Angles and trigonometry

Teo Banica

Department of Mathematics, University of Cergy-Pontoise, F-95000 Cergy-Pontoise, France. teo.banica@gmail.com

2010 *Mathematics Subject Classification.* 01A20

Key words and phrases. Angles, Trigonometry

Abstract. This is an introduction to plane geometry, angles and trigonometry, starting from zero or almost, meaning basic knowledge of numbers and fractions, and with focus on the standard applications to science and engineering questions. We provide as well an introduction to space geometry, and to advanced trigonometry too.

Preface

Measuring angles is an art, mastered by artists, as well as craftsmen, scientists and engineers, requiring you to know quite a deal of advanced mathematics, that you can hopefully learn from this book. But, before anything, why measuring angles?

Leaving arts aside, where drawing obviously requires some good knowledge of angles and perspective, unless of course you are interested in doing some low-skill work, and sell that as modern art, angles appear naturally in any question related to building, or understanding all sorts of objects, devices and phenomena, typically at big scales.

Let us take for instance, talking big scales, the question of understanding the movements of the Sun, Moon, other planets, and stars, around our Earth. With this being not that philosophical as a question as it might seem, because when sailing at sea, or even walking on unknown land, the Sun, Moon and so on can be very useful in showing you the way. Well, in relation with this, with measuring distances being barred by the big scale of our objects, you are left with observing angles, and then hopefully produce from these angles, via some tricky math computations, the direction that you need.

So, this was for the main principle of angles and trigonometry, big things can only be observed, and used, via angles. As for the applications of this principle, no need of course to go to the astronomical scales evoked above, these abound in various big scale questions from real life, and engineering. Measuring land, or even smaller things, like trees, or building various things, such as bridges, roads, big houses and so on, all this will lead you into angles and trigonometry, exactly as our ship capitain above.

As a concrete illustration, you certainly know about that amazing pyramids built by the ancient Egyptians. Well, that pyramids were built by using an advanced knowledge of trigonometry, available at that time, and which dissapeared in the present modern ages. Or at least this is how one hypothesis about the pyramids goes, and looking around, at the trigonometry knowledge of my mathematics and engineering students, I am pretty much convinced that this is indeed the true explanation for the pyramids question.

Getting now to the present book, this will be an introduction to all this, plane geometry, angles and trigonometry, starting from zero or almost, meaning basic knowledge of numbers and fractions, and with focus on the standard applications to basic science and engineering questions, along the lines evoked above. We will provide as well a brief introduction to angles and geometry in space, and to advanced trigonometry too.

More in detail now, the book is organized in 4 parts, with Part I dealing with plane geometry and angles, starting from zero, then Part II dealing with coordinates and basic trigonometry, Part III dealing with advanced trigonometry, using tools from analysis, and finally Part IV dealing with geometry and angles in three dimensions.

Many thanks to my math professors, and now that I am a professor myself, to my students. Thanks as well to my cats, for their teachings regarding the angle of attack, which is a more advanced notion, to be discussed too in this book, at the end.

Cergy, January 2025
Teo Banica

Contents

Part I

Geometry, angles

Sometimes I feel so happy
Sometimes I feel so sad
Sometimes I feel so happy
But mostly you just make me mad

CHAPTER 1

Parallel lines

1a. Parallel lines

Welcome to plane geometry. At the beginner level, which is ours for the moment, this is a story of points and lines. Here is a basic observation, to start with, and we will call this "axiom" instead of "theorem", as the statements which are true and useful are usually called, in mathematics, for reasons that will become clear in a moment:

AXIOM 1.1. *Any two distinct points $P \neq Q$ determine a line, denoted PQ.*

Obviously, our axiom holds, and looks like something very useful. Need to draw anything, for various engineering purposes, at your job, or in your garage? The rule will be your main weapon, used exactly as in Axiom 1.1, that is, put the rule on the points $P \neq Q$ that your line must unite, and then draw that line PQ. Actually, in relation with this, we are rather used in practice to draw segments PQ. But in theory, meaning some sort of idealized practice, will having that segment extended to infinity hurt? Certainly not, so this is why our lines PQ in mathematics will be infinite, as above.

Getting now to point, as already announced, why is Axiom 1.1 an axiom, instead of being a theorem? You would probably argue here that this theorem can be proved by using a rule, as indicated above. However, and with my apologies for this, although rock-solid as a scientific proof, this rule thing does not stand as a mathematical proof. This is how things are, you will have to trust me here. And for further making my case, let me mention that my theoretical physics friends agree with me, on the grounds that, when looking with a good microscope at your rule, that rule is certainly bent.

Excuse me, but cat is here, meowing something. So, what is is, cat?

CAT 1.2. *In fact, spacetime itself is bent.*

Okay, thanks cat, so looks like we have multiple problems with the "rule proof" of Axiom 1.1, so that definitely does not qualify as a proof. And so Axiom 1.1 will be indeed an axiom, that is, a true and useful mathematical statement, coming without proof.

Getting now to more discussion, around Axiom 1.1, an interesting question appears in connection with our assumption there $P \neq Q$. Indeed, given a point Q in the plane, we can come up with a sequence of points $P_n \to Q$ vertically, and in this case the lines P_nQ

will all coincide with the vertical at Q. But we can then formally say that the $n \to \infty$ limit of these lines, which makes sense to be denoted QQ, is also the vertical at Q.

However, is this a good idea, or not. The point indeed is that, when doing exactly the same trick with a series of points $P_n \to Q$ horizontally, we will obtain in this way, as our limiting line QQ, the horizontal at Q. Which does not sound very good, but since we seem however to have some sort of valuable idea here, let us formulate:

JOB 1.3. *Develop later some kind of analysis theory, generalizing plane geometry, where lines of type QQ make sense too, say as some sort of tangents.*

As a further comment now, still on Axiom 1.1, it is of course understood there that the points $P \neq Q$ appearing there, and the line PQ uniting them, lie in the given plane that we are interested in, in this Part I of the present book. However, Axiom 1.1 obviously holds too in space, and most likely, in higher dimensional spaces too.

So, the question which appears now is, on which type of spaces does Axiom 1.1 hold? And this is a quite interesting question, because if we take a sphere for instance, any two points $P \neq Q$ can be certainly united by a segment, which is by definition the shortest segment, on the sphere, uniting them. And, if we prolong this segment, in the obvious way, what we get is a circle uniting P, Q, that we can call line, and denote P, Q.

However, not so quick. There is in fact a bug with this, because if we take P to be the North Pole, and Q to be the South Pole, any meridian on the globe will do, as PQ. So, as a conclusion, Axiom 1.1 does not really hold on a sphere, but not by much.

Anyway, as before, we seem to have an idea here, so let us formulate:

JOB 1.4. *Develop later some kind of advanced geometry theory, generalizing plane geometry, where certain lines PQ can take multiple values.*

And with this, done I guess with the discussion regarding Axiom 1.1, I can only presume that you got as tired of reading this, as I got tired of writing it. Well, this is how things are, geometry is no easy business, and there are certainly plenty of things to be done, and what we will be doing here, based on Axiom 1.1, will be just a beginning.

Excuse me, but cat is meowing again. So, what is it cat, and for God's sake, in the hope that this is not in connection with Axiom 1.1. Please have mercy.

CAT 1.5. *What about $PQ = \lambda P + (1 - \lambda)Q$ proving your axiom.*

Okay, thanks cat, but I was already having this in mind, for chapter 5 below. So, Axiom 1.1 remains an axiom, please everyone disagreeing with this get out of my math class, and enjoy the sunshine outside. And well, we will see later, in chapter 5 below, how cats and physicists can prove Axiom 1.1, or at least, what their claims are.

Moving ahead now, here is an interesting observation about lines and points in the plane, coming somehow as a complement to Axiom 1.1:

OBSERVATION 1.6. *Any two distinct lines $K \neq L$ determine a point, $P = K \cap L$, unless these two lines are parallel, $K||L$.*

So, what do we have here, axiom, theorem, or something else? Not very clear, but on the bottom line, this is something which is certainly true, useful, and provable as before, with a rule. Just carefully draw K, L, and you will certainly get upon $P = K \cap L$.

However, in contrast to Axiom 1.1, there is a bit of a bug with our statement, because we do not know yet, mathematically, what parallel lines means. So, let us formulate:

DEFINITION 1.7. *We say that two lines are parallel, $K||L$, when they do not cross,*

$$K \cap L = \emptyset$$

or when they coincide, $K = L$. Otherwise, we say that K, L cross, and write $K \not\parallel L$.

Here we have tricked a bit, by agreeing to call parallel the pairs of identical lines too, and this for simplifying most of our mathematics, in what follows, trust me here.

As a first remark, with this definition in hand, Observation 1.6 makes now sense, as a formal mathematical statement, and skipping some discussion here, or rather leaving it as an exercise, for reasons which are somewhat clear, we will call this axiom:

AXIOM 1.8. *Any two crossing lines $K \not\parallel L$ determine a point, $P = K \cap L$.*

Very good, and now with Axiom 1.1 and Axiom 1.8 in hand, we are potentially ready for doing some geometry. However, this is not exactly true, and we will need as well:

AXIOM 1.9. *Given a point not lying on a line, $P \notin L$, we can draw through P a unique parallel to L. That is, we can find a line K satisfying $P \in K$, $K||L$.*

As before, we will leave as an exercise further meditating on all this.

Ready for some math? Here we go, and many things can be said here, especially about parallel lines, which are the main objects of basic geometry, as for instance:

THEOREM 1.10 (Thales). *Proportions are kept, along parallel lines.*

PROOF. This is indeed something very standard. □

Importantly, many things can be done with the parallel lines, with a suitably drawn such line hopefully solving, by some kind of miracle, your plane geometry problem.

We will see more illustrations for this general principle in the next chapter.

Switching topics, but still in relation with the parallel lines, that we constantly met in the above, you might have heard or not of projective geometry. In case you didn't yet, the general principle is that "this is the wonderland where parallel lines cross".

Which might sound a bit crazy, and not very realistic, but take a picture of some railroad tracks, and look at that picture. Do that parallel railroad tracks cross, on the picture? Sure they do. So, we are certainly not into abstractions here. QED.

Mathematically now, here are some axioms, to start with:

DEFINITION 1.11. *A projective space is a space consisting of points and lines, subject to the following conditions:*

1. *Each 2 points determine a line.*
2. *Each 2 lines cross, on a point.*

As a basic example we have the usual projective plane $P^2_{\mathbb{R}}$, which is best seen as being the space of lines in $\mathbb{R}^3$ passing through the origin. To be more precise, let us call each of these lines in $\mathbb{R}^3$ passing through the origin a "point" of $P^2_{\mathbb{R}}$, and let us also call each plane in $\mathbb{R}^3$ passing through the origin a "line" of $P^2_{\mathbb{R}}$. Now observe the following:

(1) Each 2 points determine a line. Indeed, 2 points in our sense means 2 lines in $\mathbb{R}^3$ passing through the origin, and these 2 lines obviously determine a plane in $\mathbb{R}^3$ passing through the origin, namely the plane they belong to, which is a line in our sense.

(2) Each 2 lines cross, on a point. Indeed, 2 lines in our sense means 2 planes in $\mathbb{R}^3$ passing through the origin, and these 2 planes obviously determine a line in $\mathbb{R}^3$ passing through the origin, namely their intersection, which is a point in our sense.

Thus, what we have is a projective space in the sense of Definition 1.11. More generally now, we have the following construction, in arbitrary dimensions:

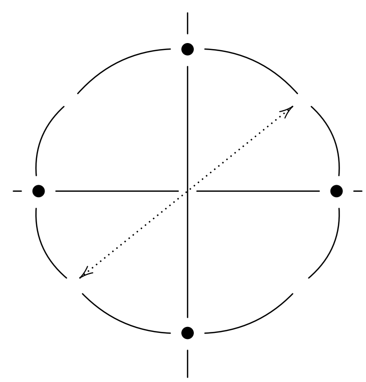
THEOREM 1.12. *We can define the projective space $P^{N-1}_{\mathbb{R}}$ as being the space of lines in $\mathbb{R}^N$ passing through the origin, and in small dimensions:*

1. *$P^1_{\mathbb{R}}$ is the usual circle.*
2. *$P^2_{\mathbb{R}}$ is some sort of twisted sphere.*

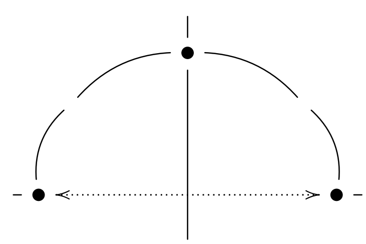
PROOF. We have several assertions here, with all this being of course a bit informal, and self-explanatory, the idea and some further details being as follows:

(1) To start with, the fact that the space $P^{N-1}_{\mathbb{R}}$ constructed in the statement is indeed a projective space in the sense of Definition 1.11 follows from definitions, exactly as in the discussion preceding the statement, regarding the case $N = 3$.

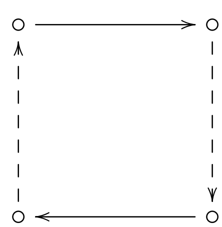
(2) At $N=2$ now, a line in $\mathbb{R}^2$ passing through the origin corresponds to 2 opposite points on the unit circle $\mathbb{T}\subset\mathbb{R}^2$, according to the following scheme:



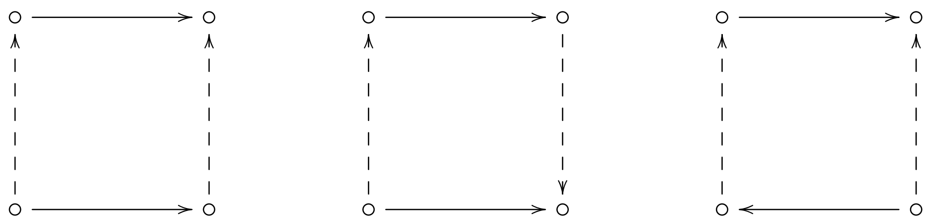
Thus, $P^1_\mathbb{R}$ corresponds to the upper semicircle of $\mathbb{T}$, with the endpoints identified, and so we obtain a circle, $P^1_\mathbb{R}=\mathbb{T}$, according to the following scheme:



(3) At $N=3$, the space $P^2_\mathbb{R}$ corresponds to the upper hemisphere of the sphere $S^2_\mathbb{R}\subset\mathbb{R}^3$, with the points on the equator identified via $x=-x$. Topologically speaking, we can deform if we want the hemisphere into a square, with the equator becoming the boundary of this square, and in this picture, the $x=-x$ identification corresponds to a "identify opposite edges, with opposite orientations" folding method for the square:



(4) Thus, we have our space. In order to understand now what this beast is, let us look first at the other 3 possible methods of folding the square, which are as follows:



Regarding the first space, the one on the left, things here are quite simple. Indeed, when identifying the solid edges we get a cylinder, and then when further identifying the dotted edges, what we get is some sort of closed cylinder, which is a torus.

(5) Regarding the second space, the one in the middle, things here are more tricky. Indeed, when identifying the solid edges we get again a cylinder, but then when further identifying the dotted edges, we obtain some sort of "impossible" closed cylinder, called Klein bottle. This Klein bottle obviously cannot be drawn in 3 dimensions, but with a bit of imagination, you can see it, in its full splendor, in 4 dimensions.

(6) Finally, regarding the third space, the one on the right, we know by symmetry that this must be the Klein bottle too. But we can see this as well via our standard folding method, namely identifying solid edges first, and dotted edges afterwards. Indeed, we first obtain in this way a Möbius strip, and then, well, the Klein bottle.

(7) With these preliminaries made, and getting back now to the projective space $P^2_{\mathbb{R}}$, we can see that this is something more complicated, of the same type, reminding the torus and the Klein bottle. So, we will call it "sort of twisted sphere", as in the statement, and exercise for you to figure out how this beast looks like, in 4 dimensions. $\square$

All this is quite exciting, and reminds childhood and primary school, but is however a bit tiring for our neurons, guess that is pure mathematics. It is possible to come up with some explicit formulae for the embedding $P^2_{\mathbb{R}} \subset \mathbb{R}^4$, which are useful in practice, allowing us to do some analysis over $P^2_{\mathbb{R}}$, and we will leave this as an instructive exercise.

All this is very nice, but we will pause our study here, because we still have many other things to say. Here is an interesting notion, that we can use for geometry:

Definition 1.13. *A field is a set F with a sum operation $+$ and a product operation $\times$, subject to the following conditions:*

(1) *$a+b=b+a$, $a+(b+c)=(a+b)+c$, there exists $0 \in F$ such that $a+0=0$, and any $a \in F$ has an inverse $-a \in F$, satisfying $a+(-a)=0$.*
(2) *$ab=ba$, $a(bc)=(ab)c$, there exists $1 \in F$ such that $a1=a$, and any $a \neq 0$ has a multiplicative inverse $a^{-1} \in F$, satisfying $aa^{-1}=1$.*
(3) *The sum and product are compatible via $a(b+c)=ab+ac$.*

The simplest possible field seems to be $\mathbb{Q}$. However, this is not exactly true, because, by a strange twist of fate, the numbers $0,1$, whose presence in a field is mandatory, $0,1\in F$, can form themselves a field, with addition as follows:

$$1+1=0$$

Let us summarize this finding, along with a bit more, obtained by suitably replacing our 2, used for addition, with an arbitrary prime number p, as follows:

THEOREM 1.14. *The following happen:*

(1) *$\mathbb{Q}$ is the simplest field having the property $1+\ldots+1\neq 0$, in the sense that any field F having this property must contain it, $\mathbb{Q}\subset F$.*
(2) *The property $1+\ldots+1\neq 0$ can hold or not, and if not, the smallest number of terms needed for having $1+\ldots+1=0$ is a certain prime number p.*
(3) *$\mathbb{F}_p=\{0,1,\ldots,p-1\}$, with p prime, is the simplest field having the property $1+\ldots+1=0$, with p terms, in the sense that this implies $\mathbb{F}_p\subset F$.*

PROOF. All this is basic number theory, the idea being as follows:

(1) This is clear, because $1+\ldots+1\neq 0$ tells us that we have an embedding $\mathbb{N}\subset F$, and then by taking inverses with respect to $+$ and $\times$ we obtain $\mathbb{Q}\subset F$.

(2) Again, this is clear, because assuming $1+\ldots+1=0$, with $p=ab$ terms, chosen minimal, we would have a formula as follows, which is a contradiction:

$$(\underbrace{1+\ldots+1}_{a\ terms})(\underbrace{1+\ldots+1}_{b\ terms})=0$$

(3) This follows a bit as in (1), with the copy $\mathbb{F}_p\subset F$ consisting by definition of the various sums of type $1+\ldots+1$, which must cycle modulo p, as shown by (2). □

Getting now to geometry over finite fields, we have here:

THEOREM 1.15. *Given a field F, we can talk about the projective space P^{N-1}_F, as being the space of lines in F^N passing through the origin. At $N=3$ we have*

$$|P^2_F|=q^2+q+1$$

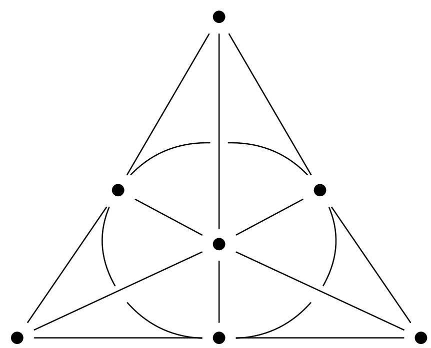
where $q=|F|$, in the case where our field F is finite.

PROOF. This is indeed clear from definitions, with the cardinality coming from:

$$|P^2_F|=\frac{|F^3-\{0\}|}{|F-\{0\}|}=\frac{q^3-1}{q-1}=q^2+q+1$$

Thus, we are led to the conclusions in the statement. □

As an example, let us see what happens for the simplest finite field that we know, namely $F = \mathbb{Z}_2$. Here our projective plane, having $4+2+1=7$ points, and 7 lines, is a famous combinatorial object, called Fano plane, which is depicted as follows:



Here the circle in the middle is by definition a line, and with this convention, the basic axioms in Definition 1.11 are satisfied, in the sense that any two points determine a line, and any two lines determine a point. And isn't this beautiful.

1b.

1c.

1d.

1e. Exercises

Exercises:

Exercise 1.16.

Exercise 1.17.

Exercise 1.18.

Exercise 1.19.

Exercise 1.20.

Exercise 1.21.

Exercise 1.22.

Exercise 1.23.

Bonus exercise.

CHAPTER 2

Triangles

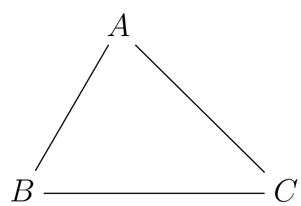
2a. Triangles

Welcome to geometry. It all started with triangles, drawn on sand. In order to get started, with some basic plane geometry, we first have the following key result:

THEOREM 2.1. *Given a triangle ABC, the following happen:*

(1) *The angle bisectors cross, at a point called incenter.*
(2) *The medians cross, at a point called barycenter.*
(3) *The perpendicular bisectors cross, at a point called circumcenter.*
(4) *The altitudes cross, at a point called orthocenter.*

PROOF. Let us first draw our triangle, with this being always the first thing to be done in geometry, draw a picture, and then thinking and computations afterwards:



Allowing us the freedom to play with some tricks, as advanced mathematicians, both students and professors, are allowed to, here is how the proof goes:

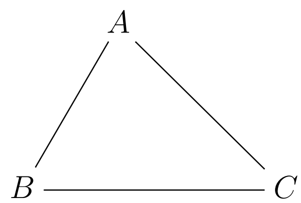
(1) Come with a small circle, inside ABC, and then inflate it, as to touch all 3 edges. The center of the circle will be then at equal distance from all 3 edges, so it will lie on all 3 angle bisectors. Thus, we have constructed the incenter, as required.

(2) This requires different techniques. Let us call $A,B,C\in\mathbb{C}$ the coordinates of A,B,C, and consider the average $P=(A+B+C)/3$. We have then:

$$P=\frac{1}{3}\cdot A+\frac{2}{3}\cdot\frac{B+C}{2}$$

Thus P lies on the median emanating from A, and a similar argument shows that P lies as well on the medians emanating from B,C. Thus, we have our barycenter.

(3) Time to draw a new triangle, for clarity, since we are now on page two:



Regarding our problem, we can use the same method as for (1). Indeed, come with a big circle, containing ABC, and then deflate it, as for it to pass through A, B, C. The center of the circle will be then at equal distance from all 3 vertices, so it will lie on all 3 perpendicular bisectors. Thus, we have constructed the circumcenter, as required.

(4) This is tougher, and I must admit that, when writing this book, I first struggled a bit with this, then ended looking it up on the internet. So, here is the trick. Draw a parallel to BC at A, and similarly, parallels to AB and AC at C and B. You will get in this way a bigger triangle, upside-down, $A'B'C'$. But then, the circumcenter of $A'B'C'$, that we know to exist from (3), will be the orthocenter of ABC, as desired. □

Along the same lines, but at a more advanced level now, we have:

Fact 2.2. *Besides the above 4 centers, many more remarkable points can be associated to a triangle ABC, and most of these lie on a line, called Euler line of ABC.*

And exercise for you of course to remember or figure out how all this works, both statement and proof. As bonus exercise, learn about the nine-point circle too.

Getting now to what we wanted to talk about in this book, angles and trigonometry, we can certain talk about angles, in the obvious way, by using triangles:

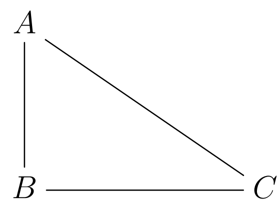
Fact 2.3. *We can talk about the angle between two crossing lines, and have some basic theory for the angles going, by using triangles.*

You might wonder of course what the values of these angles should be, say as real numbers. This is something quite tricky, that will take us some time to understand.

Getting started now with our study of angles, as a continuation of Fact 2.3, let us first talk about the simplest angle of them all, which is the right angle, denoted $90°$.

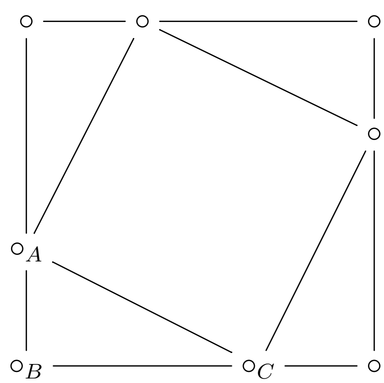
Many interesting things can be said about this right angle $90°$, in particular with:

THEOREM 2.4 (Pythagoras). *In a right triangle ABC,*



we have $AB^2 + BC^2 = AC^2$.

PROOF. This comes from the following picture, consisting of two squares, and four triangles which are identical to ABC, as indicated:



Indeed, let us compute the area S of the outer square. This can be done in two ways. First, since the side of this square is $AB + BC$, we obtain:

$$\begin{aligned} S &= (AB + BC)^2 \\ &= AB^2 + BC^2 + 2 \times AB \times BC \end{aligned}$$

On the other hand, the outer square is made of the smaller square, having side AC, and of four identical right triangles, having sizes AB, BC. Thus:

$$\begin{aligned} S &= AC^2 + 4 \times \frac{AB \times BC}{2} \\ &= AC^2 + 2 \times AB \times BC \end{aligned}$$

Thus, we are led to the conclusion in the statement. $\square$

2b.

2c.

2d.

2e. Exercises

Exercises:

Exercise 2.5.

Exercise 2.6.

Exercise 2.7.

Exercise 2.8.

Exercise 2.9.

Exercise 2.10.

Exercise 2.11.

Exercise 2.12.

Bonus exercise.

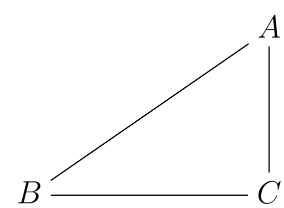
CHAPTER 3

Sine, cosine

3a. Sine, cosine

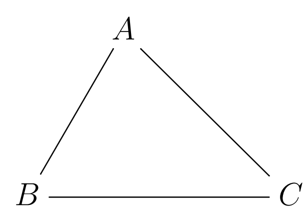
Now that we know about angles, and about Pythagoras' theorem too, it is tempting at this point to start talking about trigonometry. Let us begin with:

Definition 3.1. *We can talk about sines and cosines, by using a right triangle*



in the obvious way, and ideally, by assuming $AB = 1$.

Many interesting things can be said here, for instance regarding the sines and cosines of the angles of a triangle, which can be taken arbitrary, or of various special types:



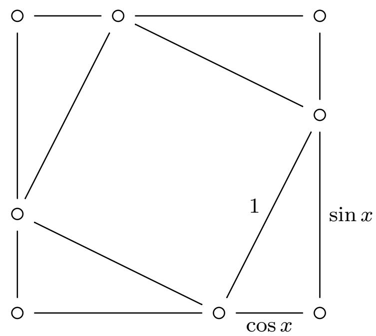
Getting now to more advanced theory, we first have:

Theorem 3.2. *The sines and cosines are subject to the formula*

$$\sin^2 x + \cos^2 x = 1$$

coming from Pythagoras' theorem.

PROOF. This is something which is certainly true, and for pure mathematical pleasure, let us reproduce the picture leading to Phythagoras, in the trigonometric setting:



When computing the area of the outer square, we obtain:

$$(\sin x + \cos x)^2 = 1 + 4 \times \frac{\sin x \cos x}{2}$$

Now when expanding we obtain $\sin^2 x + \cos^2 x = 1$, as claimed. □

It is possible to say many more things about angles and $\sin x$, $\cos x$, and also talk about some supplementary quantities, such as the tangent:

$$\tan x = \frac{\sin x}{\cos x}$$

But more on this, such as various analytic aspects, later in this book, once we will have some appropriate tools, beyond basic geometry, in order to discuss this.

Still at the level of the basics, we have the following result:

THEOREM 3.3. *The sines and cosines of sums are given by*

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

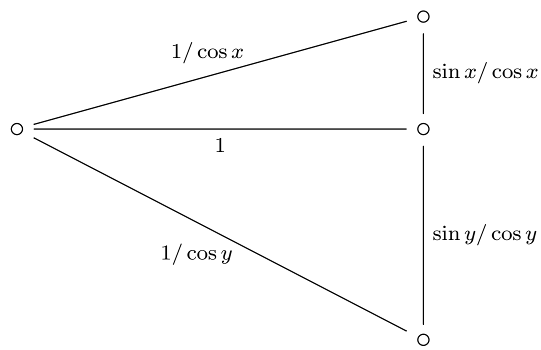
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

and these formulae give a formula for $\tan(x + y)$ *too.*

PROOF. This is something quite tricky, using the same idea as in the proof of Pythagoras' theorem, that is, computing certain areas, the idea being as follows:

(1) Let us first establish the formula for the sines. In order to do so, consider the following picture, consisting of a length 1 line segment, with angles x, y drawn on each

side, and with everything being completed, and lengths computed, as indicated:



Now let us compute the area of the big triangle, or rather the double of that area. We can do this in two ways, either directly, with a formula involving $\sin(x+y)$, or by using the two small triangles, involving functions of x, y. We obtain in this way:

$$\frac{1}{\cos x}\cdot\frac{1}{\cos y}\cdot\sin(x+y)=\frac{\sin x}{\cos x}\cdot 1+\frac{\sin y}{\cos y}\cdot 1$$

But this gives the formula for $\sin(x+y)$ from the statement.

(2) Moving ahead, no need of new tricks for cosines, because by using the formula for $\sin(x+y)$ we can deduce a formula for $\cos(x+y)$, as follows:

$$\begin{aligned}\cos(x+y) &= \sin\left(\frac{\pi}{2}-x-y\right)\\ &= \sin\left[\left(\frac{\pi}{2}-x\right)+(-y)\right]\\ &= \sin\left(\frac{\pi}{2}-x\right)\cos(-y)+\cos\left(\frac{\pi}{2}-x\right)\sin(-y)\\ &= \cos x\cos y-\sin x\sin y\end{aligned}$$

(3) Finally, in what regards the tangents, we have, according to the above:

$$\tan(x+y)=\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}$$

Thus, we are led to the conclusions in the statement. □

Observe in particular that with $x = y$ we obtain some interesting formulae for the duplication of angles. We will be back to such questions later, with better tools.

3b.

3c.

3d.

3e. Exercises

Exercises:

Exercise 3.4.

Exercise 3.5.

Exercise 3.6.

Exercise 3.7.

Exercise 3.8.

Exercise 3.9.

Exercise 3.10.

Exercise 3.11.

Bonus exercise.

CHAPTER 4

Circle, angles

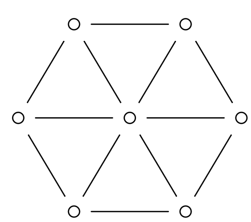
4a. Circle, angles

Let us get now into a more advanced study of the angles. For this purpose, the best is to talk first about circles, and the number π. And here, to start with, we have:

THEOREM 4.1. *The following two definitions of π are equivalent:*

(1) *The length of the unit circle is $L = 2\pi$.*
(2) *The area of the unit disk is $A = \pi$.*

PROOF. In order to prove this theorem let us cut the unit disk as a pizza, into N slices, and forgetting about gastronomy, leave aside the rounded parts:



The area to be eaten can be then computed as follows, where H is the height of the slices, S is the length of their sides, and $P = NS$ is the total length of the sides:

$$\begin{aligned} A &= N \times \frac{HS}{2} \\ &= \frac{HP}{2} \\ &\simeq \frac{1 \times L}{2} \end{aligned}$$

Thus, with $N \to \infty$ we obtain that we have $A = L/2$, as desired. □

In what regards now the precise value of π, the above picture at $N = 6$ shows that we have $\pi > 3$, but not by much. The precise figure is $\pi = 3.14159\ldots$, but we will come back to this later, once we will have appropriate tools for dealing with such questions. It is also possible to prove that π is irrational, $\pi \notin \mathbb{Q}$, but this is not trivial either.

4b.

4c.

4d.

4e. Exercises

Exercises:

Exercise 4.2.

Exercise 4.3.

Exercise 4.4.

Exercise 4.5.

Exercise 4.6.

Exercise 4.7.

Exercise 4.8.

Exercise 4.9.

Bonus exercise.

Part II

Basic trigonometry

In the clearing stands a boxer
And a fighter by his trade
And he carries the reminders
Of every glove that laid him down

CHAPTER 5

Affine coordinates

5a. Affine coordinates

Looking up, to the sky, the first thing that you see is the Sun, seemingly moving around the Earth on a circle, but a more careful study reveals that this circle is rather a deformed circle, called ellipsis. As for the other stars and planets, these have all sort of weird trajectories, but a more careful study reveals that, with due attention to what the best "center" is, replacing our Earth, the trajectories are often ellipses:

(1) Indeed, this applies to all the planets in our Solar System, which move around the biggest object in the system, which is by far the Sun, on ellipses.

(2) The same trick applies to the trajectories of various distant stars, the rule being always the same, "small moves around big, on an ellipsis".

(3) However, there are counterexamples too, such as asteroids reaching our Solar system, but then travelling outwards, never to be seen again.

Summarizing, modulo some annoying asteroids that we will leave for later, we are led in this way to ellipses, and their mathematics. And good news, a full theory of ellipses is available, and this since the ancient Greeks, whose main findings were as follows:

Theorem 5.1. *The ellipses, taken centered at the origin* 0*, and squarely oriented with respect to* Oxy*, can be defined in* 4 *possible ways, as follows:*

(1) *As the curves given by an equation as follows, with* $a,b>0$*:*

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

(2) *Or given by an equation as follows, with* $q>0$*,* $p=-q$*, and* $l\in(0,2q)$*:*

$$d(z,p)+d(z,q)=l$$

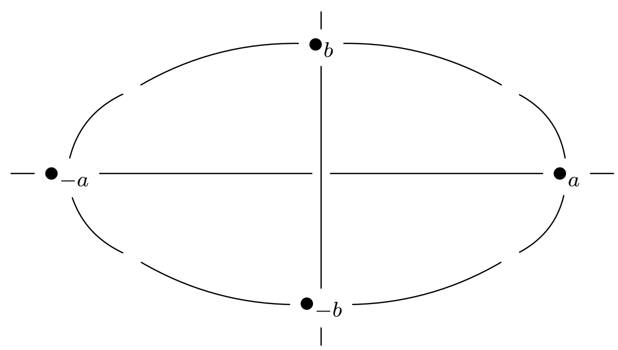
(3) *As the curves appearing when drawing a circle, from various perspectives:*

$$\bigcirc\quad\to\quad ?$$

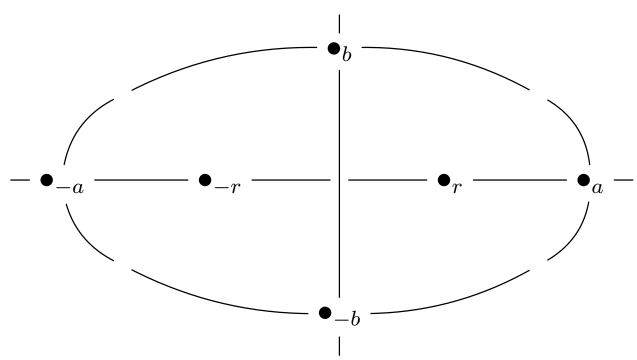
(4) *As the closed non-degenerate curves appearing by cutting a cone with a plane.*

PROOF. This might look a bit confusing, and you might say, what exactly is to be proved here. Good point, and in answer, what is to be proved is that the above constructions (1-4) give rise to the same class of curves. And this can be done as follows:

(1) To start with, let us draw a picture from what comes out of (1), which will be our main definition for the ellipses, in what follows. Here that is, making it clear what the parameters $a, b > 0$ stand for, with $2a \times 2b$ being the gift box size for our ellipsis:



(2) Let us prove now that such an ellipsis has two focal points, as stated in (2). We must look for a number $r > 0$, and a number $l > 0$, such that our ellipsis appears as $d(z, p) + d(z, q) = l$, with $p = (0, -r)$ and $q = (0, r)$, according to the following picture:



(3) Let us first compute these numbers $r, l > 0$. Assuming that our result holds indeed as stated, by taking $z = (0, a)$, we see that the length l is:

$$l = (a - r) + (a + r) = 2a$$

As for the parameter r, by taking $z = (b, 0)$, we conclude that we must have:

$$2\sqrt{b^2 + r^2} = 2a \implies r = \sqrt{a^2 - b^2}$$

(4) With these observations made, let us prove now the result. Given $l,r>0$, and setting $p=(0,-r)$ and $q=(0,r)$, we have the following computation, with $z=(x,y)$:

$$\begin{aligned}
& d(z,p)+d(z,q)=l\\
\iff\ & \sqrt{(x+r)^2+y^2}+\sqrt{(x-r)^2+y^2}=l\\
\iff\ & \sqrt{(x+r)^2+y^2}=l-\sqrt{(x-r)^2+y^2}\\
\iff\ & (x+r)^2+y^2=(x-r)^2+y^2+l^2-2l\sqrt{(x-r)^2+y^2}\\
\iff\ & 2l\sqrt{(x-r)^2+y^2}=l^2-4xr\\
\iff\ & 4l^2(x^2+r^2-2xr+y^2)=l^4+16x^2r^2-8l^2xr\\
\iff\ & 4l^2x^2+4l^2r^2+4l^2y^2=l^4+16x^2r^2\\
\iff\ & (4x^2-l^2)(4r^2-l^2)=4l^2y^2
\end{aligned}$$

(5) Now observe that we can further process the equation that we found as follows:

$$\begin{aligned}
(4x^2-l^2)(4r^2-l^2)=4l^2y^2 \iff\ & \frac{4x^2-l^2}{l^2}=\frac{4y^2}{4r^2-l^2}\\
\iff\ & \frac{4x^2-l^2}{l^2}=\frac{y^2}{r^2-l^2/4}\\
\iff\ & \left(\frac{x}{2l}\right)^2-1=\left(\frac{y}{\sqrt{r^2-l^2/4}}\right)^2\\
\iff\ & \left(\frac{x}{2l}\right)^2+\left(\frac{y}{\sqrt{r^2-l^2/4}}\right)^2=1
\end{aligned}$$

(6) Thus, our result holds indeed, and with the numbers $l,r>0$ appearing, and no surprise here, via the formulae $l=2a$ and $r=\sqrt{a^2-b^2}$, found in (3) above.

(7) Getting back now to our theorem, we have two other assertions there at the end, labelled (3,4). But, thinking a bit, these assertions are in fact equivalent, and in what concerns us, we will rather focus on (4), which looks more mathematical. And in what regards this assertion (4), this can be established indeed, by doing some 3D computations, that we will leave here as an instructive exercise, for you. And with the promise that we will come back to this in a moment, with a full proof, in a more general setting. □

All this is very nice, but before getting into physics, with some explanations for the fact that planets travel indeed on ellipses, which is something that we must surely understand, before going with some further math, let us settle as well the question of wandering asteroids. Observations show that these can travel on parabolas and hyperbolas, so what we need as mathematics is a unified theory of ellipses, parabolas and hyperbolas. And fortunately, this theory exists, also since the ancient Greeks, summarized as follows:

THEOREM 5.2. *The conics, which are the algebraic curves of degree* 2 *in the plane,*

$$C = \left\{(x,y) \in \mathbb{R}^2 \Big| P(x,y) = 0\right\}$$

with $\deg P \leq 2$*, appear modulo degeneration by cutting a* 2*-sided cone with a plane, and can be classified into ellipses, parabolas and hyperbolas.*

PROOF. This follows by further building on Theorem 5.1, as follows:

(1) Let us first classify the conics up to non-degenerate linear transformations of the plane, which are by definition transformations as follows, with $\det A \neq 0$:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to A \begin{pmatrix} x \\ y \end{pmatrix}$$

Our claim is that as solutions we have the circles, parabolas, hyperbolas, along with some degenerate solutions, namely $\emptyset$, points, lines, pairs of lines, $\mathbb{R}^2$.

(2) As a first remark, it looks like we forgot precisely the ellipses, but via linear transformations these become circles, so things fine. As a second remark, all our claimed solutions can appear. Indeed, the circles, parabolas, hyperbolas can appear as follows:

$$x^2 + y^2 = 1 \quad , \quad x^2 = y \quad , \quad xy = 1$$

As for $\emptyset$, points, lines, pairs of lines, $\mathbb{R}^2$, these can appear too, as follows, and with our polynomial P chosen, whenever possible, to be of degree exactly 2:

$$x^2 = -1 \quad , \quad x^2 + y^2 = 0 \quad , \quad x^2 = 0 \quad , \quad xy = 0 \quad , \quad 0 = 0$$

Observe here that, when dealing with these degenerate cases, assuming $\deg P = 2$ instead of $\deg P \leq 2$ would only rule out $\mathbb{R}^2$ itself, which is not worth it.

(3) Getting now to the proof of our claim in (1), classification up to linear transformations, consider an arbitrary conic, written as follows, with $a, b, c, d, e, f \in \mathbb{R}$:

$$ax^2 + by^2 + cxy + dx + ey + f = 0$$

Assume first $a \neq 0$. By making a square out of ax^2, up to a linear transformation in (x, y), we can get rid of the term cxy, and we are left with:

$$ax^2 + by^2 + dx + ey + f = 0$$

In the case $b \neq 0$ we can make two obvious squares, and again up to a linear transformation in (x, y), we are left with an equation as follows:

$$x^2 \pm y^2 = k$$

In the case of positive sign, $x^2 + y^2 = k$, the solutions are the circle, when $k \geq 0$, the point, when $k = 0$, and $\emptyset$, when $k < 0$. As for the case of negative sign, $x^2 - y^2 = k$, which reads $(x - y)(x + y) = k$, here once again by linearity our equation becomes $xy = l$, which is a hyperbola when $l \neq 0$, and two lines when $l = 0$.

(4) In the case $b \neq 0$ the study is similar, with the same solutions, so we are left with the case $a = b = 0$. Here our conic is as follows, with $c, d, e, f \in \mathbb{R}$:

$$cxy + dx + ey + f = 0$$

If $c \neq 0$, by linearity our equation becomes $xy = l$, which produces a hyperbola or two lines, as explained before. As for the remaining case, $c = 0$, here our equation is:

$$dx + ey + f = 0$$

But this is generically the equation of a line, unless we are in the case $d = e = 0$, where our equation is $f = 0$, having as solutions $\emptyset$ when $f \neq 0$, and $\mathbb{R}^2$ when $f = 0$.

(5) Thus, done with the classification, up to linear transformations as in (1). But this classification leads to the classification in general too, by applying now linear transformations to the solutions that we found. So, done with this, and very good.

(6) It remains to discuss the cone cutting. By suitably choosing our coordinate axes (x, y, z), we can assume that our cone is given by an equation as follows, with $k > 0$:

$$x^2 + y^2 = kz^2$$

In order to prove the result, we must in principle intersect this cone with an arbitrary plane, which has an equation as follows, with $(a, b, c) \neq (0, 0, 0)$:

$$ax + by + cz = d$$

(7) However, before getting into computations, observe that what we want to find is a certain degree 2 equation in the above plane, for the intersection. Thus, it is convenient to change the coordinates, as for our plane to be given by the following equation:

$$z = 0$$

(8) But with this done, what we have to do is to see how the cone equation $x^2+y^2 = kz^2$ changes, under this change of coordinates, and then set $z = 0$, as to get the (x, y) equation of the intersection. But this leads, via some thinking or computations, to the conclusion that the cone equation $x^2 + y^2 = kz^2$ becomes in this way a degree 2 equation in (x, y), which can be arbitrary, and so to the final conclusion in the statement. □

Ready for some physics? We have the following result:

THEOREM 5.3. *Planets and other celestial bodies move around the Sun on conics,*

$$C = \left\{(x, y) \in \mathbb{R}^2 \Big| P(x, y) = 0\right\}$$

with $P \in \mathbb{R}[x, y]$ being of degree 2, which can be ellipses, parabolas or hyperbolas.

PROOF. This is something quite long, due to Kepler and Newton. □

5b.

5c.

5d.

5e. Exercises

Exercises:

Exercise 5.4.

Exercise 5.5.

Exercise 5.6.

Exercise 5.7.

Exercise 5.8.

Exercise 5.9.

Exercise 5.10.

Exercise 5.11.

Bonus exercise.

CHAPTER 6

Basic trigonometry

6a. Basic trigonometry

6b.

6c.

6d.

6e. Exercises

Exercises:

EXERCISE 6.1.

EXERCISE 6.2.

EXERCISE 6.3.

EXERCISE 6.4.

EXERCISE 6.5.

EXERCISE 6.6.

EXERCISE 6.7.

EXERCISE 6.8.

Bonus exercise.

CHAPTER 7

Complex numbers

7a. Complex numbers

Let us discuss now the complex numbers. There is a lot of magic here, and we will carefully explain this material. Their definition is as follows:

Definition 7.1. *The complex numbers are variables of the form*

$$x = a + ib$$

with $a, b \in \mathbb{R}$, which add in the obvious way, and multiply according to the following rule:

$$i^2 = -1$$

Each real number can be regarded as a complex number, $a = a + i \cdot 0$.

In other words, we consider variables as above, without bothering for the moment with their precise meaning. Now consider two such complex numbers:

$$x = a + ib \quad , \quad y = c + id$$

The formula for the sum is then the obvious one, as follows:

$$x + y = (a + c) + i(b + d)$$

As for the formula of the product, by using the rule $i^2 = -1$, we obtain:

$$\begin{aligned} xy &= (a + ib)(c + id) \\ &= ac + iad + ibc + i^2bd \\ &= ac + iad + ibc - bd \\ &= (ac - bd) + i(ad + bc) \end{aligned}$$

Thus, the complex numbers as introduced above are well-defined. The multiplication formula is of course quite tricky, and hard to memorize, but we will see later some alternative ways, which are more conceptual, for performing the multiplication.

The advantage of using the complex numbers comes from the fact that the equation $x^2 = 1$ has now a solution, $x = i$. In fact, this equation has two solutions, namely:

$$x = \pm i$$

This is of course very good news. More generally, we have the following result, regarding the arbitrary degree 2 equations, with real coefficients:

THEOREM 7.2. *The complex solutions of* $ax^2 + bx + c = 0$ *with* $a, b, c \in \mathbb{R}$ *are*

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with the square root of negative real numbers being defined as

$$\sqrt{-m} = \pm i\sqrt{m}$$

and with the square root of positive real numbers being the usual one.

PROOF. We can write our equation in the following way:

$$\begin{aligned} ax^2 + bx + c = 0 \quad &\iff \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \\ &\iff \quad \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0 \\ &\iff \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \\ &\iff \quad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

We will see later that any degree 2 complex equation has solutions as well, and that more generally, any polynomial equation, real or complex, has solutions. Moving ahead now, we can represent the complex numbers in the plane, in the following way:

PROPOSITION 7.3. *The complex numbers, written as usual*

$$x = a + ib$$

can be represented in the plane, according to the following identification:

$$x = \begin{pmatrix} a \\ b \end{pmatrix}$$

With this convention, the sum of complex numbers is the usual sum of vectors.

PROOF. Consider indeed two arbitrary complex numbers:

$$x = a + ib \quad , \quad y = c + id$$

Their sum is then by definition the following complex number:

$$x + y = (a + c) + i(b + d)$$

Now let us represent x,y in the plane, as in the statement:

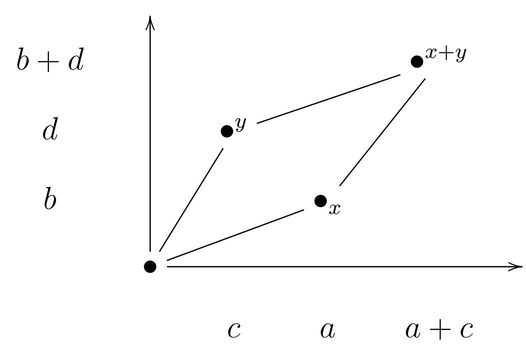
$$x=\begin{pmatrix}a\\ b\end{pmatrix}\quad,\quad y=\begin{pmatrix}c\\ d\end{pmatrix}$$

In this picture, their sum is given by the following formula:

$$x+y=\begin{pmatrix}a+c\\ b+d\end{pmatrix}$$

But this is indeed the vector corresponding to $x+y$, so we are done. $\square$

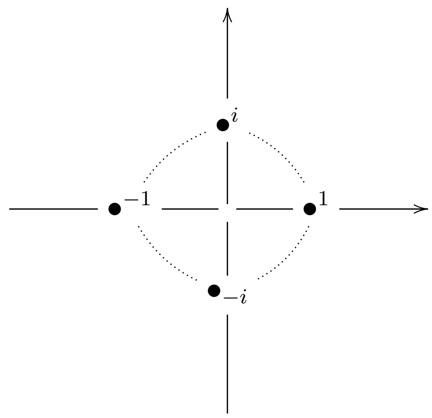
Here we have assumed that you are a bit familiar with vector calculus. If not, no problem, the idea is simply that vectors add by forming a parallelogram, as follows:



Observe that in our geometric picture from Proposition 7.3, the real numbers correspond to the numbers on the Ox axis. As for the purely imaginary numbers, these lie on the Oy axis, with the number i itself being given by the following formula:

$$i=\begin{pmatrix}0\\ 1\end{pmatrix}$$

As an illustration for this, let us record now a basic picture, with some key complex numbers, namely $1, i, -1, -i$, represented according to our conventions:



You might perhaps wonder why I chose to draw that circle, connecting the numbers $1, i, -1, -i$, which does not look very useful. More on this in a moment, the idea being that that circle can be immensely useful, and coming in advance, some advice:

ADVICE 7.4. *When drawing complex numbers, always begin with the coordinate axes Ox, Oy, and with a copy of the unit circle.*

We have so far a quite good understanding of their complex numbers, and their addition. In order to understand now the multiplication operation, we must do something more complicated, namely using polar coordinates. Let us start with:

DEFINITION 7.5. *The complex numbers $x = a + ib$ can be written in polar coordinates,*

$$x = r(\cos t + i \sin t)$$

with the connecting formulae being as follows,

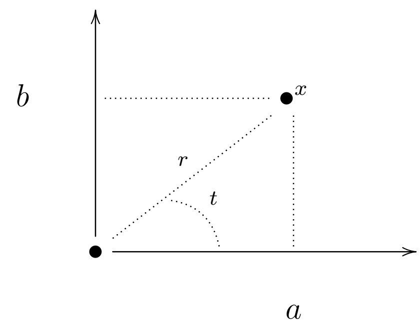
$$a = r\cos t \quad , \quad b = r\sin t$$

and in the other sense being as follows,

$$r = \sqrt{a^2 + b^2} \quad , \quad \tan t = \frac{b}{a}$$

and with r, t being called modulus, and argument.

There is a clear relation here with the vector notation from Proposition 7.3, because r is the length of the vector, and t is the angle made by the vector with the Ox axis. To be more precise, the picture for what is going on in Definition 7.5 is as follows:



As a basic example here, the number i takes the following form:

$$i=\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)$$

The point now is that in polar coordinates, the multiplication formula for the complex numbers, which was so far something quite opaque, takes a very simple form:

Theorem 7.6. *Two complex numbers written in polar coordinates,*

$$x=r(\cos s+i\sin s)\quad,\quad y=p(\cos t+i\sin t)$$

multiply according to the following formula:

$$xy=rp(\cos(s+t)+i\sin(s+t))$$

In other words, the moduli multiply, and the arguments sum up.

Proof. This follows from the following formulae, that we know well:

$$\cos(s+t)=\cos s\cos t-\sin s\sin t$$

$$\sin(s+t)=\cos s\sin t+\sin s\cos t$$

Indeed, we can assume that we have $r=p=1$, by dividing everything by these numbers. Now with this assumption made, we have the following computation:

$$\begin{aligned}xy&=(\cos s+i\sin s)(\cos t+i\sin t)\\&=(\cos s\cos t-\sin s\sin t)+i(\cos s\sin t+\sin s\cos t)\\&=\cos(s+t)+i\sin(s+t)\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

The above result, which is based on some non-trivial trigonometry, is quite powerful. As a basic application of it, we can now compute powers, as follows:

THEOREM 7.7. *The powers of a complex number, written in polar form,*

$$x = r(\cos t + i\sin t)$$

are given by the following formula, valid for any exponent $k \in \mathbb{N}$:

$$x^k = r^k(\cos kt + i\sin kt)$$

Moreover, this formula holds in fact for any $k \in \mathbb{Z}$, and even for any $k \in \mathbb{Q}$.

PROOF. Given a complex number x, written in polar form as above, and an exponent $k \in \mathbb{N}$, we have indeed the following computation, with k terms everywhere:

$$\begin{aligned}
x^k &= x \dots x \\
&= r(\cos t + i\sin t) \dots r(\cos t + i\sin t) \\
&= r^k([\cos(t + \dots + t) + i\sin(t + \dots + t)) \\
&= r^k(\cos kt + i\sin kt)
\end{aligned}$$

Thus, we are done with the case $k \in \mathbb{N}$. Regarding now the generalization to the case $k \in \mathbb{Z}$, it is enough here to do the verification for $k = -1$, where the formula is:

$$x^{-1} = r^{-1}(\cos(-t) + i\sin(-t))$$

But this number x^{-1} is indeed the inverse of x, as shown by:

$$\begin{aligned}
xx^{-1} &= r(\cos t + i\sin t) \cdot r^{-1}(\cos(-t) + i\sin(-t)) \\
&= \cos(t - t) + i\sin(t - t) \\
&= \cos 0 + i\sin 0 \\
&= 1
\end{aligned}$$

Finally, regarding the generalization to the case $k \in \mathbb{Q}$, it is enough to do the verification for exponents of type $k = 1/n$, with $n \in \mathbb{N}$. The claim here is that:

$$x^{1/n} = r^{1/n}\left[\cos\left(\frac{t}{n}\right) + i\sin\left(\frac{t}{n}\right)\right]$$

In order to prove this, let us compute the n-th power of this number. We can use the power formula for the exponent $n \in \mathbb{N}$, that we already established, and we obtain:

$$\begin{aligned}
(x^{1/n})^n &= (r^{1/n})^n\left[\cos\left(n \cdot \frac{t}{n}\right) + i\sin\left(n \cdot \frac{t}{n}\right)\right] \\
&= r(\cos t + i\sin t) \\
&= x
\end{aligned}$$

Thus, we have indeed a n-th root of x, and our proof is now complete. $\square$

We should mention that there is a bit of ambiguity in the above, in the case of the exponents $k \in \mathbb{Q}$, due to the fact that the square roots, and the higher roots as well, can take multiple values, in the complex number setting. We will be back to this.

As a basic application of Theorem 7.7, we have the following result:

PROPOSITION 7.8. *Each complex number, written in polar form,*

$$x = r(\cos t + i \sin t)$$

has two square roots, given by the following formula:

$$\sqrt{x} = \pm\sqrt{r}\left[\cos\left(\frac{t}{2}\right) + i \sin\left(\frac{t}{2}\right)\right]$$

When $x > 0$, these roots are $\pm\sqrt{x}$. When $x < 0$, these roots are $\pm i\sqrt{-x}$.

PROOF. The first assertion is clear indeed from the general formula in Theorem 7.7, at $k = 1/2$. As for its particular cases with $x \in \mathbb{R}$, these are clear from it. □

As a comment here, for $x > 0$ we are very used to call the usual $\sqrt{x}$ square root of x. However, for $x < 0$, or more generally for $x \in \mathbb{C} - \mathbb{R}_+$, there is less interest in choosing one of the possible $\sqrt{x}$ and calling it "the" square root of x, because all this is based on our convention that i comes up, instead of down, which is something rather arbitrary. Actually, clocks turning clockwise, i should be rather coming down. All this is a matter of taste, but in any case, for our math, the best is to keep some ambiguity, as above.

With the above results in hand, and notably with the square root formula from Proposition 7.8, we can now go back to the degree 2 equations, and we have:

THEOREM 7.9. *The complex solutions of $ax^2 + bx + c = 0$ with $a, b, c \in \mathbb{C}$ are*

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with the square root of complex numbers being defined as above.

PROOF. This is clear, the computations being the same as in the real case. To be more precise, our degree 2 equation can be written as follows:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Now since we know from Proposition 7.8 that any complex number has a square root, we are led to the conclusion in the statement. □

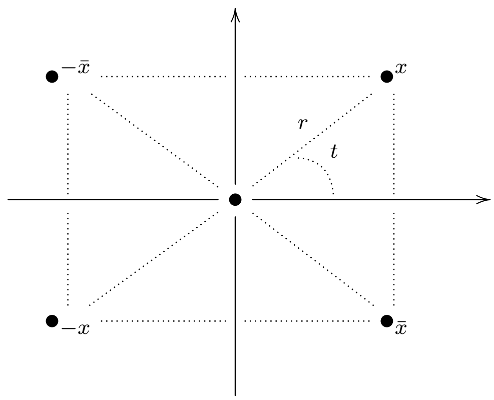
As a last general topic regarding the complex numbers, let us discuss conjugation. This is something quite tricky, complex number specific, as follows:

Definition 7.10. *The complex conjugate of $x = a + ib$ is the following number,*

$$\bar{x} = a - ib$$

obtained by making a reflection with respect to the Ox axis.

As before with other such operations on complex numbers, a quick picture says it all. Here is the picture, with the numbers $x, \bar{x}, -x, -\bar{x}$ being all represented:



Observe that the conjugate of a real number $x \in \mathbb{R}$ is the number itself, $x = \bar{x}$. In fact, the equation $x = \bar{x}$ characterizes the real numbers, among the complex numbers. At the level of non-trivial examples now, we have the following formula:

$$\bar{i} = -i$$

There are many things that can be said about the conjugation of the complex numbers, and here is a summary of basic such things that can be said:

Theorem 7.11. *The conjugation operation $x \to \bar{x}$ has the following properties:*

(1) *$x = \bar{x}$ precisely when x is real.*
(2) *$x = -\bar{x}$ precisely when x is purely imaginary.*
(3) *$x\bar{x} = |x|^2$, with $|x| = r$ being as usual the modulus.*
(4) *With $x = r(\cos t + i \sin t)$, we have $\bar{x} = r(\cos t - i \sin t)$.*
(5) *We have the formula $\overline{xy} = \bar{x}\bar{y}$, for any $x, y \in \mathbb{C}$.*
(6) *The solutions of $ax^2 + bx + c = 0$ with $a, b, c \in \mathbb{R}$ are conjugate.*

Proof. These results are all elementary, the idea being as follows:

(1) This is something that we already know, coming from definitions.

(2) This is something clear too, because with $x = a + ib$ our equation $x = -\bar{x}$ reads $a + ib = -a + ib$, and so $a = 0$, which amounts in saying that x is purely imaginary.

(3) This is a key formula, which can be proved as follows, with $x = a + ib$:

$$\begin{aligned} x\bar{x} &= (a+ib)(a-ib) \\ &= a^2 + b^2 \\ &= |x|^2 \end{aligned}$$

(4) This is clear indeed from the picture following Definition 7.10.

(5) This is something quite magic, which can be proved as follows:

$$\begin{aligned} \overline{(a+ib)(c+id)} &= \overline{(ac-bd)+i(ad+bc)} \\ &= (ac-bd)-i(ad+bc) \\ &= (a-ib)(c-id) \end{aligned}$$

However, what we have been doing here is not very clear, geometrically speaking, and our formula is worth an alternative proof. Here is that proof, which after inspection contains no computations at all, making it clear that the polar writing is the best:

$$\begin{aligned} & \overline{r(\cos s + i\sin s)\cdot p(\cos t + i\sin t)} \\ = \ & \overline{rp(\cos(s+t) + i\sin(s+t))} \\ = \ & rp(\cos(-s-t) + i\sin(-s-t)) \\ = \ & r(\cos(-s) + i\sin(-s))\cdot p(\cos(-t) + i\sin(-t)) \\ = \ & \overline{r(\cos s + i\sin s)}\cdot\overline{p(\cos t + i\sin t)} \end{aligned}$$

(6) This comes from the formula of the solutions, that we know from Theorem 7.2, but we can deduce this as well directly, without computations. Indeed, by using our assumption that the coefficients are real, $a, b, c \in \mathbb{R}$, we have:

$$\begin{aligned} ax^2 + bx + c = 0 &\implies \overline{ax^2 + bx + c} = 0 \\ &\implies \bar{a}\bar{x}^2 + \bar{b}\bar{x} + \bar{c} = 0 \\ &\implies a\bar{x}^2 + b\bar{x} + c = 0 \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Getting back to algebra, recall from Theorem 7.9 that any degree 2 equation has 2 complex roots. We can in fact prove that any polynomial equation, of arbitrary degree $N \in \mathbb{N}$, has exactly N complex solutions, counted with multiplicities:

THEOREM 7.12. *Any polynomial $P \in \mathbb{C}[X]$ decomposes as*

$$P = c(X - a_1)\dots(X - a_N)$$

with $c \in \mathbb{C}$ and with $a_1, \dots, a_N \in \mathbb{C}$.

PROOF. The problem is that of proving that our polynomial has at least one root, because afterwards we can proceed by recurrence. We prove this by contradiction. So, assume that P has no roots, and pick a number $z \in \mathbb{C}$ where $|P|$ attains its minimum:

$$|P(z)| = \min_{x \in \mathbb{C}} |P(x)| > 0$$

Since $Q(t) = P(z+t) - P(z)$ is a polynomial which vanishes at $t = 0$, this polynomial must be of the form ct^k + higher terms, with $c \neq 0$, and with $k \geq 1$ being an integer. We obtain from this that, with $t \in \mathbb{C}$ small, we have the following estimate:

$$P(z+t) \simeq P(z) + ct^k$$

Now let us write $t = rw$, with $r > 0$ small, and with $|w| = 1$. Our estimate becomes:

$$P(z+rw) \simeq P(z) + cr^k w^k$$

Now recall that we assumed $P(z) \neq 0$. We can therefore choose $w \in \mathbb{T}$ such that cw^k points in the opposite direction to that of $P(z)$, and we obtain in this way:

$$\begin{aligned} |P(z+rw)| &\simeq |P(z) + cr^k w^k| \\ &= |P(z)|(1 - |c|r^k) \end{aligned}$$

Now by choosing $r > 0$ small enough, as for the error in the first estimate to be small, and overcame by the negative quantity $-|c|r^k$, we obtain from this:

$$|P(z+rw)| < |P(z)|$$

But this contradicts our definition of $z \in \mathbb{C}$, as a point where $|P|$ attains its minimum. Thus P has a root, and by recurrence it has N roots, as stated. □

We kept the best for the end. As a last topic regarding the complex numbers, which is something really beautiful, we have the roots of unity. Let us start with:

THEOREM 7.13. *The equation $x^N = 1$ has N complex solutions, namely*

$$\left\{ w^k \middle| k = 0, 1, \ldots, N-1 \right\} \quad , \quad w = e^{2\pi i/N}$$

which are called roots of unity of order N.

PROOF. This follows from the general multiplication formula for the complex numbers in polar form. Indeed, with the notation $x = re^{it}$, our equation reads:

$$r^N e^{itN} = 1$$

Thus $r = 1$, and $t \in [0, 2\pi)$ must be a multiple of $2\pi/N$, as stated. □

As an illustration here, the roots of unity of small order, along with some of their basic properties, which are very useful for computations, are as follows:

$\underline{N=1}$. Here the unique root of unity is 1.

$\underline{N=2}$. Here we have two roots of unity, namely 1 and -1.

$\underline{N=3}$. Here we have 1, then $w = e^{2\pi i/3}$, and then $w^2 = \bar{w} = e^{4\pi i/3}$.

$\underline{N=4}$. Here the roots of unity, read as usual counterclockwise, are $1, i, -1, -i$.

$\underline{N=5}$. Here, with $w = e^{2\pi i/5}$, the roots of unity are $1, w, w^2, w^3, w^4$.

$\underline{N=6}$. Here a useful alternative writing is $\{\pm 1, \pm w, \pm w^2\}$, with $w = e^{2\pi i/3}$.

$\underline{N=7}$. Here, with $w = e^{2\pi i/7}$, the roots of unity are $1, w, w^2, w^3, w^4, w^5, w^6$.

$\underline{N=8}$. Here the roots of unity, read as usual counterclockwise, are the numbers $1, w, i, iw, -1, -w, -i, -iw$, with $w = e^{\pi i/4}$, which is also given by $w = (1+i)/\sqrt{2}$.

The roots of unity are very useful variables, and have many interesting properties. As a first application, we can now solve the ambiguity questions related to the extraction of N-th roots, that we met in the above, the statement here being as follows:

THEOREM 7.14. *Any nonzero complex number, written as*

$$x = re^{it}$$

has exactly N roots of order N, which appear as

$$y = r^{1/N} e^{it/N}$$

multiplied by the N roots of unity of order N.

PROOF. We must solve the equation $z^N = x$, over the complex numbers. Since the number y in the statement clearly satisfies $y^N = x$, our equation is equivalent to:

$$z^N = y^N$$

Now observe that we can write this equation as follows:

$$\left(\frac{z}{y}\right)^N = 1$$

We conclude that the solutions z appear by multiplying y by the solutions of $t^N = 1$, which are the N-th roots of unity, as claimed. $\square$

The roots of unity appear in connection with many other interesting questions, and there are many useful formulae relating them, which are good to know. Here is a basic such formula, very beautiful, to be used many times in what follows:

THEOREM 7.15. *The roots of unity, $\{w^k\}$ with $w = e^{2\pi i/N}$, have the property*

$$\sum_{k=0}^{N-1} (w^k)^s = N\delta_{N|s}$$

for any exponent $s \in \mathbb{N}$, where on the right we have a Kronecker symbol.

PROOF. The numbers in the statement, when written more conveniently as $(w^s)^k$ with $k = 0, \ldots, N-1$, form a certain regular polygon in the plane P_s. Thus, if we denote by C_s the barycenter of this polygon, we have the following formula:

$$\frac{1}{N}\sum_{k=0}^{N-1} w^{ks} = C_s$$

Now observe that in the case $N \not| s$ our polygon P_s is non-degenerate, circling around the unit circle, and having center $C_s = 0$. As for the case $N|s$, here the polygon is degenerate, lying at 1, and having center $C_s = 1$. Thus, we have the following formula:

$$C_s = \delta_{N|s}$$

Thus, we obtain the formula in the statement. □

As an interesting philosophical fact, regarding the roots of unity, and the complex numbers in general, we can now solve the following equation, in a "uniform" way:

$$x_1 + \ldots + x_N = 0$$

With this being not a joke. Frankly, can you find some nice-looking family of real numbers $x_1, \ldots, x_N$ satisfying $x_1 + \ldots + x_N = 0$? Certainly not. But with complex numbers we have now our answer, the sum of the N-th roots of unity being zero.

7b.

7c.

7d.

7e. Exercises

Exercises:

EXERCISE 7.16.

EXERCISE 7.17.

EXERCISE 7.18.

EXERCISE 7.19.

EXERCISE 7.20.

EXERCISE 7.21.

EXERCISE 7.22.

EXERCISE 7.23.

Bonus exercise.

CHAPTER 8

Advanced trigonometry

8a. Advanced trigonometry

Recall from before that conics are at the core of everything, mathematics, physics, life. But, what is next? A natural answer to this question comes from:

DEFINITION 8.1. *An algebraic curve in $\mathbb{R}^2$ is the vanishing set*

$$C = \left\{(x,y) \in \mathbb{R}^2 \Big| P(x,y) = 0\right\}$$

of a polynomial $P \in \mathbb{R}[X,Y]$ of arbitrary degree.

We already know well the algebraic curves in degree 2, which are the conics, and a first problem is, what results from what we learned about conics have a chance to be relevant to the arbitrary algebraic curves. And normally none, because the ellipses, parabolas and hyperbolas are obviously very particular curves, having very particular properties.

Let us record however a useful statement here, as follows:

PROPOSITION 8.2. *The conics can be written in cartesian, polar, parametric or complex coordinates, with the equations for the unit circle being*

$$x^2 + y^2 = 1 \quad , \quad r = 1 \quad , \quad x = \cos t \, , \, y = \sin t \quad , \quad |z| = 1$$

and with the equations for ellipses, parabolas and hyperbolas being similar.

PROOF. The equations for the circle are clear, those for ellipses can be found in the above, and we will leave as an exercise those for parabolas and hyperbolas. □

As a true answer to our question now, coming this time from a very modest conic, namely $xy = 0$, that we dismissed in the above as being "degenerate", we have:

THEOREM 8.3. *The following happen, for curves C defined by polynomials P:*

(1) *In degree $d = 2$, curves can have singularities, such as $xy = 0$ at $(0,0)$.*
(2) *In general, assuming $P = P_1 \ldots P_k$, we have $C = C_1 \cup \ldots\ldots \cup C_k$.*
(3) *A union of curves $C_i \cup C_j$ is generically non-smooth, unless disjoint.*
(4) *Due to this, we say that C is non-degenerate when P is irreducible.*

PROOF. All this is self-explanatory, the details being as follows:

(1) This is something obvious, just the story of two lines crossing.

(2) This comes from the following trivial fact, with the notation $z = (x, y)$:

$$P_1 \ldots P_k(z) = 0 \iff P_1(z) = 0, \text{ or } P_2(z) = 0, \ \ldots\ , \text{ or } P_k(z) = 0$$

(3) This is something very intuitive, and it actually takes a bit of time to imagine a situation where $C_1 \cap C_2 \neq \emptyset$, $C_1 \not\subset C_2$, $C_2 \not\subset C_1$, but $C_1 \cup C_2$ is smooth. In practice now, "generically" has of course a mathematical meaning, in relation with probability, and our assertion does say something mathematical, that we are supposed to prove. But, we will not insist on this, and leave this as an instructive exercise, precise formulation of the claim, and its proof, in the case you are familiar with probability theory.

(4) This is just a definition, based on the above, that we will use in what follows. □

With degree 1 and 2 investigated, and our conclusions recorded, let us get now to degree 3, see what new phenomena appear here. And here, to start with, we have the following remarkable curve, well-known from calculus, because 0 is not a maximum or minimum of the function $x \to y$, despite the derivative vanishing there:

$$x^3 = y$$

Also, in relation with set theory and logic, and with the foundations of mathematics in general, we have the following curve, which looks like the empyset $\emptyset$:

$$(x - y)(x^2 + y^2 - 1) = 0$$

But, it is not about counterexamples to calculus, or about logic, that we want to talk about here. As a first truly remarkable degree 3 curve, or cubic, we have the cusp:

PROPOSITION 8.4. *The standard cusp, which is the cubic given by*

$$x^3 = y^2$$

has a singularity at $(0, 0)$*, with only* 1 *tangent line at that singularity.*

PROOF. The two branches of the cusp are indeed both tangent to Ox, because:

$$y' = \pm\frac{3}{2}\sqrt{x} \implies y'(0) = 0$$

Observe also that what happens for the cusp is different from what happens for $xy = 0$, precisely because we have 1 line tangent at the singularity, instead of 2. □

As a second remarkable cubic, which gets the crown, and the right to have a Theorem about it, we have the Tschirnhausen curve, which is as follows:

THEOREM 8.5. *The Tschirnhausen cubic, given by the following equation,*

$$x^3 = x^2 - 3y^2$$

makes the dream of $xy = 0$ *come true, by self-intersecting, and being non-degenerate.*

PROOF. This is something self-explanatory, by drawing a picture, but there are several other interesting things that can be said about this curve, and the family of curves containing it, depending on a parameter, and up to basic transformations, as follows:

(1) Let us start with the curve written in polar coordinates as follows:

$$r\cos^3\left(\frac{\theta}{3}\right)=a$$

With $t=\tan(\theta/3)$, the equations of the coordinates are as follows:

$$x=a(1-3t^2)\quad,\quad y=at(3-t^2)$$

Now by eliminating t, we reach to the following equation:

$$(a-x)(8a+x)^2=27ay^2$$

(2) By translating horizontally by $8a$, and changing signs of variables, we have:

$$x=3a(3-t^2)\quad,\quad y=at(3-t^2)$$

Now by eliminating t, we reach to the following equation:

$$x^3=9a(x^2-3y^2)$$

But with $a=1/9$ this is precisely the equation in the statement. □

In degree 4 now, quartics, we have enough dimensions for "improving" the cusp and the Tschirnhausen curve. First we have the cardioid, which is as follows:

PROPOSITION 8.6. *The cardioid, which is a quartic, given in polar coordinates by*

$$2r=a(1-\cos\theta)$$

makes the dream of $x^3=y^2$ come true, by being a closed curve, with a cusp.

PROOF. As before with the Tschirnhausen curve, this is something self-explanatory, by drawing a picture, but there are several things that must be said, as follows:

(1) The cardioid appears by definition by rolling a circle of radius $c>0$ around another circle of same radius $c>0$. With θ being the rolling angle, we have:

$$x=2c(1-\cos\theta)\cos\theta$$

$$y=2c(1-\cos\theta)\sin\theta$$

(2) Thus, in polar coordinates we get the equation in the statement, with $a=4c$:

$$r=2c(1-\cos\theta)$$

(3) Finally, in cartesian coordinates, the equation is as follows:

$$(x^2+y^2)^2+4cx(x^2+y^2)=4c^2y^2$$

Thus, what we have is indeed a degree 4 curve, as claimed. □

Still in degree 4, the crown gets to the Bernoulli lemniscate, which is as follows:

THEOREM 8.7. *The Bernoulli lemniscate, a quartic, which is given by*

$$r^2 = a^2 \cos 2\theta$$

makes the dream of $x^3 = x^2 - 3y^2$ come true, by being closed, and self-intersecting.

PROOF. As usual, this is something self-explanatory, by drawing a picture, which looks like ∞, but there are several other things that must be said, as follows:

(1) In cartesian coordinates, the equation is as follows, with $a^2 = 2c^2$:

$$(x^2 + y^2)^2 = c^2(x^2 - y^2)$$

(2) Also, we have the following nice complex reformulation of this equation:

$$|z + c| \cdot |z - c| = c^2$$

Thus, we are led to the conclusions in in the statement. □

In degree 5, in the lack of any spectacular quintic, let us record:

THEOREM 8.8. *Unlike in degree $3, 4$, where equations can be solved, by the Cardano formula, in degree 5 this generically does not happen, an example being*

$$x^5 - x - 1 = 0$$

having Galois group S_5, not solvable. Geometrically, this tells us that the intersection of the quintic $y = x^5 - x - 1$ with the line $y = 0$ cannot be computed.

PROOF. Obviously off-topic, but with no good quintic available, and still a few more minutes before the bell ringing, I had to improvise a bit, and tell you about this:

(1) As indicated, the degree 3 equations can be solved a bit like the degree 2 ones, but with the formula, due to Cardano, being more complicated. With some square making tricks, which are non-trivial either, the Cardano formula applies to degree 4 as well.

(2) In degree 5 or higher, none of this is possible. Long story here, the idea being that in order for $P = 0$ to be solvable, the group $Gal(P)$ must be solvable, in the sense of group theory. But, unlike S_3, S_4 which are solvable, S_5 and higher are not solvable. □

Back now to our usual business, in degree 6, sextics, we first have here:

PROPOSITION 8.9. *The trefoil sextic, or Kiepert curve, which is given by*

$$r^3 = a^3 \cos 3\theta$$

looks like a trefoil, closed curve, with a triple self-intersection.

PROOF. As before, drawing a picture is mandatory. With $z = re^{i\theta}$ we have:

$$\begin{aligned}
r^3 = a^3\cos 3\theta &\iff r^3\cos 3\theta = \left(\frac{r^2}{a}\right)^3\\
&\iff z^3 + \bar{z}^3 = 2\left(\frac{z\bar{z}}{a}\right)^3\\
&\iff (x+iy)^3 + (x-iy)^3 = 2\left(\frac{x^2+y^2}{a}\right)^3\\
&\iff x^3 - 3xy^2 = \left(\frac{x^2+y^2}{a}\right)^3\\
&\iff (x^2+y^2)^3 = a^3(x^3-3xy^2)
\end{aligned}$$

Thus, we have indeed a sextic, as claimed. □

We also have in degree 6 the most beautiful of curves them all, the Cayley sextic:

THEOREM 8.10. *The Cayley sextic, given in polar coordinates by*

$$r = a\cos^3\left(\frac{\theta}{3}\right)$$

makes the dream of everyone come true, by looking like a self-intersecting heart.

PROOF. As before, picture mandatory. With $z = re^{i\theta}$ and $u = z^{1/3}$ we have:

$$\begin{aligned}
r = a\cos^3\left(\frac{\theta}{3}\right) &\iff ar\cos^3\left(\frac{\theta}{3}\right) = r^2\\
&\iff a\left(\frac{u+\bar{u}}{2}\right)^3 = r^2\\
&\iff a(u^3+\bar{u}^3+3u\bar{u}(u+\bar{u})) = 8r^2\\
&\iff 3au\bar{u}\cdot\frac{u+\bar{u}}{2} = 4r^2 - ax\\
&\iff 27a^3r^6\cdot\frac{r^2}{a} = (4r^2-ax)^3\\
&\iff 27a^2(x^2+y^2)^2 = (4x^2+4y^2-ax)^3
\end{aligned}$$

Thus, we have indeed a sextic, as claimed. □

We will be back to plane algebraic curves in chapter 15 below, with some generalizations, and more theory, when talking physics and field lines.

8b.

8c.

8d.

8e. Exercises

Exercises:

Exercise 8.11.

Exercise 8.12.

Exercise 8.13.

Exercise 8.14.

Exercise 8.15.

Exercise 8.16.

Exercise 8.17.

Exercise 8.18.

Bonus exercise.

Part III

Heavy calculus

There is a house in New Orleans
They call the Rising Sun
And it's been the ruin of many a poor boy
Dear God, I know I was one

CHAPTER 9

Functions, derivatives

9a. Functions, derivatives

The idea of calculus is very simple. We are interested in functions $f : \mathbb{R} \to \mathbb{R}$, and we already know that when f is continuous at a point x, we can write an approximation formula as follows, for the values of our function f around that point x:

$$f(x+t) \simeq f(x)$$

The problem is now, how to improve this? And a bit of thinking at all this suggests to look at the slope of f at the point x. Which leads us into the following notion:

DEFINITION 9.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is called differentiable at x when*

$$f'(x) = \lim_{t \to 0} \frac{f(x+t) - f(x)}{t}$$

called derivative of f at that point x, exists.

As a first remark, in order for f to be differentiable at x, that is to say, in order for the above limit to converge, the numerator must go to 0, as the denominator t does:

$$\lim_{t \to 0} [f(x+t) - f(x)] = 0$$

Thus, f must be continuous at x. However, the converse is not true, a basic counterexample being $f(x) = |x|$ at $x = 0$. Let us summarize these findings as follows:

PROPOSITION 9.2. *If f is differentiable at x, then f must be continuous at x. However, the converse is not true, a basic counterexample being $f(x) = |x|$, at $x = 0$.*

PROOF. The first assertion is something that we already know, from the above. As for the second assertion, regarding $f(x) = |x|$, this is something quite clear on the picture of f, but let us prove this mathematically, based on Definition 9.1. We have:

$$\lim_{t \searrow 0} \frac{|0+t| - |0|}{t} = \lim_{t \searrow 0} \frac{t - 0}{t} = 1$$

On the other hand, we have as well the following computation:

$$\lim_{t \nearrow 0} \frac{|0+t| - |0|}{t} = \lim_{t \nearrow 0} \frac{-t - 0}{t} = -1$$

Thus, the limit in Definition 9.1 does not converge, so we have our counterexample. □

Generally speaking, the last assertion in Proposition 9.2 should not bother us much, because most of the basic continuous functions are differentiable, and we will see examples in a moment. Before that, however, let us recall why we are here, namely improving the basic estimate $f(x+t) \simeq f(x)$. We can now do this, using the derivative, as follows:

THEOREM 9.3. *Assuming that f is differentiable at x, we have:*

$$f(x+t) \simeq f(x) + f'(x)t$$

In other words, f is, approximately, locally affine at x.

PROOF. Assume indeed that f is differentiable at x, and let us set, as before:

$$f'(x) = \lim_{t \to 0} \frac{f(x+t) - f(x)}{t}$$

By multiplying by t, we obtain that we have, once again in the $t \to 0$ limit:

$$f(x+t) - f(x) \simeq f'(x)t$$

Thus, we are led to the conclusion in the statement. □

All this is very nice, and before developing more theory, let us work out some examples. As a first illustration, the derivatives of the power functions are as follows:

THEOREM 9.4. *We have the differentiation formula*

$$(x^p)' = px^{p-1}$$

valid for any exponent $p \in \mathbb{R}$.

PROOF. We can do this in three steps, as follows:

(1) In the case $p \in \mathbb{N}$ we can use the binomial formula, which gives, as desired:

$$\begin{aligned} (x+t)^p &= \sum_{k=0}^{n} \binom{p}{k} x^{p-k} t^k \\ &= x^p + px^{p-1}t + \ldots + t^p \\ &\simeq x^p + px^{p-1}t \end{aligned}$$

(2) Let us discuss now the general case $p \in \mathbb{Q}$. We write $p = m/n$, with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. In order to do the computation, we use the following formula:

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-1})$$

We set in this formula $a=(x+t)^{m/n}$ and $b=x^{m/n}$. We obtain, as desired:

$$\begin{aligned}(x+t)^{m/n}-x^{m/n} &= \frac{(x+t)^m-x^m}{(x+t)^{m(n-1)/n}+\ldots+x^{m(n-1)/n}}\\ &\simeq \frac{(x+t)^m-x^m}{nx^{m(n-1)/n}}\\ &\simeq \frac{mx^{m-1}t}{nx^{m(n-1)/n}}\\ &= \frac{m}{n}\cdot x^{m-1-m+m/n}\cdot t\\ &= \frac{m}{n}\cdot x^{m/n-1}\cdot t\end{aligned}$$

(3) In the general case now, where $p\in\mathbb{R}$ is real, we can use a similar argument. Indeed, given any integer $n\in\mathbb{N}$, we have the following computation:

$$\begin{aligned}(x+t)^p-x^p &= \frac{(x+t)^{pn}-x^{pn}}{(x+t)^{p(n-1)}+\ldots+x^{p(n-1)}}\\ &\simeq \frac{(x+t)^{pn}-x^{pn}}{nx^{p(n-1)}}\end{aligned}$$

Now observe that we have the following estimate, with $[.]$ being the integer part:

$$(x+t)^{[pn]}\leq(x+t)^{pn}\leq(x+t)^{[pn]+1}$$

By using the binomial formula on both sides, for the integer exponents $[pn]$ and $[pn]+1$ there, we deduce that with $n>>0$ we have the following estimate:

$$(x+t)^{pn}\simeq x^{pn}+pnx^{pn-1}t$$

Thus, we can finish our computation started above as follows:

$$(x+t)^p-x^p\simeq\frac{pnx^{pn-1}t}{nx^{pn-p}}=px^{p-1}t$$

But this gives $(x^p)'=px^{p-1}$, which finishes the proof. $\square$

Here are some further computations, for other basic functions that we know:

Theorem 9.5. *We have the following results:*

1. $(\sin x)'=\cos x$.
2. $(\cos x)'=-\sin x$.
3. $(e^x)'=e^x$.
4. $(\log x)'=x^{-1}$.

Proof. This is quite tricky, as always when computing derivatives, as follows:

(1) Regarding sin, the computation here goes as follows:

$$\begin{aligned}(\sin x)' &= \lim_{t\to 0}\frac{\sin(x+t)-\sin x}{t}\\ &= \lim_{t\to 0}\frac{\sin x\cos t+\cos x\sin t-\sin x}{t}\\ &= \lim_{t\to 0}\sin x\cdot\frac{\cos t-1}{t}+\cos x\cdot\frac{\sin t}{t}\\ &= \cos x\end{aligned}$$

Here we have used the fact, which is clear on pictures, by drawing the trigonometric circle, that we have $\sin t \simeq t$ for $t \simeq 0$, plus the fact, which follows from this and from Pythagoras, $\sin^2+\cos^2=1$, that we have as well $\cos t \simeq 1-t^2/2$, for $t \simeq 0$.

(2) The computation for cos is similar, as follows:

$$\begin{aligned}(\cos x)' &= \lim_{t\to 0}\frac{\cos(x+t)-\cos x}{t}\\ &= \lim_{t\to 0}\frac{\cos x\cos t-\sin x\sin t-\cos x}{t}\\ &= \lim_{t\to 0}\cos x\cdot\frac{\cos t-1}{t}-\sin x\cdot\frac{\sin t}{t}\\ &= -\sin x\end{aligned}$$

(3) For the exponential, the derivative can be computed as follows:

$$\begin{aligned}(e^x)' &= \left(\sum_{k=0}^\infty\frac{x^k}{k!}\right)'\\ &= \sum_{k=0}^\infty\frac{kx^{k-1}}{k!}\\ &= e^x\end{aligned}$$

(4) As for the logarithm, the computation here is as follows, using $\log(1+y)\simeq y$ for $y \simeq 0$, which follows from $e^y \simeq 1+y$ that we found in (3), by taking the logarithm:

$$\begin{aligned}(\log x)' &= \lim_{t\to 0}\frac{\log(x+t)-\log x}{t}\\ &= \lim_{t\to 0}\frac{\log(1+t/x)}{t}\\ &= \frac{1}{x}\end{aligned}$$

Thus, we are led to the formulae in the statement. □

Speaking exponentials, we can now formulate a nice result about them:

THEOREM 9.6. *The exponential function, namely*

$$e^x = \sum_{k=0}^\infty \frac{x^k}{k!}$$

is the unique power series satisfying $f' = f$ and $f(0) = 1$.

PROOF. Consider indeed a power series satisfying $f' = f$ and $f(0) = 1$. Due to $f(0) = 1$, the first term must be 1, and so our function must look as follows:

$$f(x) = 1 + \sum_{k=1}^\infty c_k x^k$$

According to our differentiation rules, the derivative of this series is given by:

$$f(x) = \sum_{k=1}^\infty k c_k x^{k-1}$$

Thus, the equation $f' = f$ is equivalent to the following equalities:

$$c_1 = 1 \quad , \quad 2c_2 = c_1 \quad , \quad 3c_3 = c_2 \quad , \quad 4c_4 = c_3 \quad , \quad \ldots$$

But this system of equations can be solved by recurrence, as follows:

$$c_1 = 1 \quad , \quad c_2 = \frac{1}{2} \quad , \quad c_3 = \frac{1}{2\times 3} \quad , \quad c_4 = \frac{1}{2\times 3\times 4} \quad , \quad \ldots$$

Thus we have $c_k = 1/k!$, leading to the conclusion in the statement. □

Observe that the above result leads to a more conceptual explanation for the number e itself. To be more precise, $e \in \mathbb{R}$ is the unique number satisfying:

$$(e^x)' = e^x$$

Let us work out now some general results. We have here the following statement:

THEOREM 9.7. *We have the following formulae:*

(1) $(f+g)' = f' + g'$.
(2) $(fg)' = f'g + fg'$.
(3) $(f \circ g)' = (f' \circ g) \cdot g'$.

PROOF. All these formulae are elementary, the idea being as follows:

(1) This follows indeed from definitions, the computation being as follows:

$$\begin{aligned}(f+g)'(x) &= \lim_{t\to 0}\frac{(f+g)(x+t)-(f+g)(x)}{t}\\ &= \lim_{t\to 0}\left(\frac{f(x+t)-f(x)}{t}+\frac{g(x+t)-g(x)}{t}\right)\\ &= \lim_{t\to 0}\frac{f(x+t)-f(x)}{t}+\lim_{t\to 0}\frac{g(x+t)-g(x)}{t}\\ &= f'(x)+g'(x)\end{aligned}$$

(2) This follows from definitions too, the computation, by using the more convenient formula $f(x+t)\simeq f(x)+f'(x)t$ as a definition for the derivative, being as follows:

$$\begin{aligned}(fg)(x+t) &= f(x+t)g(x+t)\\ &\simeq (f(x)+f'(x)t)(g(x)+g'(x)t)\\ &\simeq f(x)g(x)+(f'(x)g(x)+f(x)g'(x))t\end{aligned}$$

Indeed, we obtain from this that the derivative is the coefficient of t, namely:

$$(fg)'(x)=f'(x)g(x)+f(x)g'(x)$$

(3) Regarding compositions, the computation here is as follows, again by using the more convenient formula $f(x+t)\simeq f(x)+f'(x)t$ as a definition for the derivative:

$$\begin{aligned}(f\circ g)(x+t) &= f(g(x+t))\\ &\simeq f(g(x)+g'(x)t)\\ &\simeq f(g(x))+f'(g(x))g'(x)t\end{aligned}$$

Indeed, we obtain from this that the derivative is the coefficient of t, namely:

$$(f\circ g)'(x)=f'(g(x))g'(x)$$

Thus, we are led to the conclusions in the statement. □

We can of course combine the above formulae, and we obtain for instance:

THEOREM 9.8. *The derivatives of fractions are given by:*

$$\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$$

In particular, we have the following formula, for the derivative of inverses:

$$\left(\frac{1}{f}\right)'=-\frac{f'}{f^2}$$

In fact, we have $(f^p)'=pf^{p-1}$, for any exponent $p\in\mathbb{R}$.

PROOF. This statement is written a bit upside down, and for the proof it is better to proceed backwards. To be more precise, by using $(x^p)' = px^{p-1}$ and Theorem 9.7 (3), we obtain the third formula. Then, with $p = -1$, we obtain from this the second formula. And finally, by using this second formula and Theorem 9.7 (2), we obtain:

$$\begin{aligned}\left(\frac{f}{g}\right)' &= \left(f \cdot \frac{1}{g}\right)' \\ &= f' \cdot \frac{1}{g} + f\left(\frac{1}{g}\right)' \\ &= \frac{f'}{g} - \frac{fg'}{g^2} \\ &= \frac{f'g - fg'}{g^2}\end{aligned}$$

Thus, we are led to the formulae in the statement. □

With the above formulae in hand, we can do all sorts of computations for other basic functions that we know, including $\tan x$, or $\arctan x$:

THEOREM 9.9. *We have the following formulae,*

$$(\tan x)' = \frac{1}{\cos^2 x} \quad , \quad (\arctan x)' = \frac{1}{1+x^2}$$

and the derivatives of the remaining trigonometric functions can be computed as well.

PROOF. For tan, we have the following computation:

$$\begin{aligned}(\tan x)' &= \left(\frac{\sin x}{\cos x}\right)' \\ &= \frac{\sin' x \cos x - \sin x \cos' x}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x}\end{aligned}$$

As for arctan, we can use here the following computation:

$$\begin{aligned}(\tan \circ \arctan)'(x) &= \tan'(\arctan x) \arctan'(x) \\ &= \frac{1}{\cos^2(\arctan x)} \arctan'(x)\end{aligned}$$

Indeed, since the term on the left is simply $x' = 1$, we obtain from this:

$$\arctan'(x) = \cos^2(\arctan x)$$

On the other hand, with $t = \arctan x$ we know that we have $\tan t = x$, and so:

$$\cos^2(\arctan x) = \cos^2 t = \frac{1}{1+\tan^2 t} = \frac{1}{1+x^2}$$

Thus, we are led to the formula in the statement, namely:

$$(\arctan x)' = \frac{1}{1+x^2}$$

As for the last assertion, we will leave this as an exercise. □

9b.

9c.

9d.

9e. Exercises

Exercises:

Exercise 9.10.

Exercise 9.11.

Exercise 9.12.

Exercise 9.13.

Exercise 9.14.

Exercise 9.15.

Exercise 9.16.

Exercise 9.17.

Bonus exercise.

CHAPTER 10

Trigonometric functions

10a. Trigonometric functions

10b.

10c.

10d.

10e. Exercises

Exercises:

Exercise 10.1.

Exercise 10.2.

Exercise 10.3.

Exercise 10.4.

Exercise 10.5.

Exercise 10.6.

Exercise 10.7.

Exercise 10.8.

Bonus exercise.

CHAPTER 11

Sums, estimates

11a. Sums, estimates

11b.

11c.

11d.

11e. Exercises

Exercises:

Exercise 11.1.

Exercise 11.2.

Exercise 11.3.

Exercise 11.4.

Exercise 11.5.

Exercise 11.6.

Exercise 11.7.

Exercise 11.8.

Bonus exercise.

CHAPTER 12

Into arithmetic

12a. Into arithmetic

12b.

12c.

12d.

12e. Exercises

Exercises:

Exercise 12.1.

Exercise 12.2.

Exercise 12.3.

Exercise 12.4.

Exercise 12.5.

Exercise 12.6.

Exercise 12.7.

Exercise 12.8.

Bonus exercise.

Part IV

Three dimensions

If you're going to San Francisco
Be sure to wear some flowers in your hair
If you're going to San Francisco
You're gonna meet some gentle people there

CHAPTER 13

Space geometry

13a. Space geometry

Getting started with some applications, here is the notion what we will need:

DEFINITION 13.1. *The vector product of two vectors in* $\mathbb{R}^3$ *is given by*

$$x \times y = ||x|| \cdot ||y|| \cdot \sin\theta \cdot n$$

where $n \in \mathbb{R}^3$ *with* $n \perp x, y$ *and* $||n|| = 1$ *is constructed using the right-hand rule:*

$$\begin{array}{c} \quad\uparrow_{x\times y} \\ \leftarrow_x \\ \swarrow_y \end{array}$$

Alternatively, in usual vertical linear algebra notation for all vectors,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$$

the rule being that of computing 2×2 *determinants, and adding a middle sign.*

Obviously, this definition is something quite subtle, and also something very annoying, because you always need this, and always forget the formula. Here are my personal methods. With the first definition, what I always remember is that:

$$||x \times y|| \sim ||x||, ||y|| \quad , \quad x \times x = 0 \quad , \quad e_1 \times e_2 = e_3$$

So, here's how it works. We are looking for a vector $x \times y$ whose length is proportional to those of x, y. But the second formula tells us that the angle θ between x, y must be involved via $0 \to 0$, and so the factor can only be $\sin\theta$. And with this we are almost there, it's just a matter of choosing the orientation, and this comes from $e_1 \times e_2 = e_3$.

As with the second definition, that I like the most, what I remember here is simply:

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = ?$$

Indeed, when trying to compute this determinant, by developing over the first column, what you get as coefficients are the entries of $x \times y$. And with the good middle sign.

In practice now, in order to get familiar with the vector products, nothing better than doing some classical mechanics. We have here the following key result:

THEOREM 13.2. *In the gravitational 2-body problem, the angular momentum*

$$J = x \times p$$

with $p = mv$ being the usual momentum, is conserved.

PROOF. There are several things to be said here, the idea being as follows:

(1) First of all the usual momentum, $p = mv$, is not conserved, because the simplest solution is the circular motion, where the moment gets turned around. But this suggests precisely that, in order to fix the lack of conservation of the momentum p, what we have to do is to make a vector product with the position x. Leading to J, as above.

(2) Regarding now the proof, consider indeed a particle m moving under the gravitational force of a particle M, assumed, as usual, to be fixed at 0. By using the fact that for two proportional vectors, $p \sim q$, we have $p \times q = 0$, we obtain:

$$\begin{aligned} \dot{J} &= \dot{x} \times p + x \times \dot{p} \\ &= v \times mv + x \times ma \\ &= m(v \times v + x \times a) \\ &= m(0 + 0) \\ &= 0 \end{aligned}$$

Now since the derivative of J vanishes, this quantity is constant, as stated. □

As another basic application of the vector products, still staying with classical mechanics, we have all sorts of useful formulae regarding rotating frames. We first have:

THEOREM 13.3. *Assume that a 3D body rotates along an axis, with angular speed w. For a fixed point of the body, with position vector x, the usual 3D speed is*

$$v = \omega \times x$$

where $\omega = wn$, with n unit vector pointing North. When the point moves on the body

$$V = \dot{x} + \omega \times x$$

is its speed computed by an inertial observer O on the rotation axis.

PROOF. We have two assertions here, both requiring some 3D thinking, as follows:

(1) Assuming that the point is fixed, the magnitude of $\omega \times x$ is the good one, due to the following computation, with r being the distance from the point to the axis:

$$||\omega \times x|| = w||x|| \sin t = wr = ||v||$$

As for the orientation of $\omega \times x$, this is the good one as well, because the North pole rule used above amounts in applying the right-hand rule for finding n, and so ω, and this right-hand rule was precisely the one used in defining the vector products $\times$.

(2) Next, when the point moves on the body, the inertial observer O can compute its speed by using a frame (u_1, u_2, u_3) which rotates with the body, as follows:

$$\begin{aligned} V &= \dot{x}_1 u_1 + \dot{x}_2 u_2 + \dot{x}_3 u_3 + x_1 \dot{u}_1 + x_2 \dot{u}_2 + x_3 \dot{u}_3 \\ &= \dot{x} + (x_1 \cdot \omega \times u_1 + x_2 \cdot \omega \times u_2 + x_3 \cdot \omega \times u_3) \\ &= \dot{x} + w \times (x_1 u_1 + x_2 u_2 + x_3 u_3) \\ &= \dot{x} + \omega \times x \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

In what regards now the acceleration, the result, which is famous, is as follows:

THEOREM 13.4. *Assuming as before that a 3D body rotates along an axis, the acceleration of a moving point on the body, computed by O as before, is given by*

$$A = a + 2\omega \times v + \omega \times (\omega \times x)$$

with $\omega = wn$ being as before. In this formula the second term is called Coriolis acceleration, and the third term is called centripetal acceleration.

PROOF. This comes by using twice the formulae in Theorem 13.3, as follows:

$$\begin{aligned} A &= \dot{V} + \omega \times V \\ &= (\ddot{x} + \dot{\omega} \times x + \omega \times \dot{x}) + (\omega \times \dot{x} + \omega \times (\omega \times x)) \\ &= \ddot{x} + \omega \times \dot{x} + \omega \times \dot{x} + \omega \times (\omega \times x) \\ &= a + 2\omega \times v + \omega \times (\omega \times x) \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

The truly famous result is actually the one regarding forces, obtained by multiplying everything by a mass m, and writing things the other way around, as follows:

$$ma = mA - 2m\omega \times v - m\omega \times (\omega \times x)$$

Here the second term is called Coriolis force, and the third term is called centrifugal force. These forces are both called apparent, or fictious, because they do not exist in the inertial frame, but they exist however in the non-inertial frame of reference, as explained above. And with of course the terms centrifugal and centripetal not to be messed up.

In fact, even more famous is the terrestrial application of all this, as follows:

THEOREM 13.5. *The acceleration of an object m subject to a force F is given by*

$$ma = F - mg - 2m\omega \times v - m\omega \times (\omega \times x)$$

with g pointing upwards, and with the last terms being the Coriolis and centrifugal forces.

PROOF. This follows indeed from the above discussion, by assuming that the acceleration A there comes from the combined effect of a force F, and of the usual g. □

We refer to any standard undergraduate mechanics book, such as Feynman [33], Kibble [57] or Taylor [91] for more on the above, including various numerics on what happens here on Earth, the Foucault pendulum, history of all this, and many other things. Let us just mention here, as a basic illustration for all this, that a rock dropped from 100m deviates about 1cm from its intended target, due to the formula in Theorem 13.5.

13b.

13c.

13d.

13e. Exercises

Exercises:

EXERCISE 13.6.

EXERCISE 13.7.

EXERCISE 13.8.

EXERCISE 13.9.

EXERCISE 13.10.

EXERCISE 13.11.

EXERCISE 13.12.

EXERCISE 13.13.

Bonus exercise.

CHAPTER 14

Solid angles

14a. Solid angles

14b.

14c.

14d.

14e. Exercises

Exercises:

Exercise 14.1.

Exercise 14.2.

Exercise 14.3.

Exercise 14.4.

Exercise 14.5.

Exercise 14.6.

Exercise 14.7.

Exercise 14.8.

Bonus exercise.

CHAPTER 15

Field lines

15a. Field lines

Time for electricity. Let us start with something very basic, namely:

Fact 15.1. *Each piece of matter has a charge $q \in \mathbb{R}$, which is normally neutral, $q = 0$, but that we can make positive or negative, by using various methods. We say that responsible for the charge is the amount of electrons present, as follows:*

1. *When the matter lacks electrons, the charge is positive, $q > 0$.*
2. *When there are more electrons than needed, the charge is negative, $q < 0$.*

As our first result, due to Coulomb, and that will come as a physics fact instead of a mathematics theorem, because, well, I must admit that what we have in Fact 15.1 is indeed more than borderline, as axiomatics for a theory, we have:

Fact 15.2 (Coulomb law). *Any pair of charges $q_1, q_2 \in \mathbb{R}$ is subject to a force as follows, which is attractive if $q_1 q_2 < 0$ and repulsive if $q_1 q_2 > 0$,*

$$||F|| = K \cdot \frac{|q_1 q_2|}{d^2}$$

where $d > 0$ is the distance between the charges, and $K > 0$ is a certain constant.

Observe the amazing similarity with the Newton law for gravity. However, as we will discover soon, passed a few simple facts, things will be far more complicated here.

As in the gravity case, the force F appearing above is understood to be parallel to the vector $x_2 - x_1 \in \mathbb{R}^3$ joining as $x_1 \to x_2$ the locations $x_1, x_2 \in \mathbb{R}^3$ of our charges, and by taking into account the attraction/repulsion rules above, we have:

Proposition 15.3. *The Coulomb force of q_1 at x_1 acting on q_2 at x_2 is*

$$F = K \cdot \frac{q_1 q_2 (x_2 - x_1)}{||x_2 - x_1||^3}$$

with $K > 0$ being the Coulomb constant, as above.

Proof. We have indeed the following computation:

$$\begin{aligned} F &= sgn(q_1q_2)\cdot||F||\cdot\frac{x_2-x_1}{||x_2-x_1||}\\ &= sgn(q_1q_2)\cdot K\cdot\frac{|q_1q_2|}{||x_2-x_1||^2}\cdot\frac{x_2-x_1}{||x_2-x_1||}\\ &= K\cdot\frac{q_1q_2(x_2-x_1)}{||x_2-x_1||^3} \end{aligned}$$

Thus, we are led to the formula in the statement. □

In analogy with the usual study of gravity, let us start with:

Definition 15.4. *Given charges $q_1,\ldots,q_k\in\mathbb{R}$ located at positions $x_1,\ldots,x_k\in\mathbb{R}^3$, we define their electric field to be the vector function*

$$E(x)=K\sum_i\frac{q_i(x-x_i)}{||x-x_i||^3}$$

so that their force applied to a charge $Q\in\mathbb{R}$ positioned at $x\in\mathbb{R}^3$ is given by $F=QE$.

More generally, we will be interested in electric fields of various non-discrete configurations of charges, such as charged curves, surfaces and solid bodies. Indeed, things like wires or metal sheets or solid bodies coming in all sorts of shapes, tailored for their purpose, play a key role, so this extension is essential. So, let us go ahead with:

Definition 15.5. *The electric field of a charge configuration $L\subset\mathbb{R}^3$, with charge density function $\rho:L\to\mathbb{R}$, is the vector function*

$$E(x)=K\int_L\frac{\rho(z)(x-z)}{||x-z||^3}\,dz$$

so that the force of L applied to a charge Q positioned at x is given by $F=QE$.

With the above definitions in hand, it is most convenient now to forget about the charges, and focus on the study of the corresponding electric fields E.

These fields are by definition vector functions $E:\mathbb{R}^3\to\mathbb{R}^3$, with the convention that they take $\pm\infty$ values at the places where the charges are located, and intuitively, are best represented by their field lines, which are constructed as follows:

Definition 15.6. *The field lines of an electric field $E:\mathbb{R}^3\to\mathbb{R}^3$ are the oriented curves $\gamma\subset\mathbb{R}^3$ pointing at every point $x\in\mathbb{R}^3$ at the direction of the field, $E(x)\in\mathbb{R}^3$.*

As a basic example here, for one charge the field lines are the half-lines emanating from its position, oriented according to the sign of the charge:

$$\begin{matrix}\nwarrow&\uparrow&\nearrow\\ \leftarrow&\oplus&\rightarrow\\ \swarrow&\downarrow&\searrow\end{matrix}\qquad\qquad\qquad\begin{matrix}\searrow&\downarrow&\swarrow\\ \rightarrow&\ominus&\leftarrow\\ \nearrow&\uparrow&\nwarrow\end{matrix}$$

For two charges now, if these are of opposite signs, $+$ and $-$, you get a picture that you are very familiar with, namely that of the field lines of a bar magnet:

$$\begin{matrix}\nearrow&\nearrow&\rightarrow&\rightarrow&\rightarrow&\rightarrow&\searrow&\searrow\\ \nwarrow&\uparrow&\nearrow&\rightarrow&\rightarrow&\searrow&\downarrow&\swarrow\\ \leftarrow&\oplus&\rightarrow&\rightarrow&\rightarrow&\rightarrow&\ominus&\leftarrow\\ \swarrow&\downarrow&\searrow&\rightarrow&\rightarrow&\nearrow&\uparrow&\nwarrow\\ \searrow&\searrow&\rightarrow&\rightarrow&\rightarrow&\rightarrow&\nearrow&\nearrow\end{matrix}$$

If the charges are $+,+$ or $-,-$, you get something of similar type, but repulsive this time, with the field lines emanating from the charges being no longer shared:

$$\begin{matrix}\leftarrow&\nwarrow&\nwarrow\\ &\uparrow&\nearrow\\ \leftarrow&\oplus&\\ &\downarrow&\searrow\\ \leftarrow&\swarrow&\swarrow\end{matrix}\qquad\qquad\qquad\begin{matrix}\nearrow&\nearrow&\rightarrow\\ \nwarrow&\uparrow&\\ &\oplus&\rightarrow\\ \swarrow&\downarrow&\\ \searrow&\searrow&\rightarrow\end{matrix}$$

These pictures, and notably the last one, with $+,+$ charges, are quite interesting, because the repulsion situation does not appear in the context of gravity. Thus, we can only expect our geometry here to be far more complicated than that of gravity.

The field lines obviously do not encapsulate the whole information about the field, with the direction of each vector $E(x) \in \mathbb{R}^3$ being there, but with the magnitude $||E(x)|| \geq 0$ of this vector missing. However, say when drawing, when picking up uniformly radially spaced field lines around each charge, and with the number of these lines proportional to the magnitude of the charge, and then completing the picture, the density of the field lines around each point $x \in \mathbb{R}$ will give you then the magnitude $||E(x)|| \geq 0$ of the field there, up to a scalar. Let us summarize these observations as follows:

Proposition 15.7. *Given an electric field $E : \mathbb{R}^3 \to \mathbb{R}^3$, the knowledge of its field lines is the same as the knowledge of the composition*

$$nE : \mathbb{R}^3 \to \mathbb{R}^3 \to S$$

where $S \subset \mathbb{R}^3$ is the unit sphere, and $n : \mathbb{R}^3 \to S$ is the rescaling map, namely:

$$n(x) = \frac{x}{||x||}$$

However, in practice, when the field lines are accurately drawn, the density of the field lines gives you the magnitude of the field, up to a scalar.

Proof. We have two assertions here, the idea being as follows:

(1) The first assertion is clear from definitions, with our usual convention that the electric field and its problematics take place outside the locations of the charges.

(2) Regarding now the last assertion, which is of course a bit informal, this follows from the above discussion. It is possible to be a bit more mathematical here, with a definition, formula and everything, but we will not need this, in what follows. □

Let us introduce now a key definition, as follows:

Definition 15.8. *The flux of an electric field $E:\mathbb{R}^3\to\mathbb{R}^3$ through a surface $S\subset\mathbb{R}^3$, assumed to be oriented, is the quantity*

$$\Phi_E(S)=\int_S<E(x),n(x)>dx$$

with $n(x)$ being unit vectors orthogonal to S, following the orientation of S. Intuitively, the flux measures the signed number of field lines crossing S.

Here by orientation of S we mean precisely the choice of unit vectors $n(x)$ as above, orthogonal to S, which must vary continuously with x. For instance a sphere has two possible orientations, one with all these vectors $n(x)$ pointing inside, and one with all these vectors $n(x)$ pointing outside. More generally, any surface has locally two possible orientations, so if it is connected, it has two possible orientations. In what follows the convention is that the closed surfaces are oriented with each $n(x)$ pointing outside.

As a first illustration, let us do a basic computation, as follows:

Proposition 15.9. *For a point charge $q\in\mathbb{R}$ at the center of a sphere S,*

$$\Phi_E(S)=\frac{q}{\varepsilon_0}$$

where the constant is $\varepsilon_0=1/(4\pi K)$, independently of the radius of S.

Proof. Assuming that S has radius r, we have the following computation:

$$\begin{aligned}\Phi_E(S)&=\int_S<E(x),n(x)>dx\\&=\int_S\left\langle\frac{Kqx}{r^3},\frac{x}{r}\right\rangle dx\\&=\int_S\frac{Kq}{r^2}\,dx\\&=\frac{Kq}{r^2}\times 4\pi r^2\\&=4\pi Kq\end{aligned}$$

Thus with $\varepsilon_0=1/(4\pi K)$ as above, we obtain the result. □

More generally now, we have the following result:

THEOREM 15.10. *The flux of a field E through a sphere S is given by*

$$\Phi_E(S)=\frac{Q_{enc}}{\varepsilon_0}$$

where Q_{enc} is the total charge enclosed by S, and $\varepsilon_0=1/(4\pi K)$.

PROOF. This can be done in several steps, as follows:

(1) Before jumping into computations, let us do some manipulations. First, by discretizing the problem, we can assume that we are dealing with a system of point charges. Moreover, by additivity, we can assume that we are dealing with a single charge. And if we denote by $q\in\mathbb{R}$ this charge, located at $v\in\mathbb{R}^3$, we want to prove that we have the following formula, where $B\subset\mathbb{R}^3$ denotes the ball enclosed by S:

$$\Phi_E(S)=\frac{q}{\varepsilon_0}\,\delta_{v\in B}$$

(2) By linearity we can assume that we are dealing with the unit sphere S. Moreover, by rotating we can assume that our charge q lies on the Ox axis, that is, that we have $v=(r,0,0)$ with $r\geq 0$, $r\neq 1$. The formula that we want to prove becomes:

$$\Phi_E(S)=\frac{q}{\varepsilon_0}\,\delta_{r<1}$$

(3) Let us start now the computation. With $u=(x,y,z)$, we have:

$$\begin{aligned}
\Phi_E(S) &= \int_S <E(u),u>du\\
&= \int_S\left\langle\frac{Kq(u-v)}{||u-v||^3},u\right\rangle du\\
&= Kq\int_S\frac{<u-v,u>}{||u-v||^3}\,du\\
&= Kq\int_S\frac{1-<v,u>}{||u-v||^3}\,du\\
&= Kq\int_S\frac{1-rx}{(1-2xr+r^2)^{3/2}}\,du
\end{aligned}$$

(4) In order to compute the above integral, we will use spherical coordinates for the unit sphere S, which are as follows, with $s\in[0,\pi]$ and $t\in[0,2\pi]$:

$$\begin{cases}x=\cos s\\ y=\sin s\cos t\\ z=\sin s\sin t\end{cases}$$

We recall that the corresponding Jacobian, computed before, is given by:

$$J = \sin s$$

(5) With the above change of coordinates, our integral from (3) becomes:

$$\begin{aligned}
\Phi_E(S) &= Kq\int_S \frac{1-rx}{(1-2xr+r^2)^{3/2}}\,du \\
&= Kq\int_0^{2\pi}\int_0^{\pi}\frac{1-r\cos s}{(1-2r\cos s+r^2)^{3/2}}\cdot\sin s\,ds\,dt \\
&= 2\pi Kq\int_0^{\pi}\frac{(1-r\cos s)\sin s}{(1-2r\cos s+r^2)^{3/2}}\,ds \\
&= \frac{q}{2\varepsilon_0}\int_0^{\pi}\frac{(1-r\cos s)\sin s}{(1-2r\cos s+r^2)^{3/2}}\,ds
\end{aligned}$$

(6) The point now is that the integral on the right can be computed with the change of variables $x=\cos s$. Indeed, we have $dx=-\sin s\,ds$, and we obtain:

$$\begin{aligned}
\int_0^{\pi}\frac{(1-r\cos s)\sin s}{(1-2r\cos s+r^2)^{3/2}}\,ds &= \int_{-1}^{1}\frac{1-rx}{(1-2rx+r^2)^{3/2}}\,dx \\
&= \left[\frac{x-r}{\sqrt{1-2rx+r^2}}\right]_{-1}^{1} \\
&= \frac{1-r}{\sqrt{1-2r+r^2}}-\frac{-1-r}{\sqrt{1+2r+r^2}} \\
&= \frac{1-r}{|1-r|}+1 \\
&= 2\delta_{r<1}
\end{aligned}$$

Thus, we are led to the formula in the statement. □

More generally now, we have the following key result, due to Gauss:

THEOREM 15.11 (Gauss law). *The flux of a field E through a surface S is given by*

$$\Phi_E(S) = \frac{Q_{enc}}{\varepsilon_0}$$

where Q_{enc} is the total charge enclosed by S, and $\varepsilon_0 = 1/(4\pi K)$.

PROOF. This basically follows from Theorem 15.10, or even from Proposition 15.9, by adding to the results there a number of new ingredients, as follows:

(1) Our first claim is that given a closed surface S, with no charges inside, the flux through it of any choice of external charges vanishes:

$$\Phi_E(S) = 0$$

This claim is indeed supported by the intuitive interpretation of the flux, as corresponding to the signed number of field lines crossing S. Indeed, any field line entering as $+$ must exit somewhere as $-$, and vice versa, so when summing we get 0.

(2) In practice now, in order to prove this rigorously, there are several ways. A standard argument, which is quite elementary, is the one used by Feynman in [34], based on the fact that, due to $F \sim 1/d^2$, local deformations of S will leave invariant the flux, and so in the end we are left with a rotationally invariant surface, where the result is clear.

(3) The point now is that, with this and Proposition 15.9 in hand, we can finish by using a standard math trick. Let us assume indeed, by discretizing, that our system of charges is discrete, consisting of enclosed charges $q_1, \ldots, q_k \in \mathbb{R}$, and an exterior total charge Q_{ext}. We can surround each of $q_1, \ldots, q_k$ by small disjoint spheres $U_1, \ldots, U_k$, chosen such that their interiors do not touch S, and we have:

$$\begin{aligned}
\Phi_E(S) &= \Phi_E(S - \cup U_i) + \Phi_E(\cup U_i)\\
&= 0 + \Phi_E(\cup U_i)\\
&= \sum_i \Phi_E(U_i)\\
&= \sum_i \frac{q_i}{\varepsilon_0}\\
&= \frac{Q_{enc}}{\varepsilon_0}
\end{aligned}$$

(4) To be more precise, in the above the union $\cup U_i$ is a usual disjoint union, and the flux is of course additive over components. As for the difference $S - \cup U_i$, this is by definition the disjoint union of S with the disjoint union $\cup(-U_i)$, with each $-U_i$ standing for U_i with orientation reversed, and since this difference has no enclosed charges, the flux through it vanishes by (2). Finally, the end makes use of Proposition 15.9. □

We have the following point of view on the Gauss formula, more conceptual:

Theorem 15.12 (Gauss). *Given an electric potential E, its divergence is given by*

$$< \nabla, E > = \frac{\rho}{\varepsilon_0}$$

where ρ denotes as usual the charge distribution. Also, we have

$$\nabla \times E = 0$$

meaning that the curl of E vanishes.

Proof. The first formula, called Gauss law in differential form, is clear. Regarding now the curl, by discretizing and linearity we can assume that we are dealing with a single charge q, positioned at 0. We have, by using spherical coordinates r, s, t:

$$\begin{aligned}\int_a^b < E(x), dx > &= \int_a^b \left\langle \frac{Kqx}{||x||^3}, dx \right\rangle \\ &= \int_a^b \left\langle \frac{Kq}{r^2} \cdot \frac{x}{||x||}, dx \right\rangle \\ &= \int_a^b \frac{Kq}{r^2}\, dr \\ &= \left[-\frac{Kq}{r} \right]_a^b \\ &= Kq \left(\frac{1}{r_a} - \frac{1}{r_b} \right)\end{aligned}$$

In particular the integral of E over any closed loop vanishes, and by using now the Stokes theorem, we conclude that the curl of E vanishes, as stated. □

In order to further advance, let us go back to the various plane curves discussed in chapter 8. Quite remarkably, most of that curves are sinusoidal spirals, in the following sense, and with actually the term "sinusoidal spiral" being a bit unfortunate:

Theorem 15.13. *The sinusoidal spirals, which are as follows,*

$$r^n = a^n \cos n\theta$$

with $a \neq 0$ and $n \in \mathbb{Q} - \{0\}$, include the following curves:

(1) $n = -1$ *line.*
(2) $n = 1$ *circle,* $n = -1/2$ *parabola,* $n = -2$ *hyperbola.*
(3) $n = -3$ *Humbert cubic,* $n = -1/3$ *Tschirnhausen curve.*
(4) $n = 1/2$ *cardioid,* $n = 2$ *Bernoulli lemniscate.*
(5) $n = 3$ *Kiepert trefoil,* $n = 1/3$ *Cayley sextic.*

Proof. We first have to prove that the sinusoidal spirals are indeed algebraic curves. But this is best done by using the complex coordinate $z = re^{i\theta}$, as follows:

$$\begin{aligned}r^n = a^n \cos n\theta &\iff r^n \cos n\theta = \left(\frac{r^2}{a} \right)^n \\ &\iff z^n + \bar{z}^n = 2 \left(\frac{z\bar{z}}{a} \right)^n \\ &\iff (x+iy)^n + (x-iy)^n = 2 \left(\frac{x^2+y^2}{a} \right)^n\end{aligned}$$

As a first observation now, in the case $n\in\mathbb N$ we can simply use the binomial formula, and we get an algebraic equation of degree $2n$, as follows:

$$\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}x^{n-2k}y^{2k}=\left(\frac{x^2+y^2}{a}\right)^n$$

In general, things are a bit more complicated, as shown for instance by our computation for the Cayley sextic. However, the same idea as there applies, and we are led in this way to the equation of an algebraic curve, as claimed. Regarding now the examples:

(1) At $n=-1$ the equation is as follows, producing a line:

$$r\cos\theta=a\iff x=a$$

(2) At $n=1$ the equation is as follows, producing a circle:

$$r=a\cos\theta\iff r^2=ax\iff x^2+y^2=ax$$

(3) At $n=-1/2$ the equation is as follows, producing a parabola:

$$a=r\cos^2(\theta/2)\iff r+x=2a\iff y^2=4a(a-x)$$

(4) At $n=-2$ the equation is as follows, producing a hyperbola:

$$a^2=r\cos^2 2\theta\iff a^2=2x^2-r^2\iff (x+y)(x-y)=a^2$$

(5) At $n=-3$ the equation is as follows, producing a curve with 3 components, which looks like some sort of "trivalent hyperbola", called Humbert cubic:

$$r^3\cos 3\theta=a^3\iff z^3+\bar z^3=2a^3\iff x^3-3xy^2=a^3$$

(6) As for the other curves, this follows from our various formulae above. □

Let us study now more in detail the sinusoidal spirals. We first have:

Proposition 15.14. *The sinusoidal spirals, which with $z=x+iy$ are*

$$z^n+\bar z^n=2\left(\frac{z\bar z}{a}\right)^n$$

with $a\neq 0$ and $n\in\mathbb Q-\{0\}$, are as follows:

1. *With $n=-m$, $m\in\mathbb N$, the equation is $z^m+\bar z^m=2a^m$, degree m.*
2. *With $n=m$, $m\in\mathbb N$, the equation is $z^m+\bar z^m=2(z\bar z/a)^m$, degree $2m$.*
3. *With $n=-1/m$, $m\in\mathbb N$, the equation is $(z^{1/m}+\bar z^{1/m})^m=2^m a$.*
4. *With $n=1/m$, $m\in\mathbb N$, the equation is $(z^{1/m}+\bar z^{1/m})^m=2^m z\bar z/a$.*

Proof. This is something self-explanatory, the details being as follows:

(1) With $n=-m$ and $m\in\mathbb N$ as in the statement, the equation is, as claimed:

$$z^{-m}+\bar z^{-m}=2\left(\frac{z\bar z}{a}\right)^{-m}\iff z^m+\bar z^m=2a^m$$

(2) This is an empty statement, just a matter of using the new variable $m=n$.

(3) With $n=-1/m$ and $m\in\mathbb N$ as in the statement, the equation is, as claimed:

$$z^{-1/m}+\bar{z}^{-1/m}=2\left(\frac{z\bar{z}}{a}\right)^{-1/m}\iff z^{1/m}+\bar{z}^{1/m}=2a^{1/m}$$
$$\iff (z^{1/m}+\bar{z}^{1/m})^m=2^m a$$

(4) With $n=1/m$ and $m\in\mathbb N$ as in the statement, the equation is, as claimed:

$$z^{1/m}+\bar{z}^{1/m}=2\left(\frac{z\bar{z}}{a}\right)^{1/m}\iff (z^{1/m}+\bar{z}^{1/m})^m=2^m\cdot\frac{z\bar{z}}{a}$$

Thus, we are led to the conclusions in the statement. □

Observe that in the fractionary cases, $n=\pm 1/m$, the equations in the above statement are not polynomial in x,y, unless at very small values of m. To be more precise:

(1) In the case $n=-1/m$, we certainly have at $m=1,2,3$ the $d=1$ line, $d=2$ parabola, and $d=3$ Tschirnhausen curve, but at $m=4$ things change, with the equation $(z^{1/4}+\bar{z}^{1/4})^4=16a$ being no longer polynomial in x,y, and requiring a further square operation to make it polynomial, and therefore leading to a curve of degree $d=8$.

(2) As for the case $n=1/m$, this is more complicated, with the data that we have at $m=1,2,3$, namely the $d=2$ circle, $d=3$ cardioid, and $d=6$ Cayley sextic, being not very good, and with things getting even more complicated at $m=4$ and higher.

In short, things quite complicated, and the general case, $n=\pm p/q$ with $p,q\in\mathbb N$, is certainly even more complicated. Instead of insisting on this, let us focus now on the simplest sinusoidal spirals that we have, namely those with $n=\pm m$, with $m\in\mathbb N$.

The point indeed is that the sinusoidal spirals with $n\in\mathbb N$ are also part of another remarkable family of plane algebraic curves, going back to Cassini, as follows:

THEOREM 15.15. *The polynomial lemniscates, which are as follows,*

$$|P(z)|=b^n$$

with $P\in\mathbb C[X]$ having n distinct roots, and $b>0$, include the following curves:

(1) *The sinusoidal spirals with $n\in\mathbb N$, including the $n=1$ circle, $n=2$ Bernoulli lemniscate, and $n=3$ Kiepert trefoil.*
(2) *The Cassini ovals, which are the quartics given by $|z+c|\cdot|z-c|=b^2$, covering too the Bernoulli lemniscate, appearing at $b=c$.*

PROOF. This is something quite self-explanatory, the details being as follows:

(1) Regarding the sinusoidal spirals with $n \in \mathbb{N}$, their equation is, with $a^n = 2c^n$:

$$\begin{aligned} z^n + \bar{z}^n = 2\left(\frac{z\bar{z}}{a}\right)^n &\iff c^n(z^n + \bar{z}^n) = (z\bar{z})^n \\ &\iff (z^n - c^n)(\bar{z}^n - c^n) = c^{2n} \\ &\iff |z^n - c^n| = c^n \end{aligned}$$

(2) Regarding the Cassini ovals, these correspond to the case where the polynomial $P \in \mathbb{C}[X]$ has degree 2, and we already know from the above that these cover the Bernoulli lemniscate. In general, the equation for the Cassini ovals is:

$$\begin{aligned} |z+c| \cdot |z-c| = b^2 &\iff |z^2 - c^2| = b^2 \\ &\iff (z^2 - c^2)(\bar{z}^2 - c^2) = b^4 \\ &\iff (z\bar{z})^2 - c^2(z^2 + \bar{z}^2) + c^4 = b^4 \\ &\iff (x^2 + y^2)^2 - c^2(x^2 - y^2) + c^4 = b^4 \\ &\iff (x^2 + y^2)^2 = c^2(x^2 - y^2) + b^4 - c^4 \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

The polynomial lemniscates can be geometrically understood as follows:

THEOREM 15.16. *The equation $|P(z)| = b$ defining the polynomial lemniscates can be written as follows, in terms of the roots $c_1, \ldots, c_n$ of the polynomial P,*

$$\sqrt[n]{\prod_{k=1}^n |z - c_i|} = b$$

telling us that the geometric mean of the distances from z to the vertices of the polygon formed by $c_1, \ldots, c_n$ must be the constant $b > 0$.

PROOF. This is something self-explanatory, and as an illustration, let us work out the case of sinusoidal spirals with $n \in \mathbb{N}$. Here with $w = e^{2\pi i/n}$ we have:

$$z^n - c^n = \prod_{k=1}^n (z - cw^k)$$

Thus, the sinusoidal spiral equation reformulates as follows:

$$|z^n - c^n| = c^n \iff \prod_{k=1}^n |z - cw^k| = c^n \iff \sqrt[n]{\prod_{k=1}^n |z - cw^k|} = c$$

Thus, for a sinusoidal spiral with positive integer parameter, the geometric mean of the distances to the vertices of a regular polygon must equal the radius of the polygon. □

Regarding now the sinusoidal spirals with $n \in -\mathbb{N}$, these are too part of another remarkable family of plane algebraic curves, constructed as follows:

THEOREM 15.17. *Given points in the plane $c_1, \ldots, c_n \in \mathbb{C}$ and a number $d \in \mathbb{R}$, construct the associated stelloid as being the set of points $z \in \mathbb{C}$ verifying*

$$\frac{1}{n}\sum_{k=1}^{n}\alpha_v(z-c_i)=d$$

with α_v denoting the angle with respect to a direction v. Then the stelloid is an algebraic curve, not depending on v, and at the level of examples we have the sinusoidal spirals with $n \in -\mathbb{N}$, including the $n=-1$ line, $n=-2$ hyperbola, and $n=-3$ Humbert cubic.

PROOF. All this is quite self-explanatory, and we will leave the verification of the various generalities regarding the stelloids, as well as the verification of the relation with the sinusoidal spirals with $n \in -\mathbb{N}$, as an instructive exercise. As a bonus exercise, try understanding the precise relation between stelloids, and polynomial lemniscates. □

So long for plane algebraic curves. Needless to say, all the above is old-style, first class mathematics, having countless applications. For instance when doing classical mechanics or electrodynamics, you will certainly meet polynomial lemniscates and stelloids, when looking at the field lines. Also, the image of any circle passing though 0 by $z \to z^2$ is a cardioid, and the famous Mandelbrot set is organized around such a cardioid.

15b.

15c.

15d.

15e. Exercises

Exercises:

EXERCISE 15.18.

EXERCISE 15.19.

EXERCISE 15.20.

EXERCISE 15.21.

EXERCISE 15.22.

EXERCISE 15.23.

EXERCISE 15.24.

EXERCISE 15.25.

Bonus exercise.

CHAPTER 16

Angle of attack

16a. Angle of attack

16b.

16c.

16d.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

Bibliography

[1] A.A. Abrikosov, Fundamentals of the theory of metals, Dover (1988).

[2] V.I. Arnold, Ordinary differential equations, Springer (1973).

[3] V.I. Arnold, Lectures on partial differential equations, Springer (1997).

[4] V.I. Arnold, Catastrophe theory, Springer (1984).

[5] N.W. Ashcroft and N.D. Mermin, Solid state physics, Saunders College Publ. (1976).

[6] T. Banica, Calculus and applications (2024).

[7] T. Banica, Linear algebra and group theory (2024).

[8] T. Banica, Introduction to modern physics (2024).

[9] G.K. Batchelor, An introduction to fluid dynamics, Cambridge Univ. Press (1967).

[10] M.J. Benton, Vertebrate paleontology, Wiley (1990).

[11] M.J. Benton and D.A.T. Harper, Introduction to paleobiology and the fossil record, Wiley (2009).

[12] S.J. Blundell and K.M. Blundell, Concepts in thermal physics, Oxford Univ. Press (2006).

[13] B. Bollobás, Modern graph theory, Springer (1998).

[14] S.M. Carroll, Spacetime and geometry, Cambridge Univ. Press (2004).

[15] P.M. Chaikin and T.C. Lubensky, Principles of condensed matter physics, Cambridge Univ. Press (1995).

[16] A.R. Choudhuri, Astrophysics for physicists, Cambridge Univ. Press (2012).

[17] J. Clayden, S. Warren and N. Greeves, Organic chemistry, Oxford Univ. Press (2012).

[18] D.D. Clayton, Principles of stellar evolution and nucleosynthesis, Univ. of Chicago Press (1968).

[19] W.N. Cottingham and D.A. Greenwood, An introduction to the standard model of particle physics, Cambridge Univ. Press (2012).

[20] A. Cottrell, An introduction to metallurgy, CRC Press (1997).

[21] C. Darwin, On the origin of species (1859).

[22] P.A. Davidson, Introduction to magnetohydrodynamics, Cambridge Univ. Press (2001).

[23] P.A.M. Dirac, Principles of quantum mechanics, Oxford Univ. Press (1930).

[24] S. Dodelson, Modern cosmology, Academic Press (2003).

[25] S.T. Dougherty, Combinatorics and finite geometry, Springer (2020).

[26] M. Dresher, The mathematics of games of strategy, Dover (1981).

[27] R. Durrett, Probability: theory and examples, Cambridge Univ. Press (1990).

[28] F. Dyson, Origins of life, Cambridge Univ. Press (1984).

[29] A. Einstein, Relativity: the special and the general theory, Dover (1916).

[30] L.C. Evans, Partial differential equations, AMS (1998).

[31] W. Feller, An introduction to probability theory and its applications, Wiley (1950).

[32] E. Fermi, Thermodynamics, Dover (1937).

[33] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics I: mainly mechanics, radiation and heat, Caltech (1963).

[34] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics II: mainly electromagnetism and matter, Caltech (1964).

[35] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics III: quantum mechanics, Caltech (1966).

[36] R.P. Feynman and A.R. Hibbs, Quantum mechanics and path integrals, Dover (1965).

[37] P. Flajolet and R. Sedgewick, Analytic combinatorics, Cambridge Univ. Press (2009).

[38] A.P. French, Special relativity, Taylor and Francis (1968).

[39] J.H. Gillespie, Population genetics, Johns Hopkins Univ. Press (1998).

[40] C. Godsil and G. Royle, Algebraic graph theory, Springer (2001).

[41] H. Goldstein, C. Safko and J. Poole, Classical mechanics, Addison-Wesley (1980).

[42] D.L. Goodstein, States of matter, Dover (1975).

[43] D.J. Griffiths, Introduction to electrodynamics, Cambridge Univ. Press (2017).

[44] D.J. Griffiths and D.F. Schroeter, Introduction to quantum mechanics, Cambridge Univ. Press (2018).

[45] D.J. Griffiths, Introduction to elementary particles, Wiley (2020).

[46] D.J. Griffiths, Revolutions in twentieth-century physics, Cambridge Univ. Press (2012).

[47] V.P. Gupta, Principles and applications of quantum chemistry, Elsevier (2016).

[48] W.A. Harrison, Solid state theory, Dover (1970).

[49] W.A. Harrison, Electronic structure and the properties of solids, Dover (1980).

[50] R.A. Horn and C.R. Johnson, Matrix analysis, Cambridge Univ. Press (1985).

[51] C.E. Housecroft and A.G. Sharpe, Inorganic chemistry, Pearson (2018).

[52] K. Huang, Introduction to statistical physics, CRC Press (2001).

[53] K. Huang, Fundamental forces of nature, World Scientific (2007).

[54] S. Huskey, The skeleton revealed, Johns Hopkins Univ. Press (2017).

[55] L. Hyman, Comparative vertebrate anatomy, Univ. of Chicago Press (1942).

[56] L.P. Kadanoff, Statistical physics: statics, dynamics and renormalization, World Scientific (2000).

[57] T. Kibble and F.H. Berkshire, Classical mechanics, Imperial College Press (1966).

[58] C. Kittel, Introduction to solid state physics, Wiley (1953).

[59] D.E. Knuth, The art of computer programming, Addison-Wesley (1968).

[60] M. Kumar, Quantum: Einstein, Bohr, and the great debate about the nature of reality, Norton (2009).

[61] T. Lancaster and K.M. Blundell, Quantum field theory for the gifted amateur, Oxford Univ. Press (2014).

[62] L.D. Landau and E.M. Lifshitz, Mechanics, Pergamon Press (1960).

[63] L.D. Landau and E.M. Lifshitz, The classical theory of fields, Addison-Wesley (1951).

[64] L.D. Landau and E.M. Lifshitz, Quantum mechanics: non-relativistic theory, Pergamon Press (1959).

[65] S. Lang, Algebra, Addison-Wesley (1993).

[66] P. Lax, Linear algebra and its applications, Wiley (2007).

[67] P. Lax, Functional analysis, Wiley (2002).

[68] P. Lax and M.S. Terrell, Calculus with applications, Springer (2013).

[69] P. Lax and M.S. Terrell, Multivariable calculus with applications, Springer (2018).

[70] S. Ling and C. Xing, Coding theory: a first course, Cambridge Univ. Press (2004).

[71] J.P. Lowe and K. Peterson, Quantum chemistry, Elsevier (2005).

[72] S.J. Marshall, The story of the computer: a technical and business history, Create Space Publ. (2022).

[73] M.L. Mehta, Random matrices, Elsevier (2004).

[74] M.A. Nielsen and I.L. Chuang, Quantum computation and quantum information, Cambridge Univ. Press (2000).

[75] R.K. Pathria and and P.D. Beale, Statistical mechanics, Elsevier (1972).

[76] T.D. Pollard, W.C. Earnshaw, J. Lippincott-Schwartz and G. Johnson, Cell biology, Elsevier (2022).

[77] J. Preskill, Quantum information and computation, Caltech (1998).

[78] R. Rojas and U. Hashagen, The first computers: history and architectures, MIT Press (2000).

[79] W. Rudin, Principles of mathematical analysis, McGraw-Hill (1964).

[80] W. Rudin, Real and complex analysis, McGraw-Hill (1966).

[81] W. Rudin, Functional analysis, McGraw-Hill (1973).

[82] B. Ryden, Introduction to cosmology, Cambridge Univ. Press (2002).

[83] B. Ryden and B.M. Peterson, Foundations of astrophysics, Cambridge Univ. Press (2010).

[84] D.V. Schroeder, An introduction to thermal physics, Oxford Univ. Press (1999).

[85] R. Shankar, Fundamentals of physics I: mechanics, relativity, and thermodynamics, Yale Univ. Press (2014).

[86] R. Shankar, Fundamentals of physics II: electromagnetism, optics, and quantum mechanics, Yale Univ. Press (2016).

[87] N.J.A. Sloane and S. Plouffe, Encyclopedia of integer sequences, Academic Press (1995).

[88] A.M. Steane, Thermodynamics, Oxford Univ. Press (2016).

[89] S. Sternberg, Dynamical systems, Dover (2010).

[90] D.R. Stinson, Combinatorial designs: constructions and analysis, Springer (2006).

[91] J.R. Taylor, Classical mechanics, Univ. Science Books (2003).

[92] J. von Neumann, Mathematical foundations of quantum mechanics, Princeton Univ. Press (1955).

[93] J. von Neumann and O. Morgenstern, Theory of games and economic behavior, Princeton Univ. Press (1944).

[94] J. Watrous, The theory of quantum information, Cambridge Univ. Press (2018).

[95] S. Weinberg, Foundations of modern physics, Cambridge Univ. Press (2011).

[96] S. Weinberg, Lectures on quantum mechanics, Cambridge Univ. Press (2012).

[97] S. Weinberg, Lectures on astrophysics, Cambridge Univ. Press (2019).

[98] H. Weyl, The theory of groups and quantum mechanics, Princeton Univ. Press (1931).

[99] H. Weyl, The classical groups: their invariants and representations, Princeton Univ. Press (1939).

[100] H. Weyl, Space, time, matter, Princeton Univ. Press (1918).

Index