.6 ms, and the maximum was around 30 ms. For the education and election datasets, the average (maximum) CPU times for computing all the orderings of a single user were 0.063 ms (0.12 ms) and 0.041 ms (4.1 ms), respectively. The microbenchmark package (Mersmann, 2023) was used for the timing.

4.1.2 Topological Orderings for Beach Preference Data

We studied how the number of orderings depends on the number of stated preferences, using a dataset containing pairwise preferences from 60 users comparing pictures of 15 beaches (Vitelli et al., 2017). We constructed a temporal order such that each user started with zero preferences, and at each timepoint one new pairwise preference from the user was randomly chosen and added to the user’s data. Figure 2 shows how the number of orderings and the CPU time developed as more preferences were added. The maximum average CPU time was 4.75 s after 14 preferences, and the maximum CPU time overall was 137 s for a single user after 13 preferences.

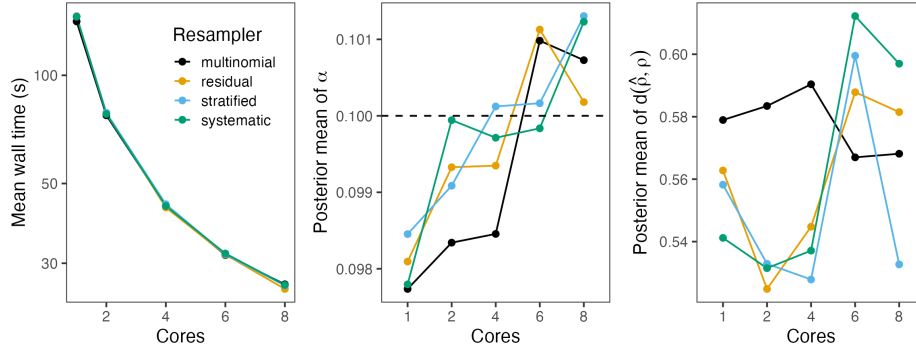


Figure 3: Average wall times (left), average posterior mean of α (center), and average posterior mean of the footrule distance to the true ranking (right), for simulations with complete rankings.

4.2 Complete Rankings

To study the performance of SMC² in a sequential inference case with complete rankings, we generated 100 datasets with complete rankings from the Mallows model with the footrule distance using the sampling algorithm of Vitelli et al. (2017, Appendix C). In all simulations there were $m = 10$ items and $N = 1000$ users, the scale parameter was $\alpha = 0.1$ and there was a single cluster. The users were assumed to enter one at a time, yielding 1,000 timepoints. For all simulated datasets the algorithm was run in parallel on P cores, with $P \in \{1, 2, 4, 6, 8\}$ and $R = 5000/P$ particles, using multinomial, residual, stratified, and systematic resampling. The gamma prior for α had shape $\gamma = 1$ and rate $\lambda = 0.5$ and the resampling threshold was $A = R/2$. Since the likelihood increments are analytically given by (14), the number of particle filters was fixed to 1 and the doubling threshold B in Algorithm 3 set to 0.

The simulation results are summarized in Figure 3. The left plot illustrates how the computing time depends on the resampling scheme and parallelization. Regarding the former, all four resampling schemes performed equally fast. Furthermore, the plot shows a clear benefit of parallelization, although with slightly diminishing returns: doubling the number of cores from 1 to 2, 2 to 4, and 4 to 8, respectively, led to factors 1.85, 1.81, and 1.67 speed-up, respectively. The average time with eight cores was 25.9 seconds. For comparison, batch estimation with MCMC with a burnin-in period of 1,000 iterations and 5,000 post-burnin iterations took on average 3 seconds. However, sequential estimation with MCMC was prohibitively slow, as the algorithm has to be rerun at each timepoint; just the last ten timepoints from 991 to 1000 took more time than the complete sequential inference with SMC on eight cores.

The center plot in Figure 3 shows the posterior means of α for different numbers of cores. MCMC batch estimation had average posterior mean at 0.100 with 95% Monte Carlo interval (0.099, 0.101), suggesting that the posterior mean on average was very close to the data generating value. SMC² had a slight negative bias⁴ when using a single core, with the

⁴With bias we mean systematic deviation from the average posterior mean, here computed using MCMC batch estimation with a sufficient number of post burn-in samples. This is not necessarily equal to the data generating value $\alpha = 0.1$, although it was very close in this case.

Table 1: Results of simulations with sequentially arriving top-3 rankings. Values are Monte Carlo averages, with standard errors in parentheses.

Proposal	Resampler	Posterior mean α	Posterior mean $d(\boldsymbol{\rho}, \hat{\boldsymbol{\rho}})$
Pseudolikelihood	Multinomial	0.299 (9e-04)	3.09 (0.058)
Pseudolikelihood	Residual	0.294 (9e-04)	2.95 (0.055)
Pseudolikelihood	Stratified	0.301 (9e-04)	2.96 (0.062)
Pseudolikelihood	Systematic	0.296 (9e-04)	2.98 (0.054)
Uniform	Multinomial	0.298 (7e-04)	3.19 (0.047)
Uniform	Residual	0.296 (8e-04)	3.55 (0.049)
Uniform	Stratified	0.293 (9e-04)	3.46 (0.048)
Uniform	Systematic	0.295 (7e-04)	3.07 (0.048)

upper limit of 95% Monte Carlo intervals lower than 0.100. With eight cores there was a tendency towards positive bias, but in this case all the Monte Carlo intervals covered the true value. Finally, the rightmost plot in Figure 3 shows the average posterior mean footrule distance to the true ranking. The differences between cores and resampling methods for the posterior mean of $d(\hat{\boldsymbol{\rho}}, \boldsymbol{\rho})$ are well within 95% Monte Carlo confidence intervals, which are not included in the plots for ease of visualization.

4.3 Top- k Rankings

We next considered the case in which users provide top-3 rankings of $m = 10$ items. As in Section 4.2, the users were assumed to arrive sequentially, one at each timepoint. A total of 30 random datasets were simulated, with $N = 200$ users, scale parameter $\alpha = 0.3$, and footrule distance. Since each user had only ranked three out of ten items, the particle filter in Algorithm 1 now had to be run to integrate over the remaining seven items for each user. The total number of particles was set to $R = 3000$ which were processed separately on eight cores and combined using (17). The resampling threshold was $A = R/2$ and the threshold for particle filter doubling was set to an average acceptance rate $B = 0.2$ in the rejuvenation step. The initial number of particle filters per core was set to $S = 20$. Both uniform and pseudolikelihood proposals were used.

Table 1 shows simulation results after the final timepoint. The third column shows that the final estimate of α was close to the true value for all algorithm settings. With uniform latent rank proposal, systematic resampling gave slightly lower posterior mean distance to the true modal ranking $\boldsymbol{\rho}$ than the other alternatives. Overall, pseudolikelihood proposal gave posterior means of $\boldsymbol{\rho}$ closer to the true ranking. Processing a single dataset on eight cores took about 12 minutes when using uniform proposal, and between 10 and 100 times more with pseudolikelihood proposal.⁵ Hence, the increase in accuracy of $\boldsymbol{\rho}$ has a cost in terms of computational demand.

Figure 4 shows the number of particle filters as a function of time (left) and the resampling probability as a function of time (right), with uniform proposal distribution. The plots for pseudolikelihood proposal were almost identical, and are not shown. The curves were

⁵Due to the longer computing time, the models with pseudolikelihood proposal were run on the Fox computing cluster, while the models with uniform proposal were run on the MacBook Pro. Thus, the computing times are not exactly comparable.

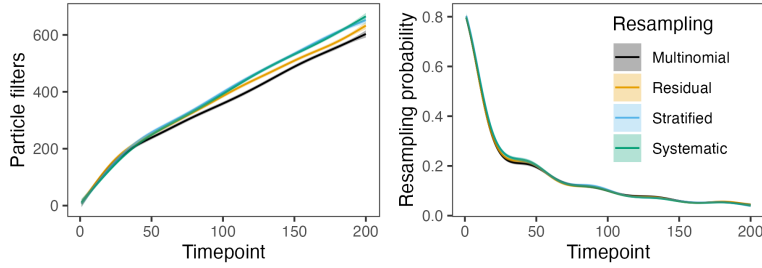


Figure 4: Number of particle filters and resampling probability for simulations with sequentially arriving top-3 rankings. Shaded regions are 95% confidence bands.

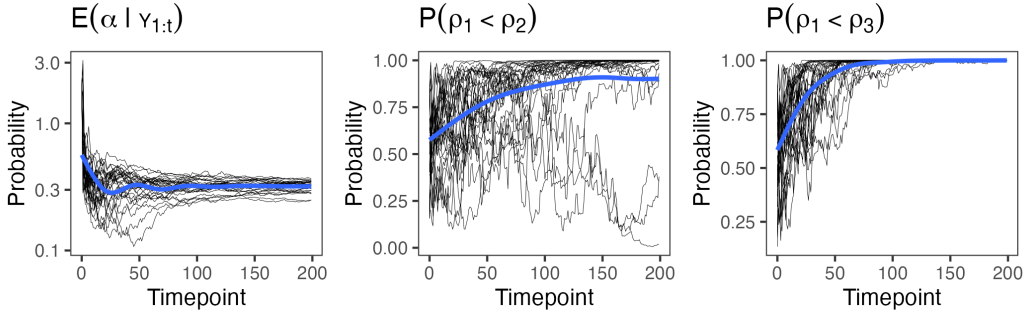


Figure 5: Trace plots of posterior means. Thin black lines shows the posterior probability for a single dataset and thick blue lines are GAM fits.

obtained by fitting generalized additive models (GAMs) (Wood, 2017) with ten thin-plate regression splines (Wood, 2003) as basis functions to the number of particle filters and a binary resampling indicator, respectively. A unit link function was used for the number of particles and a logit link for the resampling indicator, and smoothing was done with restricted maximum likelihood. The left plot shows that the number of particles grows close to linearly. This is expected from Theorem 1 of Andrieu et al. (2010), which implies that the number of particle filters must grow linearly with the number of observations for the acceptance rate to stay constant. The right plot shows that the resampling probability decreases with the number of timepoints. Since each resampling step is followed by one or more rejuvenation steps, this means that as rejuvenation becomes more computationally demanding due to more observations, it also becomes less frequent. This behavior also agrees with what is expected from theory; in particular, Theorem 1 and Section 4.3 of Chopin (2002) predicts that the time interval between each time when resampling is needed should increase geometrically in the total number of observations, which would produce an exponentially decaying curve (see also Proposition 17.1 in Chopin and Papaspiliopoulos, 2020).

4.4 Pairwise Preferences

We next simulated consistent pairwise preference data by first generating complete rankings of $m = 5$ items with $\boldsymbol{\rho} = (1, 2, \dots, 5)'$ and $\alpha = 0.3$ for 200 users, and then randomly selecting four implied pairwise preferences for each user. The users were assumed to arrive sequentially, one at each timepoint. The process was repeated 30 times, and each dataset was processed in parallel on 10 CPUs, each with $S = 300$ particles. The initial number of particle filters was 20, and all other parameters were as described in the previous sections.

Across the 30 datasets, the average posterior mean of α at the final timepoint was 0.325, with 95% Monte Carlo interval (0.314, 0.366). For comparison, the average posterior mean of α obtained by MCMC estimation with 20,000 post-burnin iterations on the same datasets was 0.300 with 95% Monte Carlo interval (0.289, 0.310), suggesting that SMC² was slightly positively biased in this case. Figure 5 (left) shows how the posterior expectation of α evolved as more data became available, with initial rapid fluctuations followed by a stabilization close to the true value of 0.3 after about 100 timepoints. The center plot in Figure 5 shows the posterior probability that item 1 is preferred to item 2 in the modal ranking, and the right plot show the posterior probability that item 1 is preferred to item 3. As expected, the latter converges more quickly to one than the former. In particular, for three of the simulated datasets $P(\rho_1 < \rho_2)$ was below 0.5 even after the final timepoint, implying that items 1 and 2 were "flipped" in the posterior distribution, while $P(\rho_1 < \rho_3)$ was close to 1 for all datasets.

4.5 Mixtures of Mallows Models

To test the conditional particle filter and the proposed algorithm's ability to estimate the number of mixture components we randomly generated ten datasets with $m = 5$ items, each having two mixture components with dispersion parameters $\boldsymbol{\alpha} = (0.3, 0.6)'$ and equal probabilities $\boldsymbol{\tau} = (0.5, 0.5)'$. Modal rankings were $\boldsymbol{\rho} = (1, 2, 3, 4, 5)'$ and $\boldsymbol{\rho} = (5, 4, 3, 2, 1)'$.

Five models were estimated for each dataset, with $C \in \{1, 2, 3, 4, 5\}$ clusters, respectively. For each model and dataset, the SMC² algorithm was run in parallel on 10 CPUs, with $S = 400$ particles, each with $R = 50$ particle filters. All other parameters were as described for the simulations above.

Figure 6 shows how the logarithm of the marginal likelihood (16) of the models estimated for each dataset varied with the number of clusters. We see a clear "elbow" at the correct number of two clusters. In terms of Bayes factors, however, models with more than two clusters were preferred for all the simulated datasets, as can be seen by the fact that the marginal likelihood keeps increasing beyond the two-cluster solution. Note that the marginal likelihood is not readily available when estimating mixtures of Mallows models using MCMC algorithms. For example, Vitelli et al. (2017) and Crispino et al. (2019) selected the number of clusters based on within-cluster distances, which requires saving the distance to the cluster centroid for each user at each MCMC step.

Previous work on estimating mixture models using SMC have either ignored the label switching problem and focused on marginal likelihood estimation (Del Moral et al., 2006; Fearnhead, 2004) or introduced identifiability constraints (Chopin, 2002; Fearnhead and Meligkotsidou, 2007) despite the known deficits of this approach (Jasra et al., 2005). Here we used Stephens' algorithm (Stephens, 2000) as implemented in the R package label.switching (Papastamoulis, 2016) for relabeling the outcomes. This has a small additional memory cost, as the cluster probabilities of each user has to be saved at the final timepoint, for each of the S particles. According to the algorithm, there was evidence of label switching in all

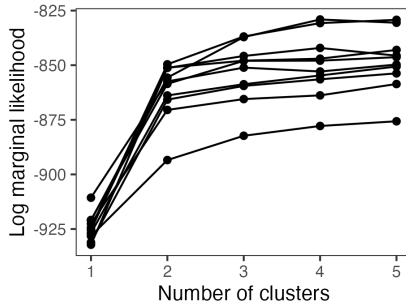


Figure 6: Logarithm of the marginal likelihood as a function of the number of clusters for ten simulated datasets.

simulated datasets for all models with $C > 1$. Figure 7 shows posterior histograms of the two components of α for the two-cluster model. With the exception of dataset 5, the two components were well separated.

For the two-cluster model, the average of the posterior mean of α across the ten simulated datasets was 0.299 for the smallest component and 0.591 for the largest component, with probabilities 0.508 and 0.492, both very close to the values in the data generating distribution. The average posterior probability of the true modal ranking ρ_c in each mixture component was 0.855 in the cluster with $\alpha_c = 0.3$ and 1.000 in the cluster with $\alpha_c = 0.6$, confirming that the modal ranking is easier to identify with a higher precision parameter.

5 Discussion

We have proposed an SMC² algorithm for sequential estimation of the Bayesian Mallows model. The algorithm naturally incorporates data in the form of partial rankings and both transitive and non-transitive pairwise preferences, and is straightforward to parallelize. Compared to MCMC, the algorithm is competitive in use cases where data arrive sequentially and the posteriors of interest need to be recomputed for each new data batch. In batch estimation problems, MCMC is typically faster, but an advantage of SMC² is that the marginal likelihood is readily available at no additional computational cost, which is useful for model comparison.

A number of future extensions are possible. First, conditioning on the full particle history in the conditional particle filter may lead to a high degree of degeneracy (Whiteley, 2010), with the consequence that a large number of particle filters is required to obtain a sufficiently accurate approximation of the conditional posterior. Backward sampling has been shown to considerably decrease this degeneracy (Lindsten and Schön, 2012), and would be interesting to consider in the current setting. Another extension of practical interest is to allow users to provide updated rankings, e.g., comparisons of previously unseen items. Stein (2023, Ch. 6) proposed an algorithm for this in the case of partial rankings, in which users whose new data contradicted their current latent rankings were removed from the pool and then reentered. An extension of this approach to SMC² would likely require a particle filter which performs this type of correction while still yielding unbiased estimated of the marginal likelihood $p(\mathbf{y}_{\mathcal{I}_{1:t}}|\theta)$, as equation (10) does in the current case where new users arrive at each timepoint. Finally, it may be of interest to let some or all of the parameters

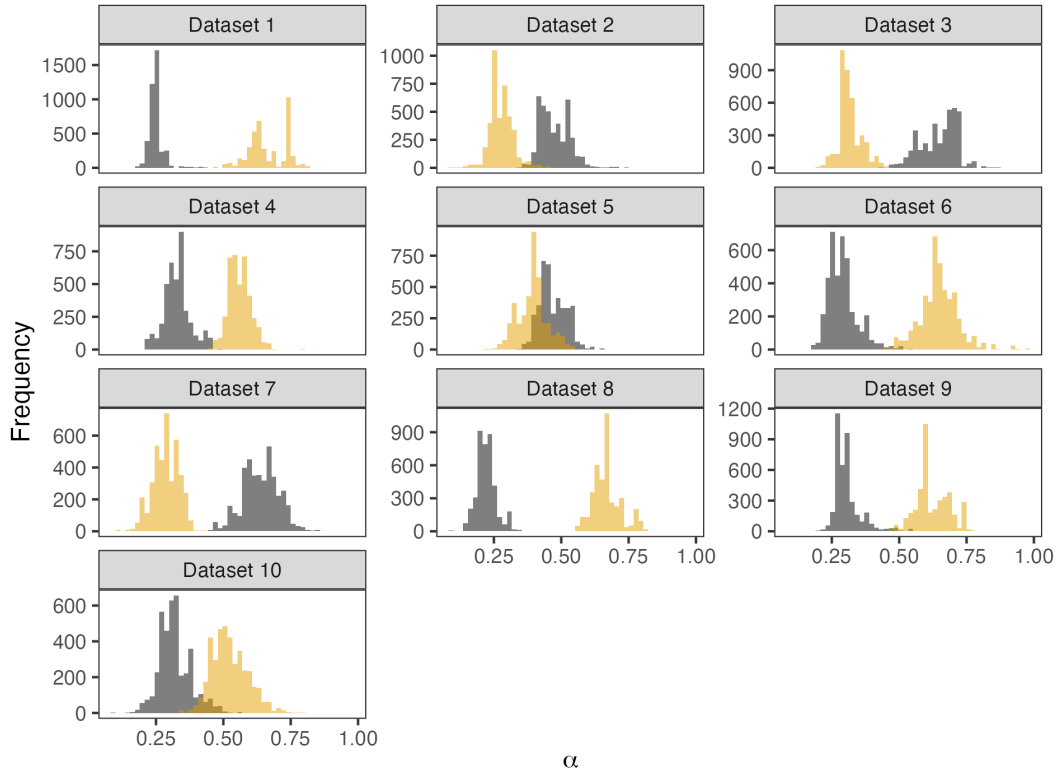


Figure 7: Posterior histograms of α for the two-cluster solution, for each simulated dataset.

in θ to depend on time, e.g., to monitor how preferences in a population evolve. An MCMC algorithm for time-varying modal ranking ρ has been proposed by [Asfaw et al. \(2017\)](#), but sequential estimation is likely a good alternative in this case.

Another interesting possibility is to use the proposed algorithm in a sequential experimental design framework. For example, at a given timepoint, the items to be ranked or compared by the next user could be determined by a utility function seeking to maximize the information about some posterior quantity of interest, e.g., whether an item A_1 is preferred to another item A_2 in the modal ranking. Examples of similar uses of SMC include estimation of generalized (non-)linear models ([Drovandi et al., 2013](#)), model selection ([Drovandi et al., 2014](#)), and hierarchical models ([McGree et al., 2016](#)).

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References

- Andrieu, C., Doucet, A., and Holenstein, R. (2010). Particle Markov Chain Monte Carlo Methods. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 72(3):269–342. [6](#), [8](#), [19](#)
- Asfaw, D., Vitelli, V., Sørensen, Ø., Arjas, E., and Frigessi, A. (2017). Time-varying rankings with the Bayesian Mallows model. *Stat*, 6(1):14–30. [22](#)
- Babington Smith, B. (1950). Discussion of professor Ross’s paper. *Journal of the Royal Statistical Society B*, 12(1):41–59. [1](#)
- Bengtsson, H. (2021). A unifying framework for parallel and distributed processing in R using futures. *R Journal*. [15](#)
- Berzuini, C. and Gilks, W. (2001). RESAMPLE-MOVE Filtering with Cross-Model Jumps. In Doucet, A., de Freitas, N., and Gordon, N., editors, *Sequential Monte Carlo Methods in Practice*, Statistics for Engineering and Information Science, pages 117–138. Springer, New York, NY. [2](#), [6](#)
- Bradley, R. A. and Terry, M. E. (1952). Rank Analysis of Incomplete Block Designs: I. The Method of Paired Comparisons. *Biometrika*, 39(3/4):324–345. [1](#)
- Brightwell, G. and Winkler, P. (1991). Counting linear extensions is $\{\#\}$ P-complete. In *Proceedings of the Twenty-Third Annual ACM Symposium on Theory of Computing*, Stoc ’91, pages 175–181, New York, NY, USA. Association for Computing Machinery. [14](#)
- Cayley, A. (1849). LXXVII. Note on the theory of permutations. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*. [5](#)
- Ceberio, J., Irurozki, E., Mendiburu, A., and Lozano, J. A. (2015). A review of distances for the Mallows and Generalized Mallows estimation of distribution algorithms. *Computational Optimization and Applications*, 62(2):545–564. [5](#)
- Chopin, N. (2002). A sequential particle filter method for static models. *Biometrika*, 89(3):539–552. [2](#), [6](#), [8](#), [19](#), [20](#)
- Chopin, N., Jacob, P. E., and Papaspiliopoulos, O. (2013). SMC2: An Efficient Algorithm for Sequential Analysis of State Space Models. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 75(3):397–426. [2](#), [6](#), [7](#), [9](#), [11](#)
- Chopin, N. and Papaspiliopoulos, O. (2020). *An Introduction to Sequential Monte Carlo*. Springer Series in Statistics. Springer International Publishing, Cham. [19](#)
- Cormen, T. H., Leiserson, C. E., Rivest, R. L., and Stein, C. (2022). *Introduction to Algorithms*. The MIT Press, Cambridge, MA, USA, 4th edition. [14](#)
- Courcoux, Ph. and Semenou, M. (1997). Preference data analysis using a paired comparison model. *Food Quality and Preference*, 8(5):353–358. [1](#)
- Crispino, M. (2018). *Bayesian Learning with the Mallows Rank Model*. PhD thesis, Bocconi University. [5](#)

- Crispino, M., Arjas, E., Vitelli, V., Barrett, N., and Frigessi, A. (2019). A Bayesian Mallows Approach to Nontransitive Pair Comparison Data: How Human Are Sounds? *The Annals of Applied Statistics*, 13(1):492–519. [3](#), [4](#), [5](#), [11](#), [20](#)
- Crispino, M., Mollica, C., Astuti, V., and Tardella, L. (2023). Efficient and accurate inference for mixtures of Mallows models with Spearman distance. *Statistics and Computing*, 33(5):98. [5](#)
- Dai, C., Heng, J., Jacob, P. E., and Whiteley, N. (2022). An Invitation to Sequential Monte Carlo Samplers. *Journal of the American Statistical Association*, 117(539):1587–1600. [6](#), [10](#)
- Del Moral, P. (2004). *Feynman-Kac Formulae*. Probability and Its Applications. Springer, New York, NY. [8](#)
- Del Moral, P., Doucet, A., and Jasra, A. (2006). Sequential Monte Carlo Samplers. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 68(3):411–436. [6](#), [20](#)
- Diaconis, P. (1988). *Group Representations in Probability and Statistics*. SPIE. [2](#), [3](#), [5](#)
- Douc, R. and Cappe, O. (2005). Comparison of resampling schemes for particle filtering. In *ISPA 2005. Proceedings of the 4th International Symposium on Image and Signal Processing and Analysis, 2005.*, pages 64–69. [8](#)
- Drovandi, C. C., McGree, J. M., and Pettitt, A. N. (2013). Sequential Monte Carlo for Bayesian sequentially designed experiments for discrete data. *Computational Statistics & Data Analysis*, 57(1):320–335. [22](#)
- Drovandi, C. C., McGree, J. M., and Pettitt, A. N. (2014). A Sequential Monte Carlo Algorithm to Incorporate Model Uncertainty in Bayesian Sequential Design. *Journal of Computational and Graphical Statistics*, 23(1):3–24. [22](#)
- Eddelbuettel, D. and François, R. (2013). *Seamless R and C++ Integration with Rcpp*. Springer, New York, NY, 1 edition. [15](#)
- Eddelbuettel, D. and Sanderson, C. (2014). RcppArmadillo: Accelerating R with high-performance C++ linear algebra. *Computational Statistics & Data Analysis*, 71:1054–1063. [15](#)
- Eliseussen, E., Fleischer, T., and Vitelli, V. (2022). Rank-based Bayesian variable selection for genome-wide transcriptomic analyses. *Statistics in Medicine*, 41(23):4532–4553. [1](#)
- Fearnhead, P. (2004). Particle filters for mixture models with an unknown number of components. *Statistics and Computing*, 14(1):11–21. [20](#)
- Fearnhead, P. and Künsch, H. R. (2018). Particle Filters and Data Assimilation. *Annual Review of Statistics and Its Application*, 5(Volume 5, 2018):421–449. [6](#)
- Fearnhead, P. and Meligkotsidou, L. (2007). Filtering Methods for Mixture Models. *Journal of Computational and Graphical Statistics*, 16(3):586–607. [20](#)
- Fligner, M. A. and Verducci, J. S. (1986). Distance Based Ranking Models. *Journal of the Royal Statistical Society. Series B (Methodological)*, 48(3):359–369. [2](#), [5](#)

- Fulop, A. and Li, J. (2013). Efficient learning via simulation: A marginalized resample-move approach. *Journal of Econometrics*, 176(2):146–161. [2](#), [6](#), [9](#), [11](#)
- Gilks, W. R. and Berzuini, C. (2001). Following a moving target—Monte Carlo inference for dynamic Bayesian models. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 63(1):127–146. [2](#), [6](#)
- Gordon, A. D. (1979). A measure of the agreement between rankings. *Biometrika*, 66(1):7–15. [5](#)
- Gordon, N. J., Salmond, D. J., and Smith, A. F. M. (1993). Novel approach to nonlinear/non-Gaussian Bayesian state estimation. *IEE Proceedings F (Radar and Signal Processing)*, 140(2):107–113. [8](#)
- Hamming, R. W. (1950). Error detecting and error correcting codes. *The Bell System Technical Journal*, 29(2):147–160. [5](#)
- Hol, J. D., Schon, T. B., and Gustafsson, F. (2006). On Resampling Algorithms for Particle Filters. In *2006 IEEE Nonlinear Statistical Signal Processing Workshop*, pages 79–82. [8](#)
- Hwang, M., Lee, G., Kee, H., Kim, C. W., Lee, K., and Oh, S. (2023). Sequential Preference Ranking for Efficient Reinforcement Learning from Human Feedback. *Advances in Neural Information Processing Systems*, 36:49088–49099. [1](#)
- Inoue, Y. and Minato, S.-i. (2014). An Efficient Method for Indexing All Topological Orders of a Directed Graph. In Ahn, H.-K. and Shin, C.-S., editors, *Algorithms and Computation*, pages 103–114, Cham. Springer International Publishing. [14](#)
- Irurozki, E. (2014). *Sampling and Learning Distance-Based Probability Models for Permutation Spaces*. PhD thesis, Department of Computer Science and Artificial Intelligence of the University of the Basque Country. [5](#)
- Irurozki, E., Calvo, B., and Lozano, J. A. (2016). PerMallows: An R Package for Mallows and Generalized Mallows Models. *Journal of Statistical Software*, 71:1–30. [5](#)
- Irurozki, E., Calvo, B., and Lozano, J. A. (2018). Sampling and Learning Mallows and Generalized Mallows Models Under the Cayley Distance. *Methodology and Computing in Applied Probability*, 20(1):1–35. [2](#), [5](#)
- Irurozki, E., Calvo, B., and Lozano, J. A. (2019). Mallows and generalized Mallows model for matchings. *Bernoulli*, 25(2):1160–1188. [5](#)
- Jasra, A., Holmes, C. C., and Stephens, D. A. (2005). Markov Chain Monte Carlo Methods and the Label Switching Problem in Bayesian Mixture Modeling. *Statistical Science*, 20(1):50–67. [20](#)
- Jun, S.-h., Wang, L., and Bouchard-côté, A. (2012). Entangled Monte Carlo. In *Advances in Neural Information Processing Systems*, volume 25. Curran Associates, Inc. [10](#)
- Kahneman, D. and Tversky, A. (1979). Prospect Theory: An Analysis of Decision under Risk. *Econometrica*, 47(2):263–291. [4](#)

- Kamishima, T. (2003). Nantonac collaborative filtering: Recommendation based on order responses. In *Proceedings of the Ninth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, KDD '03, pages 583–588, New York, NY, USA. Association for Computing Machinery. [1](#), [15](#)
- Kangas, K., Hankala, T., Niinimäki, T., and Koivisto, M. (2016). Counting linear extensions of sparse posets. In *Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence*, IJCAI'16, pages 603–609, New York, New York, USA. AAAI Press. [14](#)
- Kendall, M. G. (1938). A New Measure of Rank Correlation. *Biometrika*, 30(1/2):81–93. [2](#), [5](#)
- Kitagawa, G. (1996). Monte Carlo Filter and Smoother for Non-Gaussian Nonlinear State Space Models. *Journal of Computational and Graphical Statistics*, 5(1):1–25. [8](#)
- Krivulin, N., Prinkov, A., and Gladkikh, I. (2022). Using Pairwise Comparisons to Determine Consumer Preferences in Hotel Selection. *Mathematics*, 10(5):730. [1](#)
- Lindsten, F. and Schön, T. B. (2012). On the use of backward simulation in the particle Gibbs sampler. In *2012 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 3845–3848. [21](#)
- Liu, J. S. and Chen, R. (1998). Sequential Monte Carlo Methods for Dynamic Systems. *Journal of the American Statistical Association*, 93(443):1032–1044. [8](#)
- Liu, Q., Crispino, M., Scheel, I., Vitelli, V., and Frigessi, A. (2019a). Model-Based Learning from Preference Data. *Annual Review of Statistics and Its Application*, 6(1):329–354. [2](#)
- Liu, Q., Reiner, A. H., Frigessi, A., and Scheel, I. (2019b). Diverse personalized recommendations with uncertainty from implicit preference data with the Bayesian Mallows model. *Knowledge-Based Systems*, 186:104960. [1](#), [2](#)
- Lu, T. and Boutilier, C. (2014). Effective Sampling and Learning for Mallows Models with Pairwise-Preference Data. *Journal of Machine Learning Research*, 15(117):3963–4009. [2](#)
- Luce, R. D. (1959). *Individual Choice Behavior*. Individual Choice Behavior. John Wiley, Oxford, England. [1](#)
- Mallows, C. L. (1957). Non-Null Ranking Models. I. *Biometrika*, 44(1/2):114–130. [2](#), [3](#)
- Manuel, A., Leonhart, R., Broman, O., and Becker, G. (2015). Consumers' perceptions and preference profiles for wood surfaces tested with pairwise comparison in Germany. *Annals of Forest Science*, 72(6):741–751. [1](#)
- Marden, J. I. (1995). *Analyzing and Modeling Rank Data*, volume 64 of *Monographs on Statistics and Applied Probability*. Chapman & Hall, Cambridge, MA, USA. [2](#), [5](#)
- Mattei, N. and Walsh, T. (2013). PrefLib: A Library for Preferences <http://www.preflib.org>. In Perny, P., Pirlot, M., and Tsoukiàs, A., editors, *Algorithmic Decision Theory*, pages 259–270, Berlin, Heidelberg. Springer. [15](#)

- Mattei, N. and Walsh, T. (2017). Chapter 15: APreflib.ORG retrospective: Lessons learned and new directions. In Endriss, U., editor, *Trends in Computational Social Choice*, pages 289–309. AI Access Foundation. **15**
- McGree, J. M., Drovandi, C. C., White, G., and Pettitt, A. N. (2016). A pseudo-marginal sequential Monte Carlo algorithm for random effects models in Bayesian sequential design. *Statistics and Computing*, 26(5):1121–1136. **22**
- Meila, M. and Bao, L. (2010). An Exponential Model for Infinite Rankings. *Journal of Machine Learning Research*, 11(113):3481–3518. **2**
- Mendes, E. F., Carter, C. K., Gunawan, D., and Kohn, R. (2020). A flexible particle Markov chain Monte Carlo method. *Statistics and Computing*, 30(4):783–798. **6, 11**
- Mersmann, O. (2023). *Microbenchmark: Accurate Timing Functions*. **16**
- Mukherjee, S. (2016). Estimation in exponential families on permutations. *The Annals of Statistics*, 44(2):853–875. **5**
- Murray, L. M., Lee, A., and Jacob, P. E. (2016). Parallel Resampling in the Particle Filter. *Journal of Computational and Graphical Statistics*, 25(3):789–805. **10**
- Naesseth, C. A., Lindsten, F., and Schön, T. B. (2019). Elements of Sequential Monte Carlo. *Foundations and Trends® in Machine Learning*, 12(3):307–392. **6, 10**
- Nicholls, G. K., Lee, J. E., Karn, N., Johnson, D., Huang, R., and Muir-Watt, A. (2022). Bayesian inference for partial orders from random linear extensions: Power relations from 12th Century Royal Acta. <https://arxiv.org/abs/2212.05524v2>. **1**
- O’Neill, J. (2013). OpenSTV. **15**
- Ono, A. and Nakano, S.-i. (2005). Constant Time Generation of Linear Extensions. In Liśkiewicz, M. and Reischuk, R., editors, *Fundamentals of Computation Theory*, pages 445–453, Berlin, Heidelberg. Springer. **14**
- Papastamoulis, P. (2016). Label-switching: An R Package for Dealing with the Label Switching Problem in MCMC Outputs. *Journal of Statistical Software*, 69:1–24. **20**
- Pearce, M. and Erosheva, E. A. (2022). A Unified Statistical Learning Model for Rankings and Scores with Application to Grant Panel Review. *Journal of Machine Learning Research*, 23(210):1–33. **1**
- Plackett, R. L. (1975). The Analysis of Permutations. *Journal of the Royal Statistical Society Series C: Applied Statistics*, 24(2):193–202. **1**
- Pruesse, G. and Ruskey, F. (1994). Generating Linear Extensions Fast. *SIAM Journal on Computing*, 23(2):373–386. **14**
- R Core Team (2024). *R: A Language and Environment for Statistical Computing*. Vienna, Austria. **2**
- Rojas-Delgado, J., Ceberio, J., Calvo, B., and Lozano, J. A. (2022). Bayesian Performance Analysis for Algorithm Ranking Comparison. *IEEE Transactions on Evolutionary Computation*, 26(6):1281–1292. **1**

- Sanderson, C. and Curtin, R. (2016). Armadillo: A template-based C++ library for linear algebra. *Journal of Open Source Software*, 1(2):26. [15](#)
- Sloane, N. J. A. (2023). The Encyclopedia of Integer Sequences. [5](#)
- Sørensen, Ø., Crispino, M., Liu, Q., and Vitelli, V. (2020). BayesMallows: An R Package for the Bayesian Mallows Model. *The R Journal*, 12(1):324–342. [6](#)
- Spearman, C. (1904). The Proof and Measurement of Association between Two Things. *The American Journal of Psychology*, 15(1):72–101. [2](#), [5](#)
- Spearman, C. (1906). ‘Footrule’ for Measuring Correlation. *British Journal of Psychology*, 1904-1920, 2(1):89–108. [5](#)
- Stein, A. (2023). *Sequential Inference with the Mallows Model*. PhD thesis, Lancaster University. [2](#), [6](#), [8](#), [13](#), [14](#), [21](#)
- Stephens, M. (2000). Dealing with label switching in mixture models. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 62(4):795–809. [20](#)
- Thurstone, L. L. (1927). A law of comparative judgment. *Psychological Review*, 34(4):273–286. [1](#)
- Vaughan, D. and Dancho, M. (2021). *furrr: Apply Mapping Functions in Parallel Using Futures*. [15](#)
- Vitelli, V., Fleischer, T., Ankill, J., Arjas, E., Frigessi, A., Kristensen, V. N., and Zucknick, M. (2023). Transcriptomic pan-cancer analysis using rank-based Bayesian inference. *Molecular Oncology*, 17(4):548–563. [1](#)
- Vitelli, V., Sørensen, Ø., Crispino, M., Frigessi, A., and Arjas, E. (2017). Probabilistic preference learning with the mallows rank model. *The Journal of Machine Learning Research*, 18(1):5796–5844. [2](#), [3](#), [4](#), [5](#), [11](#), [13](#), [14](#), [16](#), [17](#), [20](#)
- Whiteley, N. (2010). Discussion on particle Markov chain Monte Carlo methods. *Journal of the Royal Statistical Society: Series B*, 72(3):306–307. [21](#)
- Wickham, H., Averick, M., Bryan, J., Chang, W., McGowan, L. D., François, R., Grolemund, G., Hayes, A., Henry, L., Hester, J., Kuhn, M., Pedersen, T. L., Miller, E., Bache, S. M., Müller, K., Ooms, J., Robinson, D., Seidel, D. P., Spinu, V., Takahashi, K., Vaughan, D., Wilke, C., Woo, K., and Yutani, H. (2019). Welcome to the Tidyverse. *Journal of Open Source Software*, 4(43):1686. [15](#)
- Wood, S. N. (2003). Thin Plate Regression Splines. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 65(1):95–114. [19](#)
- Wood, S. N. (2017). *Generalized Additive Models: An Introduction with R*. Chapman and Hall/CRC, 2 edition. [19](#)
- Yi, I. (2021). Which Firms Require More Governance? Evidence from Mutual Funds’ Revealed Preferences. [1](#)
- Yu, P. L. H., Gu, J., and Xu, H. (2019). Analysis of ranking data. *WIREs Computational Statistics*, 11(6):e1483. [2](#)