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Article *in* Contributions to Mathematics · July 2024

DOI: 10.47443/cm.2024.035

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Contributions to Mathematics
www.shahindp.com/locate/cm

Contrib. Math. **9** (2024) 56–61
DOI: 10.47443/cm.2024.035

Research Article

A note on some classes of series for the logarithmic function

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(Received: 10 June 2024. Received in revised form: 1 July 2024. Accepted: 2 July 2024. Published online: 3 July 2024.)

Abstract

A different and technical proof for a series representation of $\log s$ is provided. The mentioned series representation was used to reproduce some Bailey-Borwein-Plouffe (BBP) type formulas for a variety of numbers, including some classical BBP type formulas for π. A series representation, similar to that of $\log s$, for $\log^2(s)$ is also derived.

Keywords: iterated integrals; Bailey-Borwein-Plouffe formula; harmonic numbers.

2020 Mathematics Subject Classification: 26A09, 11A25.

1. Introduction

In [2], the authors defined the iterated primitives of the function $f(s) = \frac{1}{s}$ by $I_0(s) = \frac{1}{s}$ and

$$I_{n+1}(s) = \int_1^s I_n(u)du = \int_1^s \int_1^{u_n} \cdots \int_1^{u_0} \frac{1}{u_0}\, du_0\, du_1 \cdots du_{n-1}\, du_n \tag{1}$$

for $n \geq 1$. They called $I_n(s)$ the Laplacian-Hadamard regularization or the Laplace-Hadamard transform of $\frac{1}{t^n}$. They also proved that

$$I_n(s) = A_n(s)\log s + B_n(s), \tag{2}$$

where

$$A_n(s) = \frac{s^{n-1}}{(n-1)!},$$

$$B_n(s) = -\frac{1}{(n-1)!}\sum_{k=1}^{n-1}\binom{n-1}{k}(H_{n-1} - H_{n-1-k})(s-1)^k, \tag{3}$$

and $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, which is the n-th harmonic number. By applying (1) and the Taylor series expansion of the function $\log s$ at $s = 1$, they derived for $|s-1| < 1$ or $|s-1| = 1$ and $n \geq 2$ the following identity:

$$I_n(s) = \frac{(-1)^n}{n!}\sum_{j=0}^{\infty}\frac{(1-s)^{j+n}}{\binom{n+j}{j}}. \tag{4}$$

By combining (2) and (4), the authors in [2] derived the following series representation for $\log s$:

$$\log s = -\frac{(n-1)!}{s^{n-1}}B_n(s) + (-1)^n\frac{1}{ns^{n-1}}\sum_{j=0}^{\infty}\frac{(1-s)^{j+n}}{\binom{n+j}{j}}, \tag{5}$$

where $B_n(s)$ is as given in (3). By substituting particular values of s in (5) and considering real and imaginary parts separately, they obtained Bailey-Borwein-Plouffe (BBP) type formulas for a variety of numbers, including the following classical BBP type series for π:

$$\pi = \sum_{k=0}^{\infty}\frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right) \quad \text{(the BBP formula, see [1])},$$

and

$$\pi = \frac{1}{64}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{10k}}\left(-\frac{32}{4k+1} - \frac{1}{4k+3} + \frac{256}{10k+1} - \frac{64}{10k+3} - \frac{4}{10k+5} - \frac{4}{10k+7} + \frac{1}{10k+9}\right) \quad \text{(see [9])}.$$

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For an elementary approach to BBP type formulas for π, the reader is referred to the recent paper [11], and for some related results, the reader is referred to [12].

This article provides a completely different and novel proof for (5). A new series representation of type (5) for $\log^2(s)$ is also offered. The proofs are primarily based on the binomial theorem.

2. Preliminaries

For our purpose, we recall the following definitions and basic properties of some special functions and special numbers. The gamma function is defined by $\Gamma(z) = \int_0^\infty u^{z-1}e^{-u}du$ $(z > 0)$. The psi function or the digamma function, denoted by $\psi(x)$, is given by $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. The polygamma function $\psi^{(n)}$ is defined by

$$\psi^{(n)}(z) = \frac{d^n}{dz^n}\log(\Gamma(z)) = \frac{d^n}{dz^n}\psi(z), \quad n \geq 0,\ z \in \mathbb{C}\backslash\mathbb{Z}_{\leq 0}.$$

The harmonic numbers of order ℓ are known in the literature to be

$$H_n^{(\ell)} = \sum_{k=1}^{n} \frac{1}{k^\ell}.$$

When $\ell = 1$, they reduce to the classical harmonic numbers

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Throughout this article, $\mathbb{Z}_{\leqslant \ell}$ $(\mathbb{Z}_{\geqslant \ell})$ denotes the set of integers less than or equal to (greater than or equal to) some ℓ belonging to $\mathbb{Z}$. The following relation between the harmonic numbers and the digamma function is valid:

$$\psi(n+1) = \gamma + H_n, \tag{6}$$

for $n \in \mathbb{Z}_{\geq 0}$, where $\gamma = \lim_{n\to\infty}(H_n - \log n) = 0.57721\cdots$ is the Euler-Mascheroni constant, and

$$H_n^{(m+1)} = \zeta(m+1) + \frac{(-1)^m}{m!}\psi^{(m)}(n+1)$$

(see [13, Section 1.2]). A generalized binomial coefficient $\binom{s}{t}$ is given, in terms of the gamma function, by

$$\binom{s}{t} = \frac{\Gamma(s+t)}{\Gamma(t+1)\Gamma(s-t+1)} \quad (s, t \in \mathbb{C}). \tag{7}$$

Taking into account that $\frac{1}{\Gamma(k)} = 0$ for $k \in \mathbb{Z}_{\leq 0}$, it is clear from this definition that

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if } n \geq k \\ 0 & \text{if } k > n, \end{cases} \tag{8}$$

for $n, k \in \mathbb{Z}_{\geq 0}$. We now present a series of lemmas, which are used in the next section to prove the main results. Throughout this article, we employ the usual convention that an empty sum is taken to be zero.

Lemma 2.1. *If* $k \in \mathbb{Z}_{\geq 0}$, *then*

$$\lim_{x\to -k} \frac{\psi(x)}{\Gamma(x)} = (-1)^{k-1}k! \quad \textit{and} \tag{9}$$

$$\lim_{x\to -k} \frac{\psi^2(x) - \psi'(x)}{\Gamma(x)} = 2(-1)^{k-1}k!\,\psi(k+1). \tag{10}$$

Proof. The proof can be found in [6, Lemma 1] (see also [7]). □

Lemma 2.2. *If* n *is a non-negative integer, then*

$$\sum_{k=n+1}^{\infty} (-1)^k \binom{n}{k} \psi(n-k+1)t^k = (-1)^n \sum_{k=1}^{\infty} \frac{t^{n+k}}{k\binom{n+k}{k}} \quad \textit{and} \tag{11}$$

$$\sum_{k=n+1}^{\infty} (-1)^k \binom{n}{k} \left(\psi^2(n-k+1) - \psi'(n-k+1)\right)t^k = 2(-1)^n \sum_{k=1}^{\infty} \frac{\psi(k)t^{n+k}}{k\binom{n+k}{k}}. \tag{12}$$

Proof. By making the substitution $n-k-1=k'$ and utilizing (9), one has

$$\begin{aligned}\sum_{k=n+1}^{\infty}(-1)^k\binom{n}{k}\psi(n-k+1)t^k &= \sum_{k=0}^{\infty}(-1)^{n+k+1}\frac{n!}{(n+k+1)!}\frac{\psi(-k)}{\Gamma(-k)}t^{n+k+1}\\ &= \sum_{k=0}^{\infty}(-1)^{n+k+1}\frac{n!}{(n+k+1)!}\lim_{u\to -k}\frac{\psi(u)}{\Gamma(u)})t^{n+k+1}\\ &= \sum_{k=0}^{\infty}(-1)^{n+k+1}\frac{n!}{(n+k+1)!}(-1)^{k-1}\,k!\,t^{n+k+1}\\ &= (-1)^n\sum_{k=1}^{\infty}\frac{t^{n+k}}{k\binom{n+k}{k}},\end{aligned}$$

which proves (11). By employing (10), the identity (12) can be proved in the same way as used in the proof of (11). □

Lemma 2.3. *The following identities hold:*

$$\Gamma(z)\,\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}\quad(z\in\mathbb{C}\setminus\mathbb{Z}); \tag{13}$$

$$\psi(z+1)=\psi(z)+\frac{1}{z}; \tag{14}$$

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}=O\left(z^{\alpha-\beta}\right)\quad(|z|\to\infty,\ |\arg z|<\pi)\,; \tag{15}$$

$$\psi(z)=\log z+O\left(\frac{1}{z}\right)\quad(|z|\to\infty,\ |\arg z|<\pi)\,; \tag{16}$$

Proof. All the identities presented in this lemma are well known in the field (for example, see [7]). □

3. Main results

Theorem 3.1. *Let* $n\in\mathbb{N}$*. For* $|s-1|<1$ *or* $|s-1|=1$ *and* $n\geq 2$*, the following identity holds:*

$$\log s=-\frac{(n-1)!}{s^{n-1}}B_n(s)+\frac{(-1)^n}{ns^{n-1}}\sum_{j=0}^{\infty}\frac{(1-s)^{j+n}}{\binom{n+j}{j}}, \tag{17}$$

where $B_n(s)$ *is same as given in* (3)*.*

Proof. By the general binomial theorem, for $t\in(-1,1)$ and $x\in\mathbb{C}\backslash\mathbb{Z}_{\leq 0}$, we have

$$\sum_{k=0}^{\infty}(-1)^k\binom{x}{k}t^k=(1-t)^x. \tag{18}$$

We differentiate both sides of this equation with respect to the variable x. This gives

$$\sum_{k=0}^{\infty}(-1)^k\binom{x}{k}\big[\psi(x+1)-\psi(x-k+1)\big]t^k=(1-t)^x\log(1-t). \tag{19}$$

In order to justify termwise differentiation of the left-hand side of (18), it suffices to show that the left-hand side of (19) converges uniformly for $x>1$ (see, e.g., [14, p. 231, Theorem 7.14]; see also [10, p. 53, Theorem 2.25]). In terms of convergence, it is enough to show that the case $r=1$:

$$\sum_{k=0}^{\infty}\binom{x}{k}^2\ |\psi(x+1)-\psi(x-k+1)| \tag{20}$$

converges uniformly on $x\in(1,M]$ for any fixed $M>1$. By using (13), we find

$$\binom{x}{k}=\frac{(-1)^{k-1}}{\pi}\ \sin(\pi x)\,\Gamma(x+1)\,\frac{\Gamma(k-x)}{\Gamma(k+1)},$$

which, upon employing (15), gives

$$\binom{x}{k}=O\left(k^{-x-1}\right)\quad(k\to\infty;\ x\in(1,M])\,. \tag{21}$$

By using (14) and (16), we find

$$\psi(x-k+1)=\psi(x-k)+\frac{1}{x-k}=O\left(\log(x-k)\right)=O\left(\log k\right)=O\left(k\right)\quad (k\to\infty),$$

which leads to

$$\psi(x+1)-\psi(x-k+1)=O\left(k\right)\quad (k\to\infty,\, x\in(1,M]). \tag{22}$$

By combining (21) and (22), we find

$$\binom{x}{k}\left|\psi(x+1)-\psi(x-k+1)\right|=O\left(k^{-x-1}\right)\quad (k\to\infty,\, x\in(1,M]).$$

Therefore, the series in (20) converges uniformly on $x\in(1,M]$ for any fixed $M>1$ when $-x<-1 \quad\Leftrightarrow\quad x>1$. Let n be a positive integer. Replacing x by n in (19), we get

$$\sum_{k=0}^{\infty}(-1)^k\binom{n}{k}\left[\psi(n+1)-\psi(n-k+1)\right]t^k=(1-t)^n\log(1-t). \tag{23}$$

We split the following summation into two parts. This leads to

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k\binom{n}{k}\left[\psi(n+1)-\psi(n-k+1)\right]t^k=&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[\psi(n+1)-\psi(n-k+1)\right]t^k\\ &+\sum_{k=n+1}^{\infty}(-1)^k\binom{n}{k}\left[\psi(n+1)-\psi(n-k+1)\right]t^k.\end{aligned} \tag{24}$$

By utilizing (6), we get

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[\psi(n+1)-\psi(n-k+1)\right]t^k=\sum_{k=0}^{n}(-1)^k\binom{n}{k}(H_n-H_{n-k})t^k. \tag{25}$$

On the other hand, we have

$$\sum_{k=n+1}^{\infty}(-1)^k\binom{n}{k}\left[\psi(n+1)-\psi(n-k+1)\right]t^k=\psi(n+1)\sum_{k=n+1}^{\infty}(-1)^k\binom{n}{k}t^k-\sum_{k=n+1}^{\infty}(-1)^k\binom{n}{k}\psi(n-k+1)t^k.$$

By using (8), we have

$$\sum_{k=n+1}^{\infty}(-1)^k\binom{n}{k}t^k=0.$$

Thus, by (11), we have

$$\begin{aligned}\sum_{k=n+1}^{\infty}(-1)^k\binom{n}{k}\left[\psi(n+1)-\psi(n-k+1)\right]t^k&=-\sum_{k=n+1}^{\infty}(-1)^k\binom{n}{k}\psi(n-k+1)t^k\\ &=(-1)^{n-1}\sum_{k=0}^{\infty}\frac{t^{n+k+1}}{(n+k+1)\binom{n+k}{k}}.\end{aligned}$$

Applying the relation

$$(n+k+1)\binom{n+k}{k}=(k+1)\binom{n+k+1}{k+1},$$

and then making the replacement $k\to k-1$ gives

$$\sum_{k=n+1}^{\infty}(-1)^k\binom{n}{k}\left[\psi(n+1)-\psi(n-k+1)\right]t^k=(-1)^{n-1}\sum_{k=1}^{\infty}\frac{t^{n+k}}{k\binom{n+k}{k}}. \tag{26}$$

By combining (23), (24), (25), and (26), we have

$$(1-t)^n\log(1-t)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[H_n-H_{n-k}\right]t^k+(-1)^{n-1}\sum_{k=1}^{\infty}\frac{t^{n+k}}{k\binom{n+k}{k}}. \tag{27}$$

Differentiating both sides of (27) with respect to t yields

$$-n(1-t)^{n-1}\log(1-t)-(1-t)^{n-1}=\sum_{k=1}^{n}(-1)^k k\binom{n}{k}\left[H_n-H_{n-k}\right]t^{k-1}+(-1)^{n-1}\sum_{k=1}^{\infty}\frac{(n+k)t^{n+k-1}}{k\binom{n+k}{k}}. \tag{28}$$

Since

$$k\binom{n}{k}=n\binom{n-1}{k-1} \quad \text{and} \quad \frac{n+k}{k\binom{n+k}{k}}=\frac{1}{\binom{n+k-1}{k-1}},$$

we conclude from (28) that

$$-n(1-t)^{n-1}\log(1-t)-(1-t)^{n-1}=n\sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}\left[H_n-H_{n-k}\right]t^{k-1}+(-1)^{n-1}\sum_{k=1}^{\infty}\frac{t^{n+k-1}}{\binom{n+k-1}{k-1}}.$$

By setting $t=1-s$ and making the substitution $k\to k+1$ (and after a short computation), we have

$$\log s=-\frac{1}{n}+\frac{1}{s^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}\left[H_n-H_{n-k-1}\right](s-1)^k+\frac{(-1)^n}{ns^{n-1}}\sum_{k=0}^{\infty}\frac{(1-s)^{n+k}}{\binom{n+k}{k}}.$$

Now, the desired result is obtained by using the simple relation $H_n=H_{n-1}+\frac{1}{n}$. □

Remark 3.1. *By setting $t=1-s$ in (27), we get the following elegant alternative representation for $\log s$:*

$$\log s=\frac{1}{s^n}\sum_{k=1}^{n}\binom{n}{k}\left[H_n-H_{n-k}\right](s-1)^k-\frac{(-1)^n}{s^n}\sum_{k=1}^{\infty}\frac{(1-s)^{n+k}}{k\binom{n+k}{k}}. \tag{29}$$

Theorem 3.2. *Let $n\in\mathbb{N}$. For $|s-1|<1$ or $|s-1|=1$ and $n\geq 1$, the following identity holds:*

$$\log^2(s)=2\sum_{k=1}^{n}\frac{H_{k-1}}{k}\left(1-\frac{1}{s}\right)^k-\frac{2(-1)^n}{s^n}\sum_{k=1}^{\infty}\frac{H_{k-1}-H_n}{k\binom{n+k}{k}}(1-s)^{n+k}. \tag{30}$$

Proof. Differentiating both sides of (18) with respect to x twice yields

$$(1-t)^x\log^2(1-t)=\sum_{k=0}^{\infty}(-1)^k\binom{x}{k}\left[(\psi(x+1)-\psi(x-k+1))^2+\psi'(x+1)-\psi'(x-k+1)\right]t^k.$$

We set $x=n$ here and then split the resultant summation into two parts. This gives

$$\begin{aligned}(1-t)^n\log^2(1-t)=&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left\{(\psi(n+1)-\psi(n-k+1))^2+\psi'(n+1)-\psi'(n-k+1)\right\}t^k\\&+\sum_{k=n+1}^{\infty}(-1)^k\binom{n}{k}\left\{(\psi(n+1)-\psi(n-k+1))^2+\psi'(n+1)-\psi'(n-k+1)\right\}t^k.\end{aligned} \tag{31}$$

In view of (6), we have

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left\{\left(\psi(n+1)-\psi(n-k+1)\right)^2+\psi'(n+1)-\psi'(n-k+1)\right\}t^k\\&\qquad=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[(H_n-H_{n-k})^2+H_n^{(2)}-H_n^{(2)}\right]t^k\end{aligned} \tag{32}$$

We make use of the formula

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left\{\left(H_n-H_{n-k}\right)^2+H_{n-k}^{(2)}-H_n^{(2)}\right\}t^k=2(1-t)^n\sum_{k=1}^{n}\frac{H_{k-1}}{k(1-\frac{1}{t})^k},$$

which is proved in [3, Eq. (40)]. Using this formula in (32), we arrive at

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[(\psi(n+1)-\psi(n-k+1))^2+\psi'(n+1)-\psi'(n-k+1)\right]t^k=2(1-t)^n\sum_{k=1}^{n}\frac{H_{k-1}}{k(1-\frac{1}{t})^k}. \tag{33}$$

Since $\binom{n}{k}=0$ for $k>n$, we have

$$\sum_{k=n+1}^{\infty}(-1)^k\binom{n}{k}t^k=0.$$

Thus,

$$\sum_{k=n+1}^{\infty}(-1)^k\binom{n}{k}\left\{(\psi(n+1)-\psi(n-k+1))^2+\psi'(n+1)-\psi'(n-k+1)\right\}t^k$$

$$= -2\psi(n+1)\underbrace{\sum_{k=n+1}^{\infty}(-1)^k\binom{n}{k}\psi(n-k+1)t^k}_{A}+\underbrace{\sum_{k=n+1}^{\infty}(-1)^k\binom{n}{k}\left[\psi^2(n-k+1)-\psi'(n-k+1)\right]t^k}_{B}. \tag{34}$$

By setting $k-n-1=k'$ and then removing the prime from k', and then with the aid of (9), we have

$$A=\sum_{k=0}^{\infty}\frac{(-1)^{n+k+1}n!}{(n+k+1)!}\frac{\psi(-k)}{\Gamma(-k)}t^{n+k+1}=\sum_{k=0}^{\infty}\frac{(-1)^{n+k+1}n!}{(n+k+1)!}\lim_{u\to -k}\frac{\psi(u)}{\Gamma(u)}t^{n+k+1}$$

$$=(-1)^n\sum_{k=0}^{\infty}\frac{t^{n+k+1}}{(n+k+1)\binom{n+k}{k}}$$

$$=(-1)^n\sum_{k=1}^{\infty}\frac{t^{n+k}}{k\binom{n+k}{k}}. \tag{35}$$

Similarly, from (10), it follows that

$$B=\sum_{k=n+1}^{\infty}\frac{(-1)^k n!}{(n+k+1)!}\frac{\psi^2(n-k+1)-\psi'(n-k+1)}{\Gamma(n-k+1)}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^{n+k+1}n!}{(n+k+1)!}\lim_{u\to -k}\frac{\psi^2(u)-\psi'(u)}{\Gamma(u)}t^{n+k+1}$$

$$=(-1)^n\sum_{k=0}^{\infty}\frac{\psi(k)}{k\binom{n+k}{k}}t^{n+k}. \tag{36}$$

From (31), (33), (34), (35), and (36), the required identity follows. □

Remark 3.2. *Note that the starting point of the proofs of the main results (of this article) is the binomial theorem. This once again shows how important the binomial theorem is in mathematics. These results are interesting as they establish a link between the binomial theorem and the BBP type series. The approach used in this article enables one to establish formulas of type* (29) *and* (30) *for higher powers of* $\log s$. *Many finite sums of the type of* (3) *can be found in [4, 8]. Many series similar to those given in Theorem 3.1 and Theorem 3.2 can be found in [5].*

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