elle est développable en série entière sur cet intervalle et on a pour tout $x \in]-1,1[$:

$$\arcsin(x) = \sum_{n=0}^{+\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1}. \quad (1)$$

3. On pose $f_n(x) = \dfrac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1}$ et $\|f_n\|_\infty = \sup_{x\in]-1,1[} |f_n(x)|$. Par la formule de Stirling on voit que $\|f_n\|_\infty = O\left(\dfrac{1}{n^2}\right)$. Par suite, la série de fonctions $\sum_n f_n$ converge normalement sur $]-1,1[$. Lorsque x tend vers 1^- et en utilisant le théorème d'interversion des limites on déduit de (1) que :

$$\frac{\pi}{2} = \sum_{n=0}^{+\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)}.$$

Exercice 7.17

Formule de Simon Plouffe (1995)

On considère une suite $(a_n)_{n\geq 1}$ de réels telle que :

$$\forall k \in \mathbb{N} \ : \ a_{8k+1} = \frac{4}{16^k}, \qquad a_{8k+4} = -\frac{2}{16^k}, \qquad a_{8k+5} = a_{8k+6} = -\frac{1}{16^k}$$

et $a_n = 0$ dans tous les autres cas.

1. Déterminer le rayon de convergence R de la série $\sum_{n\geq 1} a_n \dfrac{x^n}{n}$.

2. Pour $|x| < R$ on pose $f(x) = \sum_{n=1}^{+\infty} a_n \dfrac{x^n}{n}$. Montrer que

$$f'(x) = \frac{16(x-1)}{(x^2-2)(x^2-2x+2)}.$$

3. En exprimant $f(x)$ à l'aide des fonctions usuelles, déduire la formule de Plouffe :

$$\sum_{n=0}^{+\infty} \frac{1}{16^n}\left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}\right) = \pi.$$

1. Il est facile de voir que les séries $\sum_{n\geq 1} a_n \dfrac{x^n}{n}$ et $\sum_{n\geq 1} a_n x^n$ ont le même rayon de convergence.

9. The jumbled calculation of π digits

In 1995 Peter Borwein and Simon Plouffe realized that the series

$$\ln 2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \frac{1}{4\cdot 2^4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n2^n}$$

could be used to compute one of the binary digits of $\ln 2$ without calculating the previous digits, by means of the following ingenious formula, where $\{x\}$ means the fractional part of x:

$$\left\{2^k \ln 2\right\} = \left\{ \left\{ \sum_{n=1}^{k} \frac{2^{k-n} \bmod n}{n} \right\} + \sum_{n=k+1}^{\infty} \frac{1}{n2^{n-k}} \right\}.$$

It lets us efficiently obtain the binary digit of $\ln 2$ in position $k+1$. The main remark to carry out all the involved calculation is that the numerator of the first sum, that is, $2^{k-n} \bmod n$, can be computed very quickly. For example, the calculation of 2^{65} does not require 65 multiplications but only 7 if we do it as $((((((2^2)^2)^2)^2)^2)^2)\cdot 2$ and a very efficient algorithm is used to obtain for example $2^{65} \bmod 7$. It consists of reducing modulo 7 after each multiplication instead of doing them in a row.

In 1989 the Borwein brothers claimed that obtaining just one binary digit of π could not be easier than computing all the previous digits. However, after this discovery for $\ln 2$, it was not so clear and the team consisting of D. Bailey, P. Borwein and S. Plouffe began to search, using the PSLQ algorithm, for a series for π that would allow them to calculate isolatedly its base 2 digits (see [**14**], chapter 8).

The PSLQ algorithm is a numerical algorithm which operates to a prearranged accuracy. It links an output of n integers $a_1, a_2, \ldots, a_n$, where $a_i \neq 0$ for any i, with an input of n real numbers $x_1, x_2, \ldots, x_n$, and

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

is verified.

Success arrived taking

$$x_j = \sum_{n=0}^{\infty} \frac{1}{(8n+j)16^n}, \qquad j = 1, 2, \ldots, 7,$$

and trying to find, using PSLQ, a linear relation with integer coefficients using as input π, x_1, x_2, x_3, x_4, x_5, x_6, x_7. The output obtained was this sequence of integers: -1, 4, 0, 0, -2, -1, -1, 0. That is, they found a series [**2**] which they called BBP type (BBP are the initials of the team members' names)

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right) \tag{9.1}$$

that lets us calculate π's hexadecimal digits (hence also its binary ones) in isolation. Later, a proof of this formula was obtained from the easy result

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8} dx = \int_0^{1/\sqrt{2}} \sum_{n=0}^{\infty} x^{k-1+8n} dx = \frac{1}{2^{k/2}} \sum_{n=0}^{\infty} \frac{1}{16^n(8n+k)}.$$

Le mot de la fin

[...] ces choses-là sont rudes.
Il faut pour les comprendre avoir fait des études.

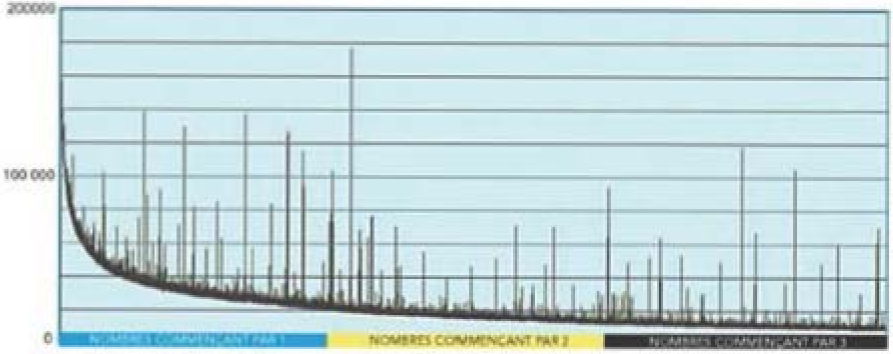
— ***Victor Hugo***, *La Légende des Siècles.*

Comment conclure cet ouvrage ? Bien sûr, nous pourrions reprendre tout ce qui a été présenté et en faire une synthèse pour rappeler les deux voies que nous avons suivies pour faire de l'algorithmique : la programmation impérative et la programmation fonctionnelle. Nous préférons cependant montrer qu'il y a bien d'autres choses à découvrir et à explorer. On pourrait citer plusieurs nouveaux langages, notamment orientés objet, faire référence aux nouvelles méthodes de conception et de réalisation de projets informatiques d'envergure ou encore parler des nouveaux algorithmes qui sont mis au point pour améliorer l'efficacité des programmes et des machines. Nous allons plutôt présenter très rapidement un petit exemple de ce que l'on peut faire avec quelques techniques un peu plus évoluées que celles développées dans cet ouvrage. Cet exemple fait intervenir de l'algorithmique, du calcul numérique et de la programmation impérative en environnement distribué. Autrement dit, nous faisons marcher plusieurs processeurs voire plusieurs machines à la fois pour faire un unique calcul : on appelle cela du parallélisme.

Cette conclusion survole une méthode récente de calcul de π, méthode que l'on doit à David Bailey, Peter Borwein et Simon Plouffe. Le nouvel algorithme de calcul de π a l'avantage de pouvoir être exécuté sur plusieurs machines à la fois comme nous allons le découvrir dans les lignes qui suivent. Jusqu'à présent, on croyait que calculer la k-ième décimale d'un nombre transcendant comme π était aussi difficile que de calculer π lui-même. En fait, il n'en est rien et nos trois mathématiciens ont pu calculer le 10 milliardième chiffre du développement hexadécimal de π en utilisant leur nouvelle formule :

$$\pi = \sum_{n=0}^{\infty} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right) \left(\frac{1}{16} \right)^n .$$

Pour les autres chiffres, le constat est analogue et donc, en un sens très particulier (celui des moyennes itérées de Césaro), on peut dire que la fréquence des nombres commençants par c dans l'ensemble de tous les entiers est $\log(1 + 1/c)$. Il s'agit là d'une démonstration partielle de la loi de Benford.



La table des constantes numériques (e, π, log π, exp (2), etc. de Simon Plouffe contient plusieurs milliards de constantes et suit approximativement la loi de Benford.

Aussi merveilleux que soient les résultats purement mathématiques, ils ne suffisent pas à expliquer tous les cas concrets de rencontres avec des données statistiques conformes à la loi de Benford. D'autres explications doivent exister.

Un fait a depuis longtemps étonné les statisticiens : la loi de Benford est vérifiée pour les longueurs des fleuves quand on mesure ces longueurs en kilomètres, mais aussi quand on les mesure en miles ou en utilisant n'importe quelle autre unité de longueur. Cette invariance quand on multiplie les données par une constante a été étudiée et Roger Pinkham a démontré en 1961 que la seule loi sur les mantisses qui soit invariante par multiplication (c'est-à-dire telle que $\Pr(a < \text{mantisse}(x) < b) = \Pr(a < c.\text{mantisse}(x) < b)$ est celle générale de Benford. S'il existe une loi pour les mantisses et que, comme on s'y attend, cette loi ne dépend pas des unités de mesure utilisées pour collecter les données, alors cette loi ne peut être que la loi de Benford. Un résultat analogue concernant le changement de base de numération a été démontré par T. Hill en 1995.

Ces résultats et quelques autres plus récents établissent de manière concordante que si une loi doit être envisagée pour le premier chiffre significatif, cela ne peut être que la loi de Benford et surtout pas la loi équitable du 1/9 que nous soufflait notre intuition. Les travaux se poursuivent pour expliquer et comprendre cette loi paradoxale que maintenant on utilise pour le repérage des statistiques

Usual functions on vectors	
Vector construction	`vector`
Cross product	`cross_product`
Scalar product	`dot_product`
Norm of a vector	`norm`

TABLE 2.5 – Vector computations.

3. Prove the BBP formula:

$$\sum_{n=0}^{+\infty}\left(\frac{4}{8n+1}-\frac{2}{8n+4}-\frac{1}{8n+5}-\frac{1}{8n+6}\right)\left(\frac{1}{16}\right)^n=\pi.$$

This fabulous formula was found on September 19, 1995 by Simon Plouffe, in collaboration with David Bailey and Peter Borwein. Thanks to computation derived from the BBP formula, the 4 000 000 000 000 000-th digit of π in radix 2 was computed in 2001.

2.4 Basic Linear Algebra (*)

In this section, we describe the basic useful functions in linear algebra: first operations on vectors, then on matrices. For more details, we refer the reader to Chapter 8 for symbolic linear algebra, and to Chapter 13 for numerical linear algebra.

2.4.1 Solving Linear Systems

To solve a linear system, we can use the `solve` function, already seen above.

Exercise 8 (Polynomial approximation of the sine function). Determine the polynomial of degree at most 5 which approximates best, in the least squares sense, the sine function on the interval $[-\pi,\pi]$:

$$\alpha_5=\min\left\{\int_{-\pi}^{\pi}|\sin x-P(x)|^2\,\mathrm{d}x\;\middle|\;P\in\mathbb{R}_5[x]\right\}.$$

2.4.2 Vector Computations

The basic functions for manipulating vectors are summarized in Table 2.5.

We can use those functions to deal with the following exercise.

Exercise 9 (Gauss' problem). Consider a satellite in orbit around the Earth, and

Si vous vous intéressez aux nombres réels, en plus de Wikipédia qui parle bien sûr de π, e, $\sqrt{2}$ (voir http://fr.wikipedia.org/wiki/Nombre_réel), il sera utile d'essayer l'Inverseur de Simon Plouffe à l'adresse : http://pi.lacim.uqam.ca/eng/

Ce programme est construit sur une base de données de constantes mathématiques soigneusement collectées et calculées par Simon Plouffe depuis 1986. Si vous rencontrez le nombre 8,53973422 dans un problème de physique, l'inverseur vous indiquera qu'il s'agit du nombre π.e. Si cette valeur ne vous semble pas convenir, l'inverseur vous en propose d'autres voisines.

La table des constantes numériques de Simon Plouffe contient plusieurs dizaines de millions de constantes mathématiques. Elle est obtenue en prenant bien sûr π, e, $\sqrt{2}$, $\sqrt{3}$, $\sin(1)$, $\log(\pi)$, $\exp(\sqrt{2})$ etc. mais aussi des sommes de séries et des valeurs de fonctions plus ou moins basiques dont de nombreuses séries entières ($\Sigma a(n)x^n$) évaluées en divers points, etc.

Des statistiques sur le premier chiffre, menées il y a quelques années, donnaient des résultats assez proches de ce que la loi de Benford prévoit, comme on le voit sur la figure suivante. Celle-ci donne la courbe observée avec les données de la base de Plouffe, celle de la base de Sloane dont nous parlons juste après, ainsi que la courbe théorique que donnerait la loi de Benford. Pour la base de Sloane, nous n'avons conservé que les valeurs inférieures à 25 000, ce qui induit une légère surreprésentation des 1. Les données de Plouffe présentent un écart faible mais non négligeable à la courbe théorique. La raison de cet écart n'est pas bien comprise.

L'histoire aurait pu s'arrêter là mais, entre-temps, la nouvelle de l'existence de la collection de Sloane s'était répandue auprès de ses collègues. L'idée fut accueillie avec enthousiasme, et beaucoup de mathématiciens et mathématiciennes réalisèrent combien il pouvait être pratique pour leurs propres travaux d'avoir un tel dictionnaire de suites sous la main.

Alors Sloane décida de poursuivre son travail. Il continua à agréger de plus en plus de suites et, en 1973, il publia son premier ouvrage *A Handbook of Integer Sequences* (« Manuel des suites entières »), dans lequel 2 372 suites numériques sont méthodiquement répertoriées. Le livre fut un succès, et Sloane se mit à recevoir une multitude de messages pour l'encourager et lui suggérer de nouvelles suites à ajouter au catalogue. Avec l'aide de son collègue Simon Plouffe, Sloane mit au point une deuxième version plus complète qui fut publiée en 1995 sous le titre *The Encyclopedia of Integer Sequences* (« L'encyclopédie des suites entières »). Cette dernière contenait alors 5 487 suites différentes.

Mais, dans les années 1990, Internet est en pleine expansion et va donner à la collection une dimension inédite. Parallèlement à la préparation du livre, Sloane décida de mettre en place une adresse mail, permettant aux chercheurs et chercheuses du monde entier d'accéder au contenu de l'encyclopédie. Les messages affluent, les suggestions de nouvelles suites arrivent par milliers, et, un an seulement après la parution du livre, la base de données a déjà triplé de volume : en 1996, 16 000 suites sont répertoriées. Sloane lance alors un site Web dédié : l'OEIS®, Online Encyclopedy of Integer Sequences (« Encyclopédie en ligne des suites entières »). En 1999, l'OEIS en répertoriait plus de 50 000.

Dix ans plus tard, Sloane décida de confier la gestion de l'OEIS à une organisation collégiale de mathématiciens bénévoles afin d'assurer sa pérennité. En 2020, à l'heure où nous écrivons cet article, Neil Sloane est âgé de quatre-vingts ans et fait toujours partie du conseil d'organisation : sa base de données contient désormais plus de 334 000 suites différentes. Elle continue d'être régulièrement alimentée et abondamment utilisée par les mathématiciens du monde entier. Elle est d'ailleurs accessible à tout le monde sur le site Web www.oeis.org et, si cela vous amuse, libre à vous d'aller la consulter.

Ce site fonctionne comme un moteur de recherche classique : donnez-lui une succession de nombres quelconques et il vous dira si elle apparait dans l'une des suites de la base. Vous pourrez par exemple y chercher la séquence « 0, 1, 8, 78, 944, 13800, 237432 » et vous verrez apparaitre pour résultat la suite numéro A000435 accompagnée du commentaire suivant : « *This is the sequence that started it all : the first sequence in the database !* » (« Ceci est la suite avec laquelle tout a commencé : la première dans la base de données »). Vous y lirez également les termes venant après, ceux que Sloane cherchait en 1963 : le nombre 237 432 est suivi de 4 708 144, 105 822 432, 2 660 215 680, et ainsi de suite.

Mais revenons un peu en arrière, car ce n'est pas tout. À la fin des années 2000, une autre étape a été franchie pour l'OEIS. Alors qu'elle n'était jusque-là qu'un outil, facilitant le travail des chercheuses et chercheurs de nombreux domaines, elle est devenue à son tour un objet d'étude.

En 2008, Philippe Guglielmetti, un amateur de mathématiques tenant un blog dénommé « Pourquoi Comment Combien », se pose une étrange question : quels sont les nombres les moins intéressants ? Et, pour y répondre, il va utiliser l'OEIS et regarder dans combien de suites différentes apparait chaque nombre. Par exemple, si l'on cherche 1 089, le moteur de recherche nous annonce 1 190 résultats à ce jour. Le premier d'entre eux est la suite des nombres carrés, en effet, 1 089 est un nombre carré puisqu'il est égal à 33^2. Si, au contraire, on cherche 7 133, il n'y a que 24 résultats à ce jour. De ce point de vue, il est donc possible de considérer le nombre 7 133 comme moins intéressant que 1 089 (*voir* Mille quatre-vingt-neuf).

À la suite de ce premier article, Guglielmetti va contacter le mathématicien et vulgarisateur Jean-Paul Delahaye qui a l'idée de tracer le graphe de l'intérêt d'un nombre. Ils obtiennent alors le résultat suivant :

FIGURE 17.4

Neil Sloane (1939–).

Courtesy of Neil J. A. Sloane.

The On-Line Encyclopedia of Integer Sequences was created by Neil Sloane (Figure 17.4). The first incarnation was a book, published in 1973.[25] A new book, with over twice as many sequences, coauthored with Simon Plouffe, came out in 1995.[26] With the rapid growth of the database that followed, thanks to reader contributions, the move to an online format made sense.

In addition to his work on the Encyclopedia, Sloane has published in many areas, including combinatorics, error-correcting codes, sphere packing, rock climbing, and cryptography![27]

Generating Functions

> A generating function is a clothesline on which we hang up a sequence of numbers for display.

24 http://oeis.org/A003412.
25 Sloane, Neil J. A., *A Handbook of Integer Sequences*, Academic Press, New York and London, 1973.
26 Sloane, Neil J. A. and Simon Plouffe, *The Encyclopedia of Integer Sequences*, Academic Press, San Diego, CA, 1995.
27 See http://neilsloane.com/and https://en.wikipedia.org/wiki/Neil_Sloane.

> Herbert S. Wilf[28]

Given a finite sequence $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$, the generating function for it is the polynomial

$$G(x)=a_0+a_1x+a_2x^2+\cdots+a_{n-1}x^{n-1}+a_nx^n=\sum_{i=0}^{n}a_ix^i$$

If we are dealing with an infinite sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ the generating function is given by

$$G(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots=\sum_{i=0}^{\infty}a_ix^i$$

You should be aware that z is often used in place of x and the generating function $G(z)$ may be referred to as the z transform of the sequence $\{a_n\}$.

Example 8 – Power sets

Although the phrase "generating function" wasn't used at the time, an example was seen in Chapter 4, when the following was shown as a means of generating the elements of the power set of $S=\{a, b, c\}$:

$$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(bc+ac+ab)x+abc.$$

28 Wilf, Herbert S., Generatingfunctionology, Second edition, Academic Press, 1994, p. 1.

Recall: The constant term is abc, corresponding to the set $\{a, b, c\}$. The linear term has coefficient $bc+ac+ab$, corresponding to the sets $\{b, c\}$, $\{a, c\}$, and $\{a, b\}$. The quadratic term has coefficient $(a+b+c)$, corresponding to the sets $\{a\}$, $\{b\}$, and $\{c\}$. Finally, the cubic term has coefficient 1, which does not involve any of the element of our set S, so it corresponds to the empty set, $\varnothing$.

An alternative multiplication to achieve the same sort of result was also given:

$$(1+ax)(1+bx)(1+cx)=1+(a+b+c)x+(bc+ac+ab)x^2+(abc)x^3.$$

As was mentioned in Chapter 4, these results generalize to n-element sets.

It's now time to put a twist on these results. Suppose that instead of generating the elements of a power set, we merely want to know how many elements there are of any given size?

To answer this new question, simply let $a=b=c=1$ in the alternative multiplication given above. This gives

$$(1+x)^3=1+3x+3x^2+x^3.$$

shortcut that avoided the necessity of computing all digits up to and including the n-th digit. He and Simon Plouffe found a way to compute the n-th binary digit of the natural logarithm of two, namely $\ln 2 = 0.693147\ldots$, by manipulating the following well-known formula for $\ln 2$:

$$\ln 2 = \sum_{k=1}^{\infty} \frac{1}{k2^k} = \frac{1}{2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \frac{1}{5 \cdot 2^5} + \cdots. \tag{4}$$

After this discovery, Borwein and Plouffe immediately asked whether they could do the same mathematical "trick" for π. It all depended on finding a similar formula for π. Peter Borwein, who was very familiar with the mathematical literature regarding π, was not aware of any such formula for π, and it seemed exceedingly unlikely that such a formula would have escaped detection by the many thousands of great mathematicians who have studied π through the ages. But Plouffe embarked on a computer search for such a formula, using a 200-digit computer implementation (provided by the present author) of mathematician-sculptor Helaman Ferguson's integer relation "PSLQ" algorithm, which finds integer linear relations among an input set of numerical values. After several months of fits and starts, Plouffe and his computer found formula (1). The rest, as they say, is history.

Since 1995, researchers have discovered similar digit-calculating formulas for numerous other fundamental constants of mathematics, in most cases by similar computer searches using the PSLQ algorithm. See [10, Chap. 3] or [7] for details.

4 Ramanujan's Continued Fraction

Srinivasa Ramanujan (1887–1920), born to a poor family in India, learned mathematics mostly by studying math books on his own. His genius was recognized by British mathematician G. H. Hardy, who invited him to come work with him in Cambridge. Ramanujan's mathematical achievements have been recognized as among the greatest of all time, in spite of the fact that he died at the tender age of 32.

One of the many topics that he addressed in his notebooks is the following class of continued fractions. Given $a, b, \eta > 0$, define

$$R_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}}. \tag{5}$$

This complicated-looking expression simply means to evaluate the indicated compound fraction out to some level, and then take the limit as more and more terms are included. Ramanujan discovered the beautiful fact that

$$\frac{R_\eta(a,b) + R_\eta(b,a)}{2} = R_\eta\left(\frac{a+b}{2}, \sqrt{ab}\right), \tag{6}$$

~njas/sequences/. If the sequence 1, 2035, 74316, 302995, 190575 is entered, the response is A060843: *Triangle of associated Stirling numbers of second kind*. A hard-copy published version is authored by Neil J. A. Sloane and Simon Plouffe. Incidentally, Professor Plouffe broke the world record in 1975 for memorizing digits of π by reciting 4096 digits; alas, he remained the champion until 1977. And, I sometimes forget a $\TeX$command!! Perhaps a little *Aricept* might help.

2.9 ROUCHÉ'S THEOREM AND THE LAGRANGE'S INVERSION FORMULA

No discussion of generating functions would be complete without describing and illustrating the following:

- *Rouché's theorem* [Ahlfors 1953, p. 124][7] gives a set of conditions guaranteeing root(s) of an analytic function in some set.
- The *Lagrange inversion formula* [Whittaker and Watson 1952, p. 153][8] often can be used to give an explicit solution for the root.

We begin with a simplified, plain vanilla version of Rouché's theorem.

Theorem 2.5 (*Rouché's theorem*).

If $f(z)$ and $g(z)$ are both analytic in $\mathcal{C}_r(0)$, the closed circle of radius R (about 0) and $|f(z)| < |g(z)|$ for $|z| = R$, then $f(z)$ has exactly as many zeros as $g(z)$ in $\mathcal{C}_r(0)$, the open circle of radius R (about 0).

Example 2.6.

Suppose $f(z) = \sum_{n=0}^{\infty} p_n z^n$ is a probability generating function (§2.2)

H1. $p_n \geq 0$ $(n \geq 0)$ and $1 = \sum_{n=0}^{\infty} p_n = 1$

H2. $\mu \equiv \sum_{n=0}^{\infty} n p_n < 1$.

The equation $z^m - f(z) = 0$ certainly has a zero at $z = 1$. Setting $g(z) = z^m$, Rouché's theorem implies that $f(z)$ has m zeros (counting multiplicity) in $\mathcal{C}_1(0)$.

`isirreduc` procedure that the p we have found is indeed irreducible, confirming this conclusion theoretically. ◁

Example 5.3.9: Finding formulas for π. As investigated in Exercise 5.38, the classical formula

$$\ln(2) = \sum_{k=1}^{\infty} \frac{1}{k2^k} \tag{5.39}$$

can be used to compute individual binary digits of ln(2) without first calculating the preceding digits. An algorithm capable of producing individual digits in this manner is called an *arbitrary digit algorithm*. This is clearly a very desirable feature if one is interested in finding a large number of digits because it makes it possible to continue a previous calculation, to check other calculations without redoing them completely, and because it facilitates parallelization.

Inspired by this application of (5.39), Peter Borwein and Simon Plouffe began to search for an arbitrary digit algorithm for π in 1994. As discussed in Exercise 5.39, any constant α satisfying

$$\alpha = \sum_{k=1}^{\infty} \frac{p(k)}{b^k q(k)}, \tag{5.40}$$

where $b > 1$ is an integer, p and q are integer polynomials and q has no zeroes at positive integers can be computed with an arbitrary digit algorithm. More generally, it is possible to develop an arbitrary digit algorithm for any linear combination of constants obeying (5.40).

Unfortunately, no known formula for π had the required form, so Borwein and Plouffe began to search for one using an implementation of PSLQ written by David Bailey. Together, they identified a number of constants of the required type and used PSLQ to attempt to find an integer relation between π and these constants. After months of computations (and several restarts necessitated by the addition of new constants that had been identified in the literature), the program found the following integer relation:

$$\pi = {}_2F_1\left(\left.\begin{matrix}\frac{1}{4}, \frac{5}{4}\\ 1\end{matrix}\right| \frac{-1}{4}\right) + 2\arctan\left(\frac{1}{2}\right) - \ln 5,$$

where the first term is a complicated hypergeometric function. Converting this equation to a series gives the Bailey–Borwein–Plouffe formula:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right). \tag{5.41}$$

After discovering (5.41) through this carefully constructed experiment, Borwein, Bailey, and Plouffe were able to prove that the equation holds using traditional methods [BBP97]. In fact, this proof is not complicated, and the key to the result was clearly to establish *what* to prove. In this way, the BOX P. 249 ← example showcases experimental mathematics at its best. ◁

8.6.3 Backhouse-Lucas Formula

In [5, 42], and [71] the following integral is discussed.

```
i[m_Integers,n_Integers]:=∫_0^1 x^m (1-x)^n/(1+x^2) dx
i[m,n]
FullSimplify[Assuming[m>0,i[m,2m]]]
i[m,n]
i[m,2 m]
```

On the same line we mention four more results from [42].

$$\left\{\int_0^1 \left(\frac{x^{10}(1-x)^8}{4(1+x^2)} + \frac{5}{138450312}\right) dx == \frac{355}{113} - \pi,\right.$$
$$\int_0^1 \left(\frac{8192}{114291} \times \frac{x^9(1-x)^8}{\sqrt{1-x^2}} + \frac{15409}{219772564011}\right) dx == \frac{355}{113} - \pi,$$
$$\int_0^1 \frac{x^{14}(1-x)^{12}(124360+77159x^2)}{755216(1+x^2)} dx == \pi - \frac{103993}{33102},$$
$$\left.\int_0^1 \frac{x^{12}(1-x)^{12}(1349-1060x^2)}{38544(1+x^2)} dx == \frac{104348}{33215} - \pi\right\}$$

{True, True, True, True}

8.7 BBP and Adamchik-Wagon Formulas

This section is mainly based on [2, 3, 6], and [69]. A *BBP formula* allows to compute the n-th decimal of π without knowing its previous decimals. The acronym BBP comes from the names of three mathematicians called (David) Bailey, (Peter) Borwein, and (Simon) Plouffe.

An interesting result of this kind is proven in [2].

We underline that the next result, called an Adamchik-Wagon formula (AW formula), is true for any complex number r.

FullSimplify[PowerExpand[

$$\sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4+8r}{8k+1} - \frac{8r}{8k+2} - \frac{4r}{8k+3} - \frac{2+8r}{8k+4} - \frac{1+2r}{8k+5} - \frac{1+2r}{8k+6} + \frac{r}{8k+7}\right)]]$$

π

shortcut that avoided the necessity of computing all digits up to and including the n-th digit. He and Simon Plouffe found a way to compute the n-th binary digit of the natural logarithm of two, namely $\ln 2 = 0.693147\ldots$, by manipulating the following well-known formula for $\ln 2$:

$$\ln 2 = \sum_{k=1}^{\infty} \frac{1}{k2^k} = \frac{1}{2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \frac{1}{5 \cdot 2^5} + \cdots. \quad (4)$$

After this discovery, Borwein and Plouffe immediately asked whether they could do the same mathematical "trick" for π. It all depended on finding a similar formula for π. Peter Borwein, who was very familiar with the mathematical literature regarding π, was not aware of any such formula for π, and it seemed exceedingly unlikely that such a formula would have escaped detection by the many thousands of great mathematicians who have studied π through the ages. But Plouffe embarked on a computer search for such a formula, using a 200-digit computer implementation (provided by the present author) of mathematician-sculptor Helaman Ferguson's integer relation "PSLQ" algorithm, which finds integer linear relations among an input set of numerical values. After several months of fits and starts, Plouffe and his computer found formula (1). The rest, as they say, is history.

Since 1995, researchers have discovered similar digit-calculating formulas for numerous other fundamental constants of mathematics, in most cases by similar computer searches using the PSLQ algorithm. See [10, Chap. 3] or [7] for details.

4 Ramanujan's Continued Fraction

Srinivasa Ramanujan (1887–1920), born to a poor family in India, learned mathematics mostly by studying math books on his own. His genius was recognized by British mathematician G. H. Hardy, who invited him to come work with him in Cambridge. Ramanujan's mathematical achievements have been recognized as among the greatest of all time, in spite of the fact that he died at the tender age of 32.

One of the many topics that he addressed in his notebooks is the following class of continued fractions. Given $a, b, \eta > 0$, define

$$R_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}}. \quad (5)$$

This complicated-looking expression simply means to evaluate the indicated compound fraction out to some level, and then take the limit as more and more terms are included. Ramanujan discovered the beautiful fact that

$$\frac{R_\eta(a,b) + R_\eta(b,a)}{2} = R_\eta\left(\frac{a+b}{2}, \sqrt{ab}\right), \quad (6)$$

shortcut that avoided the necessity of computing all digits up to and including the n-th digit. He and Simon Plouffe found a way to compute the n-th binary digit of the natural logarithm of two, namely $\ln 2 = 0.693147\ldots$, by manipulating the following well-known formula for $\ln 2$:

$$\ln 2 = \sum_{k=1}^{\infty} \frac{1}{k2^k} = \frac{1}{2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \frac{1}{5 \cdot 2^5} + \cdots. \tag{4}$$

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milliardième chiffre derrière la virgule est un 5...). Au passage, on a fait bien des découvertes qui ont de quoi stupéfier. Un exemple ? Vous vous souvenez sans doute de cette curiosité évoquée dans l'introduction. La centième décimale de Pi est un 9. Jusque-là, rien d'extraordinaire. Mais tout se corse lorsqu'on réalise que la millième décimale est encore un 9 ! Enfin, on ne peut que froncer les sourcils en observant que la milliardième décimale – un milliard de décimales derrière la virgule – est à nouveau un 9 ! Le mathématicien canadien Simon Plouffe est l'un des meilleurs spécialistes mondiaux de Pi. Il en a récité par cœur en 1977 les 4 096 premières décimales, décrochant ainsi le record Guinness de l'année. Il a aussi découvert une nouvelle formule pour le calculer. Et après ces harassantes années d'exploration, pour lui comme pour les frères Chudnovsky, les répétitions de chiffres dans Pi ne sont pas de simples accidents. Ils veulent dire « quelque chose ». Mais quoi ? Seule solution pour espérer en savoir un peu plus un jour ou l'autre : continuer à creuser. A calculer de plus en plus loin l'effarante suite de décimales. Or, même si on comprend de moins en moins le sens de ce calcul démentiel, *impensable*, à mesure qu'on s'enfonce dans l'infini, la course aux chiffres ne rencontrera jamais aucune limite. Combien de décimales aura-t-on calculé dans un siècle ? Dans mille ans ? Si nous devions écrire le nombre Pi sur une feuille de papier, celle-ci ferait mille fois, un milliard de fois, une infinité de fois le tour de l'Univers sans que nous puissions jamais atteindre le dernier chiffre. Pourtant, ce nombre infini ne représente qu'un point – un point minuscule – entre 3 et 4 sur la droite des nombres réels. Mais il y a encore plus vertigineux. Quoi donc ? A première vue, les milliards de milliards de décimales de Pi semblent se succéder au hasard. Un 7 après un 2. Un 4 devant un 9, etc. Mais est-ce bien le cas ? Pour trouver la réponse – et quelle réponse – nous allons nous tourner vers un extraordinaire mathématicien russe, qui a vécu à peu près à la même époque que Claude Shannon. Son nom ? Andreï Kolmogorov. Ce qu'il va nous apprendre sur Pi va peut-être au-delà de tout ce que vous pouvez imaginer.

10

J'accompagnai Fabien afin de rencontrer le professeur Andrieux dans son bureau, au Centre de cryptologie de l'École normale, situé au sous-sol du bâtiment adjacent au cloître. Lorsque je demandai à mon ami de me renseigner sur la cryptologie, en cheminant à travers le dédale d'escaliers et de couloirs, il me dit que c'était une science vieille comme l'art militaire, qui avait aujourd'hui des applications et des débouchés importants, dans l'informatique, les cartes à puce, en plus de l'industrie de défense et de sécurité. Le laboratoire de Normale Sup était à la pointe de ce domaine. Pourtant il y faisait sombre, et il n'y avait personne dans cet endroit vétuste qui paraissait presque désaffecté.

Enfin, nous sommes arrivés devant la porte du bureau du professeur Andrieux. Après avoir frappé, nous sommes entrés dans un antre rempli de papiers, de livres, d'ordinateurs anciens et nouveaux à travers des entrelacs inextricables de fils électriques et de câbles. Sur les murs, des affiches représentaient des fractales psychédéliques et multicolores.

Avec sa longue barbe, sa longue écharpe multicolore sous une sorte de curieux veston à motifs mexicains, le professeur Andrieux ressemblait à un devin ou un mage. Il ne portait ni chaussures ni chaussettes. Cela ne semblait pas étonner Fabien, qui m'avait expliqué qu'il était capable de donner de véritables shows pendant les cours, avec ses accoutrements excentriques. Il aimait faire des blagues, des canulars. Un jour, il était même arrivé déguisé en professeur anglais avec une toge académique et un accent oxfordien.

– I, Pi, Pi, pourra ! s'exclama le professeur Andrieux, lorsque je lui posai la question sur le dénombrement des décimales. Figurez-vous qu'en prenant les six premiers milliards, nous avons les quantités suivantes : à peu après autant de 0 que de 1, 2, 3, etc., autour de 600 000 environ. Une chose étrange, n'est-ce pas, jeune homme ?

– Comment avez-vous obtenu ce résultat, professeur ?

– Je me suis fondé sur le travail d'une équipe de chercheurs, celle de David Bailey, Peter Borwein et Simon Plouffe, qui a mis au jour la possibilité de calculer n'importe quelle décimale de Pi en base 2 sans avoir à calculer les chiffres qui précèdent. Leur travail a permis d'avancer considérablement dans la connaissance de Pi. On a pu remarquer aussi la présence de séquences répétées étonnantes : des 999999 entre les positions 762 et 767. On trouve aussi la séquence 123456789. Vous ne trouvez pas ça formidable ?

– Mais quel sens donner à cela ?

– La question du sens ne concerne pas le mathématicien mais le philosophe, dit le professeur Andrieux, qui enleva ses lunettes de presbyte, pour revêtir d'énormes lunettes rondes en écailles. Vous disiez, jeune homme ? Ah oui, la question du sens ! Bien sûr ! À travers Pi, on aborde les questions les plus profondes des mathématiques et de la philosophie… Comme le hasard, par exemple… Savez-vous ce qu'est une suite de chiffres tirés au hasard ? me demanda-t-il.

– Non.

– Ce sont des suites où tout arrive : c'est-à-dire toute séquence possible apparaît, tôt ou tard. De telles suites sont appelées des suites-univers. Eh bien, croyez-le ou non, Pi fait partie de ces suites !

– Le professeur Andrieux veut dire que quelque part dans Pi, expliqua Fabien, il y a ta date de naissance, ton nom et ton adresse codés, ton numéro de Sécurité sociale, les codes numériques qui permettraient de faire le film de ta vie mais aussi des versions de

Understanding the LDL receptor Structure through Probabilistic Models (using HMMs of the LDL receptor – MIT Computational Biophysics Laboratory – October 2005)
A Web-Based System for Public-Private Sector Collaborative Ecosystem Management (construction of probabilistic models of ecosystems processes – Timothy C. Haas – University of Wisconsin-Milwaukee)
Probabilistic Basecalling (Speed, Li, Nelson, Cawley – University of California, Berkeley – January 1999)
A probabilistic analysis of a greedy algorithm arising from computational biology (Daniel G. Brown – Cornel University)
A probabilistic model of mosaicism based on thee histological analysis of chimaeric rat liver (Iannaccone, Weinberg, Berkwits – Northwestern University Medical School, Chicago II)

Mathématiques :

Le hasard n'est pas absent de cette branche "pure" des recherches scientifiques :
« Ces concepts de hasard et d'imprédictibilité, qui sont fondamentaux en physique classique et en physique quantique, sont aussi au cœur des mathématiques pures « (Chaitin – La Recherche Décembre 2004 N° 381)

« … Il semble bien que les décimales de µ (Pi) apparaissent de façon aléatoire : ceux qui ont cherché des régularités dans la répartition des décimales n'ont rien trouvé, même avec des tests statistiques très poussés » (Simon Plouffe – La Recherche - Décembre 2005 – N° 392)

Transcendance de pi

En 1766,Adrien-Marie Legendre a démontré l'irrationalité de pi.

Il est donc impossible d'écrire pi sous la forme d'une fraction.

En 1882, Ferdinand von Lindeman a démontré la transcendance de pi.

C'est-à-dire que pi ne peut être la racine d'une équation polynomiale (équation de la forme $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$).

En conséquence, il est également impossible de résoudre la quadrature du cercle, c'est-à-dire de tracer un segment de droite ayant la même longueur que la circonférence d'un cercle donné en utilisant uniquement la règle et le compas.

Calcul des décimales de pi

Grâce aux ordinateurs, il est possible de calculer un grand nombre de décimales de pi. On en est aujourd'hui à plus de 200 milliards.

En 1995, Simon Plouffe découvrit une formule permettant de calculer une décimale binaire[1] de pi sans calculer les décimales qui précèdent.

Aujourd'hui, on a pu calculer la 40 000 milliardième décimale binaire.

L'inverse de pi

L'école nous a rendu familière la révolution de 1830 qui permit de renverser les Bourbons en 3 jours mais amenèrent un nouveau monarque, Louis-Philippe.

---

[1] *Etymologiquement, les décimales, ce sont les chiffres après la virgule en base dix. Mais on utilise tout de même ce terme pour les autres bases !*

13.5 The hunt for single π digits

We have already quoted the observation by Adamchik and Wagon that the "2000-year search [for ever more π digits is changing] direction". Today more and more is heard about a new type of π record, which was proclaimed with the words "The forty trillionth bit of π is 0".

The background to this is the fascinating "BBP algorithm", which is named after the initial letters of its authors, David Bailey, Peter Borwein and Simon Plouffe, and was only published recently, in September 1995. With this algorithm it is possible to calculate individual hexadecimal positions *in the middle* of π without having to calculate all the preceding digits. This method is described in detail in Chapter 10.

This algorithm has opened up a new stage in the quest for π digits, namely the search for digits as far removed from the beginning of π as possible. The first landmark was set by the three researchers themselves with the 10 billionth hexadecimal π digit, corresponding to the 12 billionth decimal place. At the time of publication of their data (September 1995), this digit was almost twice as far into π as the last known digit obtained by the conventional route.

Within only a year later a student called Fabrice Bellard[6] had caught up. He discovered an even better formula and used it to calculate the 100 billionth (October 1996) and the 250 billionth (September 1997) hexadecimal digits of π, which were a factor of 10 and 25, respectively, more distant. But even these achievements were overtaken almost immediately by the disclosure of some hexadecimal digits in the trillionth area such as the 1.25 trillionth (August 1998), the 10 trillionth (February 1999) and the 250 trillionth (11 September 2000)[7], which in turn bettered the previous records by factors, respectively, of 5, 20 and 1000. The holder of both records is 19-year-old Colin Percival, a student at the Simon Fraser University in Burnaby, Canada, the same establishment where the Borwein brothers and Simon Plouffe work. No doubt we will hear many more great things about this young man. He began his university studies in mathematics at the age of 13 in parallel to his secondary school education, and has successfully completed an unusually major project while still in his teens.

Although the formula Percival used was not new (in fact, it was the Bellard formula, 10.8), for the calculation itself he used an unusual

[6] `http://www-stud.enst.fr/~bellard/`

[7] `http://www.cecm.sfu.ca/projects/pihex/announce1q.html`

Perl code development consumes most of the developer time, closely followed by manual generation of HTML code.

- C/C++
  Since interpreted scripts can be often somewhat inefficient or insecure, C or C++ is used for all exectuables. In particular, these are most helpful for compute-intensive or memory-intensive operations and also for programs composed of highly complex operations or those needing a high level of security. Only a small percentage of the development was devoted to C/C++ coding.

8.3.3 Computer Algebra Systems

This volume uses several computer algebra systems to "activate" the papers, but principally relies on Maple. There are several reasons for this, but the main one is that the [33]CECM is a "Maple shop". Most of us have used Maple for many years, and the Maple expertise locally available is by any standard considerable, with Michael Monagan (one of the Maple designers), Greg Fee, Simon Plouffe, and Rob Corless present and helping. [34]Simon Fraser University has had Maple in its mathematics courses for many years, so there was a considerable level of background Maple ability in the grad students, post-docs, and faculty associated with the project as well.

That said, the Organic Mathematics Project is not limited to Maple, using Pari, GAP, Mathematica, and AXIOM, as well. Simon Plouffe has the most expertise with Pari of the people at the CECM, but one of our invited authors (Henri Cohen) is among the principal designers of Pari. [35] Stan Wagon is known for his Mathematica expertise, and [36] Stan Devitt is a prime mover at Waterloo Maple Inc.

The main use of computer algebra in these papers is to enliven the examples. Many of the papers discuss algorithms, and it is reasonable to present the reader with an already-implemented version of the algorithm, both so that the reader may test the claims of the paper and so that the algorithm/code can be used by the reader in their own work.

Dave Fayegh has written a system that allows people to use Maple on our server from a remote site while reading one or more of these papers. This Maple Form Interface, though in its infancy, has proved already to be a valuable tool for mathematical exposition. If you try out some of the examples, we are sure you will agree. We note that there are similar form interfaces for other computer algebra systems, notably Mathematica, and that this is an idea whose time has come. We believe that our [37] Maple Form Interface is currently the best available, and we also believe that it is the most highly-developed from the point of view of serious use. The examples and algorithms displayed in the Maple annotation system (which uses the form interface) have been carefully chosen to make a difference in the understanding of the reader, not just in an attempt to impress the onlookers. (It is

[33]http://www.cecm.sfu.ca/
[34]http://www.sfu.ca/
[35]http://www.cecm.sfu.ca/organics/authors/wagon/index.html
[36]http://www.cecm.sfu.ca/organics/authors/devitt/index.html
[37]http://www.cecm.sfu.ca/projects/OMP/mmpform.html

faster thanks to the huge denominators that appear.

No one has done better using pencil-and-paper arithmetic, but mechanical calculators and electronic computers made the computations faster and eliminated mistakes. Attention switched to finding formulas that gave very good approximations using only a few terms. The Chudnovsky series

$$\frac{1}{\pi} = 12\sum_{k=0}^{\infty}\frac{(-1)^k(6k)!(545,140,134k+13,591,409)}{(3k)!(k!)^3 640,320^{3k+\frac{3}{2}}}$$

found by the brothers David and Gregory Chudnovsky, produces 14 new decimal digits per term. Here the summation sign $\sum$ means: add the values of stated expression as k runs through all whole numbers starting at 0 and going on forever.

There are many other methods for computing π, and new discoveries are still being made. In 1997 Fabrice Bellard announced that the trillionth digit of π, in binary notation, is 1. Amazingly, he didn't calculate any of the earlier digits. In 1996, David Bailey, Peter Borwein, and Simon Plouffe had discovered a very curious formula:

$$\pi = \sum_{n=0}^{\infty}\frac{1}{2^{4n}}\left(\frac{4}{8n+1}-\frac{2}{8n+4}-\frac{1}{8n+5}-\frac{1}{8n+6}\right)$$

Bellard used a similar formula, more efficient for computations:

$$\pi = \frac{1}{64}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{10n}}$$

$$\left(-\frac{32}{4n+1}-\frac{1}{4n+3}+\frac{256}{10n+1}-\frac{64}{10n+3}-\frac{4}{10n+5}-\frac{4}{10n+7}+\frac{1}{10n+9}\right)$$

With some clever analysis, the method gives individual binary digits. The key feature of the formula is that many of the numbers in it, such as 4, 32, 64, 256, 2^{4n}, and 2^{10n}, are powers of 2, which are very simple in the binary system used for the internal workings of computers. The record for finding a single binary digit of π is broken regularly: in 2010 Yahoo's Nicholas Sze computed the two quadrillionth binary digit of π, which turns out to be 0.

The same formulas can be used to find isolated digits of π in arithmetic to the bases 4, 8, and 16. Nothing of the kind is known for any other base; in particular, we can't compute decimal digits in isolation. Do such formulas exist? Until the Bailey-Borwein-Plouffe formula was found, no one imagined it could be done in binary.

Squaring the Circle

The ancient Greeks sought a geometric construction for squaring the circle: finding the side of a square with the same area as a given circle. Eventually it was proved that, just as for angle trisection and cube duplication, no construction with ruler and compass exists [see 3]. The proof depends upon knowing what kind of number π is.

We've seen that π is not a rational number. The next step up from rational numbers is to algebraic numbers, which satisfy a polynomial equation with integer coefficients. For instance, $\sqrt{2}$ is algebraic, satisfying the equation $x^2 - 2 = 0$. A number that is not algebraic is called transcendental, and in 1761 Lambert, who first proved π to be irrational, conjectured that it is actually transcendental.

It took 112 years until Charles Hermite made the first big breakthrough in 1873, by proving that the *other* famous curious number in mathematics, the base e of natural logarithms [see e], is transcendental. In 1882 Ferdinand von Lindemann improved Hermite's method, and proved that if a nonzero number is algebraic, then e raised to the power of that number is transcendental. He then took advantage of Euler's formula $e^{i\pi} = -1$, like this. Suppose that π is algebraic. Then so is iπ. Therefore Lindemann's theorem implies that −1 does *not* satisfy an algebraic equation. However, it obviously does, namely $x + 1 = 0$. The only way out of this logical contradiction is that π does not satisfy an algebraic equation; that is, it is transcendental.

One important consequence of this theorem is the answer to the ancient geometric problem of squaring the circle, that is, constructing a square of the same area as a circle using ruler and compass only. This is equivalent to constructing a line of length π starting from a line of length 1. Coordinate

1836.1526661 (relative error is 3×10^{-9}). In the table it would be placed between number 13 and 14.

3. It was Werner Heisenberg in 1935 [14] who suggested to use number $2^4 3^3$ (which is equal to 432) to calculate alpha as $\alpha^{-1} = 432/\pi$, so m_p/m_e ratio can be also obtained approximately via 432. The expression can be rewritten as $1836 = 17 \cdot 108$ (the number 108 was considered to be sacred by ancients). There are other possible representations for the number 1836 which were noticed in the past, for example: $1836 = (136 \cdot 135)/10$ (see review in [5] and [22]).
4. This expression has some certain theoretical base related to original R. Fürth ideas [6], but it won't be discussed here. The precision has the same order as famous $6\pi^5$.
5. This is a Lenz's formula and it remains the favorite among the physicists. Recently Simon Plouffe also suggested yet another adjustment to this formula as following: $\mu = \frac{1}{5\cosh(\pi)} + 6\pi^5 + \frac{1}{5\sinh(\pi)}$ which looks remarkably symmetric and natural. The relative error is also extremely good: 4×10^{-9}. This formula has not been published before, it definitely has to attract further attention of the researchers.
6. The simplest way to approximate m_p/m_e ratio using powers of 2 and 7. Similar formula: $\mu = \frac{3^5 7^{16}}{2^{42}}$.
7. The elegant expression which uses almost 'kabalistic' numbers 22, 5, 3 and fine structure constant. Other possible expression with similar look and with the same precision: $\mu = \frac{5^{76}}{2^{127}3^{23}}$. Being combined together one can derive approximation for fine structure constant as 137.035999761 (with good relative deviation of 5×10^{-9}): $\alpha^{-2} = \frac{5^{78}}{11 \cdot 2^{127}3^{23}}$.
8. Parker-Rhodes in 1981, see [21] and review in [5]. McGoveran D.O. [20] claimed that this formula does not have anything in common with numerology as it was derived entirely from their discrete theory.
9. This elegant expression uses only the fine structure constant α, powers of 2, 3, 5 and the number 103. As J.I. Good said: "the favoured integers seem all to be of the form $2^a 3^b$" [5].
10. By unknown source. No comment.
11. The expression can be also rewritten in more symmetric form: $\mu = 2\left(\frac{20}{3}\alpha^{-1} + \left(\frac{20}{3\pi}\right)^2\right)$. It can be noted that the number (20/3) appears in the author previous work [18] in the expression for the gravitational constant G.
12. One of the found expressions by author's specialized program. The search was performed for the expression of the view: $\mu = p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}$, where p_i — some prime numbers, n_i — some natural numbers. Also:
$\mu = \left(\frac{19}{5}\right)^{21} \frac{1}{13^8}$.
13. Number 2267 has many interesting properties; it is a prime of the form $(30n - 13)$ and $(13n + 5)$, it is congruent to 7 mod 20. It is father primes of order 4 and 10 etc. In the divisor of this formula there are sequential primes 5, 7, 11. There are other possible expressions of the similar form with such precision (10^{-8}), for example: $\mu = \frac{45 * 49 * 53^2}{8 * 29 * \alpha^{-1}} 5\pi$. It is also hard to justify why in expressions 9 and 13 α^{-1} stays opposite to π as by definition they supposed to be on the same side: $\alpha^{-1} = hc/ke^2$ or $(2\pi\alpha^{-1}) = hc/ke^2$. But the author did not succeed in finding similar expressions with α and π on the same side with the same uncertainty. There are some few other nice looking formulas which the use of big prime numbers, for example: $\mu = \sqrt{4^3 \cdot 52679}$ (9×10^{-8}).
14. Another possible expression was found using web based program Wolframalpha [23]. The precision is the same as in next formula.
15. Simon Plouffe's approximation using Fibonacci and Lucas numbers [8] - slightly adjusted from its original look. Another elegant form for this expression is following: $\mu^{32} = \frac{11^{47}5^{80}}{\phi^2}$.
16. This formula has the best precision alone the listed. Though, powers of π and e seem to despoil its possible physical meaning.

5 Conclusions

At the present moment big attention is paid to experimental verification of possible proton-electron mass ratio variation. If experimental data will provide evidence for the ratio constancy then only few expressions (14-16 from the listed) may pretend to express proton-electron mass ratio as they fall closely into current experimental uncertainty range (4.1×10^{-10} as per CODATA 2010). Of course Simon Plouffe's formula (14) seems as a pure winner among them in terms of the balance between it simplicity and precision. However, some future hope for the other formulas remains if the variability of the proton to electron mass ratio is confirmed. Important to note that there could be unlimited numbers of numerical approximations for dimensionless constant. Some of them may look more simple and "natural" than others. It is easy to see that expression simplicity and explain-ability in opposite determines its precision. As all formulas with uncertainty 10^{-8} and better become obviously more complex. And at the end: "What is the chance that seemingly impressive formulae arise

En revanche, aucune des suites ($b \times n$) avec $b > 0$, (n^b) avec $b > 0$, ($\log_b(n)$), (p_n) (n-ième nombre premier), ($\log(p_n)$) ne vérifient la loi de Benford, tout simplement parce que les fréquences associées aux chiffres ne convergent pas.

Le cas de la table de constantes mathématiques que Simon Plouffe collecte depuis de nombreuses années, et qui contient aujourd'hui plus d'un milliard de nombres, est intéressant, car elle satisfait assez bien la loi de Benford. Simon Plouffe exploite d'ailleurs cette loi pour détecter les erreurs qui pourraient s'y glisser.

Reste que les résultats qu'on obtient dans le monde parfait des mathématiques ne justifient pas ce que Newcomb et Benford ont vu pour les séries de nombres provenant du monde réel. C'est pourquoi de nombreuses études tentent d'expliquer rationnellement l'étrange phénomène. Cinq types d'explications sont proposés ; nous allons les exposer rapidement.

CONVERGENCE DES FRÉQUENCES

Pour les numéros des rues qu'on trouve dans des adresses collectées dans un annuaire – et qui vérifient assez bien la loi de Benford –, une idée simple vient immédiatement à l'esprit. Si une rue possède 50 numéros, alors plus d'un cinquième des numéros commencent par un '**1**' (à cause de l'enchaînement de numéros 10, 11, 12, ..., 19) ; si elle en possède 20 ou 200, plus de la moitié des numéros commencent par un '**1**'. Il est donc parfaitement normal que dans une rue dont la longueur est inconnue, ou que lorsqu'on considère une multitude de rues, on trouve en moyenne plus souvent des numéros commençant par '**1**' que par '**9**' (et plus généralement par le chiffre c que par $c + 1$).

Réfléchissons à cette observation en étudiant les chiffres qui apparaissent dans les entiers compris entre 1 et n. Notons $fr_1(n)$ la fréquence des entiers commençant par '**1**' parmi les entiers compris entre 1 et n, $fr_2(n)$ la fréquence pour le '**2**', $fr_3(n)$ la fréquence pour le 3, etc.

par des étudiants indiens d'un peu plus de 70 000 décimales énoncées en 9 h 27 min et 17 h14 min.

Fou des constantes, Simon Plouffe invente « l'inverseur de Plouffe », un site internet fermé maintenant, qui répertoriait un nombre incalculable de constantes mathématiques et qui permettait donc de retrouver une constante via son développement décimal.

Il co-démontre en 1995 la formule de Bailey-Borwein-Plouffe :

$$\pi = \sum_{k\geq 0} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$$

qui permet d'avoir accès à la n^e décimale de π en base 2 ou 16. Il existe des formules plus simples à mémoriser pour retrouver π. Par exemple, en sommant les fractions de dénominateur impair, en alternant les signes, on trouve :

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots = \sum_{n\geq 0} \frac{(-1)^n}{2n+1}.$$

Cette formule est nommée formule de Madhava, Gregory et Leibniz, en l'honneur du mathématicien indien Madhava de Sangamagrama et des Européens James Gregory et Gottfried Wilhelm Leibniz. Elle se démontre à partir d'un développement limité de la fonction arctan et bien que plus facile à retenir que la formule de Bailey-Borwein-Plouffe, elle n'est pas très utile en pratique. Une autre formule simple à retenir pour retrouver π est donnée par le produit de Wallis :

$$\frac{\pi}{2} = \frac{2\times 2\times 4\times 4\times \ldots}{3\times 3\times 5\times 5\times \ldots}$$

Notons qu'en 2016, on connaissait 22 600 milliards de décimales de π. Pour les curieux, la salle Pi du Palais de la Découverte affiche sur ses murs 704 décimales de π. Elles ont été calculées « à la main » (sans ordinateur ni calculatrice) par William Shanks en 1873. Cependant ce dernier avait fait une erreur à partir de la 528^e décimale. Cette faute, repérée en 1945 dans la galerie, a maintenant été corrigée.

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(the brother of your first author), David Bailey, and Simon Plouffe, and named the BBP formula after them:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \tag{2.1}$$

[Bailey et al. 97]. (It is the nature of much experimental mathematics to date that it involves formulas that occupy a fair amount of space on the page. The above formula is but a gentle introduction to what is to follow. In most of the examples we present, however, the formulas involve only very simple notions, as is the case here.)

Using formula (2.1), you can directly calculate binary or hexadecimal digits of π beginning with the nth digit, without having to first compute the previous $n-1$ digits. All you need to carry out the computation is a simple algorithm using standard 64-bit or 128-bit arithmetic. We'll describe how this calculation is carried out, but our real interest is in how the BBP formula came to be discovered.

The story began with the well-known classical formula

$$\log 2 = \sum_{k=1}^{\infty} \frac{1}{k2^k}. \tag{2.2}$$

Around 1994, Peter Borwein and Simon Plouffe of Simon Fraser University in Canada realized that you could use this formula to calculate individual binary digits of $\log 2$. Suppose you want to compute a few binary digits starting at position $d+1$ for some positive integer d. This is equivalent to computing $\{2^d \log 2\}$, where $\{\dots\}$ denotes fractional part. From (2.2) we get

$$\begin{aligned} \{2^d \log 2\} &= \left\{ \left\{ \sum_{k=0}^{d} \frac{2^{d-k}}{k} \right\} + \left\{ \sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k} \right\} \right\} \\ &= \left\{ \left\{ \sum_{k=0}^{d} \frac{2^{d-k} \bmod k}{k} \right\} + \left\{ \sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k} \right\} \right\}. \end{aligned} \tag{2.3}$$

(Take this one step at a time; there is nothing deep going on here, just notational complexity. In calculating the fractional part of a sum, we can (the brother of your first author), David Bailey, and Simon Plouffe, and named the BBP formula after them:

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- 1983 – Gerd Faltings proves the Mordell conjecture and thereby shows that there are only finitely many whole number solutions for each exponent of Fermat's Last Theorem.
- 1985 – Louis de Branges de Bourcia proves the Bieberbach conjecture.
- 1986 – Ken Ribet proves Ribet's theorem.
- 1987 – Yasumasa Kanada, David Bailey, Jonathan Borwein, and Peter Borwein use iterative modular equation approximations to elliptic integrals and a NEC SX-2 supercomputer to compute π to 134 million decimal places.
- 1991 – Alain Connes and John W. Lott develop non-commutative geometry.
- 1992 – David Deutsch and Richard Jozsa develop the Deutsch–Jozsa algorithm, one of the first examples of a quantum algorithm that is exponentially faster than any possible deterministic classical algorithm.
- 1994 – Andrew Wiles proves part of the Taniyama–Shimura conjecture and thereby proves Fermat's Last Theorem.
- 1994 – Peter Shor formulates Shor's algorithm, a quantum algorithm for integer factorization.
- 1995 – Simon Plouffe discovers Bailey–Borwein–Plouffe formula capable of finding the nth binary digit of π.
- 1998 – Thomas Callister Hales (almost certainly) proves the Kepler conjecture.
- 1999 – the full Taniyama–Shimura conjecture is proven.
- 2000 – the Clay Mathematics Institute proposes the seven Millennium Prize Problems of unsolved important classic mathematical questions.

16.12.2 Les travaux de Simon Plouffe

Pour clore ce livre, plongeons-nous dans une démarche amusante. Dans son *Mémoire présenté comme exigence partielle de la maîtrise en mathématiques* de 1992, Simon Plouffe explique comment il a trouvé des fonctions génératrices pour un certain nombre de suites de la base de suites réunie par Sloane (voir 5.1.1). Il prenait les premiers termes de la suite, cherchait un approximant de Padé dont le développement en série entière avait pour premiers coefficients ceux de la suite étudiée (il s'agit de résoudre un système linéaire, voir exercice 12.3). Puis il regardait si, par hasard, les coefficients suivants étaient les mêmes pour la suite et pour l'approximant de Padé trouvé, autrement dit, il vérifiait si l'approximant de Padé convenait pour la suite entière. Dans les 4568 suites qu'il avait à sa disposition, 614 se sont révélées avoir pour fonction génératrice une fraction rationnelle. Simon Plouffe a ensuite cherché si d'autres suites de la base de Sloane pouvaient être décrites par d'autres types de fonctions génératrices, un vaste problème.

Notons que la suite des nombres premiers n'a pas de fonction génératrice sous forme de fraction rationnelle.

about the nature of mathematical things. To what extent are the decimal places of pi that we haven't yet figured out real? It would be hard to argue that, say, the trillion trillionth digit of pi doesn't exist or that it doesn't have a specific fixed value, even though we don't yet know what it is. But in what sense or form *does* it exist until, at the end of an immensely long calculation, still to be carried out, it pops into a computer's memory?

The first couple of hundred digits of pi.

(ARTWORK BY JEFF DARLING)

As a curious aside, it's worth mentioning a discovery made by researchers David Bailey, Peter Borwein, and Simon Plouffe in 1996. They found a fairly simple formula—the sum of an infinite series of terms—for pi that allows any digit of pi to be calculated *without knowing any of the preceding digits*. (Strictly speaking, the digits calculated by the Bailey-Borwein-Plouffe formula are hexadecimal—base-16—digits as opposed to decimal digits.) That seems, at first sight, impossible, and it certainly came as a surprise to other mathematicians. What's more, a computation of, say, the billionth digit of pi, using this method, can be done on an ordinary laptop in less time than it takes to eat a meal at a restaurant. Variations on the Bailey-Borwein-Plouffe formula can be used to find other "irrational" numbers like pi whose decimal extensions go on forever without repeating.

The question of whether anything in pure mathematics is truly random is a valid one. Randomness implies the complete absence of pattern or predictability. Something is unpredictable only if it's cannot be random. But what about the digits lying beyond those that have been computed? Assuming pi is normal in base 10 they remain essentially statistically random to us. In other words, if someone asked you for a random string of a thousand digits it would be a valid response to build a computer to calculate pi to 1,000 places more than is presently known and use those places as the random string. Asked for another random string of the same length, you could compute the next (previously unknown) thousand digits. This raises an interesting philosophical question about the nature of mathematical things. To what extent are the decimal places of pi that we haven't yet figured out real? It would be hard to argue that, say, the trillion trillionth digit of pi doesn't exist or that it doesn't have a specific fixed value, even though we don't yet know what it is. But in what sense or form *does* it exist until, at the end of an immensely long calculation, still to be carried out, it pops into a computer's memory?

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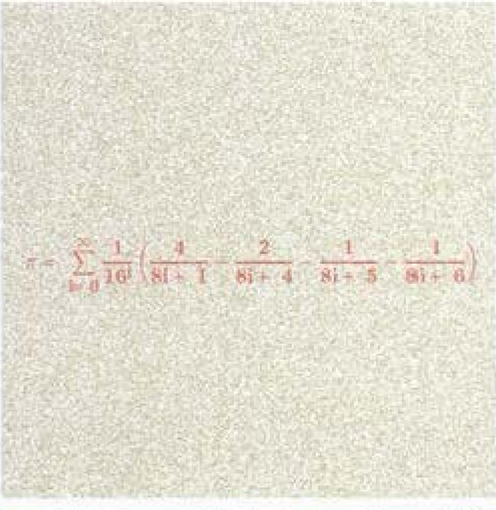
beginning with the trillion-and-first. The result (B4466E8D21 5388C4E014, if you must know) agreed with the main computation.

The algorithm for computing isolated digits of π is based on a remarkable formula discovered in 1996 by David Bailey, now at the Lawrence Berkeley Laboratory, Peter Borwein at Simon Fraser University in Burnaby, British Columbia, and Simon Plouffe, now at the University of Montreal. The BBP formula, as it's called, looks on the surface like many other infinite series approximations for π (see Figure 5), but it represents another major breakthrough in π computation. For the first time, a computer can compute the hexadecimal digits of π starting anywhere. That is, to obtain the trillion-and-first hexadecimal digit, it is not necessary to compute any of the first trillion digits. Until Bailey, Borwein and Plouffe discovered their formula, no one had even dreamed such a thing was possible.

The other unusual aspect of the BBP formula is the way it was discovered. Most formulas for π—indeed, most formulas of any sort—are derived by hand, starting from other known formulas. The BBP formula, though, was the product of a computer search. In 1995, Borwein and Plouffe realized that any infinite sum whose terms are of the form $P(k)/b^k Q(k)$, where P and

Q are polynomials, permits the rapid determination of individual base-b digits. Their starting point was a familiar formula for the natural logarithm of 2, $\ln 2 = \frac{1}{2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \cdots$, but they were soon able to find similar formulas in the mathematical literature for many other well and lesser known numbers. However, their literature search did not turn up anything for π. They had better luck looking in the computer.



$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right)$$

Figure 5. *The Borwein-Bailey-Plouffe formula for pi, which for the first time made it possible to compute any given digit of pi without knowing all the previous digits. However, it works only in base 16 (or more generally, a power of 2). In the background is a computer image of the first 250,000 binary digits of pi, with each 0 rendered as a dark pixel and each 1 rendered as a light pixel. The image is 500 pixels by 500 pixels. (Figure courtesy of Jonathan Borwein.)*

Borwein and Plouffe turned to an algorithm called PSLQ, developed by Bailey and Helaman Ferguson, who was then at the Supercomputing Research Center in Lanham, Maryland. PSLQ is a streamlined version of an algorithm discovered