On analytic properties and canonical constructions of absolute zeta functions

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Contents

List of Notations

$\mathbb{N}$	The set of positive integers
$\mathbb{N}_0$	The set of non-negative integers
$\mathbb{Z}$	The ring of integers
$\mathbb{Q}$	The field of rational numbers
$\overline{\mathbb{Q}}$	The field of algebraic numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers
$\mathbb{P}$	The set of prime numbers
$\mathbb{P}^{\mathbb{N}}$	The set of prime powers p^m ($p \in \mathbb{P}$, $m \in \mathbb{N}$)
$\mathbb{P}_S^{\mathbb{N}}$	The set of prime powers p^m ($p \in \mathbb{P} \setminus S$, $m \in \mathbb{N}$), where S is a subset of $\mathbb{P}$
$\mathbb{F}_q$	The finite field with q elements ($q \in \mathbb{P}^{\mathbb{N}}$)
$\mathbb{D}$	The open unit disk $\{z \in \mathbb{C} \mid \lvert z \rvert < 1\}$
$M_n(R)$	The set of $n \times n$ matrices over a ring R
$\mathrm{Ob}(\mathcal{C})$	The collection of objects of a category $\mathcal{C}$
$\mathfrak{Set}$	The category of sets
$\mathfrak{Ab}$	The category of abelian groups
$\mathfrak{CRing}$	The category of commutative rings
$\mathfrak{M}_0$	The category of monoids, where a monoid in this thesis means a commutative multiplicative semigroup with 1 and 0
$\mathfrak{Alg}_R$	The category of R-algebras, where R is a commutative ring
$\mathfrak{Mod}_R$	The category of R-module, where R is a commutative ring
Sch	The category of schemes over $\mathbb{Z}$
MSch	The category of monoid schemes
$\lceil x \rceil$	The ceiling function ($x \in \mathbb{R}$), i.e. $\lceil x \rceil := \min\{n \in \mathbb{Z} \mid x \leq n\}$
$\lfloor x \rfloor$	The floor function ($x \in \mathbb{R}$), i.e. $\lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid n \leq x\}$
$\mu(n)$	The Möbius function, which sends $n \in \mathbb{N}$ to $(-1)^k$ if n is square-free and has k distinct prime factors or 0 if n has a squared prime factor
ζ_n	The primitive n-th root $\exp(\frac{2\pi\sqrt{-1}}{n})$ of unity ($n \in \mathbb{N}$)
$M^\times$	The group of invertible elements of a monoid M
$\chi_{\mathrm{top}}(X)$	The Euler characteristic of a complex manifold X

Let R be a commutative ring, A be an R-algebra and $\mathcal{X}$ be a scheme over R.

$\mathcal{X}_A$	The base change $\mathcal{X} \times_{\operatorname{Spec} R} \operatorname{Spec} A$

Let $X = (X, \mathcal{O}_X)$ be a monoid scheme of finite type and $x \in X$.

r_x, l_x, $t_{x,j}$	The integers satisfying $\mathcal{O}_{X,x}^\times \cong \mathbb{Z}^{r_x} \times \prod_{j=1}^{l_x} \mathbb{Z}/t_{x,j}\mathbb{Z}$ with $t_{x,j} \mid t_{x,j+1}$
T_x	The product of $t_{x,1}, \ldots, t_{x,l_x}$
R_X	The maximum of r_x's
T_X	The product of T_x's

Chapter 1

Introduction

In this thesis, we study analytic properties of an absolute zeta function and give a simpler construction of the absolute zeta function of a geometric object.

An absolute zeta function is the zeta function associated with an "$\mathbb{F}_1$-scheme", which would be a geometric object expected to be a strong tool in an approach to solving the Riemann hypothesis. In this approach, it is pivotal to discover the relationship between geometric operations of $\mathbb{F}_1$-schemes such as fibre products and function-theoretic information such as poles and zeros of an absolute zeta function.

In this thesis, we give a further understanding of absolute zeta functions in terms of their analytic properties and relationship with $\mathbb{F}_1$-schemes. In this chapter, we review the background of absolute zeta functions and state three main results of this thesis.

In what follows, we denote the set of prime numbers and that of prime powers by $\mathbb{P}$ and $\mathbb{P}^{\mathbb{N}}$, respectively.

1.1 Background of absolute zeta functions

In number theory, it is traditionally important to study the solutions over $\mathbb{Z}$ of algebraic equations. One of the approaches to such a problem is to investigate the set $\mathcal{X}(\mathbb{F}_{p^m})$ of $\mathbb{F}_{p^m}$-rational points of a scheme $\mathcal{X}$ of finite type over $\mathbb{Z}$ for each $p \in \mathbb{P}$ and unify information on $\mathcal{X}(\mathbb{F}_{p^m})$'s. In particular, the *congruent zeta function*, or the *local zeta function*,

$$Z(\mathcal{X}_{\mathbb{F}_p}, p^{-s}) := \exp\left(\sum_{m=1}^{\infty} \frac{\#\mathcal{X}(\mathbb{F}_{p^m})}{m} p^{-ms}\right)$$

of $\mathcal{X}_{\mathbb{F}_p} := \mathcal{X} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{F}_p$, defined as the generating function of the number of $\mathbb{F}_{p^m}$-rational points of $\mathcal{X}_{\mathbb{F}_p}$, has been studied as exemplified by the Weil Conjecture since the 20th century.

1.1.1 Absolute zeta functions

Soulé [46] studied the limit

$$\lim_{p\to 1}(p-1)^{f_{\mathcal{X}}(1)} Z(\mathcal{X}_{\mathbb{F}_p}, p^{-s}) = \lim_{p\to 1}(p-1)^{f_{\mathcal{X}}(1)} \exp\left(\sum_{m=1}^{\infty} \frac{f_{\mathcal{X}}(p^m)}{m} p^{-ms}\right)$$

when there exists a polynomial $f_{\mathcal{X}}(t) = \sum_{j=0}^{R} a_j t^j \in \mathbb{Z}[t]$ satisfying that $\#\mathcal{X}(\mathbb{F}_q) = f_{\mathcal{X}}(q)$ for any $q \in \mathbb{P}^{\mathbb{N}}$. More precisely, he found the fact that

$$\lim_{p\to 1}(p-1)^{f_{\mathcal{X}}(1)} \exp\left(\sum_{m=1}^{\infty} \frac{f_{\mathcal{X}}(p^m)}{m} p^{-ms}\right) = \prod_{j=0}^{R}(s-j)^{-a_j}$$

and called it the *absolute zeta function of* $\mathcal{X}$, motivated by the ambition to define the zeta function of an $\mathbb{F}_1$-geometric object (see Chapter 2 for the details of $\mathbb{F}_1$-geometry). Later, Deitmar [11, 12] introduced a *monoid scheme* and realised the above rational function as an invariant of a certain monoid scheme. After Deitmar's work, Connes and Consani generalised the above definition of absolute zeta functions as follows.

Definition 1.1.1 (Connes and Consani [8, §2.1])**.** Let $f\colon [1,\infty) \to \mathbb{C}$ be a function satisfying that $|f(t)| \leq Ct^d$ for some $C > 0$ and $d > 0$. Then, the *absolute zeta function* of f is defined by the limit

$$\zeta_f^{\lim}(s) := \lim_{p\to 1+}(p-1)^{f(1)} \exp\left(\sum_{m=1}^{\infty} \frac{f(p^m)}{m} p^{-ms}\right)$$

for $\mathrm{Re}(s) > d$ when the right-hand side converges.

To define an absolute zeta function even for a function with a pole at $t = 1$, Kurokawa gave another definition of an absolute zeta function using zeta regularisation. This enables us to regard Barnes' multiple gamma function as an absolute zeta function.

Definition 1.1.2 (Kurokawa and Ochiai [34], Deitmar, Koyama and Kurokawa [14])**.** Let $f\colon (1,\infty) \to \mathbb{C}$ be a measurable function. The function f is said to be *admissible* if there exist a constant $d > 0$ and a (non-empty open) domain D of $\mathbb{C}$ such that

$$Z_f(w,s) := \frac{1}{\Gamma(w)} \int_1^{\infty} f(t) t^{-s} (\log t)^{w-1} \frac{dt}{t} = \frac{1}{\Gamma(w)} \int_0^{\infty} f(e^x) e^{-sx} x^{w-1} dx$$

converges for every $(w,s) \in D \times \{s \in \mathbb{C} \mid \mathrm{Re}(s) > d\}$ and $Z_f(w,s)$ admits a unique holomorphic extension to $w = 0$. If f is admissible, then we define the *absolute zeta function* of f by

$$\zeta_f(s) := \exp\left(\frac{\partial}{\partial w} Z_f(w,s)\Big|_{w=0}\right).$$

It is known that absolute zeta functions have the following functoriality.

Proposition 1.1.3 (Kurokawa and Ochiai [34, Theorem B]).

(1) Let f_1*,* f_2 *be admissible functions on* $(1,\infty)$*. Then, it holds that*

$$\zeta_{f_1+f_2}(s) = \zeta_{f_1}(s)\zeta_{f_2}(s).$$

(2) Let Φ_1*,* Φ_2 *be finite subsets of* $\mathbb{C}$*. Put* $f_1(t) := \sum_{\rho_1\in\Phi_1} a_{\rho_1} t^{\rho_1}$ $(a_{\rho_1}\in\mathbb{C})$ *and* $f_2(t) := \sum_{\rho_2\in\Phi_2} b_{\rho_2} t^{\rho_2}$ $(b_{\rho_2}\in\mathbb{C})$*. Then, it holds that*

$$\zeta_{f_1 f_2}(s) = \prod_{\rho_1\in\Phi_1}\prod_{\rho_2\in\Phi_2}(s-(\rho_1+\rho_2))^{-a_{\rho_1}b_{\rho_2}}.$$

Remark 1.1.4. Kurokawa may have introduced the above definition with the following ambition to treat the *Riemann zeta function* $\zeta(s)$ as the absolute zeta function of "Spec $\mathbb{Z}$ over $\mathbb{F}_1$" (see Section 2.1 for some details on the Riemann zeta function). Let $\mathcal{Z}$ be the set of nontrivial zeros of $\zeta(s)$. According to Deninger [17, Theorem 3.3], we could formally have the decomposition of the completed Riemann zeta function

$$\widehat{\zeta}(s) := 2^{-\frac{1}{2}}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \zeta_{f_{\mathbb{Z}}}\left(\frac{s}{2\pi}\right) = \frac{\zeta_{f_\xi}\left(\frac{s}{2\pi}\right)}{\frac{s}{2\pi}\cdot\frac{s-1}{2\pi}}$$

if $f_\xi(t) := \widetilde{f}_\xi(t^{\frac{1}{2\pi}})$ were admissible, where $f_{\mathbb{Z}}(t) := \widetilde{f}_{\mathbb{Z}}(t^{\frac{1}{2\pi}})$ and

$$\widetilde{f}_{\mathbb{Z}}(t) := t - \widetilde{f}_\xi(t) + 1, \quad \widetilde{f}_\xi(t) := \sum_{\rho\in\mathcal{Z}} t^\rho.$$

On the other hand, Kurokawa suggests that "$\widehat{\zeta}(s) = \zeta_{\widetilde{f}_{\mathbb{Z}}}(s)$" (see e.g. [30, p. 42]).

Example 1.1.5 (Kurokawa and Ochiai [34, Theorem A])**.** Let Φ be a finite subset of $\mathbb{C}$ and put

$$f_\Phi(t) = \sum_{\rho\in\Phi} a_\rho t^\rho \quad (a_\rho\in\mathbb{Z}).$$

Then, it holds that

$$\zeta_{f_\Phi}^{\lim}(s) = \prod_{\rho\in\Phi}(s-\rho)^{-a_\rho} = \zeta_{f_\Phi}(s).$$

Thus, both definitions coincide with each other if f is a linear combination of t^ρ's.

For example, motivated by Remark 1.1.4, Kurokawa [31] introduced the following function called the *counting function of an absolute Riemann surface*

$$f_{\boldsymbol{\alpha}}(t) := t - 2\sqrt{t}\sum_{k=1}^{g}\cos(\alpha_k\log t) + 1 = t - \sum_{k=1}^{g}\left(t^{\rho_k} + t^{\overline{\rho_k}}\right) + 1,$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g) \in [0, \infty)^g$ and $\rho_k := \frac{1}{2} + \sqrt{-1}\alpha_k$. Note that this is comparable with "the finite version of $\widetilde{f_{\mathbb{Z}}}$" in Remark 1.1.4. Then, its absolute zeta function is

$$\zeta_{f_{\boldsymbol{\alpha}}}(s) = \frac{1}{s(s-1)} \prod_{k=1}^{g} \left((s - \rho_k)(s - \overline{\rho_k}) \right).$$

Thus, we might consider this as "the finite version of $\widehat{\zeta}(s)$" in Remark 1.1.4. However, the definition of "absolute Riemann surfaces" has not yet been established.

Example 1.1.6 (Kurokawa and Ochiai [34]). Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r) \in (0, \infty)^r$ and put

$$f_{\boldsymbol{\omega}}(t) := \frac{1}{(1 - t^{-\omega_1}) \cdots (1 - t^{-\omega_r})}.$$

Then, it holds that

$$\zeta_{f_{\boldsymbol{\omega}}}(s) = \Gamma_r(s, \boldsymbol{\omega}),$$

where $\Gamma_r(s, \boldsymbol{\omega})$ is Barnes' multiple gamma function. In particular, when $r = 1$, it holds that

$$\zeta_{f_\omega}(s) = \Gamma_1(s, \omega) = \frac{\omega^{\frac{s}{\omega} - \frac{1}{2}}}{\sqrt{2\pi}} \Gamma\left(\frac{s}{\omega}\right).$$

Thus, the gamma function $\Gamma(s)$ is also an absolute zeta function.

For a function f which satisfies the conditions of two definitions, it is not clear whether $\zeta_f^{\lim}(s)$ and $\zeta_f(s)$ always coincide with each other. However, in joint work with Y. Hirakawa, we found that we can create infinitely many simple examples in which they do not coincide by using Connes-Consani's integral formula (see Example 4.3.7).

1.1.2 Absolute Euler products

Many zeta functions including the Riemann zeta function have an infinite product representation running over all prime numbers (see e.g. Kurokawa [29, §11.1]). This infinite product is called an *Euler product*. For example, the Riemann zeta function has the following Euler product representation:

$$\zeta(s) = \prod_{p \in \mathbb{P}} (1 - p^{-s})^{-1}.$$

As a generalisation of this Euler product, the zeta function $\zeta_R(s)$ of a finitely generated $\mathbb{Z}$-algebra R, which is called the Hasse zeta function of $\operatorname{Spec} R$, has the following Euler product representation:

$$\zeta_R(s) := \prod_{\mathfrak{a} \in \text{m-Spec}R} (1 - (\#R/\mathfrak{a})^{-s})^{-1} = \prod_{p \in \mathbb{P}} \prod_{n=1}^{\infty} (1 - p^{-ns})^{-\kappa(p,n;R)},$$

where m-Spec$R$ is the set of maximal ideals of R and

$$\kappa(p,n;R) := \#\{\mathfrak{a} \in \text{m-Spec}R \mid \#R/\mathfrak{a} = p^n\}.$$

Similarly, the congruent zeta function $Z(X, p^{-s})$ of a smooth projective variety X over $\mathbb{F}_p$ has the following Euler product representation:

$$Z(X,p^{-s}) = \prod_{l \in \mathbb{P}} \prod_{n=1}^{\infty} (1 - l^{-ns})^{-\kappa(l,n;X)}, \tag{1.1}$$

where $k(x)$ is the residue field of x, the set $|X|$ is the set of closed points of X, and

$$\kappa(l,n;X) = \begin{cases} \#\{x \in |X| \mid \#k(x) = p^n\} & \text{if } l = p, \\ 0 & \text{if } l \neq p. \end{cases}$$

However, since there should be no prime numbers in $\mathbb{F}_1$-theory, absolute zeta functions cannot have a similar Euler product representation. In [30, §7.3], Kurokawa calculated certain infinite products of absolute zeta functions for some specific schemes of finite type over $\mathbb{Z}$ according to Soulé's definition. Then, he suggested that the "absolute zeta function $\zeta_{\mathcal{X}}(s)$" of a general scheme $\mathcal{X}$ of finite type over $\mathbb{Z}$ has the following infinite product structure, which he called the *absolute Euler product*. Note that "$\zeta_{\mathcal{X}}(s)$" has not generally been defined yet.

Conjecture 1.1.7 (Kurokawa's suggestion [30, §7.3])**.** *For a scheme $\mathcal{X}$ of finite type over $\mathbb{Z}$, there should exist the absolute Euler product of "$\zeta_{\mathcal{X}}(s)$" of the form*

$$\zeta_{\mathcal{X}}(s) = \left(\frac{1}{s}\right)^{\chi_{abs}(\mathcal{X})} \prod_{n=1}^{\infty} \left(1 - \left(\frac{1}{s}\right)^n\right)^{-\kappa(n,\mathcal{X})}, \tag{1.2}$$

where $\kappa(n,\mathcal{X})$ is an integer for any $n \in \mathbb{N}$, the integer $\chi_{abs}(\mathcal{X}) := f_{\mathcal{X}}(1)$ is the absolute Euler characteristic*, and $f_{\mathcal{X}}(t)$ is the function associated with $\mathcal{X}$. Moreover, the infinite product (1.2) converges absolutely for* $\mathrm{Re}(s) > \dim \mathcal{X}$.

Remark 1.1.8. Assume that $\mathcal{X}_{\mathbb{F}_p}$ is a smooth projective variety. Let $f_{\mathcal{X}}$ be a function which satisfies $f_{\mathcal{X}}(q) = \#\mathcal{X}(\mathbb{F}_q)$ for any $q \in \mathbb{P}^{\mathbb{N}}$. Then, by the Lefschetz trace formula (see e.g. Hartshorne [22, p. 454]), it holds that

$$f_{\mathcal{X}}(p^m) = \#\mathcal{X}(\mathbb{F}_{p^m}) = \sum_{i=0}^{2\dim \mathcal{X}_{\mathbb{F}_p}} (-1)^i (\alpha_{i,1}^m + \cdots + \alpha_{i,b_i}^m),$$

where the integer $\dim \mathcal{X}_{\mathbb{F}_p}$ is the dimension of $\mathcal{X}_{\mathbb{F}_p}$, the complex numbers $\alpha_{i,1}, \ldots, \alpha_{i,b_i}$ are eigenvalues of the induced map of the Frobenius morphism on the i-th étale cohomology of $\mathcal{X}_{\overline{\mathbb{F}_p}}$, and the integer b_i is the i-th Betti number. If we substituted $m = 0$ in this equality, we could formally obtain

$$f_{\mathcal{X}}(1) = \text{“}\#\mathcal{X}(\mathbb{F}_1)\text{”} = \sum_{i=0}^{2\dim \mathcal{X}_{\mathbb{F}_p}} (-1)^i b_i = \chi_{\mathrm{top}}(\mathcal{X}(\mathbb{C})),$$

where $\chi_{\mathrm{top}}(\mathcal{X}(\mathbb{C}))$ is the Euler characteristic of the complex manifold $\mathcal{X}(\mathbb{C})$. This is the reason why $f_{\mathcal{X}}(1)$ is called the absolute Euler characteristic.

This philosophy that the number of "$\mathbb{F}_1$-rational points" of a scheme and the value at 1 of the original function f of the absolute zeta function ζ_f associated with it coincide with its Euler characteristic also appears in [46, Théorème 2], [28, Remark 2], [12, p. 141], and the proof of [14, Theorem 2.1].

Equation (1.2) in Conjecture 1.1.7 does not seem like the Euler products which we mentioned above. Kurokawa himself did not explain the validity of the name "absolute Euler product". However, in joint work with Y. Hirakawa, we found one of the reasons why it is reasonable to call Equation (1.2) an Euler product. We explain this in Subsection 4.4.2.

1.2 Main results

Originally, an absolute zeta function was expected to be the zeta function associated with an $\mathbb{F}_1$-scheme. Hence, we are interested in the relationship between absolute zeta functions and $\mathbb{F}_1$-schemes. However, most previous works study $\mathbb{F}_1$-geometry and absolute zeta functions separately.

In this thesis, we mainly give three results. The first main result is on the absolute Euler product of the absolute zeta function of a certain $\mathbb{F}_1$-scheme defined by Connes and Consani. The second is on three analytic properties of absolute zeta functions: the series expansion, an integral formula, and the absolute Euler product. The third is on a "canonical" construction of the absolute zeta function from a scheme over $\mathbb{Z}$ or $\mathbb{Q}$ using *ceiling and floor Puiseux polynomials*. Here, the second and the third results are based on joint work with Y. Hirakawa.

1.2.1 Absolute Euler products and $\mathbb{F}_1$-schemes

Let f be a polynomial with integer coefficients. Kurokawa [33, Theorem 11.1] formally gave the absolute Euler product of $\zeta_f(s)$ which is similar to Equation (1.2) and described the exponent corresponding to $\kappa(n, \mathcal{X})$ explicitly. However, he did not give the region where the absolute Euler product converges absolutely.

In this thesis, we formulate Kurokawa's suggestion (Conjecture 1.1.7) for a torsion free Noetherian $\mathbb{F}_1$-scheme, whose $\mathbb{Z}$-lift is almost a toric variety (see Chapter 2 for its definition). Moreover, we prove that the region where the absolute Euler product converges absolutely is $|s| > \operatorname{rel.dim} X_{\mathbb{Z}}/\mathbb{Z}$, where $\operatorname{rel.dim} X_{\mathbb{Z}}$ is the relative dimension of $X_{\mathbb{Z}}$ over $\mathbb{Z}$, and it is the widest possible region.

Theorem 1.2.1 (Theorem 3.1.5, [49, Theorem 3.8]). *Let $\mathcal{X}$ be a torsion free Noetherian $\mathbb{F}_1$-scheme. Then, there exists a polynomial $N_{\mathcal{X}} \in \mathbb{Z}[t]$ associated with $\mathcal{X}$. Let $X_{\mathfrak{M}_0}$ (resp. $X_{\mathbb{Z}}$) be the scheme (resp. the monoid scheme) obtained from $\mathcal{X}$. Then,* *it holds that*

$$\zeta_{\mathcal{X}/\mathbb{F}_1}(s) = s^{-N_{\mathcal{X}}(1)} \prod_{n=1}^{\infty} \left(1 - s^{-n}\right)^{-\kappa(n, X_{\mathfrak{M}_0})},$$

where $\kappa(n, X_{\mathfrak{M}_0})$ is a certain integer which is explicitly given by the information of $X_{\mathfrak{M}_0}$. Moreover, if $X_{\mathbb{Z}}$ is of finite type over $\mathbb{Z}$, the region of absolute convergence of this absolute Euler product is $\{s \in \mathbb{C} \mid |s| > \operatorname{rel.dim} X_{\mathbb{Z}}/\mathbb{Z}\}$.

Remark 1.2.2. By combining Theorem 4.4.6 and a method in Subsection 1.2.3, which we explain later, we can remove the condition "torsion free" in Theorem 1.2.1 and generalise Theorem 1.2.1.

1.2.2 Three analytic properties of absolute zeta functions

In previous works, absolute zeta functions were mainly investigated when f was a relatively specific function such as a polynomial or a product of $f_{\boldsymbol{\omega}}$'s. In this thesis, we investigate three analytic properties of the absolute zeta functions of elements of the following class of analytic functions:

$$\mathcal{A}_d := \{f \in C^{\omega}([1,\infty), \mathbb{C}) \mid \exists C > 0,\ \forall n \in \mathbb{N}_0,\ |\alpha_n(f)| \leq C d^n\}$$

for $d > 0$, where

$$\alpha_n(f) := \left(t\frac{\partial}{\partial t}\right)^n f(t)\bigg|_{t=1} = \left(\frac{\partial}{\partial x}\right)^n f(e^x)\bigg|_{x=0}$$

for any $n \in \mathbb{N}_0$. In these results, we take the branch of $\log s$ with the branch cut along $\mathbb{C} \setminus (-\infty, 0]$, and define $s^w := e^{w \log s}$ for $s, w \in \mathbb{C}$.

The first analytic property is that the logarithmic derivative of an absolute zeta function can be identified with the generating function of the iterative Euler derivatives of a given function. Note that we can determine the region where the absolute zeta function of $f \in \mathcal{A}_d$ can be defined based on its proof, while it is originally defined for a sufficiently large $\operatorname{Re}(s)$ by its definition.

Moreover, this expression of the absolute zeta function is useful for obtaining its asymptotic behaviour. According to Kurokawa [33, p. 116], it is classically known that the Hasse zeta function and the congruent zeta function tend to 1 as $\operatorname{Re}(s) \to \infty$. In [31, §1.7], Kurokawa observed that the absolute zeta function of $f_{\Phi}(t) = \sum_{\rho\in\Phi} a_{\rho} t^{\rho}$ for some $a_{\rho} \in \mathbb{Z}$ and a finite set $\Phi \subset \mathbb{C}$ satisfies

$$s^{f_{\Phi}(1)} \zeta_{f_{\Phi}}(s) = \prod_{\rho\in\Phi} \left(1 - \frac{\rho}{s}\right)^{-a_{\rho}} \to 1 \quad (s \to +\infty).$$

The following theorem is a generalisation of this observation.

Theorem 1.2.3 (Theorem 4.2.1)**.** *Let $d > 0$ and $f \in \mathcal{A}_d$.*

(1) For $\mathrm{Re}(s) > d$*, the absolute zeta function of* f *is*

$$\zeta_f(s) = s^{-f(1)} \exp\left(\sum_{n=1}^{\infty} \frac{\alpha_n(f)}{ns^n}\right).$$

In particular, this can be analytically continued to a single-valued holomorphic function on $\{s \in \mathbb{C} \mid |s| > d\}$*. Moreover, we have*

$$\lim_{s \to \infty} s^{f(1)} \zeta_f(s) = 1.$$

(2) For $\mathrm{Re}(s) > d$*, the logarithmic derivative of the absolute zeta function* $\zeta_f(s)$ *is*

$$\frac{\zeta_f'(s)}{\zeta_f(s)} = -\sum_{n=1}^{\infty} \alpha_{n-1}(f) s^{-n}.$$

The second analytic property is a corollary of Theorem 1.2.3, which asserts that the logarithmic derivative of the absolute zeta function of $f \in \mathcal{A}_d$ has a similar integral formula to Connes-Consani's integral formula.

Theorem 1.2.4 (Corollary 4.3.3)**.** *Let* $d > 0$ *and* $f \in \mathcal{A}_d$*. Then, it holds that*

$$\frac{\zeta_f'(s)}{\zeta_f(s)} = -\int_0^{\infty} f(e^x) e^{-sx} dx = -\mathcal{L}[f \circ \exp](s)$$

for $\mathrm{Re}(s) > d$*, where* $\mathcal{L}$ *is the Laplace transform.*

The third analytic property is also a corollary of Theorem 1.2.3, which is that the absolute zeta function of $f \in \mathcal{A}_d$ has the absolute Euler product expression. Let $\mathcal{M} \colon \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}^{\mathbb{N}_0}$ be the linear map defined by $\mathcal{M}(a)_0 := a_0$ and

$$\mathcal{M}(\alpha)_n := \frac{1}{n} \sum_{m \mid n} \mu\left(\frac{n}{m}\right) a_m \quad (n \in \mathbb{N})$$

for a sequence $a = \{a_n\}_{n=0}^{\infty}$ of complex numbers. Then, we define the linear map $M_n \colon \mathcal{A}_d \to \mathbb{C}$ by

$$M_n(f) := \mathcal{M}(\{\alpha_n(f)\}_{n=0}^{\infty})_n.$$

Theorem 1.2.5 (Theorem 4.4.6)**.** *Let* $d > 0$ *and* $f \in \mathcal{A}_d$*. Then, the following statements hold.*

(1) The series

$$S_f(s) := \sum_{n=1}^{\infty} M_n(f) \log\left(1 - s^{-n}\right)$$

converges absolutely for $|s| > \max\{d, 1\}$*.*

(2) It holds that

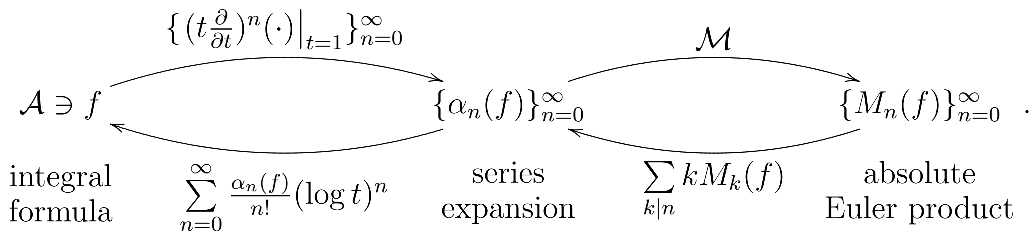
$$\log \zeta_f(s) = -f(1) \log s - S_f(s)$$

for $|s| > \max\{d, 1\}$. *In particular, it holds that*

$$\zeta_f(s) = s^{-M_0(f)} \prod_{n=1}^{\infty} (1 - s^{-n})^{-M_n(f)}.$$

According to the theorem, the series $S_f(s)$ converges absolutely at least for $|s| > \max\{d(f), 1\}$, where $d(f) := \inf\{d \mid f \in \mathcal{A}_d\}$. We wonder whether this region is the widest possible region where $S_f(s)$ converges absolutely. In this thesis, we prove it affirmatively at least if f is a linear combination of t^ρ's satisfying some technical assumptions by using ergodicity of an irrational rotation (Theorem 4.4.12).

Note that the relationship of a function $f \in \mathcal{A}_d$, the value $\alpha_n(f)$, and the linear map M_n which appeared in the above three theorems is described in the following diagram:



1.2.3 Ceiling and floor Puiseux polynomials

As we mentioned before, Soulé obtained the absolute zeta function from a scheme X of finite type over $\mathbb{Z}$, assuming that the sequence $(\#X(\mathbb{F}_q))_{q \in \mathbb{P}^{\mathbb{N}}}$ is interpolated by a polynomial. In addition, we can obtain the canonical polynomial from a torsion free Noetherian $\mathbb{F}_1$-scheme as we mentioned in Theorem 1.2.1. In these cases, we can define the canonical absolute zeta function of the geometric objects as the absolute zeta function of the polynomials.

However, there are normally infinitely many continuous functions which interpolate the number of $\mathbb{F}_q$-rational points of a geometric object. Hence, we cannot generally obtain such a canonical continuous function from a geometric object. For example, let $X = (X, \mathcal{O}_X)$ be a monoid scheme of finite type and $X_\mathbb{Z}$ be the $\mathbb{Z}$-lift of X (see Section 2.2 for their definitions). Then, Connes and Consani showed that

$$\#X_\mathbb{Z}(\mathbb{F}_q) = \sum_{x \in X} (q-1)^{r_x} \prod_{j=1}^{l_x} \gcd(q-1, t_{x,j})$$

for any prime power q (see Propositions 2.2.7 and 2.2.10), where the non-negative integers r_x, l_x and the positive integers $t_{x,j}$ are taken so that $\mathcal{O}_{X,x}^\times \cong \mathbb{Z}^{r_x} \times \prod_{j=1}^{l_x} \mathbb{Z}/t_{x,j}\mathbb{Z}$

with $t_{x,j} \mid t_{x,j+1}$ for each $x \in X$. In this case, there is no clue to determine what is a "canonical" interpolation of $(\#X_{\mathbb{Z}}(\mathbb{F}_q))_{q\in\mathbb{P}^{\mathbb{N}}}$.

By using the Fourier expansion of the periodic function $\gcd(q-1, t_{x,j})$ in q, Deitmar, Koyama and Kurokawa [14, pp. 61–63] interpolated $\#X_{\mathbb{Z}}(\mathbb{F}_q)$ to a certain continuous function $N_{X_{\mathbb{Z}}}$ on $[1,\infty)$ and then regarded $\zeta^{\lim}_{N_{X_{\mathbb{Z}}}}(s)$ as the absolute zeta function of X.

Theorem 1.2.6 (Deitmar, Koyama and Kurokawa [14, Theorem 2.1])**.** *For the above function $N_{X_{\mathbb{Z}}}$, it holds that*

$$\zeta^{\lim}_{N_{X_{\mathbb{Z}}}}(s) = \prod_{k=0}^{R_X}(s-k)^{\sum_{x\in X} T_x(-1)^{r_x-k+1}\binom{r_x}{k}},$$

where $T_x := \prod_{j=1}^{l_x} t_{x,j}$ and $R_X := \max_{x\in X} r_x$. Moreover, if $X_{\mathbb{Z}}$ is a smooth projective variety of relative dimension d, it holds that $N_{X_{\mathbb{Z}}}(1) = \chi_{\mathrm{top}}(X_{\mathbb{Z}}(\mathbb{C}))$ and $\zeta_{N_{X_{\mathbb{Z}}}}(d-s) = (-1)^{\chi_{\mathrm{top}}(X_{\mathbb{Z}}(\mathbb{C}))}\zeta_{N_{X_{\mathbb{Z}}}}(s)$, where $\chi_{\mathrm{top}}(X_{\mathbb{Z}}(\mathbb{C}))$ is the Euler characteristic of the complex manifold $X_{\mathbb{Z}}(\mathbb{C})$.

Remark 1.2.7. In [14], Deitmar, Koyama and Kurokawa took $t_{x,j}$'s as prime powers instead of the above integers satisfying $t_{x,j} \mid t_{x,j+1}$.

Despite this simple result, the proof of Theorem 1.2.6 involves relatively complicated calculations. In fact, comparing with Example 1.1.5 and Theorem 1.2.6, we see that the absolute zeta function $\zeta_{N_{X_{\mathbb{Z}}}}(s)$ of $N_{X_{\mathbb{Z}}}$ coincides with the absolute zeta function $\zeta_{\mathfrak{C}_{X_{\mathbb{Z}}}}(s)$ of the polynomial

$$\mathfrak{C}_{X_{\mathbb{Z}}}(t) = \sum_{x\in X} T_x(t-1)^{r_x}.$$

This polynomial $\mathfrak{C}_{X_{\mathbb{Z}}}$ is characterised as the *ceiling polynomial* of $X_{\mathbb{Z}}$, which is defined as the unique polynomial in $\mathbb{R}[t]$ satisfying the following conditions:

(1) The inequality $\mathfrak{C}_{X_{\mathbb{Z}}}(q) \geq \#X_{\mathbb{Z}}(\mathbb{F}_q)$ holds for every $q \in \mathbb{P}^{\mathbb{N}}$.

(2) There exist infinitely many prime powers q such that $\mathfrak{C}_{X_{\mathbb{Z}}}(q) = \#X_{\mathbb{Z}}(\mathbb{F}_q)$.

Thus, we have a simpler way to obtain the above absolute zeta function $\zeta_{N_{X_{\mathbb{Z}}}}(s)$ without using the periodicity of $\gcd(q-1, t_{x,j})$. This simple observation is notable in extending Soulé's idea to a more general scheme of finite type over $\mathbb{Z}$ for which we do not have any formula like Connes-Consani's formula of $\#X_{\mathbb{Z}}(\mathbb{F}_q)$.

Similarly, by replacing $\geq$ with $\leq$ in the first condition, we can recover the polynomial

$$\mathfrak{F}_{X_{\mathbb{Z}}}(t) = \sum_{x\in X}(t-1)^{r_x},$$

introduced by Deitmar [12, Theorem 1]. We call it the *floor polynomial* of $X_{\mathbb{Z}}$.

The above conditions satisfied by the ceiling polynomial suggest that it is not necessary to interpolate the whole sequence $(\#X_{\mathbb{Z}}(\mathbb{F}_q))_{q\in\mathbb{P}^{\mathbb{N}}}$ for the definition of an absolute zeta function of $X_{\mathbb{Z}}$, at least in view of the result of Deitmar, Koyama and Kurokawa [14]. Therefore, it is more natural to start from a general (separated) scheme of finite type over $\mathbb{Q}$ instead of the $\mathbb{Z}$-lift of a monoid scheme of finite type. Moreover, since the polynomial condition is too strict for most schemes of finite type over $\mathbb{Z}[S^{-1}]$, where S is a finite subset of $\mathbb{P}$, we generalise the ceiling polynomial by means of Puiseux polynomial. For example, a desired Puiseux polynomial exists uniquely for every elliptic curve E over $\mathbb{Q}$ as follows; this fact leads us to a provisional definition of the absolute zeta function of E.

Theorem 1.2.8 (Corollary 5.2.15, [23, Definition 3.4 and Corollary 3.15])**.** *Let E be an elliptic curve defined over $\mathbb{Q}$. Then, the Puiseux polynomial $\mathfrak{C}_E(t) := t+2t^{1/2}+1$ is characterised as the unique element in $\mathbb{R}[t^{1/\infty}] = \bigcup_{n\in\mathbb{N}} \mathbb{R}[t^{1/n}]$ satisfying the following condition: for any separated scheme $\mathcal{E}$ of finite type over $\mathbb{Z}$ satisfying that $\mathcal{E}_{\mathbb{Q}} \cong E$, there exists a finite set $S_{\mathcal{E}}$ of prime numbers such that for any finite set S of prime numbers containing $S_{\mathcal{E}}$, the Puiseux polynomial $\mathfrak{C}_E$ satisfies the following conditions:*

(1) The inequality $\mathfrak{C}_E(p^m) \geq \#\mathcal{E}(\mathbb{F}_{p^m})$ holds for every $p^m \in \mathbb{P}^{\mathbb{N}}$, where $p \notin S$.

(2) There exist infinitely many prime powers p^m such that $p \notin S$ and the equality $\lfloor \mathfrak{C}_E(p^m) \rfloor = \#\mathcal{E}(\mathbb{F}_{p^m})$ holds.

(3) $\mathfrak{C}_E(1) \in \mathbb{Z}$.

Moreover, the absolute zeta function of $\mathfrak{C}_E$ is

$$\zeta_{\mathfrak{C}_E}(s) = \frac{1}{s\left(s-\frac{1}{2}\right)^2(s-1)}.$$

We call $\mathfrak{C}_E$ the *ceiling Puiseux polynomial* of E. A drawback of $\mathfrak{C}_E$ is that the special value $\mathfrak{C}_E(1)$ does not coincide with the Euler characteristic of the complex torus $E(\mathbb{C})$. This is not consistent with the philosophy which we mentioned in Remark 1.1.8. Indeed, if X is a monoid scheme of finite type such that $T_x = 1$ for each $x \in X$ and $X_{\mathbb{Z}}$ is a smooth projective variety, then it holds that $N_{X_{\mathbb{Z}}}(1) = \mathfrak{C}_{X_{\mathbb{Z}}}(1) = \mathfrak{F}_{X_{\mathbb{Z}}}(1) = \chi_{\mathrm{top}}(X_{\mathbb{Z}}(\mathbb{C}))$. However, it holds that $\mathfrak{C}_E(1) = 4 \neq 0 = \chi_{\mathrm{top}}(E(\mathbb{C}))$.

On the other hand, by replacing $\geq$ in (1) (resp. $\lfloor \mathfrak{C}_E(p^m) \rfloor = \#\mathcal{E}(\mathbb{F}_{p^m})$ in (2)) in Theorem 1.2.8 with $\leq$ (resp. $\lceil \mathfrak{C}_E(p^m) \rceil = \#\mathcal{E}(\mathbb{F}_{p^m})$), we can naturally define the *floor Puiseux polynomial* of E and determine it as follows.

Theorem 1.2.9 (Corollary 5.2.15, [23, Corollary 3.15])**.** *Let E be an elliptic curve defined over $\mathbb{Q}$. Then, the floor Puiseux polynomial $\mathfrak{F}_E(t)$ of E coincides with $t - 2t^{1/2} + 1$ and its absolute zeta function is*

$$\zeta_{\mathfrak{F}_E}(s) = \frac{\left(s-\frac{1}{2}\right)^2}{s(s-1)}.$$

Here, note that the special value $\mathfrak{F}_E(1)$ coincides with the Euler characteristic of $E(\mathbb{C})$, which is consistent with the above philosophy. In this view, it is fair to say that $\zeta_{\mathfrak{F}_E}(s)$ is better than $\zeta_{\mathfrak{C}_E}(s)$.

Remark 1.2.10. While Deitmar, Koyama, and Kurokawa constructed the absolute zeta function of a scheme over $\mathbb{Z}$ associated with a monoid scheme of finite type such as a toric variety, our method enables the construction of a pair of the absolute zeta functions of a general scheme over $\mathbb{Z}$ or $\mathbb{Q}$ whose corresponding $\mathbb{F}_1$-scheme has not been established yet such as an elliptic curve. According to Remark 5.2.18, the absolute zeta function $\zeta_{\mathfrak{C}_E}(s)$ or $\zeta_{\mathfrak{F}_E}(s)$ might be regarded as the absolute zeta function of "the reduction modulo 1" of an elliptic curve over $\mathbb{Q}$. This observation might help guess an $\mathbb{F}_1$-scheme associated with an elliptic curve.

1.3 Outline of this thesis

The outline of this thesis is as follows.

As a preliminary, we survey the background of $\mathbb{F}_1$-theory and the $\mathbb{F}_1$-schemes by following Deitmar and Connes-Consani in Chapter 2. In particular, we review the definition of monoid schemes defined by Deitmar and $\mathbb{F}_1$-schemes defined by Connes and Consani, and introduce their properties related to their rational points.

In Chapter 3, we formulate Kurokawa's suggestion for a torsion free Noetherian $\mathbb{F}_1$-scheme $\mathcal{X}$ defined by Connes and Consani and give the absolute Euler product representation of its absolute zeta function. Moreover, we show that its region of absolute convergence is purely determined by the relative dimension of the scheme obtained from $\mathcal{X}$.

In Chapter 4, we introduce the commutative $\mathbb{C}$-algebra $\mathcal{A}$ consisting of certain analytic functions. Then, we investigate three analytic properties of the absolute zeta function of $f \in \mathcal{A}$: the series expansion, the integral formula according to Connes and Consani, and the absolute Euler product expression.

In Chapter 5, we study the ceiling and floor (Puiseux) polynomials of schemes and their absolute zeta functions. First, we characterise the polynomial which Deitmar, Koyama and Kurokawa implicitly used as the ceiling polynomial of the $\mathbb{Z}$-lift of a monoid scheme. Then, we introduce the ceiling and floor Puiseux polynomials of a separated scheme X of finite type over $\mathbb{Q}$ and provide a certain pair of the absolute zeta functions of X. In particular, we investigate their existence when X is an elliptic curve.

Chapter 2

$\mathbb{F}_1$-schemes

In this chapter, we review the background of $\mathbb{F}_1$-geometry and introduce monoid schemes, defined by Deitmar [11], and $\mathbb{F}_1$-schemes, defined by Connes and Consani [8]. In this thesis, we mean a *monoid* to be a commutative multiplicative semigroup with the identity 1 and the absorbing element 0 which maps any element to 0 by multiplication. Note that a morphism between monoids is a semigroup homomorphism which preserves both 0 and 1.

2.1 Background of $\mathbb{F}_1$-geometry

In number theory, the *Riemann zeta function*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\mathrm{Re}(s) > 1)$$

is one of the classical research objects. This function is meromorphically continued to the entire complex plane $\mathbb{C}$ and has a simple pole at $s = 1$. It is also well-known that $\zeta(s)$ has a simple zero at any negative even integer and $\zeta(s) \neq 0$ unless $s \in \mathbb{C}$ is a negative even integer or in $0 < \mathrm{Re}(s) < 1$ (see e.g. Serre [42, Chapter VI, §3.2]). These negative even integers are called the *trivial zeros* and all other zeros of $\zeta(s)$ are called the *nontrivial zeros*. The following conjecture on nontrivial zeros of $\zeta(s)$ is well-known as the Riemann hypothesis.

Conjecture 2.1.1 (Riemann hypothesis)**.** *All nontrivial zeros of $\zeta(s)$ lie on the line $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = \frac{1}{2}\}$.*

This conjecture is of great importance since it implies numerous conjectures in number theory. For example, von Koch [50] proved that it implies a sharp estimate

$$\pi(x) = \mathrm{Li}(x) + O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

of the number $\pi(x)$ of prime numbers up to a sufficiently large x for any $\varepsilon > 0$, where $\mathrm{Li}(x)$ is the logarithmic integral function.

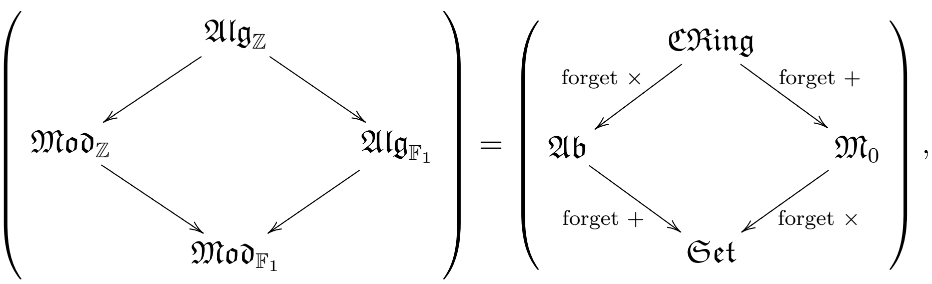
Despite various attempts to solve the Riemann hypothesis, it has remained unsolved for more than a century. However, in the early 1990s, a new approach to solving the Riemann hypothesis was proposed by Deninger and Kurokawa using $\mathbb{F}_1$-geometry. The concept of $\mathbb{F}_1$ was first introduced by Tits [48] independently from the context of the Riemann hypothesis. Deninger and Kurokawa suggested that the Riemann hypothesis could be proved similarly to Deligne's proof of the Weil conjecture if we introduced appropriate $\mathbb{F}_1$-geometry and absolute zeta functions (cf. Manin [36]).

The Weil conjecture is a function field analogue of the Riemann hypothesis, which was proved by Deligne in 1974. Fix a prime number p and let X be a smooth projective variety over $\mathbb{F}_p$. The Weil conjecture states that the congruent zeta function $Z(X, p^{-s})$ is a rational function of p^{-s} and all zeros and poles of $Z(X, p^{-s})$ lie in $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \in \frac{1}{2}\mathbb{Z}\}$, where $\frac{1}{2}\mathbb{Z}$ is the set of half integers (see e.g. Hartshorne [22, p. 450]). Deligne [15] proved this conjecture by using the étale cohomology of $\overline{X} := X \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$. The key of this proof is to narrow the bound of the absolute value of an eigenvalue of the étale cohomology of the fibre product $\overline{X} \times_{\overline{\mathbb{F}_p}} \cdots \times_{\overline{\mathbb{F}_p}} \overline{X}$.

Deninger [16, 17] pointed out that the Riemann zeta function would have had a determinant expression (cf. Remark 1.1.4) similar to the congruent zeta function of a smooth projective curve over $\mathbb{F}_p$ if there were a "cohomology theory of $\mathbb{F}_1$-geometry" [36]. On the other hand, Kurokawa introduced a tensor product for zeta functions which is called the *Kurokawa tensor product* (see Definition 3.1.3 for the definition) and proposed an idea to solve the Riemann hypothesis similar to the Weil conjecture by using the Kurokawa tensor product (see Kurokawa [32, §11.3] [33, §2.1]). While both ideas are a little different, they at least require constructing an $\mathbb{F}_1$-scheme X whose fibre product "$X \times_{\mathbb{F}_1} X$" is not trivial, i.e. $X \times_{\mathbb{F}_1} X \ncong X$. In particular, the latter idea requires the compatibility of the fibre product "$X \times_{\mathbb{F}_1} X$" and the Kurokawa tensor product of the zeta functions of X, which are called *absolute zeta functions*.

This inspired many mathematicians to consider candidates of $\mathbb{F}_1$-geometry (see e.g. Peña and Lorscheid [39]). Any theory constructs $\mathbb{F}_1$-algebras so that the tensor product $\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}$ is not trivial, i.e. $\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z} \ncong \mathbb{Z}$. However, there is still no definitive definition of $\mathbb{F}_1$-geometry. In particular, the candidates of $\mathbb{F}_1$-geometry are not currently equipped with a good cohomology theory.

Kurokawa, Ochiai and Wakayama [35] defined the category $\mathfrak{Alg}_{\mathbb{F}_1}$ of $\mathbb{F}_1$-*algebras* and the category $\mathfrak{Mod}_{\mathbb{F}_1}$ of $\mathbb{F}_1$-*modules* by the following identification of the diagrams



where $\mathfrak{CRing}$, $\mathfrak{Ab}$, $\mathfrak{M}_0$ and $\mathfrak{Set}$ are the categories of commutative rings, abelian groups, monoids and sets, respectively.

This idea became the basis of many candidates for $\mathbb{F}_1$-geometry. Deitmar [11] extended this idea to define schemes over $\mathbb{F}_1$ using $\mathbb{F}_1$-algebras, i.e. monoids. Later, Connes and Consani [8] generalised Deitmar's $\mathbb{F}_1$-scheme to a functor which encodes a monoid scheme and a usual scheme over $\mathbb{Z}$. In this thesis, we treat these two candidates of $\mathbb{F}_1$-geometry.

2.2 Monoid schemes

Deitmar defined monoid schemes as $\mathbb{F}_1$-schemes (see Deitmar [11], where monoid schemes are called schemes over $\mathbb{F}_1$). In short, a *monoid scheme* is a topological space together with a sheaf of monoids, which is constructed by gluing spectra of monoids just like a scheme. Precisely, it is constructed as follows.

Let $\mathbb{F}_1[\cdot]\colon \mathfrak{Ab} \to \mathfrak{M}_0$ be the covariant functor which sends a multiplicative abelian group G to a monoid $G \cup \{0\}$. We put $\mathbb{F}_{1^n} := \mathbb{F}_1[C_n]$, where $C_n := \langle \zeta_n \rangle$ is the cyclic group of order $n \in \mathbb{N}$ generated by the primitive n-th root ζ_n of unity. In particular, we abbreviate $\mathbb{F}_{1^1} = \{0, 1\}$ to $\mathbb{F}_1$.

For a commutative ring R, we define the base extension functor $\cdot \otimes_{\mathbb{F}_1} R\colon \mathfrak{M}_0 \to \mathfrak{Alg}_R$ by $M \otimes_{\mathbb{F}_1} R := R[M]$, where $R[M]$ is the monoidal ring which is defined by

$$R[M] := \left\{ \sum_{m \in M} n_m m \;\middle|\; n_m \in R,\ n_m = 0 \text{ for all but finitely many } m \in M \right\}.$$

Note that this functor $\cdot \otimes_{\mathbb{F}_1} \mathbb{Z}$ is left adjoint to the forgetful functor $\mathfrak{CRing} \to \mathfrak{M}_0$ [11, Theorem 1.1].

2.2.1 Definition of monoid schemes

Let M be a monoid. A nonempty subset $\mathfrak{a}$ of M is called an *ideal* of M if it satisfies that $\mathfrak{a}A \subset \mathfrak{a}$. An ideal $\mathfrak{a} \neq M$ is to be *prime* if $M \setminus \mathfrak{a}$ is multiplicatively closed, i.e. $a, b \notin \mathfrak{a} \;\Rightarrow\; ab \notin \mathfrak{a}$.

For a subsemigroup S of M with the unit 1, i.e. $1 \in S$ and $st \in S$ for all $s, t \in S$, the *localisation* of M at S is defined by $S^{-1}M := S \times M/\sim$, where $(s, m) \sim (s', m')$ if and only if there exists $t \in S$ such that $ts'm = tsm'$. We denote $[(s, m)] \in S^{-1}M$ by $\frac{m}{s}$. In particular, for a prime ideal $\mathfrak{p}$, we define the localisation $M_{\mathfrak{p}}$ at $\mathfrak{p}$ by $S_{\mathfrak{p}}^{-1}M$, where $S_{\mathfrak{p}} := M \setminus \mathfrak{p}$.

We introduce a topology on the set $\operatorname{spec} M$ of prime ideals of M[1] like the Zariski topology by defining the closed sets as sets of the form $V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{spec} M \mid \mathfrak{p} \supset \mathfrak{a}\}$ for any ideal $\mathfrak{a}$ of M.

[1] In this thesis, we use "spec" for the spectrum of a monoid to distinguish it from "Spec", the spectrum of a commutative ring.

Definition 2.2.1 (Deitmar [11, §2.1]). Let M be a monoid. For each open subset U of $\operatorname{spec} M$, we define

$$\mathcal{O}_{\operatorname{spec} M}(U) := \left\{ s\colon U \to \coprod_{\mathfrak{p}\in U} M_{\mathfrak{p}} \;\middle|\; \begin{array}{c} \forall \mathfrak{p} \in U,\ \bullet\ s(\mathfrak{p}) \in M_{\mathfrak{p}}, \\ \bullet\ \exists V \subset U \text{ a neighbourhood of } \mathfrak{p},\ \exists a, b \in M \\ \text{s.t. } \forall \mathfrak{q} \in V,\ b \notin \mathfrak{q} \text{ and } s(\mathfrak{q}) = \dfrac{a}{b} \text{ in } M_{\mathfrak{q}}. \end{array} \right\}.$$

Then, we call the functor $\mathcal{O}_{\operatorname{spec} M}\colon \operatorname{Op}(\operatorname{spec} M) \to \mathfrak{M}_0$ the *structure sheaf* of $\operatorname{spec} M$, where $\operatorname{Op}(\operatorname{spec} M)$ is the category of open sets of $\operatorname{spec} M$.

A *monoidal space* is a topological space together with a sheaf of monoids. A pair of the morphisms $(f, f^{\#})\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the morphism of monoidal spaces if $f\colon X \to Y$ is a continuous function and $f^{\#}\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of monoids on Y. Note that the pair $(\operatorname{spec} M, \mathcal{O}_{\operatorname{spec} M})$ is a monoidal space and spec satisfies the functoriality [11, Proposition 2.2]. Then, we define monoid schemes as follows.

Definition 2.2.2 (Deitmar [11, §2.3]). A monoidal space X is an *affine monoid scheme* if there exists a monoid M such that X is isomorphic to the spectrum $\operatorname{spec} M$ of M. Moreover, a monoidal space X is a *monoid scheme* if for any $x \in X$ there exists an open neighbourhood U of X such that $(U, \mathcal{O}_X|_U)$ is an affine monoid scheme.

A morphism of monoid schemes is a *local* morphism of monoidal spaces, i.e. a morphism $(f, f^{\#})\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of monoidal spaces which induces the morphism $f^{\#}_x\colon \mathcal{O}_{Y,f(x)} \to f_*\mathcal{O}_{X,x}$ satisfying $(f^{\#}_x)^{-1}(\mathcal{O}^{\times}_{X,x}) = \mathcal{O}^{\times}_{Y,f(x)}$ for each $x \in X$. For a monoid scheme X, we define $X(M) := \operatorname{Hom}(\operatorname{spec} M, X)$ for each $M \in \mathfrak{M}_0$.

Let R be a commutative ring and X be a monoid scheme with an affine covering $X = \bigcup_{i\in I} \operatorname{spec} M_i$. Through the base extension functor $\cdot \otimes_{\mathbb{F}_1} R\colon \mathfrak{M}_0 \to \mathfrak{Alg}_R$, we obtain the scheme $X_R := \bigcup_{i\in I} \operatorname{Spec}(M_i \otimes_{\mathbb{F}_1} R)$ over R and call X_R as the *R-lift* of X. Here, the isomorphism class of X_R does not depend on the choices of affine coverings of X due to compatibility with gluing [11].

Example 2.2.3. Let G be a multiplicative abelian group. Then, the $\mathbb{Z}$-lift of $\operatorname{spec} \mathbb{F}_1[G] = \{(0)\}$ is $\operatorname{Spec} \mathbb{Z}[G]$. Note that the monoid $\mathbb{F}_1[G]$ is sometimes called an *$\mathbb{F}_1$-field* because of the similarity to a field K such as the fact that $\operatorname{Spec} K = \{(0)\}$ (see Kurokawa [33, p. 141]). In particular, for each $q \in \mathbb{P}^{\mathbb{N}}$, the monoid scheme $\operatorname{spec} \mathbb{F}_{1^{q-1}} = \operatorname{spec} \mathbb{F}_1[\mathbb{F}_q^{\times}]$ plays a similar role to $\operatorname{Spec} \mathbb{F}_q$ in counting rational points of a monoid scheme.

Definition 2.2.4 (Deitmar [12, §1]). Let X be a monoid scheme. We say X to be *of finite type* if it has a finite covering by affine monoid schemes $\operatorname{spec} M_i$ such that each M_i is finitely generated.

Proposition 2.2.5 (Deitmar [12, Lemma 2]). *A monoid scheme X is of finite type if and only if the $\mathbb{Z}$-lift $X_{\mathbb{Z}}$ is of finite type over $\mathbb{Z}$.*

A monoid scheme is said to be *integral* if it is covered by affine monoid schemes $\operatorname{spec} M$'s, where M is integral, i.e. $ab = ac$ implies $b = c$ in M. The following proposition shows that integral monoid schemes of finite type are essentially toric.

Proposition 2.2.6 (Deitmar [13, Theorem 4.1])**.** *Every irreducible component of the $\mathbb{C}$-lift of a connected integral monoid scheme of finite type is a toric variety.*

2.2.2 Properties of monoid schemes of finite type

Now, we review the basic properties of monoid schemes of finite type which we use in this thesis.

Proposition 2.2.7 (Deitmar [12, Remark 1])**.** *Let X be a monoid scheme of finite type. Then, it holds that*

$$X(\mathbb{F}_{1^{q-1}}) \cong X_{\mathbb{Z}}(\mathbb{F}_q)$$

for any $q \in \mathbb{P}^{\mathbb{N}}$. In particular, the underlying set of X is finite, i.e. $\#X = \#X(\mathbb{F}_1) = \#X_{\mathbb{Z}}(\mathbb{F}_2) < \infty$.

Proof. As in Example 2.2.3, $\operatorname{spec} \mathbb{F}_{1^{q-1}}$ consists of one point, the generic point (0). Since any morphism of monoid schemes sends it to the generic point, it holds that

$$X(\mathbb{F}_{1^{q-1}}) = \bigcup_{i \in I} U_i(\operatorname{spec} \mathbb{F}_{1^{q-1}}),$$

where we take an affine covering of X as $\bigcup_{i \in I} U_i$. Hence, it suffices to show the statement in the affine case, which follows from the fact that

$$X(\mathbb{F}_{1^{q-1}}) \cong \operatorname{Hom}(M, \mathbb{F}_{1^{q-1}}) \cong \operatorname{Hom}(\mathbb{Z}[M], \mathbb{F}_q) \cong X_{\mathbb{Z}}(\mathbb{F}_q)$$

by the functoriality of spec and $\cdot \otimes_{\mathbb{F}_1} \mathbb{Z}$ when we put $X = \operatorname{spec} M$. □

Connes and Consani explicitly described the right-hand side of Proposition 2.2.7. Before stating their formula, we introduce some notations used hereafter.

Definition 2.2.8. Let $X = (X, \mathcal{O}_X)$ be a monoid scheme of finite type. For each $x \in X$, we define $r_x, l_x \in \mathbb{N}_0$ and $t_{x,j} \in \mathbb{N}$ by the integers satisfying

$$\mathcal{O}_{X,x}^{\times} \cong \mathbb{Z}^{r_x} \times \prod_{j=1}^{l_x} \mathbb{Z}/t_{x,j}\mathbb{Z} \quad \text{with} \quad t_{x,j} \mid t_{x,j+1}$$

and put $T_x := \#(\mathcal{O}_{X,x}^{\times})_{\mathrm{tors}} = \prod_{j=1}^{l_x} t_{x,j}$. Here, $\mathcal{O}_{X,x}^{\times}$ denotes the group of invertible elements of the monoid $\mathcal{O}_{X,x}$ and $(\mathcal{O}_{X,x}^{\times})_{\mathrm{tors}}$ denotes its torsion subgroup. In addition, we put $R_X := \max_{x \in X} r_x$ and $T_X := \prod_{x \in X} T_x$.

Lemma 2.2.9 (Connes and Consani [8, Proposition 3.22])**.** *For an abelian group G, it holds that*

$$X(\mathbb{F}_1[G]) = \bigsqcup_{x \in X} \operatorname{Hom}_{\mathfrak{Ab}}\left(\mathcal{O}_{X,x}^{\times}, G\right).$$

The following property follows form Lemma 2.2.9 by putting $G = \mathbb{Z}/n\mathbb{Z}$.

Proposition 2.2.10 (Connes and Consani [9, Lemma 4.4])**.** *Let X be a monoid scheme of finite type. Then, it holds that*

$$\#X(\mathbb{F}_{1^n}) = \sum_{x \in X} n^{r_x} \prod_{j=1}^{l_x} \gcd(n, t_{x,j})$$

for any $n \in \mathbb{N}$.

2.3 $\mathbb{F}_1$-schemes by Connes-Consani

Next, we review $\mathbb{F}_1$-schemes defined by Connes and Consani [8]. They constructed the category $\mathfrak{MR}$, which glues together the category $\mathfrak{M}_0$ of monoids and the category $\mathfrak{CRing}$ of commutative rings using adjoint functors. Then, they defined an $\mathbb{F}_1$-scheme as a functor $\mathfrak{MR} \to \mathfrak{Set}$ which satisfies certain conditions.

2.3.1 $\mathbb{F}_1$-schemes

As we mentioned above, the functor $\mathbb{Z}[\cdot]\colon \mathfrak{M}_0 \to \mathfrak{CRing}$ is left adjoint to the forgetful functor $U\colon \mathfrak{CRing} \to \mathfrak{M}_0$, which forgets the addition of rings [11, Theorem 1.1]. Using these functors, we glue $\mathfrak{M}_0$ and $\mathfrak{CRing}$.

Definition 2.3.1 (Connes and Consani [8, §4.1])**.** We define the category $\mathfrak{MR} := \mathfrak{M}_0 \cup_{\mathbb{Z}[\cdot],U} \mathfrak{CRing}$ by the category which consists of the following data:

- The collection of objects of $\mathfrak{MR}$ is the disjoint union of the collections of objects of $\mathfrak{M}_0$ and $\mathfrak{CRing}$.
- For any $M, N \in \mathfrak{M}_0$ and $R, S \in \mathfrak{CRing}$, we set

$$\begin{aligned}&\mathrm{Hom}_{\mathfrak{MR}}(M,N) := \mathrm{Hom}_{\mathfrak{M}_0}(M,N), \quad \mathrm{Hom}_{\mathfrak{MR}}(R,S) := \mathrm{Hom}_{\mathfrak{CRing}}(R,S),\\ &\mathrm{Hom}_{\mathfrak{MR}}(M,R) := \mathrm{Hom}_{\mathfrak{CRing}}(\mathbb{Z}[M],R) \cong \mathrm{Hom}_{\mathfrak{M}_0}(M,U(R)),\\ &\mathrm{Hom}_{\mathfrak{MR}}(R,M) := \emptyset.\end{aligned}$$

- For $\phi \in \mathrm{Hom}_{\mathfrak{MR}}(M,R)$, $f \in \mathrm{Hom}_{\mathfrak{M}_0}(N,M)$ and $g \in \mathrm{Hom}_{\mathfrak{CRing}}(R,S)$, we define $\phi \circ f \in \mathrm{Hom}_{\mathfrak{MR}}(N,R)$ and $g \circ \phi \in \mathrm{Hom}_{\mathfrak{MR}}(M,S)$ as $\phi \circ \mathbb{Z}[f] \in \mathrm{Hom}_{\mathfrak{CRing}}(\mathbb{Z}[N],R)$ and $g \circ \phi \in \mathrm{Hom}_{\mathfrak{CRing}}(\mathbb{Z}[M],R)$, respectively.

An $\mathbb{F}_1$-scheme is a functor $\mathfrak{MR} \to \mathfrak{Set}$ which combines information of an $\mathfrak{M}_0$-scheme, a $\mathbb{Z}$-scheme and a natural transformation which binds them. Here, we introduce $\mathfrak{M}_0$-schemes and $\mathbb{Z}$-schemes.

Definition 2.3.2 (Connes and Consani [8, Definition 3.5])**.** Let $\mathcal{F}$ be a covariant functor $\mathfrak{M}_0 \to \mathfrak{Set}$.

- A covariant functor $\mathcal{G}$ is a *subfunctor* of $\mathcal{F}$ if $\mathcal{G}(M) \subset \mathcal{F}(M)$ for all $M \in \mathrm{Ob}(\mathfrak{M}_0)$ and $\mathcal{G}(f) = \mathcal{F}(f)|_{\mathcal{G}(M)}$ for all $f \in \mathrm{Hom}_{\mathfrak{M}_0}(M, M')$.
- A subfunctor $\mathcal{G} \subset \mathcal{F}$ is said to be *open* if for any monoid M and any morphism $\phi\colon \mathrm{Hom}_{\mathfrak{M}_0}(M, \cdot) \to \mathcal{F}$, there exists an ideal $\mathfrak{a} \subset M$ such that

$$\phi(\rho) \in \mathcal{G}(N)\,(\subset \mathcal{F}(N)) \;\Leftrightarrow\; \rho(\mathfrak{a})N = N$$

  for any $N \in \mathrm{Ob}(\mathfrak{M}_0)$ and any $\rho \in \mathrm{Hom}_{\mathfrak{M}_0}(M, N)$.
- Let $\mathcal{F}$ be a covariant functor $\mathfrak{M}_0 \to \mathfrak{Set}$ and $\{\mathcal{F}_i\}_{i\in I}$ be a family of open subfunctors of $\mathcal{F}$. Then, $\{\mathcal{F}_i\}_{i\in I}$ is an *open covering* of $\mathcal{F}$ if it holds that

$$\mathcal{F}(\mathbb{F}_1[G]) = \bigcup_{i\in I} \mathcal{F}_i(\mathbb{F}_1[G])$$

  for any multiplicative abelian group G.

Definition 2.3.3 (Connes and Consani [8, Definition 3.10])**.** An $\mathfrak{M}_0$-*scheme* is a covariant functor $\mathfrak{M}_0 \to \mathfrak{Set}$ which admits an open covering by representable subfunctors. A $\mathbb{Z}$-*scheme* is defined similarly by exchanging $\mathfrak{M}_0$ for $\mathfrak{CRing}$.

Next, we introduce the natural transformation which is one of the data of an $\mathbb{F}_1$-scheme. For each $R \in \mathrm{Ob}(\mathfrak{CRing})$, put

$$\alpha'_R := \Phi^{-1}(\mathrm{id}_{U(R)}) \in \Phi^{-1}(\mathrm{Hom}_{\mathfrak{M}_0}(U(R), U(R)) = \mathrm{Hom}_{\mathfrak{MR}}(U(R), R),$$

where $\Phi\colon \mathrm{Hom}_{\mathfrak{CRing}}(\mathbb{Z}[M], S) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{M}_0}(M, U(S))$ is a natural bijection in $M \in \mathrm{Ob}(\mathfrak{M}_0)$ and $S \in \mathrm{Ob}(\mathfrak{CRing})$ which is determined by the adjoint functors $\mathbb{Z}[\cdot]\colon \mathfrak{M}_0 \to \mathfrak{CRing}$ and $U\colon \mathfrak{CRing} \to \mathfrak{M}_0$. Then, we have the map $e_R := \mathcal{X}(\alpha'_R)\colon \mathcal{X}|_{\mathfrak{M}_0} \circ U(R) \to \mathcal{X}|_{\mathfrak{CRing}}(R)$ for any covariant functor $\mathcal{X}\colon \mathfrak{MR} \to \mathfrak{Set}$. Thus, we obtain the natural transformation $e\colon \mathcal{X}|_{\mathfrak{M}_0} \circ U \to \mathcal{X}|_{\mathfrak{CRing}}$.

Definition 2.3.4 (Connes and Consani [8, Definition 4.7])**.** A functor $\mathcal{X}\colon \mathfrak{MR} \to \mathfrak{Set}$ is an $\mathbb{F}_1$-*scheme* if it satisfies the following conditions:

- The restriction $\mathcal{X}|_{\mathfrak{M}_0}$ is an $\mathfrak{M}_0$-scheme.
- The restriction $\mathcal{X}|_{\mathfrak{CRing}}$ is a $\mathbb{Z}$-scheme.
- The map $e_K\colon \mathcal{X}|_{\mathfrak{M}_0} \circ U(K) \to \mathcal{X}|_{\mathfrak{CRing}}(K)$ is bijective for any field K.

Remark 2.3.5 (Connes and Consani [8, Proposition 3.17])**.** For a monoid scheme $\mathfrak{X}$ and a scheme X over $\mathbb{Z}$, we put

$$\underline{\mathfrak{X}} := \mathrm{Hom}_{\mathbf{MSch}}(\mathrm{spec}(\cdot), \mathfrak{X}) \quad \text{and} \quad \underline{X} := \mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}(\cdot), X),$$

where **MSch** and **Sch** are the categories of monoid schemes and schemes over $\mathbb{Z}$, respectively. Since the category of $\mathfrak{M}_0$-schemes is equivalent as a category to **MSch**, for any $\mathfrak{M}_0$-scheme $\mathcal{F}\colon \mathfrak{M}_0 \to \mathfrak{Set}$, there exists the unique monoid scheme $X_{\mathfrak{M}_0}$ (up to natural isomorphism) such that $\mathcal{F} = \underline{X}_{\mathfrak{M}_0}$. We call the monoid scheme $X_{\mathfrak{M}_0}$ the *geometric realisation of* $\mathcal{F}$. Similarly, for any $\mathbb{Z}$-scheme $\mathcal{G}\colon \mathfrak{CRing} \to \mathfrak{Set}$, there exists the unique scheme $X_{\mathbb{Z}}$ over $\mathbb{Z}$ (up to natural isomorphism) such that $\mathcal{G} = \underline{X}_{\mathbb{Z}}$. We also call the scheme $X_{\mathbb{Z}}$ the *geometric realisation of* $\mathcal{G}$.

Remark 2.3.6 (cf. Connes and Consani [8, Remark 4.10])**.** It is important that e_K is bijective not for an arbitrary *rings* but for an arbitrary *fields*, since $\mathcal{X}|_{\mathfrak{M}_0} \circ U$ is not always a $\mathbb{Z}$-scheme. For example, when $\mathcal{X}$ is an $\mathbb{F}_1$-scheme satisfying that the geometric realisation of $\mathcal{X}|_{\mathfrak{M}_0}$ (resp. $\mathcal{X}|_{\mathfrak{CRing}}$) is a monoid scheme X (resp. its $\mathbb{Z}$-lift $X_{\mathbb{Z}}$), the map e_R is bijective for any rings R. On the other hand, when $\mathcal{X}$ is the $\mathbb{F}_1$-scheme associated with the projective line or a Chevalley group, the map e_R is *not* bijective for any rings R.

2.3.2 Torsion free Noetherian $\mathbb{F}_1$-schemes

Next, we show some properties of $\mathbb{F}_1$-schemes under certain conditions. First of all, we review the definition of a torsion free Noetherian $\mathbb{F}_1$-scheme.

Definition 2.3.7 (Connes and Consani [8, Definition 4.12])**.** An $\mathfrak{M}_0$-scheme $\mathcal{F}$ is *Noetherian* if there exists a finite open covering by representable subfunctors, each of which is naturally isomorphic to $\mathrm{Hom}_{\mathfrak{M}_0}(M, \cdot)$ for some Noetherian monoid M. A *Noetherian* $\mathbb{Z}$-scheme is also defined similarly.

An $\mathbb{F}_1$-scheme $\mathcal{X}$ is *Noetherian* if $\mathcal{X}|_{\mathfrak{M}_0}$ is a Noetherian $\mathfrak{M}_0$-scheme and $\mathcal{X}|_{\mathfrak{CRing}}$ is a Noetherian $\mathbb{Z}$-scheme.

Remark 2.3.8. The following conditions on a monoid M are equivalent.

- M is Noetherian.
- M is finitely generated.
- $\mathbb{Z}[M]$ is a Noetherian ring.

This implies that an $\mathfrak{M}_0$-scheme is Noetherian if and only if its geometric realisation is of finite type.

Definition 2.3.9 (Connes and Consani [8, §4.4])**.** A monoid scheme X is *torsion free* if the group $\mathcal{O}_{X,x}^{\times}$ is torsion free for any $x \in X$. We call an $\mathbb{F}_1$-scheme $\mathcal{X}$ to be *torsion free* if the geometric realisation of $\mathcal{X}|_{\mathfrak{M}_0}$ is torsion free.

The following theorem is the important property of a torsion free Noetherian $\mathbb{F}_1$-scheme to define its absolute zeta function defined by Soulé.

Theorem 2.3.10 (Connes and Consani [8, Theorem 4.13])**.** *Let $\mathcal{X}$ be a torsion free Noetherian $\mathbb{F}_1$-scheme. Then, there exists a unique polynomial $N_{\mathcal{X}} \in \mathbb{Z}[t]$ satisfying the following conditions:*

- *It holds that* $\#\mathcal{X}(\mathbb{F}_{1^n}) = N_{\mathcal{X}}(n+1)$ *for any* $n \in \mathbb{N}$.
- *It holds that* $\#\mathcal{X}(\mathbb{F}_q) = N_{\mathcal{X}}(q)$ *for any* $q \in \mathbb{P}^{\mathbb{N}}$.

We call this polynomial $N_{\mathcal{X}}$ *the* counting function *of* $\mathcal{X}$.

Remark 2.3.11 (Connes and Consani [8, Theorem 4.13])**.** It is easy to check that the counting function $N_{\mathcal{X}}(t)$ in Theorem 2.3.10 is given by

$$N_{\mathcal{X}}(t) := \sum_{x \in X_{\mathfrak{M}_0}} (t-1)^{r_x} = \sum_{x \in X_{\mathfrak{M}_0}} \sum_{j=0}^{r_x} (-1)^{r_x - j} \binom{r_x}{j} t^j.$$

Note that the underlying set of $X_{\mathfrak{M}_0}$ is finite by Proposition 2.2.7.

The degree of the counting function of a torsion free Noetherian $\mathbb{F}_1$-scheme coincides with the relative dimension of the scheme associated with the $\mathbb{F}_1$-scheme. We use this property later when we determine the region of absolute convergence of the absolute Euler product using the relative dimension in Theorem 3.1.5.

Theorem 2.3.12 ([49, Theorem 2.10])**.** *Let* $\mathcal{X}$ *be a torsion free Noetherian* $\mathbb{F}_1$*-scheme and* $N_{\mathcal{X}}$ *be its counting function. Assume that the geometric realisation* $X_{\mathbb{Z}}$ *of* $\mathcal{X}|_{\mathfrak{CRing}}$ *is of finite type over* $\mathbb{Z}$*. Then, it holds that*

$$\deg N_{\mathcal{X}} = \operatorname{rel.dim} X_{\mathbb{Z}}/\mathbb{Z},$$

where $\operatorname{rel.dim} X_{\mathbb{Z}}/\mathbb{Z}$ *is the relative dimension of* $X_{\mathbb{Z}}$ *over* $\mathbb{Z}$.

Proof. Fix a prime power $q \in \mathbb{P}^{\mathbb{N}}$. Put $Y = \operatorname{Spec} \mathbb{Z}$ and take $y \in Y(\mathbb{F}_q)$. When we put $d := \operatorname{rel.dim} X_{\mathbb{Z}}/\mathbb{Z}$, we have $d = \dim(X_{\mathbb{Z}})_y$ for any $y \in Y(\mathbb{F}_q)$, where $(X_{\mathbb{Z}})_y := X_{\mathbb{Z}} \times_Y \operatorname{Spec} \mathbb{F}_q$. Here, it holds that

$$\#(X_{\mathbb{Z}})_y(\mathbb{F}_q) = O(q^d)$$

by Lang and Weil (cf. Poonen [40, Theorem 7.7.1 (i)]). Since we have $\#X_y(\mathbb{F}_q) = N_{\mathcal{X}}(q)$ by Theorem 2.3.10 and $N_{\mathcal{X}}$ is a polynomial, we have

$$\deg N_{\mathcal{X}} = d = \operatorname{rel.dim} X_{\mathbb{Z}}/\mathbb{Z}.$$

□

Corollary 2.3.13 ([49, Corollary 2.12])**.** *Let* $\mathcal{X}$ *be a torsion free Noetherian* $\mathbb{F}_1$*-scheme and* $X_{\mathfrak{M}_0}$ *(resp.* $X_{\mathbb{Z}}$*) be the geometric realisation of* $\mathcal{X}|_{\mathfrak{M}_0}$ *(resp.* $\mathcal{X}|_{\mathfrak{CRing}}$*). Assume that* $X_{\mathbb{Z}}$ *is of finite type over* $\mathbb{Z}$*. Then the counting function of* $\mathcal{X}$ *is given explicitly as*

$$N_{\mathcal{X}}(t) = \sum_{j=0}^{d} \left(\sum_{l=j}^{d} (-1)^{l-j} \binom{l}{j} \#I_l \right) t^j,$$

where $d := \operatorname{rel.dim} X_{\mathbb{Z}}/\mathbb{Z}$, $r_x := \operatorname{rank} \mathcal{O}^{\times}_{X_{\mathfrak{M}_0},x}$ *and* $I_l := \{x \in X_{\mathfrak{M}_0} \mid r_x = l\}$.

Proof. By Remark 2.3.11 and Theorem 2.3.12, we have

$$\max_{x \in X_{\mathfrak{m}_0}} r_x = \deg N_{\mathcal{X}} = \mathrm{rel.dim}\, X_{\mathbb{Z}}/\mathbb{Z} = d.$$

Since $I_l = \emptyset$ for $l > d$, we have

$$\begin{aligned} N_{\mathcal{X}}(t) &= \sum_{x \in X_{\mathfrak{m}_0}} (t-1)^{r_x} = \sum_{l=0}^{d} \# I_l (t-1)^l \\ &= \sum_{l=0}^{d} \# I_l \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} t^j = \sum_{j=0}^{d} \left(\sum_{l=j}^{d} (-1)^{l-j} \binom{l}{j} \# I_l \right) t^j \end{aligned}$$

by Remark 2.3.11. □

Chapter 3

Absolute Euler products and $\mathbb{F}_1$-schemes

In this chapter, we prove the first main result of this thesis. Before stating the main result, we introduce the absolute zeta function of a torsion free Noetherian $\mathbb{F}_1$-scheme. Then, we state the main theorem asserting that it has the absolute Euler product and its region of absolute convergence is determined by the relative dimension of the scheme obtained by the torsion free Noetherian $\mathbb{F}_1$-scheme.

3.1 The absolute zeta function of a torsion free Noetherian $\mathbb{F}_1$-scheme

As we mentioned in Subsection 1.1.1, Soulé [46] defined the absolute zeta function for a scheme X of finite type over $\mathbb{Z}$ with the condition on rational points as the limit of $Z(X, p^{-s})$ as $p \to 1$. Note that absolute zeta functions for general schemes of finite type over $\mathbb{Z}$ cannot be defined similarly.

Extending this definition, Connes and Consani [8] defined the absolute zeta function $\zeta_{\mathcal{X}/\mathbb{F}_1}(s)$ for a torsion free Noetherian $\mathbb{F}_1$-scheme $\mathcal{X}$ as the absolute zeta function for the geometric realisation of $\mathcal{X}|_{\mathfrak{CRing}}$.

Definition 3.1.1 (Connes and Consani [8, §2])**.** Let $\mathcal{X}$ be a torsion free Noetherian $\mathbb{F}_1$-scheme. Note that the counting function $N_{\mathcal{X}} \in \mathbb{Z}[t]$ of $\mathcal{X}$ exists by Theorem 2.3.10. We define the function

$$\zeta_{\mathcal{X}/\mathbb{F}_1}(s) := \zeta^{\lim}_{N_{\mathcal{X}}}(s) = \lim_{p \to 1} (p-1)^{N_{\mathcal{X}}(1)} \exp\left(\sum_{m=1}^{\infty} \frac{N_{\mathcal{X}}(p^m)}{m} p^{-sm} \right)$$

and call it the *absolute zeta function for $\mathcal{X}$*.

By this definition, we immediately obtain the following proposition.

Proposition 3.1.2 (Soulé [46, Lemme 1]). *Let $\mathcal{X}$ be a torsion free Noetherian $\mathbb{F}_1$-scheme and $N_{\mathcal{X}}$ be the counting function of $\mathcal{X}$. Put $N_{\mathcal{X}}(t) = \sum_{k=0}^{r} a_k t^k$ (cf. Theorem 2.3.10). Then, the absolute zeta function of $\mathcal{X}$ is expressed as*

$$\zeta_{\mathcal{X}/\mathbb{F}_1}(s) = \prod_{k=0}^{r} (s-k)^{-a_k},$$

and can be continued meromorphically to the whole complex plane $\mathbb{C}$.

In [8], Connes and Consani gave a finite product representation of the absolute zeta function of a torsion free Noetherian $\mathbb{F}_1$-scheme $\mathcal{X}$ which runs over the points of the geometric realisation of $\mathcal{X}|_{\mathfrak{M}_0}$. Each factor of the product is expressed using the Kurokawa tensor product as defined below.

Definition 3.1.3 (Kurokawa [27], Manin [36]). For $i \in \{1, \ldots, r\}$, let Φ_i be a finite subset of $\mathbb{C}$ and $m_i \colon \Phi_i \to \mathbb{Z}$ be a function and put

$$Z_i(s) := \prod_{\rho \in \Phi_i} (s-\rho)^{m_i(\rho)}.$$

We define the *Kurokawa tensor product* by

$$Z_1(s) \otimes \cdots \otimes Z_r(s) := \prod_{(\rho_1, \ldots, \rho_r) \in \Phi_1 \times \cdots \times \Phi_r} (s - (\rho_1 + \cdots + \rho_r))^{m(\rho_1, \ldots, \rho_r)},$$

where

$$m(\rho_1, \ldots, \rho_r) := \begin{cases} m_1(\rho_1) \cdots m_r(\rho_r) & \text{if } \operatorname{Im}(\rho_i) \geq 0 \text{ for each } i, \\ (-1)^{r-1} m_1(\rho_1) \cdots m_r(\rho_r) & \text{if } \operatorname{Im}(\rho_i) < 0 \text{ for each } i, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.1.4 (Connes and Consani [8, Theorem 4.13]). *Let $\mathcal{X}$ be a torsion free Noetherian $\mathbb{F}_1$-scheme. Then we have*

$$\zeta_{\mathcal{X}/\mathbb{F}_1}(s) = \prod_{x \in X_{\mathfrak{M}_0}} \frac{1}{\left(1 - \frac{1}{s}\right)^{\otimes r_x}},$$

where $r_x := \operatorname{rank} \mathcal{O}^{\times}_{X_{\mathfrak{M}_0}, x}$ and $\otimes$ is the Kurokawa tensor product.

Using this theorem, we obtain the following absolute Euler product of the absolute zeta function of a torsion free Noetherian $\mathbb{F}_1$-scheme. This is a formulation of Kurokawa's suggestion (Conjecture 1.1.7) using torsion free Noetherian $\mathbb{F}_1$-schemes. Moreover, we give an explicit form of $\kappa(n, X)$ in Conjecture 1.1.7 using the points of the monoid scheme X associated with the $\mathbb{F}_1$-scheme and determine the region of absolute convergence. This is the main theorem of this chapter.

Theorem 3.1.5 ([49, Theorem 3.8]). *Let $\mathcal{X}$ be a torsion free Noetherian $\mathbb{F}_1$-scheme and $N_{\mathcal{X}} \in \mathbb{Z}[t]$ be its counting function. Let $X_{\mathfrak{M}_0}$ (resp. $X_{\mathbb{Z}}$) be the geometric realisation of $\mathcal{X}|_{\mathfrak{M}_0}$ (resp. $\mathcal{X}|_{\mathfrak{CRing}}$).*

(1) It holds that

$$\zeta_{\mathcal{X}/\mathbb{F}_1}(s) = s^{-N_{\mathcal{X}}(1)} \prod_{n=1}^{\infty} \left(1 - s^{-n}\right)^{-\kappa(n, X_{\mathfrak{M}_0})},$$

where

$$\kappa(n, X_{\mathfrak{M}_0}) := \sum_{x \in X_{\mathfrak{M}_0}} \sum_{j=0}^{r_x} (-1)^{r_x - j} \binom{r_x}{j} \kappa_j(n), \quad \kappa_j(n) := \frac{1}{n} \sum_{m|n} \mu\left(\frac{n}{m}\right) j^m.$$

Here, the map $\mu\colon \mathbb{N} \to \{-1, 0, 1\}$ is the Möbius function*, which is defined by*

$$\mu(n) := \begin{cases} 0 & \text{if } n \text{ has a squared prime factor,} \\ (-1)^k & \text{if } n \text{ has } k \text{ distinct prime factors.} \end{cases}$$

(2) If $X_{\mathbb{Z}}$ is of finite type over $\mathbb{Z}$, the region of absolute convergence of this absolute Euler product is $\{s \in \mathbb{C} \mid |s| > \operatorname{rel.dim} X_{\mathbb{Z}}/\mathbb{Z}\}$.

We give this proof in the next section. In fact, $\kappa(n, X_{\mathfrak{M}_0})$ is given as the image of the counting function of $\mathcal{X}$ with respect to the following homomorphism.

Definition 3.1.6 ([49, Definition 3.9]). For any $n \in \mathbb{N}$, we define the homomorphism of $\mathbb{Z}$-modules

$$M_n \colon \mathbb{Z}[t] \to \mathbb{Z}$$

such that $M_n(t^a) := \kappa_a(n)$ for $a \in \mathbb{N}_0$. Note that we show that $\kappa_a(n) \in \mathbb{Z}$ later.

Since we have $\kappa(n, X_{\mathfrak{M}_0}) = M_n(N_{\mathcal{X}})$ by Remark 2.3.11, Theorem 3.1.5 can be represented only by the counting function $N_{\mathcal{X}}$.

Corollary 3.1.7 ([49, Corollary 3.10]). *Let $\mathcal{X}$ be a torsion free Noetherian $\mathbb{F}_1$-scheme and $N_{\mathcal{X}}(t)$ be its counting function. Then, it holds that*

$$\zeta_{\mathcal{X}/\mathbb{F}_1}(s) = s^{-N_{\mathcal{X}}(1)} \prod_{n=1}^{\infty} \left(1 - s^{-n}\right)^{-M_n(N_{\mathcal{X}})}.$$

Remark 3.1.8. As demonstrated in the proof of Theorem 3.1.5 in Subsection 3.2.2, the region of absolute convergence of the above absolute Euler product is $\{s \in \mathbb{C} \mid |s| > \deg N_{\mathcal{X}}\}$.

3.2 Proof of Theorem 3.1.5

3.2.1 Lemmas

To prove Theorem 3.1.5, we first prove the core formula of the absolute Euler product representation and its region of absolute convergence.

Lemma 3.2.1 (cf. Kurokawa [30, Exercise 7.1])**.** *Let $a \in \mathbb{Z}$, $n \in \mathbb{N}$, and*

$$\kappa_a(n) := \frac{1}{n} \sum_{m|n} \mu\left(\frac{n}{m}\right) a^m.$$

Then, we have $\kappa_a(n) \in \mathbb{Z}$ for any $n \in \mathbb{N}$ and $a \in \mathbb{Z}$, and it holds that in $\mathbb{Z}[\![u]\!]$

$$\prod_{n=1}^{\infty} (1 - u^n)^{\kappa_a(n)} = 1 - au. \tag{3.1}$$

Remark 3.2.2. When a is a variable, the polynomial $\kappa_a(n)$ is called the *necklace polynomial*, which was introduced by Moreau [38].

Moreover, the inverse of Equation (3.1) coincides with the *cyclotomic identity* (cf. Metropolis and Rota [37]), which is the same as the condition that $\{a^n\}_{n=1}^{\infty}$ is the Euler transform of $\{\kappa_a(n)\}_{n=1}^{\infty}$ (cf. Sloane and Plouffe [45]).

Proof. First of all, we show $\kappa_a(n) \in \mathbb{Z}$. We prove this in a different way from Kurokawa's proof [30, Exercise 7.1], using the following property of the unit group of $\mathbb{Z}/p^{e+1}\mathbb{Z}$:

$$\left(\mathbb{Z}/p^{e+1}\mathbb{Z}\right)^{\times} \cong \begin{cases} \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p^e\mathbb{Z} & \text{if } p \text{ is odd,} \\ \{1\} & \text{if } p = 2 \text{ and } e = 0, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{e-1}\mathbb{Z} & \text{if } p = 2 \text{ and } e \geq 1, \end{cases}$$

for any $e \in \mathbb{N}_0$ and $p \in \mathbb{P}$ (see e.g. Serre [42, Chapter II, Theorem 2]). Since $y^{p^{e+1}} - y^{p^e} = y^{p^e}(y^{p^e(p-1)} - 1)$, we have

$$y^{p^{e+1}} \equiv y^{p^e} \pmod{p^{e+1}} \tag{3.2}$$

for any $y \in \mathbb{Z}$, $e \in \mathbb{N}_0$ and $p \in \mathbb{P}$.

Since $\mathbb{Z} = \bigcap_{p \in \mathbb{P}} \mathbb{Z}_{(p)}$, it suffices to show that $\kappa_a(n) \in \mathbb{Z}_{(p)}$ for any $p \in \mathbb{P}$. Fix any a and p. If $p \nmid n$, $\kappa_a(n) \in \mathbb{Z}\left[\frac{1}{n}\right] \subset \mathbb{Z}_{(p)}$ holds by definition. We assume that $p \mid n$. By Equation (3.2), putting $n = p^{\nu} u$ ($\nu = v_p(n)$, $u \in \mathbb{Z}$),

$$\begin{aligned} \kappa_a(n) &= \frac{1}{n} \sum_{m|n} \mu\left(\frac{n}{m}\right) a^m = \frac{1}{p^{\nu} u} \sum_{m|p^{\nu} u} \mu\left(\frac{p^{\nu} u}{m}\right) a^m \\ &\overset{(*)}{=} \sum_{d|u} \frac{1}{p^{\nu} u} \mu\left(\frac{u}{d}\right) \left(a^{p^{\nu} d} - a^{p^{\nu-1} d}\right) \\ &\in \mathbb{Z}_{(p)} \qquad \left(\because\ a^{p^{\nu} d} - a^{p^{\nu-1} d} \equiv 0 \pmod{p^{\nu} \mathbb{Z}_{(p)}}\right). \end{aligned}$$

Here, $(*)$ follows by dividing into two cases: $m = p^{\nu-1}d$ and $m = p^{\nu}d$, where $\frac{p^{\nu}u}{m}$ does not have any squared prime factor.

Next, we formally calculate the infinite product representation of $1 - au$:

$$\begin{aligned}
\log\left(\prod_{n=1}^{\infty}(1-u^n)^{\kappa_a(n)}\right) &= \sum_{n=1}^{\infty}\kappa_a(n)\log(1-u^n) = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{n\kappa_a(n)}{nk}u^{nk} \\
&\overset{\text{(a)}}{=} -\sum_{m=1}^{\infty}\frac{1}{m}\left(\sum_{n|m}n\kappa_a(n)\right)u^m \\
&= -\sum_{m=1}^{\infty}\frac{1}{m}\left(\sum_{n|m}\sum_{l|n}\mu\left(\frac{n}{l}\right)a^l\right)u^m \\
&\overset{\text{(b)}}{=} -\sum_{m=1}^{\infty}\frac{1}{m}a^m u^m = \log(1-au).
\end{aligned}$$

Here, we put $m = nk$ in (a) and we use the Möbius inversion formula in (b). □

The exponent $\kappa_a(n)$ has the following properties.

Proposition 3.2.3 ([49, Proposition 4.4 and Corollary 4.5])**.** *For any $a \in \mathbb{Z}$ and $n \in \mathbb{N}$, it holds that*

$$\left|\kappa_a(n) - \frac{a^n}{n}\right| \le \frac{|a|^{\lfloor n/2\rfloor+1}}{n}.$$

Thus, the values and signs of $\kappa_a(n)$'s are as in Table 3.1.

$n\backslash a$	$\cdots$	-2	-1	0	1	2	$\cdots$
1	$\cdots$	-2	-1	0	1	2	$\cdots$
2	$+$	(if $2 \mid n$)	1	0	0		
3	$-$	(if $2 \nmid n$)	0	0	0	$+$	
$\vdots$			$\vdots$	$\vdots$	$\vdots$		

Table 3.1: the values and signs of $\kappa_a(n)$'s

Proof. First, we consider the easy cases. When $a = 0$, we have $\kappa_0(n) = 0$. When $a = 1$, we have

$$\kappa_1(n) = \frac{1}{n}\sum_{m|n}\mu\left(\frac{n}{m}\right) = \frac{1}{n}\delta_{1n} = \begin{cases}1 & \text{if } n = 1,\\ 0 & \text{if } n \ge 2,\end{cases}$$

by a standard property of the Möbius function.

Let $a = -1$. We have $\kappa_{-1}(1) = -1$ and $\kappa_{-1}(2) = 1$. Hence, we consider the case when $n \geq 3$. Then, we have

$$\kappa_{-1}(n) = \frac{1}{n}\sum_{m|n}\mu\left(\frac{n}{m}\right)(-1)^m = 0.$$

Indeed, since the number of divisors of n is at most $\left\lfloor \frac{n}{2} \right\rfloor + 1$, it holds that $|\kappa_{-1}(n)| < \frac{1}{2} + \frac{1}{n} < 1$. Thus, we have $\kappa_{-1}(n) = 0$ since $\kappa_{-1}(n) \in \mathbb{Z}$ by Lemma 3.2.1.

Let $|a| \geq 2$. We have $\kappa_a(1) = a$. For $n \geq 2$, we have

$$|n\kappa_a(n) - a^n| \leq \sum_{\substack{m|n \\ m \neq n}} |a|^m \leq \sum_{m=0}^{\lfloor n/2 \rfloor} |a|^m = \frac{|a|^{\lfloor n/2 \rfloor + 1} - 1}{|a| - 1} < |a|^{\lfloor n/2 \rfloor + 1}.$$

Note that this implies the inequality

$$\frac{a^n}{n} - \frac{|a|^{\lfloor n/2 \rfloor + 1}}{n} < \kappa_a(n) < \frac{a^n}{n} + \frac{|a|^{\lfloor n/2 \rfloor + 1}}{n}.$$

Hence, when $a \leq -2$, it holds that $\kappa_a(n) > 0$ if $2 \mid n$ and $\kappa_a(n) < 0$ if $2 \nmid n$.

Thus, the proposition follows. $\square$

Using the above properties of $\kappa_a(n)$, we give the region of absolute convergence of the infinite product in Lemma 3.2.1, which is the key lemma for Theorem 3.1.5.

Lemma 3.2.4 ([49, Lemma 4.3])**.** *Let $a \in \mathbb{Z}$ and $u \in \mathbb{C}$. If $a \geq 2$, then the region of absolute convergence of the infinite product in Lemma 3.2.1*

$$\prod_{n=1}^{\infty}(1 - u^n)^{\kappa_a(n)}$$

is $\{u \in \mathbb{C} \mid |u| < \frac{1}{a}\}$. If $a = 1$, then its region of absolute convergence is $\mathbb{C}$.

Proof. When $a = 1$, $\kappa_1(n) = \delta_{1n}$ and thus the infinite product converges for any $u \in \mathbb{C}$. We assume $a \geq 2$ in the following. First, we show that the infinite product converges absolutely at least for $|u| < \frac{1}{a}$. Since

$$\prod_{n=1}^{\infty}(1 - u^n)^{\kappa_a(n)} = \prod_{n=1}^{\infty}\left(1 + ((1 - u^n)^{\kappa_a(n)} - 1)\right),$$

it suffices to show that

$$\sum_{n=1}^{\infty}|(1 - u^n)^{\kappa_a(n)} - 1|$$

converges for $|u| < \frac{1}{a}$. First of all, we calculate the upper bound of the binomial coefficient $\binom{\kappa_a(n)}{m}$ as follows:

$$\binom{\kappa_a(n)}{m} \leq \frac{\kappa_a(n)^m}{m!} < \frac{\kappa_a(n)^m}{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m} = \frac{\kappa_a(n)^m e^m}{\sqrt{2\pi} m^{m+\frac{1}{2}}} \leq \frac{1}{\sqrt{2\pi}}\left(\frac{e\kappa_a(n)}{m}\right)^m.$$

Since $\kappa_a(n) \le \dfrac{a^n + a^{\lfloor n/2 \rfloor + 1}}{n} \le \dfrac{2a^n}{n}$ by Proposition 3.2.3, it holds that

$$\binom{\kappa_a(n)}{m} < \frac{1}{\sqrt{2\pi}} \left(\frac{e\kappa_a(n)}{m} \right)^m \le \frac{1}{\sqrt{2\pi}} \left(\frac{2ea^n}{nm} \right)^m .$$

Putting $|u| = \frac{1}{a} - \varepsilon$ $\left(0 < \varepsilon < \frac{1}{a}\right)$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} |(1-u^n)^{\kappa_a(n)} - 1| &= \sum_{n=1}^{\infty} \left| \sum_{m=1}^{\kappa_a(n)} (-1)^m \binom{\kappa_a(n)}{m} u^{nm} \right| \\
&\le \sum_{n=1}^{\infty} \sum_{m=1}^{\kappa_a(n)} \binom{\kappa_a(n)}{m} |u|^{nm} = \sum_{n=1}^{\infty} \sum_{m=1}^{\kappa_a(n)} \binom{\kappa_a(n)}{m} \left(\frac{1}{a} - \varepsilon \right)^{nm} \\
&< \sum_{n=1}^{\infty} \sum_{m=1}^{\kappa_a(n)} \frac{1}{\sqrt{2\pi}} \left(\frac{2ea^n}{nm} \right)^m \left(\frac{1-\varepsilon a}{a} \right)^{nm} \\
&\le \sum_{n=1}^{\infty} \sum_{m=1}^{\kappa_a(n)} \frac{1}{\sqrt{2\pi}} \left(\frac{2e(1-\varepsilon a)^n}{n} \right)^m .
\end{aligned}$$

By setting

$$c_\varepsilon := \frac{1}{1-\varepsilon a} > 1, \quad r_n := \frac{2e(1-\varepsilon a)^n}{n} = \frac{2e}{nc_\varepsilon^n},$$

it holds that

$$\sum_{n=1}^{\infty} |(1-u^n)^{\kappa_a(n)} - 1| < \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}} \sum_{m=1}^{\kappa_a(n)} r_n^m = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{r_n^{\kappa_a(n)+1} - r_n}{r_n - 1}.$$

We take and fix sufficiently large $N \in \mathbb{N}$ satisfying $c_\varepsilon^n > n$ and $r_n < 1$ for any $n \ge N$.

$$\begin{aligned}
\sum_{n=1}^{\infty} |(1-u^n)^{\kappa_a(n)} - 1| &< \frac{1}{\sqrt{2\pi}} \left(\sum_{n=1}^{N-1} \frac{r_n^{\kappa_a(n)+1} - r_n}{r_n - 1} + \sum_{n=N}^{\infty} \frac{r_n - r_n^{\kappa_a(n)+1}}{1 - r_n} \right) \\
&\le \frac{1}{\sqrt{2\pi}} \left(\sum_{n=1}^{N-1} \frac{r_n^{\kappa_a(n)+1} - r_n}{r_n - 1} + \sum_{n=N}^{\infty} \frac{r_n}{1 - r_N} \right) \\
&\le \frac{1}{\sqrt{2\pi}} \left(\sum_{n=1}^{N-1} \frac{r_n^{\kappa_a(n)+1} - r_n}{r_n - 1} + \frac{1}{1 - r_N} \sum_{n=N}^{\infty} \frac{2e}{n^2} \right) \\
&\le \frac{1}{\sqrt{2\pi}} \left(\sum_{n=1}^{N-1} \frac{r_n^{\kappa_a(n)+1} - r_n}{r_n - 1} + \frac{2e}{1 - r_N} \zeta(2) \right) < \infty.
\end{aligned}$$

Therefore, the series above converges. Thus, the infinite product converges absolutely at least for $|u| < \frac{1}{a}$ when $a \ge 2$.

Next, we show that the infinite product does not converge absolutely for $|u| \geq \frac{1}{a}$ when $a \geq 2$. It is easy to show that $\sum_{n=1}^{\infty} |(1-u^n)^{\kappa_a(n)} - 1|$ diverges when $|u| \geq 1$. Hence, we may assume that $\frac{1}{a} \leq |u| < 1$.

Firstly, we prove that it diverges when $u = \frac{1}{a}$. It holds that

$$\sum_{n=1}^{\infty} |(1-u^n)^{\kappa_a(n)} - 1| = \sum_{n=1}^{\infty} \left(1 - \left(1 - \frac{1}{a^n}\right)^{\kappa_a(n)}\right)$$
$$= \sum_{n=1}^{\infty} \frac{\kappa_a(n)}{a^n} \sum_{m=1}^{\kappa_a(n)} (-1)^{m+1} \frac{1}{\kappa_a(n)} \binom{\kappa_a(n)}{m} \frac{1}{a^{n(m-1)}}.$$

For $n \geq 3$, put

$$I_n := \sum_{m=1}^{\kappa_a(n)} (-1)^{m+1} \frac{1}{\kappa_a(n)} \binom{\kappa_a(n)}{m} \frac{1}{a^{n(m-1)}}.$$

Then, we have

$$|I_n - 1| = \left| \sum_{m=2}^{\kappa_a(n)} (-1)^{m+1} \frac{\kappa_a(n) - 1}{a^n} \cdots \frac{\kappa_a(n) - m + 1}{a^n} \cdot \frac{1}{m!} \right|$$
$$\leq \sum_{m=2}^{\kappa_a(n)} \left| \frac{\kappa_a(n) - 1}{a^n} \right| \cdots \left| \frac{\kappa_a(n) - m + 1}{a^n} \right| \cdot \frac{1}{m!}.$$

Here, we have

$$\left| \frac{\kappa_a(n) - i}{a^n} \right| \leq \frac{\kappa_a(n)}{a^n} \leq \frac{1}{n} + \frac{1}{na^{n - \lfloor n/2 \rfloor - 1}} \leq \frac{2}{n}$$

for any $i \in \{1, \ldots, m-1\}$ by Proposition 3.2.3. Hence, it holds that

$$|I_n - 1| \leq \sum_{m=2}^{\kappa_a(n)} \frac{2^{m-1}}{n^{m-1}} \cdot \frac{1}{m!} \leq \sum_{m=2}^{\kappa_a(n)} \frac{1}{n^{m-1}} = \frac{1 - n^{1-\kappa_a(n)}}{n-1} < \frac{1}{n-1} \leq \frac{1}{2}$$

for $n \geq 3$. Therefore, we have $I_n \geq \frac{1}{2}$ for $n \geq 3$ and then

$$\sum_{n=1}^{\infty} \left(1 - \left(1 - \frac{1}{a^n}\right)^{\kappa_a(n)}\right) \geq \sum_{n=3}^{\infty} \frac{\kappa_a(n)}{a^n} I_n \geq \sum_{n=3}^{\infty} \frac{1}{2n} \cdot \frac{1}{2} = \infty, \tag{3.3}$$

since for $n \geq 3$

$$\frac{\kappa_a(n)}{a^n} \geq \frac{1}{n} - \frac{1}{na^{n - \lfloor n/2 \rfloor - 1}} \geq \frac{1}{2n}.$$

Secondly, we prove that $\sum_{n=1}^{\infty} |(1-u^n)^{\kappa_a(n)} - 1|$ diverges for $\frac{1}{a} \leq |u| < 1$. Let $r = \frac{\operatorname{Arg} u}{2\pi}$ $(0 \leq r < 1)$. Assume that $r \in \mathbb{Q}$. We put $r = \frac{k_1}{k_2}$ $(k_1 \in \mathbb{Z}, k_2 \in \mathbb{N})$. Let

$N = \{k_2 m \mid m \in \mathbb{N}\}$, then it holds that $u^n = |u|^n \geq \frac{1}{a^n}$ for any $n \in N$. Therefore, we have

$$\sum_{n=1}^{\infty} |(1-u^n)^{\kappa_a(n)} - 1| \geq \sum_{n \in N} \left(1 - (1-|u|^n)^{\kappa_a(n)}\right) \geq \sum_{n \in N} \left(1 - \left(1 - \frac{1}{a^n}\right)^{\kappa_a(n)}\right).$$

In a similar way to Equation (3.3), we have

$$\sum_{n \in N} \left(1 - \left(1 - \frac{1}{a^n}\right)^{\kappa_a(n)}\right) \geq \frac{1}{4} \sum_{n \in N \cap [3,\infty)} \frac{1}{n}.$$

Here, since the natural density of N is $\frac{1}{k_2} > 0$ and coincides with the Dirichlet density of N, the last infinite sum diverges. Thus, the series $\sum_{n=1}^{\infty} |(1-u^n)^{\kappa_a(n)} - 1|$ diverges if $r \in \mathbb{Q}$.

Assume that $r \notin \mathbb{Q}$. If $n \in \mathbb{N}$ satisfies $|1-u^n| \geq 1 + \frac{1}{2a^n}$, it holds that

$$\begin{aligned} \left|(1-u^n)^{\kappa_a(n)} - 1\right| &\geq \left||1-u^n|^{\kappa_a(n)} - 1\right| \geq \left(1 + \frac{1}{2a^n}\right)^{\kappa_a(n)} - 1 \\ &> \frac{\kappa_a(n)}{2a^n} \overset{(*)}{\geq} \frac{1}{2n}\left(1 - \frac{1}{a^{n-\lfloor n/2 \rfloor - 1}}\right) \geq \frac{1}{4n} \end{aligned}$$

by applying Proposition 3.2.3 to $(*)$. Hence, for every $N \subset \mathbb{N}$ which consists of $n \in \mathbb{N}$ satisfying $|1-u^n| \geq 1 + \frac{1}{2a^n}$, we have

$$\sum_{n=1}^{\infty} |(1-u^n)^{\kappa_a(n)} - 1| \geq \sum_{n \in N} |(1-u^n)^{\kappa_a(n)} - 1| > \frac{1}{4} \sum_{n \in N} \frac{1}{n}.$$

Here, if the natural density of N is positive, then the last infinite sum diverges and hence the series $\sum_{n=1}^{\infty} |(1-u^n)^{\kappa_a(n)} - 1|$ diverges. Therefore, it suffices to show that there exists $N \subset \mathbb{N}$ with positive natural density such that $|1-u^n| \geq 1 + \frac{1}{2a^n}$ for any $n \in N$.

As described in Figure 3.1, let $p_n \in \mathbb{C}$ be the intersection of $|z| = |u|^n$ and $|z-1| = 1 + \frac{1}{2a^n}$ whose imaginary part is positive and put $\theta_n := \operatorname{Arg} p_n$. Let $\Theta_n := \{\theta \in \mathbb{R} \mid \theta_n \leq \theta \leq 2\pi - \theta_n\}$. Note that if $\operatorname{Arg}(u^n) \in \Theta_n$, then such an n satisfies that $|1-u^n| \geq 1 + \frac{1}{2a^n}$. Since $\frac{1}{a} \leq |u| < 1$, we have

$$\cos\theta_n = \frac{|u|^n}{2} - \frac{1}{2a^n|u|^n} - \frac{1}{8a^{2n}|u|^n} \geq \frac{1}{2a^n} - \frac{1}{2} - \frac{1}{8a^n} = \frac{3}{8a^n} - \frac{1}{2} \geq -\frac{1}{2}$$

for any $n \in \mathbb{N}$. Hence, we have $\theta_n \leq \frac{2}{3}\pi$ for any $n \in \mathbb{N}$. Therefore, it holds that $\Theta_n \supset \left[\frac{2}{3}\pi, \frac{4}{3}\pi\right] =: \Theta_\infty$ for any $n \in \mathbb{N}$. Let $N := \{n \in \mathbb{N} \mid \operatorname{Arg}(u^n) \in \Theta_\infty\}$. Here, any $n \in N$ satisfies $|1-u^n| \geq 1 + \frac{1}{2a^n}$, since $N \subset \{n \in \mathbb{N} \mid \operatorname{Arg}(u^n) \in \Theta_n\}$. Therefore, it suffices to show that the natural density of N is positive. Put $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Let

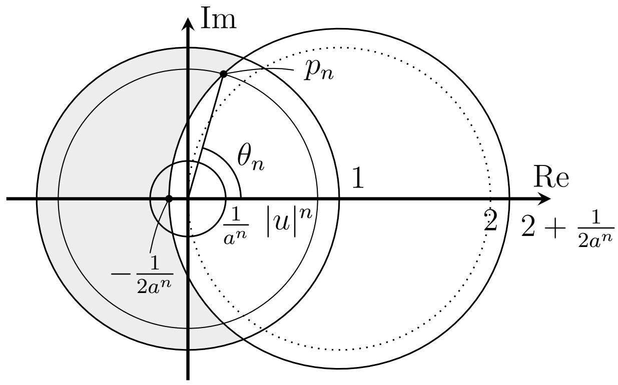


Figure 3.1: the definition of p_n and θ_n in the complex plane

$\iota\colon \mathbb{R} \to \mathbb{T}$ be the natural isomorphism defined by $\iota(x) := x \bmod \mathbb{Z}$ and $R_r\colon \mathbb{T} \to \mathbb{T}$ be the map satisfying that $R_r(x) = \iota(x+r)$. Put $\overline{\Theta}_\infty := \iota\left(\left[\frac{1}{3}, \frac{2}{3}\right]\right)$. Then, it holds that $N = \{n \in \mathbb{N} \mid R_r^n(0) \in \overline{\Theta}_\infty\}$. Since $r \notin \mathbb{Q}$, the continuous map R_r on $\mathbb{T}$ is uniquely ergodic (cf. Einsiedler and Ward [18, Example 1.3 and Example 4.11]). Therefore, since $\overline{\Theta}_\infty$ is an interval in $\mathbb{T}$, we have

$$\lim_{M\to\infty} \frac{1}{M} \#\{n \in \mathbb{N}_0 \cap [0, M) \mid R_r^n(x) \in \overline{\Theta}_\infty\} = m_{\mathbb{T}}(\overline{\Theta}_\infty) = \frac{1}{3},$$

where $m_{\mathbb{T}}$ is the Lebesgue measure on $\mathbb{T}$, for every $x \in \mathbb{T}$ by [18, Example 4.18 and Lemma 4.17] (also see Einsiedler and Ward [18, Example 1.3]). By putting $x = 0$, we have

$$\lim_{M\to\infty} \frac{\#(N \cap [1, M])}{M} = \lim_{M\to\infty} \frac{1}{M} \#\{n \in \mathbb{N}_0 \cap [0, M) \mid R_r^n(0) \in \overline{\Theta}_\infty\} = \frac{1}{3} > 0.$$

Therefore, the natural density of N is positive.

Thus, the infinite product does not converge absolutely for $|u| \geq \frac{1}{a}$. $\square$

3.2.2 Proof

First, we derive the infinite product representation as an element of $\mathbb{Z}[\![\frac{1}{s}]\!]$. By Theorem 3.1.4, we have

$$\begin{aligned}
\zeta_{\mathcal{X}/\mathbb{F}_1}(s) &= \prod_{x \in X_{\mathfrak{M}_0}} \frac{1}{\left(1 - \frac{1}{s}\right)^{\otimes r_x}} = \prod_{x \in X_{\mathfrak{M}_0}} \prod_{j=0}^{r_x} (s - r_x + j)^{(-1)^{j+1}\binom{r_x}{j}} \\
&= \prod_{x \in X_{\mathfrak{M}_0}} \prod_{j=0}^{r_x} s^{(-1)^{j+1}\binom{r_x}{j}} \left(1 - \frac{r_x - j}{s}\right)^{(-1)^{j+1}\binom{r_x}{j}} \\
&= \left(\frac{1}{s}\right)^{\sum_{x \in X_{\mathfrak{M}_0}} \sum_{j=0}^{r_x} (-1)^j \binom{r_x}{j}} \prod_{x \in X_{\mathfrak{M}_0}} \prod_{j=0}^{r_x} \left(1 - \frac{r_x - j}{s}\right)^{(-1)^{j+1}\binom{r_x}{j}}.
\end{aligned}$$

Since the counting function $N_{\mathcal{X}}(t)$ of $\mathcal{X}$ satisfies

$$N_{\mathcal{X}}(t) = \sum_{x \in X_{\mathfrak{M}_0}} (t-1)^{r_x} \in \mathbb{Z}[t]$$

by Remark 2.3.11, we have

$$\sum_{x \in X_{\mathfrak{M}_0}} \sum_{j=0}^{r_x} (-1)^j \binom{r_x}{j} = \sum_{x \in X_{\mathfrak{M}_0}} (1-1)^{r_x} = N_{\mathcal{X}}(1).$$

By Lemma 3.2.1, it holds that

$$1 - \frac{r_x - j}{s} = \prod_{n=1}^{\infty} \left(1 - \left(\frac{1}{s}\right)^n\right)^{\kappa_{r_x - j}(n)}.$$

Hence, it holds that

$$\begin{aligned}
\zeta_{\mathcal{X}/\mathbb{F}_1}(s) &= \left(\frac{1}{s}\right)^{N_{\mathcal{X}}(1)} \prod_{x \in X_{\mathfrak{M}_0}} \prod_{j=0}^{r_x} \left(1 - \frac{r_x - j}{s}\right)^{(-1)^{j+1}\binom{r_x}{j}} \\
&= \left(\frac{1}{s}\right)^{N_{\mathcal{X}}(1)} \prod_{x \in X_{\mathfrak{M}_0}} \prod_{j=0}^{r_x} \left(\prod_{n=1}^{\infty} \left(1 - \left(\frac{1}{s}\right)^n\right)^{\kappa_{r_x - j}(n)}\right)^{(-1)^{j+1}\binom{r_x}{j}} \\
&= \left(\frac{1}{s}\right)^{N_{\mathcal{X}}(1)} \prod_{n=1}^{\infty} \left(1 - \left(\frac{1}{s}\right)^n\right)^{-\sum_{x \in X_{\mathfrak{M}_0}} \sum_{j=0}^{r_x} (-1)^j \binom{r_x}{j} \kappa_{r_x - j}(n)}.
\end{aligned}$$

Put

$$\kappa(n, X_{\mathfrak{M}_0}) := \sum_{x \in X_{\mathfrak{M}_0}} \sum_{j=0}^{r_x} (-1)^j \binom{r_x}{j} \kappa_{r_x - j}(n) = \sum_{x \in X_{\mathfrak{M}_0}} \sum_{j=0}^{r_x} (-1)^{r_x - j} \binom{r_x}{j} \kappa_j(n).$$

Then, we get the desired infinite product

$$\zeta_{\mathcal{X}/\mathbb{F}_1}(s) = \left(\frac{1}{s}\right)^{N_{\mathcal{X}}(1)} \prod_{n=1}^{\infty} \left(1 - \left(\frac{1}{s}\right)^n\right)^{-\kappa(n, X_{\mathfrak{M}_0})}.$$

Next, we show that the infinite product converges absolutely for $|s| > \mathrm{rel.dim}\, X_{\mathbb{Z}}/\mathbb{Z}$ if $X_{\mathbb{Z}}$ is of finite type over $\mathbb{Z}$. In the above calculation of the infinite product representation, the point which is relevant to its convergence area is the following equality for which we use Lemma 3.2.1:

$$1 - \frac{r_x - j}{s} = \prod_{n=1}^{\infty} \left(1 - \left(\frac{1}{s}\right)^n\right)^{\kappa_{r_x - j}(n)}.$$

Let $|s| > \operatorname{rel.dim} X_{\mathbb{Z}}/\mathbb{Z}$. Since $X_{\mathbb{Z}}$ is of finite type over $\mathbb{Z}$, we have

$$\deg N_{\mathcal{X}} = \operatorname{rel.dim} X_{\mathbb{Z}}/\mathbb{Z}$$

by Theorem 2.3.12. Since

$$|s| > \operatorname{rel.dim} X_{\mathbb{Z}}/\mathbb{Z} = \deg N_{\mathcal{X}} = R_{X_{\mathfrak{M}_0}} \geq r_x \geq r_x - j$$

for any $x \in X_{\mathfrak{M}_0}$, we have $\frac{1}{|s|} < \frac{1}{r_x - j}$ when $0 \leq j < r_x$. Therefore, when $j \neq r_x$,

$$\prod_{n=1}^{\infty} \left(1 - \left(\frac{1}{s}\right)^n\right)^{\kappa_{r_x - j}(n)} \tag{3.4}$$

converges absolutely by Lemma 3.2.4. Also, when $j = r_x$, it converges absolutely since $\kappa_0(n) = 0$. Thus, the infinite product

$$\prod_{n=1}^{\infty} \left(1 - \left(\frac{1}{s}\right)^n\right)^{-\kappa(n, X_{\mathfrak{M}_0})} \tag{3.5}$$

converges absolutely for $|s| > \operatorname{rel.dim} X_{\mathbb{Z}}/\mathbb{Z}$ if $X_{\mathbb{Z}}$ is of finite type over $\mathbb{Z}$.

Lastly, we show that the infinite product (3.5) diverges for $|s| \leq \operatorname{rel.dim} X_{\mathbb{Z}}/\mathbb{Z}$ if $X_{\mathbb{Z}}$ is of finite type over $\mathbb{Z}$. By Theorem 2.3.12, we have $|s| \leq R_{X_{\mathfrak{M}_0}}$. When $x \in X_{\mathfrak{M}_0}$ and j satisfy $r_x - j = R_{X_{\mathfrak{M}_0}}$, the infinite product (3.4) diverges by Lemma 3.2.4. Thus, the infinite product (3.5) diverges for $|s| \leq \operatorname{rel.dim} X_{\mathbb{Z}}/\mathbb{Z}$.

3.3 Applications of Theorem 3.1.5

In this section, we see some applications of Theorem 3.1.5 and Corollary 3.1.7 to the cases of $\mathbb{A}^r$, $\mathbb{G}_m^r$ and toric varieties to obtain the absolute Euler products of the absolute zeta functions for the $\mathbb{F}_1$-schemes associated with these cases. In fact, for the cases of $\mathbb{A}^r$ and $\mathbb{G}_m^r$, our result coincides with Kurokawa's calculations [30, Exercise 7.2] in a different method.

3.3.1 Fundamental $\mathbb{F}_1$-schemes

Example 3.3.1 ([49, Example 5.1])**.** Let $r \in \mathbb{N}$. Put $\mathbb{F}_1[t_1, \ldots, t_r] := \mathbb{F}_1[\{t_1^{u_1} \cdots t_r^{u_r} \mid u_i \in \mathbb{N}_0\}]$ and $\mathbf{A}^r := \operatorname{spec} \mathbb{F}_1[t_1, \ldots, t_r]$. Then, by the extension of the functors $\underline{\mathbf{A}^r} \colon \mathfrak{M}_0 \to \mathfrak{Set}$, $\underline{\mathbb{A}^r} \colon \mathfrak{CRing} \to \mathfrak{Set}$, we obtain the functor $\mathcal{A}^r \colon \mathfrak{MR} \to \mathfrak{Set}$ satisfying that the geometric realisation of $\mathcal{A}^r|_{\mathfrak{M}_0}$ (resp. $\mathcal{A}^r|_{\mathfrak{CRing}}$) is $\mathbf{A}^r$ (resp. $\mathbb{A}^r$) (see Connes and Consani [8, §4.2] for the extension of the functors). Moreover, $\mathcal{A}^r$ is a torsion free Noetherian $\mathbb{F}_1$-scheme.

Since $\#\mathbb{A}^r(\mathbb{F}_{p^m}) = p^{mr}$ for any $m \in \mathbb{N}$ and $p \in \mathbb{P}$, the counting function of $\mathcal{A}^r$ is $N_{\mathcal{A}^r}(t) = t^r \in \mathbb{Z}[t]$. Hence, we have $\chi_{\text{abs}}(\mathbb{A}^r) = N_{\mathcal{A}^r}(1) = 1$ and

$$\zeta_{\mathcal{A}^r/\mathbb{F}_1}(s) = \zeta_{N_{\mathcal{A}^r}}^{\lim}(s) = \frac{1}{s - r}.$$

Since the prime ideals of $\mathbb{F}_1[t_1, \ldots, t_r]$ are of the form

$$\mathfrak{p}_I = \bigcup_{i \in I} t_i \mathbb{F}_1[t_1, \ldots, t_r],$$

where $I \subset \{1, \ldots, r\}$ and $\mathfrak{p}_\emptyset = (0)$, we have $r_{\mathfrak{p}_I} = r - \#I$ for $\mathfrak{p}_I \in \mathbf{A}^r$. We put

$$\begin{aligned}
\kappa(n, \mathbf{A}^r) &:= \sum_{\mathfrak{p}_I \in \mathbf{A}^r} \sum_{j=0}^{r_{\mathfrak{p}_I}} (-1)^{r_{\mathfrak{p}_I} - j} \binom{r_{\mathfrak{p}_I}}{j} \kappa_j(n) = \sum_{i=0}^{r} \binom{r}{i} \sum_{j=0}^{r-i} (-1)^{r-i-j} \binom{r-i}{j} \kappa_j(n) \\
&= \sum_{j=0}^{r} \kappa_j(n) \sum_{i=0}^{r-j} (-1)^{r-j-i} \binom{r}{i} \binom{r-i}{j} \\
&= \sum_{j=0}^{r} \binom{r}{j} \kappa_j(n) \sum_{i=0}^{r-j} (-1)^{r-j-i} \binom{r-j}{i} = \sum_{j=0}^{r} \binom{r}{j} \kappa_j(n) (1-1)^{r-j} \\
&= \kappa_r(n).
\end{aligned}$$

By Theorem 3.1.5, we obtain the absolute Euler product

$$\zeta_{\mathcal{A}^r/\mathbb{F}_1}(s) = \frac{1}{s} \prod_{n=1}^{\infty} \left(1 - \left(\frac{1}{s}\right)^n\right)^{-\kappa_r(n)}$$

and this infinite product converges absolutely for $|s| > \mathrm{rel.dim}\, \mathbb{A}^r/\mathbb{Z} = r$.

Example 3.3.2 ([49, Example 5.2])**.** Let $r \in \mathbb{N}$. Put $\mathbb{F}_1[t_1^{\pm 1}, \ldots, t_r^{\pm 1}] := \mathbb{F}_1[\{t_1^{u_1} \cdots t_r^{u_r} \mid u_i \in \mathbb{Z}\}]$ and $\mathbf{G}_m^r := \mathrm{spec}\, \mathbb{F}_1[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$. Then, by the extension of the functors $\underline{\mathbf{G}_m^r} \colon \mathfrak{M}_0 \to \mathfrak{Set}$, $\underline{\mathbb{G}_m^r} \colon \mathfrak{CRing} \to \mathfrak{Set}$, we obtain the functor $\mathcal{G}_m^r \colon \mathfrak{MR} \to \mathfrak{Set}$ satisfying that the geometric realisation of $\mathcal{G}_m^r|_{\mathfrak{M}_0}$ (resp. $\mathcal{G}_m^r|_{\mathfrak{CRing}}$) is $\mathbf{G}_m^r$ (resp. $\mathbb{G}_m^r$). Moreover, $\mathcal{G}_m^r$ is a torsion free Noetherian $\mathbb{F}_1$-scheme.

Since $\#\mathbb{G}_m^r(\mathbb{F}_{p^m}) = (p^m - 1)^r$ for any $m \in \mathbb{N}$ and $p \in \mathbb{P}$, the counting function of $\mathcal{G}_m^r$ is $N_{\mathcal{G}_m^r}(t) = (t-1)^r \in \mathbb{Z}[t]$. Hence, we have $\chi_{\mathrm{abs}}(\mathbb{G}_m^r) = N_{\mathcal{G}_m^r}(1) = 0$ and

$$\zeta_{\mathcal{G}_m^r/\mathbb{F}_1}(s) = \zeta^{\lim}_{N_{\mathcal{G}_m^r}}(s) = \prod_{k=0}^{r} (s-k)^{(-1)^{r-k+1}\binom{r}{k}}.$$

Since $\mathbf{G}_m^r = \{(0)\}$ and $r_{(0)} = r$, we put

$$\kappa(n, \mathbf{G}_m^r) := \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} \kappa_k(n).$$

By Theorem 3.1.5, we get the absolute Euler product

$$\zeta_{\mathcal{G}_m^r/\mathbb{F}_1}(s) = \prod_{n=1}^{\infty} \left(1 - \left(\frac{1}{s}\right)^n\right)^{-\sum_{k=0}^{r} (-1)^{r-k}\binom{r}{k}\kappa_k(n)}$$

and this infinite product converges absolutely for $|s| > \mathrm{rel.dim}\, \mathbb{G}_m^r/\mathbb{Z} = r$.

3.3.2 Toric varieties

Lastly, we calculate the absolute Euler product of the absolute zeta function for the $\mathbb{F}_1$-scheme associated with a toric variety, using their counting functions calculated by Deitmar [13].

First, we review the notation of cones and fans according to Peña and Lorscheid [39, §2.1]. Let N be a *lattice*, i.e. $N \cong \mathbb{Z}^r \subset \mathbb{R}^r$ as additive groups. Fixing this isomorphism, we put $N_{\mathbb{R}} := N \otimes \mathbb{R} \cong \mathbb{R}^r$. An additive semi-group $\sigma \subset \mathbb{R}^r$ is called a *(strongly convex rational polyhedral) cone* if $\sigma \cap (-\sigma) = \{0\}$ and there exists a linearly independent set $\{v_1, \ldots, v_k\} \subset N$ such that $\sigma = v_1\mathbb{R}_{\geq 0} + \cdots + v_k\mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0} := [0, \infty)$. A *face* τ of a cone σ is a cone of the form $\tau = v_{i_1}\mathbb{R}_{\geq 0} + \cdots + v_{i_m}\mathbb{R}_{\geq 0}$ for some $\{i_1, \ldots, i_m\} \subset \{1, \ldots, k\}$. We denote a face τ of σ by $\tau \prec \sigma$. A nonempty set Φ of cones in $N_{\mathbb{R}}$ is called a *fan* if every face of a cone in Φ is also in Φ and $\sigma_1 \cap \sigma_2$ is a face of each σ_1 and σ_2 for any $\sigma_1, \sigma_2 \in \Phi$. Also, we call Φ to be *finite* if $|\Phi| < \infty$.

Next, we review the definition of the $\mathbb{F}_1$-scheme associated with a cone [39, §2.1]. Let N be a lattice and σ be a cone in $N_{\mathbb{R}}$. Let $N^\vee := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^r$ be the dual of N and $\langle \cdot, \cdot \rangle \colon N^\vee \times N \to \mathbb{Z}$ be the natural pairing. We define the dual of σ by $\sigma^\vee := \{u \in N^\vee \otimes \mathbb{R} \mid \forall v \in \sigma,\ \langle u, v \rangle \geq 0\}$. Put $A_\sigma := \sigma^\vee \cap N^\vee$. Then, we have the scheme $\mathrm{Spec}\,\mathbb{Z}[A_\sigma]$ of finite type over $\mathbb{Z}$ and the monoid scheme $\mathrm{spec}\,A_\sigma$. The $\mathbb{F}_1$-scheme $\mathcal{X}_\sigma$ associated with the cone σ is defined as the $\mathbb{F}_1$-scheme satisfying that the geometric realisation of $\mathcal{X}_\sigma|_{\mathfrak{Mo}_0}$ is $X_\sigma^{\mathfrak{Mo}_0} := \mathrm{spec}\,A_\sigma$ and that of $\mathcal{X}_\sigma|_{\mathfrak{CRing}}$ is $X_\sigma^{\mathbb{Z}} := \mathrm{Spec}\,\mathbb{Z}[A_\sigma]$. Since A_σ is finitely generated, the $\mathbb{F}_1$-scheme $\mathcal{X}_\sigma$ is Noetherian.

Lastly, we review the definition of the $\mathbb{F}_1$-scheme associated with a fan [39, §2.1]. Let Φ be a finite fan in $N_{\mathbb{R}}$. An inclusion $\tau \subset \sigma$ of cones induces an open immersions $\mathrm{spec}\,A_\tau \hookrightarrow \mathrm{spec}\,A_\sigma$ and $\mathrm{Spec}\,\mathbb{Z}[A_\tau] \hookrightarrow \mathrm{Spec}\,\mathbb{Z}[A_\sigma]$. We define the monoid scheme and the scheme associated with the fan Φ by

$$X_\Phi^{\mathfrak{Mo}_0} := \varinjlim_{\sigma \in \Phi} \mathrm{spec}\,A_\sigma, \quad X_\Phi^{\mathbb{Z}} := \varinjlim_{\sigma \in \Phi} \mathrm{Spec}\,\mathbb{Z}[A_\sigma],$$

and we call $(X_\Phi^{\mathbb{Z}}, \Phi)$ the *toric variety* of the fan Φ of dimension r. The $\mathbb{F}_1$-scheme $\mathcal{X}_\Phi$ associated with Φ is the $\mathbb{F}_1$-scheme satisfying that the geometric realisation of $\mathcal{X}_\Phi|_{\mathfrak{Mo}_0}$ is $X_\Phi^{\mathfrak{Mo}_0}$ and that of $\mathcal{X}_\Phi|_{\mathfrak{CRing}}$ is $X_\Phi^{\mathbb{Z}}$. Since Φ is finite and $\mathrm{spec}\,A_\sigma$ is Noetherian, $\mathcal{X}_\Phi$ is Noetherian.

Deitmar calculated the counting functions of the schemes associated with cones and finite fans, which are equal to the counting functions of the $\mathbb{F}_1$-scheme associated with them. Hereinafter, let N be a lattice of dimension r, σ be a cone in $N_{\mathbb{R}}$ and Φ be a finite fan in $N_{\mathbb{R}}$. Let $\mathcal{X}_\sigma$ (resp. $\mathcal{X}_\Phi$) be the $\mathbb{F}_1$-scheme associated with σ (resp. Φ).

Proposition 3.3.3 (Deitmar [13, Proposition 4.3])**.** *In the above setting, the counting functions $N_{\mathcal{X}_\sigma}$ and $N_{\mathcal{X}_\Phi}$ are*

$$N_{\mathcal{X}_\sigma}(t) = \sum_{k=0}^{\dim\sigma} i^\sigma_{\dim\sigma - k}(t-1)^k = \sum_{j=0}^{\dim\sigma} \left(\sum_{k=j}^{\dim\sigma} (-1)^{k-j} \binom{k}{j} i^\sigma_{\dim\sigma - k} \right) t^j$$

and

$$N_{\mathcal{X}_\Phi}(t) = \sum_{k=0}^{r} i_{r-k}^{\Phi}(t-1)^k = \sum_{j=0}^{r}\left(\sum_{k=j}^{r}(-1)^{k-j}\binom{k}{j} i_{r-k}^{\Phi}\right) t^j,$$

where $i_k^\sigma := \#\{\eta \prec \sigma \mid \dim \eta = k\}$ *and* $i_k^\Phi := \#\{\tau \in \Phi \mid \dim \tau = k\}$.

Example 3.3.4 ([49, Example 5.4])**.** By Proposition 3.3.3, the absolute zeta functions of $\mathcal{X}_\sigma$ and $\mathcal{X}_\Phi$ are

$$\zeta_{\mathcal{X}_\sigma/\mathbb{F}_1}(s) = \prod_{j=0}^{\dim\sigma}(s-j)^{\sum_{k=j}^{\dim\sigma}(-1)^{k-j}\binom{k}{j} i_{\dim\sigma-k}^{\sigma}},$$

$$\zeta_{\mathcal{X}_\Phi/\mathbb{F}_1}(s) = \prod_{j=0}^{r}(s-j)^{\sum_{k=j}^{r}(-1)^{k-j}\binom{k}{j} i_{r-k}^{\Phi}},$$

where i_k^σ and i_k^Φ are defined in Proposition 3.3.3. Using the homomorphism M_n (Corollary 3.1.7), we can easily obtain the absolute Euler products of those absolute zeta functions. Since $N_{\mathcal{X}_\sigma}(1) = 1$ and $N_{\mathcal{X}_\Phi}(1) = i_r^\Phi$ by Proposition 3.3.3, we have the absolute Euler products

$$\zeta_{\mathcal{X}_\sigma/\mathbb{F}_1}(s) = \frac{1}{s}\prod_{n=1}^{\infty}\left(1-\left(\frac{1}{s}\right)^n\right)^{-\sum_{j=0}^{\dim\sigma}\left(\sum_{k=j}^{\dim\sigma}(-1)^{k-j}\binom{k}{j} i_{\dim\sigma-k}^{\sigma}\right)\kappa_j(n)},$$

$$\zeta_{\mathcal{X}_\Phi/\mathbb{F}_1}(s) = \left(\frac{1}{s}\right)^{\#I_r}\prod_{n=1}^{\infty}\left(1-\left(\frac{1}{s}\right)^n\right)^{-\sum_{j=0}^{r}\left(\sum_{k=j}^{r}(-1)^{k-j}\binom{k}{j} i_{r-k}^{\Phi}\right)\kappa_j(n)},$$

and $\zeta_{\mathcal{X}_\sigma/\mathbb{F}_1}(s)$ (resp. $\zeta_{\mathcal{X}_\Phi/\mathbb{F}_1}(s)$) converges absolutely for $|s| > \dim\sigma$ (resp. $|s| > r$).

Chapter 4

Analytic properties of absolute zeta functions

In this chapter, we prove the second main result of this thesis. That is, we show three analytic properties of absolute zeta functions: the series expansion, the integral formula, and the absolute Euler product. In this chapter, we take the branch of $\log s$ with the branch cut along $\mathbb{C} \setminus (-\infty, 0]$, and define $s^w := e^{w \log s}$ for $s, w \in \mathbb{C}$.

In the previous chapter, we studied the absolute Euler product of the absolute zeta functions for $\mathbb{F}_1$-schemes defined by Connes and Consani and introduced the homomorphism $M_n \colon \mathbb{Z}[t] \to \mathbb{Z}$ in Definition 3.1.6. This morphism can be easily generalised to the morphism $\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}$. Putting $f(t) := \sum_{j=-d_-}^{d_+} a_j t^j \in \mathbb{Z}[t, t^{-1}]$ $(d_\pm \in \mathbb{N}_0)$, we define $\mathcal{D} \colon \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}^{\mathbb{N}_0}$ by $\mathcal{D}(f) := \{\alpha_n(f)\}_{n=0}^\infty$, where

$$\alpha_n(f) := \left(t\frac{d}{dt}\right)^n f(t)\bigg|_{t=1} = \sum_{j=-d_-}^{d_+} a_j j^n,$$

and the linear map $\mathcal{M} \colon \mathbb{Z}^{\mathbb{N}_0} \to \mathbb{Z}^{\mathbb{N}_0}$ by $\mathcal{M}(\alpha)_0 := \alpha_0$ and

$$\mathcal{M}(\alpha)_n := \frac{1}{n} \sum_{m|n} \mu\left(\frac{n}{m}\right) \alpha_m \quad (n \in \mathbb{N})$$

for an integer sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$. Then, we define $M_n \colon \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}$ by

$$M_n(f) := (\mathcal{M} \circ \mathcal{D}(f))_n = \sum_{j=-d_-}^{d_+} a_j \kappa_j(n).$$

Thus, we have the following series expression.

Proposition 4.0.1. *For $f(t) := \sum_{j=-d_-}^{d_+} a_j t^j \in \mathbb{Z}[t, t^{-1}]$, we put*

$$\zeta_f(s) := s^{-M_0(f)} \prod_{n=1}^\infty (1 - s^{-n})^{-M_n(f)} \in \mathbb{Q}[\![s^{-1}]\!].$$

Then, it holds that in $\mathbb{Q}[\![s^{-1}]\!]$

$$\frac{\zeta_f'(s)}{\zeta_f(s)} = -\sum_{n=1}^{\infty}\left(\left(t\frac{d}{dt}\right)^{n-1} f(t)\Big|_{t=1}\right)s^{-n} = -\left(s - t\frac{d}{dt}\right)^{-1} f(t)\Big|_{t=1}.$$

In particular, the left-hand side lies in $\mathbb{Z}[\![s^{-1}]\!]$. *Moreover, the series in the middle converges absolutely at least for* $|s| > \max\{d_+, d_-\}$.

Proof. Since

$$\frac{d}{ds}\log(1 - s^{-n}) = \frac{ns^{-n-1}}{1 - s^{-n}} = \frac{n}{s^{n+1} - s},$$

we have

$$\begin{aligned}\frac{\zeta_f'(s)}{\zeta_f(s)} &= -\frac{f(1)}{s} - \sum_{n=1}^{\infty}\frac{nM_n(f)}{s^{n+1} - s} = -\frac{1}{s}\left(f(1) + \sum_{n=1}^{\infty}\frac{nM_n(f)}{s^n - 1}\right)\\ &= -\frac{1}{s}\left(f(1) + \sum_{n=1}^{\infty} nM_n(f)\sum_{m=1}^{\infty}\frac{1}{s^{nm}}\right) = -\frac{1}{s}\left(f(1) + \sum_{N=1}^{\infty}\frac{1}{s^N}\sum_{n|N} nM_n(f)\right).\end{aligned}$$

Here, it holds that

$$\begin{aligned}\sum_{n|N} nM_n(f) &= \sum_{n|N}\sum_{j=-d_-}^{d_+} a_j n\kappa_j(n) = \sum_{n|N}\sum_{j=-d_-}^{d_+} a_j \sum_{k|n}\mu\left(\frac{n}{k}\right) j^k\\ &= \sum_{j=-d_-}^{d_+} a_j \sum_{n|N}\sum_{k|n}\mu\left(\frac{n}{k}\right) j^k \overset{(*)}{=} \sum_{j=-d_-}^{d_+} a_j j^N = \left(t\frac{d}{dt}\right)^N f(t)\Big|_{t=1}\end{aligned}$$

by applying the Möbius inversion formula to $(*)$. Therefore, we have

$$\frac{\zeta_f'(s)}{\zeta_f(s)} = -\frac{1}{s}\sum_{N=0}^{\infty}\left(\left(t\frac{d}{dt}\right)^N f(t)\Big|_{t=1}\right)\frac{1}{s^N} = -\sum_{N=1}^{\infty}\left(\left(t\frac{d}{dt}\right)^{N-1} f(t)\Big|_{t=1}\right)\frac{1}{s^N}.$$

For $|s| > \max\{d_+, d_-\}$, it holds that

$$\begin{aligned}\left|\sum_{n=1}^{\infty}\left(\left(t\frac{d}{dt}\right)^{n-1} f(t)\Big|_{t=1}\right)\frac{1}{s^n}\right| &\le \sum_{n=1}^{\infty}\left|\left(\sum_{j=-d_-}^{d_+} a_j j^{n-1}\right)\frac{1}{s^n}\right|\\ &\le \sum_{j=-d_-}^{d_+}\frac{|a_j|}{|s|}\sum_{n=1}^{\infty}\frac{j^{n-1}}{|s|^{n-1}} = \sum_{j=-d_-}^{d_+}\frac{|a_j|}{|s|}\frac{1}{1 - \frac{j}{|s|}} < \infty.\end{aligned}$$

Thus, the series converges for $|s| > \max\{d_+, d_-\}$. □

In the following sections, we generalise these simple observations to a certain class of analytic functions.

4.1 The class $\mathcal{A}$ of analytic functions

Before stating the main results, we introduce a commutative $\mathbb{C}$-algebra consisting of analytic functions on $[1, \infty)$ with a nice condition.

Definition 4.1.1. Let $d > 0$. We define $\mathcal{A}_d$ by the set of analytic functions on $[1, \infty)$ satisfying that there exists $C > 0$ such that for any $n \in \mathbb{N}_0$

$$|\alpha_n(f)| \leq Cd^n,$$

where

$$\alpha_n(f) := \left(\frac{\partial}{\partial x}\right)^n f(e^x)\bigg|_{x=0} = \left(t\frac{\partial}{\partial t}\right)^n f(t)\bigg|_{t=1}$$

for each $n \in \mathbb{N}_0$. We define $\mathcal{A} := \bigcup_{d>0} \mathcal{A}_d$.

Remark 4.1.2. The class $\mathcal{A}$ requires that f is defined at $t = 1$. Therefore, functions with a pole at $t = 1$ such as the function whose absolute zeta function is Barnes' multiple gamma function are not included in $\mathcal{A}$.

Example 4.1.3. We easily see that $t^\rho \in \mathcal{A}_{|\rho|}$ $(\rho \in \mathbb{C})$. Let $c \in \mathbb{C}$ be a constant. Then, we also see that c, $\log t \in \mathcal{A}_\varepsilon$ for any $\varepsilon > 0$.

Proposition 4.1.4. *The classes $\mathcal{A}_d$'s satisfy the following properties:*

- *If $0 < d_1 < d_2$, then it holds that $\mathcal{A}_{d_1} \subset \mathcal{A}_{d_2}$.*
- *Let $f \in \mathcal{A}_{d_1}$, $g \in \mathcal{A}_{d_2}$. Then, we have $f + g \in \mathcal{A}_{\max\{d_1,d_2\}}$, $fg \in \mathcal{A}_{d_1+d_2}$.*

Therefore, $\mathcal{A}$ is a continuously filtered commutative $\mathbb{C}$-algebra.

Proof. Let $d_1 < d_2$. For $f \in \mathcal{A}_{d_1}$, there exists $C > 0$ such that

$$|\alpha_n(f)| \leq Cd_1^n < Cd_2^n$$

for any $n \in \mathbb{N}_0$. Thus, it follows that $f \in \mathcal{A}_{d_2}$.

Let $f \in \mathcal{A}_{d_1}$, $g \in \mathcal{A}_{d_2}$. Then, we have

$$\exists C_1 > 0, \quad \forall n \in \mathbb{N}_0, \quad |\alpha_n(f)| \leq C_1 d_1^n,$$

$$\exists C_2 > 0, \quad \forall n \in \mathbb{N}_0, \quad |\alpha_n(g)| \leq C_2 d_2^n.$$

Since $\alpha_n(\cdot)$ has the linearity $\alpha_n(f + g) = \alpha_n(f) + \alpha_n(g)$ by definition, it holds that

$$|\alpha_n(f + g)| \leq C_1 d_1^n + C_2 d_2^n \leq (C_1 + C_2)(\max\{d_1, d_2\})^n.$$

Thus, we have $f + g \in \mathcal{A}_{\max\{d_1,d_2\}}$. Since

$$\alpha_n(fg) = \sum_{m=0}^{n} \binom{n}{m} \alpha_m(f)\alpha_{n-m}(g)$$

by the Leibniz rule, it holds that

$$|\alpha_n(fg)| \le \sum_{m=0}^{n} \binom{n}{m} \cdot C_1 d_1^m \cdot C_2 d_2^{n-m} = C_1 C_2 (d_1 + d_2)^n$$

by the Leibniz rule. Thus, we have $fg \in \mathcal{A}_{d_1+d_2}$. □

Example 4.1.5. For a finite subset Φ of $\mathbb{C}$, it holds that

$$\sum_{\rho\in\Phi} c_\rho t^\rho \in \mathcal{A}_{\max_{\rho\in\Phi}|\rho|} \quad (c_\rho \in \mathbb{C}\setminus\{0\}),$$

since $t^\rho \in \mathcal{A}_{|\rho|}$ by Example 4.1.3. Moreover, it holds that $t\log t \in \mathcal{A}_{1+\varepsilon}$ for any $\varepsilon > 0$, since $t \in \mathcal{A}_1$ and $\log t \in \mathcal{A}_\varepsilon$ by Example 4.1.3.

Example 4.1.6 (cf. Kurokawa [30, Theorem 2.1] [31, §5.2, p. 56])**.** Let $n_\pm \in \mathbb{N}_0$ and put $f_A(t) := \mathrm{Tr}(t^{A_+}) - \mathrm{Tr}(t^{A_-}) + a$, where $A = (A_+, A_-, a) \in M_{n_+}(\mathbb{C}) \times M_{n_-}(\mathbb{C}) \times \mathbb{C}$. Since it holds that

$$f_A(t) = \sum_{j=1}^{n_+} t^{\lambda_j^+} - \sum_{j=1}^{n_-} t^{\lambda_j^-} + a,$$

where $\lambda_1^\pm, \ldots, \lambda_{n_\pm}^\pm$ are eigenvalues of $A_\pm$, we have $f_A \in \mathcal{A}_{\max\{\rho(A_+),\ \rho(A_-)\}}$, where $\rho(A_\pm)$ is the spectral radius of $A_\pm$.

Remark 4.1.7 (Manin [36, p. 134])**.** Set $f_1(t) := (1 - t^{-1})^{-1}$. Then, it holds that $\zeta_{f_1}(s) = \frac{1}{\sqrt{2\pi}}\Gamma(s)$ as in Example 1.1.6. Here, the equality

$$f_1(t) = \frac{1}{1 - t^{-1}} = \sum_{n=0}^{\infty} t^{-n},$$

suggests that f_1 is derived from the trace of a trace class operator whose eigenvalues are all non-negative (or non-positive) integers. This is compatible with Manin's suggestion that $\zeta_{f_1}(s)$ is the zeta function of "the dual infinite dimensional projective space over $\mathbb{F}_1$".

By Example 4.1.6, it holds that

$$\zeta_{f_A}(s) = s^{-a} \frac{\det(sI_{n_-} - A_-)}{\det(sI_{n_+} - A_+)} = s^{-f_A(1)} \frac{\det(I_{n_-} - s^{-1}A_-)}{\det(I_{n_+} - s^{-1}A_+)}. \tag{4.1}$$

In joint work with Y. Hirakawa, we found that some functions which have been classically investigated such as congruent zeta functions and Ihara zeta functions are absolute zeta functions by using this example.

Let X be a smooth projective variety over $\mathbb{F}_q$. By the Weil conjecture (see e.g. Hartshorne [22, p. 450]), there exist a finite subset $\Phi(X/\mathbb{F}_q)$ of $\mathbb{C}$ and a map $m\colon \Phi(X/\mathbb{F}_q) \to \mathbb{Z}\setminus\{0\}$ such that

$$Z(X/\mathbb{F}_q, u) = \prod_{\lambda\in\Phi(X/\mathbb{F}_q)} (1 - \lambda u)^{-m(\lambda)}. \tag{4.2}$$

Setting

$$A_{X/\mathbb{F}_q}^{+} := \bigoplus_{\substack{\lambda \in \Phi(X/\mathbb{F}_q) \\ m(\lambda)>0}} \bigoplus_{j=1}^{m(\lambda)} (\lambda) \quad \text{and} \quad A_{X/\mathbb{F}_q}^{-} := \bigoplus_{\substack{\lambda \in \Phi(X/\mathbb{F}_q) \\ m(\lambda)<0}} \bigoplus_{j=1}^{-m(\lambda)} (\lambda),$$

where $\oplus$ is the direct sum of matrices. By Equation (4.1), we have the following.

Corollary 4.1.8. *Under the above notations, the congruent zeta function of X is expressed by an absolute zeta function as follows:*

$$Z(X/\mathbb{F}_q, q^{-s}) = q^{\chi_{\mathrm{top}}(X)s} \zeta_{f_{X/\mathbb{F}_q}}(q^s)$$

for $\mathrm{Re}(s) > \dim X$*, where* $\chi_{\mathrm{top}}(X)$ *is the Euler characteristic of* X *and*

$$f_{X/\mathbb{F}_q}(t) := f_{\left(A_{X/\mathbb{F}_q}^{+}, A_{X/\mathbb{F}_q}^{-}, 0\right)}(t) = \mathrm{Tr}\left(t^{A_{X/\mathbb{F}_q}^{+}}\right) - \mathrm{Tr}\left(t^{A_{X/\mathbb{F}_q}^{-}}\right) = \sum_{\lambda \in \Phi(X/\mathbb{F}_q)} m(\lambda) t^{\lambda}.$$

Proof. This follows from Equation (4.1), Equation (4.2) and the equality $f_{X/\mathbb{F}_q}(1) = \chi_{\mathrm{top}}(X)$, since

$$\sum_{\lambda \in \Phi(X/\mathbb{F}_q)} m(\lambda) = \sum_{i=1}^{2\dim X} \sum_{\substack{\lambda \in \Phi(X/\mathbb{F}_q) \\ |\lambda| = q^{\frac{i}{2}}}} (-1)^i = \sum_{i=1}^{2\dim X} (-1)^i b_i = \chi_{\mathrm{top}}(X),$$

where b_i is the i^{th}-Betti number of X. □

Let $G = (V, E)$ be a finite connected graph without degree-1 vertices. The *Ihara zeta function* of G is defined by

$$Z_G(u) := \prod_{P \in \mathrm{Prim}(G)} \left(1 - u^{l(P)}\right)^{-1},$$

where $\mathrm{Prim}(G)$ is the set of primes (equivalent classes of primitive closed paths) in G and $l(P)$ is the length of a path P (see e.g. Terras [47, Definition 2.2]). It is known that the Ihara zeta function has the following two-term determinant formula:

$$Z_G(u) = \det\left(I - uW_1\right)^{-1}, \tag{4.3}$$

where $W_1 \in M_{2|E|}(\mathbb{Z})$ is the edge adjacency matrix of G [4]. Let $\Phi(G)$ be the set of eigenvalues of W_1 and $m(\lambda)$ be the multiplicity of each eigenvalue λ of W_1. Then, we have the following by Equation (4.1).

Corollary 4.1.9. *Under the above notations, the Ihara zeta function of G is expressed by an absolute zeta function as follows:*

$$Z_G(u) = u^{-\chi_{\mathrm{top}}(G)} \zeta_{f_G}(u^{-1})$$

for $\mathrm{Re}(u^{-1}) > \max_{\lambda\in\Phi(G)} |\lambda|$*, where* $\chi_{\mathrm{top}}(G) = |V| - |E|$ *is the Euler characteristic of* G *and*

$$f_G(t) := f_{(W_1,0,|V|-3|E|)}(t) = \mathrm{Tr}(t^{W_1}) + (|V| - 3|E|) = \sum_{\lambda\in\Phi(G)} m(\lambda)t^{\lambda} + (|V| - 3|E|)\,.$$

Proof. This follows from Equation (4.3), Equation (4.1) and the equality $f_G(1) = \chi_{\mathrm{top}}(G)$. □

Example 4.1.10. Let $z \in \mathbb{C}$, $a \in \mathbb{C} \setminus \mathbb{Z}_{<0}$. Then, we put

$$\Phi^{\exp}(z, a, s) := \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!\,(a+n)^s}.$$

This function converges locally uniformly absolutely for each $s \in \mathbb{C}$. Note that it was introduced by Hardy [21] and Boyadzhiev [5], who called it a *polyexponential*. Let $k \in \mathbb{N}$ and

$$P_k(t) := -t\Phi^{\exp}(-\log t, 0, k+1) = -t\sum_{j=1}^{\infty} \frac{(-\log t)^j}{j!\,j^k}.$$

By [2, p. 589] (the special case of Hoffman's identity [24, Theorem 4.2]), we have

$$-\sum_{j=1}^{\infty}(-1)^j\binom{n}{j}\frac{1}{j^k} = \sum_{0<j_1\le\cdots\le j_k\le n}\frac{1}{j_1\cdots j_k} =: H_n^{\star}(\{1\}^k),$$

where $H_n^{\star}(s_1,\ldots,s_k)$ is the multiple harmonic star sum. Thus, it holds that

$$\begin{aligned} P_k(e^x) &= -\left(\sum_{m=0}^{\infty}\frac{x^m}{m!}\right)\left(\sum_{j=1}^{\infty}\frac{(-x)^j}{j!\,j^k}\right) = \sum_{m=0}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}x^{m+j}}{m!\,j!\,j^k} \\ &= \sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}x^n}{(n-j)!\,j!\,j^k} = \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{j=1}^{\infty}(-1)^{j+1}\binom{n}{j}\frac{1}{j^k} = \sum_{n=0}^{\infty}\frac{x^n}{n!}H_n^{\star}(\{1\}^k). \end{aligned}$$

Since $H_n^{\star}(\{1\}^k) \le (H_n)^k = O((\log n)^k)$, where $H_n := \sum_{k=1}^{n}\frac{1}{k}$ is the n-th harmonic number, it holds that $P_k \in \mathcal{A}_{1+\varepsilon}$ for any $\varepsilon > 0$.

4.2 Series expansion at the infinity

The absolute zeta function $\zeta_f(s)$ of $f \in \mathcal{A}$ is defined if $\mathrm{Re}(s) \gg 1$ by definition. The following theorem gives the explicit region in which the absolute zeta function is defined. Also, its analytic continuation enables us to discuss whether a functional equation for $\zeta_f(s)$ holds or not. Moreover, this is a generalisation of Proposition 4.0.1.

Theorem 4.2.1. *Let $d > 0$ and $f \in \mathcal{A}_d$.*

(1) For $\mathrm{Re}(s) > d$, the absolute zeta function of f is

$$\zeta_f(s) = s^{-f(1)} \exp\left(\sum_{n=1}^{\infty} \frac{\alpha_n(f)}{ns^n}\right).$$

In particular, this can be analytically continued to a single-valued holomorphic function on $\{s \in \mathbb{C} \mid |s| > d\}$. Moreover, we have

$$\lim_{s\to\infty} s^{f(1)}\zeta_f(s) = 1.$$

(2) For $\mathrm{Re}(s) > d$, the logarithmic derivative of the absolute zeta function $\zeta_f(s)$ is

$$\frac{\zeta_f'(s)}{\zeta_f(s)} = -\sum_{n=1}^{\infty} \alpha_{n-1}(f)s^{-n}.$$

In particular, this can be analytically continued to a single-valued holomorphic function on $\{s \in \mathbb{C} \mid |s| > d\}$.

Proof. In this proof, we abbreviate $\alpha_n(f)$ to α_n.
(1) It holds that

$$f(e^x) = \sum_{n=0}^{\infty} \frac{\alpha_n x^n}{n!}$$

by the definition of α_n. Since $f \in \mathcal{A}_d$, there exists $C > 0$ such that $|\alpha_n| \le Cd^n$ for each $n \in \mathbb{N}_0$. For $\mathrm{Re}(s) > d$ and $\mathrm{Re}(w) > 0$, it holds that

$$\begin{aligned}\int_0^\infty \left|\sum_{n=0}^{\infty} \frac{\alpha_n x^n}{n!} e^{-sx} x^{w-1}\right| dx &\le \int_0^\infty \sum_{n=0}^{\infty} \frac{Cd^n x^n}{n!} e^{-\mathrm{Re}(s)x} x^{\mathrm{Re}(w)} \frac{dx}{x} \\ &\le C\int_0^\infty e^{-(\mathrm{Re}(s)-d)x} x^{\mathrm{Re}(w)} \frac{dx}{x} = \frac{C\,\Gamma(\mathrm{Re}(w))}{(\mathrm{Re}(s)-d)^{\mathrm{Re}(w)}} < \infty.\end{aligned}$$

By Lebesgue's dominated convergence theorem, for $\mathrm{Re}(s) > d$ and $\mathrm{Re}(w) > 0$

$$Z_f(w,s) = \frac{1}{\Gamma(w)} \sum_{n=0}^{\infty} \frac{\alpha_n}{n!} \int_0^\infty e^{-sx} x^{n+w} \frac{dx}{x} = \frac{1}{\Gamma(w)} \sum_{n=0}^{\infty} \frac{\alpha_n}{n!} \frac{\Gamma(n+w)}{s^{n+w}} = \sum_{n=0}^{\infty} \frac{\alpha_n (w)_n}{n!\; s^{n+w}},$$

where $(w)_n := w(w+1)\cdots(w+n-1)$ is the Pochhammer symbol. If $|s| > d$ and $|w| < 1$, it holds that

$$\sum_{n=0}^{\infty} \left|\frac{\alpha_n (w)_n}{n!\; s^{n+w}}\right| \le \frac{Ce^{\mathrm{Im}(w)\arg(s)}}{|s|^{\mathrm{Re}(w)}} \sum_{n=0}^{\infty} \frac{d^n}{|s|^n} \prod_{m=0}^{n-1} \frac{|w+m|}{m+1} < \frac{Ce^{\mathrm{Im}(w)\arg(s)}}{|s|^{\mathrm{Re}(w)}} \cdot \frac{1}{1-\frac{d}{|s|}} < \infty$$

since $|w+m| < m+1$ for each $m \in \mathbb{N}_0$. Therefore, $Z_f(w,s)$ can be analytically continued to $|s|>d$, $\{w \mid \mathrm{Re}(w)>0\} \cup \{w \mid |w|<1\}$. Then, we have

$$\begin{aligned}\frac{\partial}{\partial w} Z_f(w,s) &= \frac{\partial}{\partial w}\left(\frac{\alpha_0}{s^w} + w\sum_{n=1}^{\infty} \frac{\alpha_n}{s^{n+w}} \prod_{m=1}^{n-1} \frac{w+m}{m+1}\right) \\ &= -\frac{\alpha_0}{s^w}\log s + \sum_{n=1}^{\infty} \frac{\alpha_n}{s^{n+w}} \prod_{m=1}^{n-1} \frac{w+m}{m+1} + w\frac{\partial}{\partial w}\left(\sum_{n=1}^{\infty} \frac{\alpha_n}{s^{n+w}} \prod_{m=1}^{n-1} \frac{w+m}{m+1}\right)\end{aligned}$$

for $|s|>d$ and $|w|<1$. Since

$$\sum_{n=0}^{\infty}\left|\frac{\alpha_n}{s^{n+w}} \prod_{m=1}^{n-1} \frac{w+m}{m+1}\right| \le \frac{Ce^{\mathrm{Im}(w)\arg(s)}}{|s|^{\mathrm{Re}(w)}} \sum_{n=0}^{\infty} \frac{d^n}{|s|^n} \prod_{m=0}^{n-1} \frac{|w+m|}{m+1} < \infty$$

for $|s|>d$ and $|w|<1$, it holds that

$$\left|\frac{\partial}{\partial w}\left(\sum_{n=1}^{\infty} \frac{\alpha_n}{s^{n+w}} \prod_{m=1}^{n-1} \frac{w+m}{m+1}\right)\right| < \infty$$

for $|s|>d$ and $|w|<1$. Since the derivative of the holomorphic function defined by a convergent series is given by the termwise differentiation and is particularly finite, we have

$$\log\zeta_f(s) = \frac{\partial}{\partial w} Z_f(w,s)\Big|_{w=0} = -\alpha_0\log s + \sum_{n=1}^{\infty} \frac{\alpha_n}{n!} s^{-n}(n-1)! = -\alpha_0 \log s + \sum_{n=1}^{\infty}\frac{\alpha_n}{n}s^{-n}$$

for $|s|>d$ and $|w|<1$. Note that this can be analytically continued to a single-valued holomorphic function on $\{s \in \mathbb{C} \mid |s|>d\} \setminus (-\infty, 0]$. Since $\log\left(s^{f(1)}\zeta_f(s)\right) = \log\zeta_f(s) + f(1)\log s$, it holds that

$$\log\left(s^{f(1)}\zeta_f(s)\right) = \sum_{n=1}^{\infty}\frac{\alpha_n}{n}s^{-n}$$

and it can be analytically continued to $\{s\in\mathbb{C} \mid |s|>d\}$. Thus, the claim follows.

Moreover, the statement $s^{f(1)}\zeta_f(s) \to 1$ $(s\to\infty)$ follows since

$$\left|\log\left(s^{f(1)}\zeta_f(s)\right)\right| = \left|\sum_{n=1}^{\infty}\frac{\alpha_n(f)}{ns^n}\right| \le \sum_{n=1}^{\infty}\frac{Cd^n}{n|s|^n} \le \frac{Cd}{|s|-d} \longrightarrow 0 \quad (s\to\infty).$$

(2) By (1), it holds that

$$\frac{\zeta_f'(s)}{\zeta_f(s)} = \frac{\partial}{\partial s}\log\zeta_f(s) = -\sum_{n=0}^{\infty}\alpha_n s^{-n-1}$$

for $\mathrm{Re}(s) > d$. Moreover, since

$$\sum_{n=1}^{\infty} \left|\alpha_{n-1} s^{-n}\right| = \sum_{n=1}^{\infty} |\alpha_{n-1}||s|^{-n} \leq \sum_{n=1}^{\infty} C d^{n-1} |s|^{-n} = \frac{C}{d} \frac{d|s|^{-1}}{1-d|s|^{-1}} = \frac{C}{|s|-d} < \infty$$

for $|s| > d$, the series $\sum_{n=1}^{\infty} \alpha_{n-1} s^{-n}$ converges absolutely for $|s| > d$. □

Example 4.2.2. Let Φ be a finite subset of $\mathbb{C}$ and put

$$f_\Phi(t) = \sum_{\rho \in \Phi} a_\rho t^\rho \quad (a_\rho \in \mathbb{Z}).$$

Since $f_\Phi \in \mathcal{A}_{\max_{\rho\in\Phi}|\rho|}$ by Example 4.1.5, it holds that

$$\frac{\zeta'_{f_\Phi}(s)}{\zeta_{f_\Phi}(s)} = -\sum_{n=1}^{\infty} \left(\sum_{\rho\in\Phi} a_\rho \rho^{n-1} \right) s^{-n}$$

for $\mathrm{Re}(s) > \max_{\rho\in\Phi}|\rho|$ by Theorem 4.2.1.

In particular, the counting function of an absolute Riemann surface of genus g

$$f_{\boldsymbol{\alpha}}(t) := t - 2\sqrt{t} \sum_{k=1}^{g} \cos(\alpha_k \log t) + 1 = t - \sum_{k=1}^{g} \left(t^{\rho_k} + t^{\overline{\rho_k}}\right) + 1,$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g) \in [0,\infty)^g$ and $\rho_k := \frac{1}{2} + \sqrt{-1}\alpha_k$, satisfies that

$$f_{\boldsymbol{\alpha}} \in \mathcal{A}_{\max\{1,|\rho_1|,\ldots,|\rho_g|\}}.$$

By Theorem 4.2.1, we have

$$\zeta_{f_{\boldsymbol{\alpha}}}(s) = \frac{1}{s^{2-2g}} \exp\left(\sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{g} (\rho_k^n + \overline{\rho_k}^n) \right) \frac{1}{n s^n} \right)$$

for $\mathrm{Re}(s) > \max\{1, |\rho_1|, \ldots, |\rho_g|\}$. Note that the Euler characteristic $2-2g$ of a Riemann surface of genus g appears in the exponent of $\frac{1}{s}$.

While the above examples are the series expansion of rational functions, the following functions have not been known as absolute zeta functions.

Example 4.2.3. By Example 4.1.10, it holds that $P_k(t) := -t\Phi^{\exp}(-\log t, 0, k+1) \in \mathcal{A}_{1+\varepsilon}$ for any $\varepsilon > 0$ and $P_k(1) = 0$. By the proof of Theorem 4.2.1, we have

$$\log \zeta_{P_k}(s) = \sum_{n=1}^{\infty} \frac{H_n^\star(\{1\}^k)}{n s^n} = \sum_{n=1}^{\infty} \sum_{0<j_1\leq\cdots\leq j_k\leq n} \frac{\left(\frac{1}{s}\right)^n}{j_1\cdots j_k n} = \sum_{|\boldsymbol{m}|=k+1} \mathrm{Li}_{\boldsymbol{m}}\left(\frac{1}{s}\right)$$

for $\mathrm{Re}(s) > 1$, where $\boldsymbol{m} \in \mathbb{N}^r$ is a multi-index, $|\boldsymbol{m}| := \sum m_i$ is its weight, and

$$\mathrm{Li}_{\boldsymbol{m}}(z) := \sum_{0<n_1<\cdots<n_r} \frac{z^{n_r}}{n_1^{m_1} \cdots n_r^{m_r}}$$

is the multiple polylogarithm. Therefore, the absolute zeta function of P_k is

$$\zeta_{P_k}(s) = \prod_{|\boldsymbol{m}|=k+1} \exp\left(\mathrm{Li}_{\boldsymbol{m}}\left(\frac{1}{s}\right)\right)$$

for $\mathrm{Re}(s) > 1$. Moreover, it holds that

$$\begin{aligned}
\frac{\zeta'_{P_k}(s)}{\zeta_{P_k}(s)} &= -\sum_{n=1}^{\infty} \frac{H_n^{\star}(\{1\}^k)}{s^{n+1}} = -\frac{1}{s} \sum_{0<j_1\leq\cdots\leq j_k\leq n} \frac{\left(\frac{1}{s}\right)^n}{j_1 \cdots j_k} \\
&= -\frac{1}{s} \sum_{0<j_1\leq\cdots\leq j_k} \frac{1}{j_1 \cdots j_k} \sum_{n=j_k}^{\infty} \left(\frac{1}{s}\right)^n = -\frac{1}{s} \sum_{0<j_1\leq\cdots\leq j_k} \frac{1}{j_1 \cdots j_k} \frac{\left(\frac{1}{s}\right)^{j_k}}{1-\frac{1}{s}} \\
&= -\frac{1}{s\left(1-\frac{1}{s}\right)} \sum_{0<j_1\leq\cdots\leq j_k} \frac{\left(\frac{1}{s}\right)^{j_k}}{j_1 \cdots j_k} = \frac{1}{1-s} \sum_{|\boldsymbol{m}|=k} \mathrm{Li}_{\boldsymbol{m}}\left(\frac{1}{s}\right).
\end{aligned}$$

Example 4.2.4. Put $f(t) = t \log t$. By Example 4.1.5, it holds that $f \in \mathcal{A}_{1+\varepsilon}$ for any $\varepsilon > 0$. Since $\alpha_0(f) = f(1) = 0$ and $\alpha_n(f) = n$ for each $n \in \mathbb{N}$, it holds that

$$\zeta_f(s) = \exp\left(\sum_{n=1}^{\infty} \frac{n}{ns^n}\right) = \exp\left(\frac{1}{s-1}\right) = \exp\left(\mathrm{Li}_0\left(\frac{1}{s}\right)\right).$$

This corresponds to "$\zeta_{P_{-1}}(s)$" in Example 4.2.3. In addition, if $f(t) = t-1$, then we have $\zeta_f(s) = \frac{s}{s-1} = \exp\left(-\log(1-\frac{1}{s})\right) = \exp\left(\mathrm{Li}_1(\frac{1}{s})\right)$. Thus, this corresponds to "$\zeta_{P_0}(s)$" in Example 4.2.3.

4.3 Integral formula in the sense of Connes and Consani

Connes and Consani [8] proved the following integral formula of an absolute zeta function $\zeta_f^{\lim}(s)$.

Theorem 4.3.1 (Connes and Consani [8, Lemma 2.1] [9, Lemma 4.10])**.** *Let $f\colon [1,\infty) \to \mathbb{R}$ be a continuous function satisfying that $|f(t)| \leq Ct^d$ for some $C > 0$ and $d \in \mathbb{N}$. Then, it holds that*

$$\frac{(\zeta_f^{\lim})'(s)}{\zeta_f^{\lim}(s)} = -\int_1^{\infty} f(t) t^{-s} \frac{dt}{t} \quad \left(= -\int_0^{\infty} f(e^x) e^{-sx} dx\right)$$

for $\mathrm{Re}(s) > d$*. Connes and Consani* [9] *called it the* integral formula.

Remark 4.3.2. By the definition of $\zeta_f^{\lim}(s)$, it holds that

$$\frac{(\zeta_f^{\lim})'(s)}{\zeta_f^{\lim}(s)} = \lim_{q\to 1}\left(-\frac{\partial}{\partial s}\sum_{m=1}^{\infty} f(q^m)\frac{q^{-ms}}{m}\right)$$

if we can interchange differentiation and integration. Note that Connes and Consani treated this as the definition of $\zeta_f^{\lim}(s)$. Thus, it is essential to prove

$$\lim_{q\to 1}\left(-\frac{\partial}{\partial s}\sum_{m=1}^{\infty} f(q^m)\frac{q^{-ms}}{m}\right) = \int_1^{\infty} f(t)t^{-s}\frac{dt}{t}$$

in order to prove Theorem 4.3.1.

The following corollary is an analogy of Connes-Consani's integral formula and is the corollary of Theorem 4.2.1.

Corollary 4.3.3. *Let $d > 0$ and $f \in \mathcal{A}_d$. Then, it holds that*

$$\frac{\zeta_f'(s)}{\zeta_f(s)} = -\int_0^{\infty} f(e^x)e^{-sx}dx = -\mathcal{L}[f\circ\exp](s)$$

for $\mathrm{Re}(s) > d$, where $\mathcal{L}$ is the Laplace transform.

Proof. Since $f \in \mathcal{A}_d$, there exists $C > 0$ such that $|\alpha_n(f)| \le Cd^n$ for $n \in \mathbb{N}_0$. Since

$$\begin{aligned}\left|\int_0^{\infty}\sum_{n=0}^{\infty}\frac{\alpha_n(f)x^n}{n!}e^{-sx}dx\right| &\le \int_0^{\infty}\sum_{n=0}^{\infty}\frac{Cd^nx^n}{n!}e^{-\mathrm{Re}(s)x}dx\\ &= C\int_0^{\infty}e^{-(\mathrm{Re}(s)-d)x}dx = \frac{C}{\mathrm{Re}(s)-d} < \infty\end{aligned}$$

for $\mathrm{Re}(s) > d$, it holds that

$$\begin{aligned}\frac{\zeta_f'(s)}{\zeta_f(s)} &= -\sum_{n=1}^{\infty}\alpha_{n-1}(f)s^{-n} = -\sum_{n=1}^{\infty}\alpha_{n-1}(f)\frac{1}{(n-1)!}\int_0^{\infty}e^{-sx}x^{n-1}dx\\ &= -\int_0^{\infty}e^{-sx}\sum_{n=1}^{\infty}\frac{\alpha_{n-1}(f)x^{n-1}}{(n-1)!}dx = -\int_0^{\infty}f(e^x)e^{-sx}dx\end{aligned}$$

by Theorem 4.2.1 (2) and Lebesgue's dominated convergence theorem. □

The following proposition compares the assumption of Connes-Consani's integral formula (Theorem 4.3.1) with that of Corollary 4.3.3.

Proposition 4.3.4. *Let $f \in C^{\omega}([1,\infty))$. If $f \in \mathcal{A}_d$, then there exists $C > 0$ such that $|f(t)| \le Ct^d$.*

Proof. Since $f \in \mathcal{A}_d$, there exists $C > 0$ such that $|\alpha_n(f)| \leq Cd^n$ for each $n \in \mathbb{N}_0$. Then, it holds that

$$|f(e^x)| = \left| \sum_{n=0}^{\infty} \frac{\alpha_n(f)}{n!} x^n \right| \leq C \sum_{n=0}^{\infty} \frac{(dx)^n}{n!} = Ce^{dx}$$

for $x \in [0, \infty)$. Hence, we have $|f(t)| \leq Ct^d$ by substituting $e^x = t$. $\square$

Remark 4.3.5. The converse of Proposition 4.3.4 does not hold. For example, let $\frac{1}{2} \leq a < 1$ and $f(t) = \frac{t}{t-a}$. Then, it holds that $|f(t)| \leq \frac{1}{1-a} t^{\varepsilon}$ for any $\varepsilon > 0$. However, the inequality $|\alpha_n(f)| \leq Cd^n$ cannot hold for any $C > 0$ and $d > 0$. Indeed, it holds that

$$f(e^x) = \sum_{m=0}^{\infty} a^m \sum_{n=0}^{\infty} \frac{(-mx)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \sum_{m=0}^{\infty} a^m m^n = \sum_{n=0}^{\infty} \frac{(-1)^n \operatorname{Li}_{-n}(a)}{n!} x^n,$$

where $\operatorname{Li}_m(z)$ is the polylogarithm, which is the multiple polylogarithm with $r = 1$ (cf. Example 4.2.3). Here, the order of $\operatorname{Li}_{-n}(a)$ is

$$\operatorname{Li}_{-n}(a) = \frac{1}{(1-a)^{n+1}} \sum_{k=0}^{n-1} A(n,k) a^{n-k} \geq \frac{a^n}{(1-a)^{n+1}} \sum_{k=0}^{n-1} A(n,k) \geq n!,$$

where $A(n,k)$ is the Eulerian number, the number of permutations in $\mathfrak{S}_n$ in which k elements are less than the previous elements. Thus, the absolute value of $\alpha_n(f)$ cannot be bound by Cd^n for any $C, d > 0$.

By Proposition 4.3.4, we can obtain the relation between an absolute zeta function of Soulé-Connes-Consani's type and that of Kurokawa's type.

Corollary 4.3.6. *Let $d > 0$ and $f \in \mathcal{A}_d$ be a real-valued function. Then, we have*

$$\frac{(\zeta_f^{\lim})'(s)}{\zeta_f^{\lim}(s)} = \frac{\zeta_f'(s)}{\zeta_f(s)}$$

for $\operatorname{Re}(s) > d$. In particular, there exists $C \in \mathbb{C} \setminus \{0\}$ such that $\zeta_f^{\lim}(s) = C\zeta_f(s)$.

Proof. By Proposition 4.3.4, we can define $\zeta_f^{\lim}(s)$ for $f \in \mathcal{A}_d$. Therefore, by Theorem 4.3.1 and Corollary 4.3.3, we have

$$\frac{\zeta_f'(s)}{\zeta_f(s)} = -\int_0^{\infty} f(e^x) e^{-sx} dx = \frac{(\zeta_f^{\lim})'(s)}{\zeta_f^{\lim}(s)}.$$

Moreover, since $\frac{\partial}{\partial s} \log \left(\zeta_f^{\lim}(s) \zeta_f(s)^{-1} \right) = 0$, there exists $C \in \mathbb{C} \setminus \{0\}$ such that $\zeta_f^{\lim}(s) = C\zeta_f(s)$. $\square$

By Corollary 4.3.6, we see that $\zeta_f^{\lim}(s)$ and $\zeta_f(s)$ "nearly" coincide with each other at least if $f \in \mathcal{A}$. Hence, Corollary 4.3.3 gives another proof of Connes-Consani's integral formula (Theorem 4.3.1).

We may think that $\zeta_f^{\lim}(s)$ and $\zeta_f(s)$ always coincide with each other if a function f satisfies the conditions of Definition 1.1.1 and Definition 1.1.2. However, this is not true as in the following example.

Example 4.3.7. Let $a > 0$ and $f, g \in \mathcal{A}$ be distinct two functions satisfying that $f(e^a) = g(e^a)$. Assume that $\frac{1}{s} f(e^{1/s})$ is not equal to $\int_0^\infty f(e^x)e^{-sx}dx$ for $\mathrm{Re}(s) > 0$. We define

$$F(t) := \begin{cases} f(t) & \text{if } 1 \le t \le e^a, \\ g(t) & \text{if } e^a \le t. \end{cases}$$

For example, the case where $a = 1$, $f(t) = -t + 2e$ and $g(t) = t$ is one of the examples. This continuous function is not included in $\mathcal{A}$. By Theorem 4.3.1, we have

$$\frac{(\zeta_F^{\lim})'(s)}{\zeta_F^{\lim}(s)} = -\int_0^\infty F(e^x)e^{-sx}dx = -\int_0^a f(e^x)e^{-sx}dx - \int_a^\infty g(e^x)e^{-sx}dx.$$

On the other hand, we calculate $\zeta_F(s)$. It holds that

$$\begin{aligned} Z_F(w,s) &= \frac{1}{\Gamma(w)}\left(\int_0^a \sum_{n=0}^\infty \frac{\alpha_n(f)}{n!} e^{-sx}x^{n+w}\frac{dx}{x} + \int_a^\infty \sum_{n=0}^\infty \frac{\alpha_n(g)}{n!} e^{-sx}x^{n+w}\frac{dx}{x}\right) \\ &= \sum_{n=0}^\infty \frac{1}{n!}\left(\frac{\alpha_n(f)}{\Gamma(w)}\int_0^a e^{-sx}x^{n+w}\frac{dx}{x} + \frac{\alpha_n(g)}{\Gamma(w)}\int_a^\infty e^{-sx}x^{n+w}\frac{dx}{x}\right). \end{aligned}$$

Here, we have

$$\begin{aligned} \frac{\partial}{\partial w}\left(\frac{1}{\Gamma(w)}\int_a^\infty e^{-sx}x^{n+w}\frac{dx}{x}\right) &= -\frac{\Gamma'(w)}{\Gamma(w)^2}\int_a^\infty e^{-sx}x^{n+w}\frac{dx}{x} \\ &\quad + \frac{1}{\Gamma(w)}\int_a^\infty e^{-sx}x^{n+w-1}\log x\, dx. \end{aligned}$$

Since $\Gamma(w) = \frac{1}{w} + O(1)$ $(w \to 0)$, we have

$$\left.\frac{\partial}{\partial w}\left(\frac{1}{\Gamma(w)}\int_a^\infty e^{-sx}x^{n+w}\frac{dx}{x}\right)\right|_{w=0} = \int_a^\infty e^{-sx}x^n\frac{dx}{x}.$$

Thus, we have

$$\begin{aligned} \left.\frac{\partial}{\partial w}\left(\frac{1}{\Gamma(w)}\int_0^a e^{-sx}x^{n+w}\frac{dx}{x}\right)\right|_{w=0} &= \left.\frac{\partial}{\partial w}\left(s^{-n-w} - \frac{1}{\Gamma(w)}\int_a^\infty e^{-sx}x^{n+w}\frac{dx}{x}\right)\right|_{w=0} \\ &= -\frac{\log s}{s^n} - \int_a^\infty e^{-sx}x^n\frac{dx}{x}. \end{aligned}$$

Therefore, it holds that

$$\begin{aligned}\log \zeta_F(s) &= \frac{\partial}{\partial w} Z_F(w,s)\Big|_{w=0} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\alpha_n(f) \left(-\frac{\log s}{s^n} - \int_a^{\infty} e^{-sx} x^n \frac{dx}{x} \right) + \alpha_n(g) \left(\int_a^{\infty} e^{-sx} x^n \frac{dx}{x} \right) \right) \\ &= -f(e^{1/s}) \log s - \int_a^{\infty} f(e^x) e^{-sx} \frac{dx}{x} + \int_a^{\infty} g(e^x) e^{-sx} \frac{dx}{x}.\end{aligned}$$

By differentiating $\log \zeta_F(s)$, we have

$$\begin{aligned}\frac{\zeta_F'(s)}{\zeta_F(s)} &= -\frac{1}{s} f(e^{1/s}) + \int_a^{\infty} f(e^x) e^{-sx} dx - \int_a^{\infty} g(e^x) e^{-sx} dx \\ &= -\frac{1}{s} f(e^{1/s}) + \int_0^{\infty} f(e^x) e^{-sx} dx + \frac{(\zeta_F^{\lim})'(s)}{\zeta_F^{\lim}(s)}.\end{aligned}$$

Since $\frac{1}{s} f(e^{1/s})$ is not equal to $\int_0^{\infty} f(e^x) e^{-sx} dx$, we have

$$\frac{(\zeta_F^{\lim})'(s)}{\zeta_F^{\lim}(s)} \neq \frac{\zeta_F'(s)}{\zeta_F(s)}.$$

This implies that $\zeta_F^{\lim}(s) \neq \zeta_F(s)$. Thus, this is an example of a function which satisfies the conditions of Definition 1.1.1 and Definition 1.1.2 but absolute zeta functions are different.

4.4 Absolute Euler products

According to the introduction of this chapter, each factor of the absolute Euler product of the absolute zeta function of a Laurent polynomial can be described by using the linear map $M_n \colon \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}$. In this section, we generalise this map and give the absolute Euler product of the absolute zeta function of $f \in \mathcal{A}$.

4.4.1 Absolute Euler product in the case of $f \in \mathcal{A}$

Definition 4.4.1. We define the map $\mathcal{M} \colon \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}^{\mathbb{N}_0}$ by $\mathcal{M}(a)_0 := a_0$ and

$$\mathcal{M}(a)_n := \frac{1}{n} \sum_{m|n} \mu\left(\frac{n}{m}\right) a_m \quad (n \in \mathbb{N})$$

for a sequence $a = \{a_n\}_{n=0}^{\infty}$ of complex numbers. In particular, we put

$$\kappa_\rho(n) := \mathcal{M}(\{\rho^n\}_{n=0}^{\infty})_n$$

for $\rho \in \mathbb{C}$ and $n \in \mathbb{N}$.

Remark 4.4.2. The map $\mathcal{M}\colon \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}^{\mathbb{N}_0}$ is a linear map.

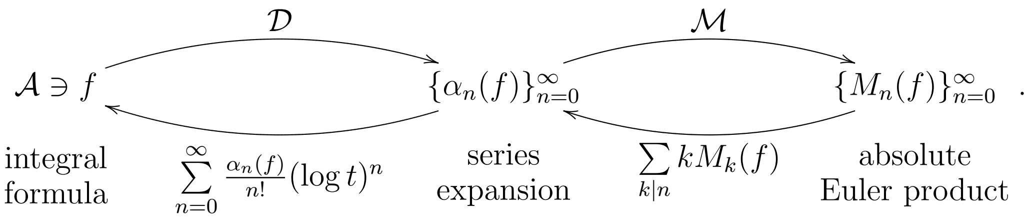
Definition 4.4.3. We define M_n by

$$\{M_n(f)\}_{n=0}^{\infty} := \mathcal{M}(\mathcal{D}(f))$$

for $f \in \mathcal{A}$, where $\mathcal{D}\colon \mathcal{A} \to \mathbb{C}^{\mathbb{N}_0}$ is defined by $\mathcal{D}(f) := \{\alpha_n(f)\}_{n=0}^{\infty}$.

Remark 4.4.4. Since $\mathcal{M}\colon \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}^{\mathbb{N}_0}$ and $\mathcal{D}\colon \mathcal{A} \to \mathbb{C}^{\mathbb{N}_0}$ is linear, the map $M_n\colon \mathcal{A} \to \mathbb{C}$ is linear.

We can summarise the relationship between a function f, the coefficient $\alpha_n(f)$ of the Taylor series of $f(e^x)$, and the linear map M_n into the following diagram:



Kurokawa's core formula, Lemma 3.2.1, can be generalised as follows. Note that $\log(1-z) = -\sum_{m=1}^{\infty} \frac{z^m}{m}$ for any $z \in \mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$.

Lemma 4.4.5. *It holds that*

$$\log(1-\rho u) = \sum_{n=1}^{\infty} \kappa_\rho(n) \log\left(1-u^n\right) \ \in \mathbb{Q}[\![\rho, u]\!].$$

Moreover, for $\rho \in \mathbb{C}$, the region of absolute convergence of the series in the right-hand side is

$$\begin{cases} \left\{u \in \mathbb{D} \;\middle|\; |u| < \frac{1}{|\rho|}\right\} & \text{if } \rho \neq 0, 1, \\ \mathbb{C} & \text{if } \rho = 0, \\ \mathbb{D} & \text{if } \rho = 1. \end{cases}$$

Proof. If $\rho = 0$, then both sides become 0. If $\rho = 1$, then both sides become $\log(1-u)$ since $\kappa_1(n) = \delta_{1n}$. We may assume that $\rho \neq 0, 1$. Then, the equality as formal series holds since

$$\begin{aligned} \sum_{n=1}^{\infty} \kappa_\rho(n) \log(1-u^n) &\overset{(*)}{=} -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{n\kappa_\rho(n)}{nk} u^{nk} \overset{(a)}{=} -\sum_{m=1}^{\infty} \frac{1}{m}\left(\sum_{n|m} n\kappa_\rho(n)\right) u^m \\ &= -\sum_{m=1}^{\infty} \frac{1}{m}\left(\sum_{n|m}\sum_{l|n} \mu\left(\frac{n}{l}\right)\rho^l\right) u^m \overset{(b)}{=} -\sum_{m=1}^{\infty} \frac{1}{m}\rho^m u^m \\ &= \log(1-\rho u) \end{aligned}$$

by substituting $m = nk$ in (a) and using the Möbius inversion formula in (b). Note that the equality $(*)$ holds for $u \in \mathbb{D}$.

Next, we show that the series $\sum_{n=1}^{\infty} \kappa_\rho(n) \log(1-u^n)$ converges absolutely for $|u| < \min\left\{\frac{1}{|\rho|}, 1\right\}$. It holds that

$$\sum_{n=1}^{\infty} |\kappa_\rho(n) \log(1-u^n)| \leq \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) \rho^d \right| \sum_{m=1}^{\infty} \frac{|u|^{nm}}{m}$$
$$\leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{|u|^{nm}}{nm} \sum_{d|n} |\rho|^d = \sum_{N=1}^{\infty} \frac{|u|^N}{N} \sum_{n|N} \sum_{d|n} |\rho|^d.$$

By [1, Theorem 13.12], there exists $n_0 \in \mathbb{N}$ such that $\sum_{d|n} 1 < n^{\frac{1}{2}}$ for each $n \geq n_0$. Hence, there exists $C > 0$ such that $\sum_{d|n} 1 < C\sqrt{n}$ for each $n \in \mathbb{N}$. Therefore, it holds that

$$\sum_{n|N} \sum_{d|n} |\rho|^d \leq (\max\{|\rho|, 1\})^N \sum_{n|N} \sum_{d|n} 1 < C (\max\{|\rho|, 1\})^N \sum_{n|N} \sqrt{n}$$
$$\leq C\sqrt{N} (\max\{|\rho|, 1\})^N \sum_{n|N} 1 < C^2 N (\max\{|\rho|, 1\})^N .$$

Thus, we have

$$\sum_{n=1}^{\infty} |\kappa_\rho(n) \log(1-u^n)| \leq C^2 \sum_{N=1}^{\infty} |u|^N (\max\{|\rho|, 1\})^N = \frac{C^2 |u| \max\{|\rho|, 1\}}{1 - |u| \max\{|\rho|, 1\}} < \infty$$

since $|u| \max\{|\rho|, 1\} < 1$. Therefore, the series $\sum_{n=1}^{\infty} \kappa_\rho(n) \log(1-u^n)$ converges absolutely for $|u| < \min\left\{\frac{1}{|\rho|}, 1\right\}$.

Next, we show that the series $\sum_{n=1}^{\infty} \kappa_\rho(n) \log(1-u^n)$ does not converge absolutely for $|u| \geq \min\left\{\frac{1}{|\rho|}, 1\right\}$. If $|\rho| \leq 1$, then the series $\log(1-u^n)$ does not converge absolutely since $|u| \geq 1$. We may assume that $|\rho| > 1$. It is sufficient to show that the series does not converge absolutely for $\frac{1}{|\rho|} \leq |u| < 1$. By [1, Theorem 13.12], there exists $n_1 \in \mathbb{N}$ such that $\sum_{d|n} 1 < \frac{1}{2}|\rho|^{\frac{n}{2}}$ for $n \geq n_1$. Therefore, it holds that

$$|n\kappa_\rho(n)| \geq |\rho|^n - \left| \sum_{n \neq d|n} \mu\left(\frac{n}{d}\right) \rho^d \right| \geq |\rho|^n - |\rho|^{\frac{n}{2}} \sum_{n \neq d|n} 1 > |\rho|^n - \frac{1}{2}|\rho|^n = \frac{|\rho|^n}{2}$$

for each $n \geq n_1$. Moreover, since $|u| < 1$ and $\frac{3}{2}x + \log(1-x) \geq 0$ at least for $0 < x \leq \frac{1}{3}$, it holds that

$$|\log(1-u^n)| \geq |u|^n - \sum_{m=2}^{\infty} \frac{|u|^{nm}}{m} \geq |u|^n - \sum_{m=2}^{\infty} \frac{|u|^m}{m} = 2|u|^n + \log(1-|u|^n) \geq \frac{|u|^n}{2}$$

for $n \geq n_2 := \frac{\log 3}{\log |u|^{-1}}$. Therefore, since $|\rho||u| \geq 1$, we have

$$\sum_{n=1}^{\infty} |\kappa_\rho(n) \log(1-u^n)| \geq \sum_{n \geq \max\{n_1, n_2\}} \frac{|\rho|^n}{2n} \cdot \frac{|u|^n}{2} \geq \frac{1}{4} \sum_{n \geq \max\{n_1, n_2\}} \frac{1}{n} = \infty.$$

□

By Theorem 4.2.1, we obtain the logarithmic version of an absolute Euler product

$$s^{-M_0(f)} \prod_{n=1}^{\infty} (1-s^{-n})^{-M_n(f)}.$$

Theorem 4.4.6. *Let $d > 0$ and $f \in \mathcal{A}_d$. Then, the following statements hold.*

(1) The series

$$S_f(s) := \sum_{n=1}^{\infty} M_n(f) \log\left(1-s^{-n}\right)$$

converges absolutely for $|s| > \max\{d, 1\}$.

(2) It holds that

$$\log \zeta_f(s) = -f(1) \log s - S_f(s)$$

for $|s| > \max\{d, 1\}$.

Proof. (1) Since $|M_n(f)| \leq \frac{1}{n} \sum_{l|n} |\alpha_l(f)| \leq \frac{C}{n} \sum_{l|n} d^l \leq C(\max\{d, 1\})^n$, we have

$$\sum_{n=1}^{\infty} \left|M_n(f) \log\left(1-s^{-n}\right)\right| \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{|M_n(f)|}{m|s|^{nm}} \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{|M_n(f)|}{|s|^{n+m-1}}$$
$$\leq \sum_{n=1}^{\infty} \frac{(\max\{d, 1\})^n}{|s|^n} \sum_{m=1}^{\infty} \frac{C}{|s|^{m-1}} = \frac{\frac{\max\{d,1\}}{|s|}}{1-\frac{\max\{d,1\}}{|s|}} \frac{C}{1-\frac{1}{|s|}} < \infty$$

for $|s| > \max\{d, 1\}$. Thus, the series $S_f(s)$ converges absolutely in the region.

(2) By Theorem 4.2.1, it holds that

$$\frac{\zeta_f'(s)}{\zeta_f(s)} = -\sum_{N=1}^{\infty} \frac{\alpha_{N-1}(f)}{s^N} = -\frac{1}{s} \sum_{N=0}^{\infty} \frac{\alpha_N(f)}{s^N}$$

for $\mathrm{Re}(s) > d$. Note that the region of absolute convergence of this series is $\{s \in \mathbb{C} \mid |s| > d\}$. Since it holds that

$$\sum_{n|N} n M_n(f) = \sum_{n|N} \sum_{m|n} \mu\left(\frac{n}{m}\right) \alpha_m(f) = \alpha_N(f)$$

for each $N \in \mathbb{N}$ by the Möbius inversion formula, we have

$$\begin{aligned}\frac{\zeta_f'(s)}{\zeta_f(s)} &= -\frac{1}{s}\left(\alpha_0(f) + \sum_{N=1}^{\infty} \frac{1}{s^N} \sum_{n|N} nM_n(f)\right) = -\frac{1}{s}\left(\alpha_0(f) + \sum_{n=1}^{\infty} nM_n(f) \sum_{m=1}^{\infty} \frac{1}{s^{nm}}\right) \\ &= -\frac{1}{s}\left(\alpha_0(f) + \sum_{n=1}^{\infty} \frac{nM_n(f)}{s^n - 1}\right) = -\frac{\alpha_0(f)}{s} - \sum_{n=1}^{\infty} \frac{nM_n(f)}{s^{n+1} - s}.\end{aligned}$$

Since $\frac{d}{ds}\log(1 - s^{-n}) = \frac{ns^{-n-1}}{1-s^{-n}} = \frac{n}{s^{n+1}-s}$, we have

$$\begin{aligned}\frac{\zeta_f'(s)}{\zeta_f(s)} &= -\frac{\alpha_0(f)}{s} - \sum_{n=1}^{\infty} M_n(f) \frac{d}{ds} \log(1 - s^{-n}) \\ &= \frac{d}{ds}\left(-\alpha_0(f)\log s - \sum_{n=1}^{\infty} M_n(f) \log\left(1 - s^{-n}\right)\right)\end{aligned}$$

for $|s| > \max\{d, 1\}$. Therefore, it holds that

$$\frac{d}{ds}\left(-\log\zeta_f(s) - \alpha_0(f)\log s - \sum_{n=1}^{\infty} M_n(f)\log\left(1 - s^{-n}\right)\right) = 0.$$

Then, there exists a constant $C \in \mathbb{C}$ which is independent of s such that

$$\log\zeta_f(s) = -f(1)\log s - \sum_{n=1}^{\infty} M_n(f)\log\left(1 - s^{-n}\right) + C$$

since $\alpha_0(f) = f(1)$. As $s \to \infty$ in $s \notin \mathbb{D}$, we have

$$C = \lim_{\substack{s\to\infty \\ s\notin\mathbb{D}}} (\log\zeta_f(s) + f(1)\log s) = \lim_{\substack{s\to\infty \\ s\notin\mathbb{D}}} \log\left(s^{f(1)}\zeta_f(s)\right) = 0$$

by Theorem 4.2.1 (1). □

4.4.2 Examples of absolute Euler products

Example 4.4.7. Let Φ be a finite subset of $\mathbb{C}$ and put $d(\Phi) := \max_{\rho\in\Phi} |\rho|$ and

$$f_\Phi(t) = \sum_{\rho\in\Phi} a_\rho t^\rho \in \mathcal{A}_{d(\Phi)} \quad (a_\rho \in \mathbb{Z}).$$

Then, it holds that $M_0(f_\Phi) = f_\Phi(1) = \sum_{\rho\in\Phi} a_\rho$ and

$$M_n(f_\Phi) = \sum_{\rho\in\Phi} a_\rho \kappa_\rho(n)$$

for $n \in \mathbb{N}$. By Theorem 4.4.6, we have the absolute Euler product

$$\zeta_{f_\Phi}(s) = \prod_{\rho \in \Phi} (s-\rho)^{-a_\rho} = s^{-\sum_{\rho\in\Phi} a_\rho} \prod_{n=1}^{\infty} \left(1-s^{-n}\right)^{-\sum_{\rho\in\Phi} a_\rho \kappa_\rho(n)}$$

for $|s| > \max\{d(\Phi), 1\}$.

Example 4.4.8. Let $k \in \mathbb{N}$. By Example 4.1.10, it holds that

$$P_k(t) := -t\Phi^{\exp}(-\log t, 0, k+1) \in \mathcal{A}_{1+\varepsilon}$$

for any $\varepsilon > 0$ and $M_0(P_k) = P_k(1) = 0$. By Theorem 4.4.6, we have the absolute Euler product

$$\zeta_{P_k}(s) = \prod_{|\boldsymbol{m}|=k+1} \exp\left(\mathrm{Li}_{\boldsymbol{m}}\left(\frac{1}{s}\right)\right) = \prod_{n=1}^{\infty} (1-s^{-n})^{-M_n(P_k)}$$

for $\mathrm{Re}(s) > 1$, where

$$M_n(P_k) = \frac{1}{n}\sum_{m|n} \mu\left(\frac{n}{m}\right) H_m^{\star}(\{1\}^k) = \frac{1}{n}\sum_{m|n}\mu\left(\frac{n}{m}\right) \sum_{0<j_1\le\cdots\le j_k\le m} \frac{1}{j_1\cdots j_k}.$$

Example 4.4.9. Put $f(t) = t\log t$. By Example 4.1.5, it holds that $f \in \mathcal{A}_{1+\varepsilon}$ for any $\varepsilon > 0$. Since $f(e^x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$, it holds that $M_0(f) = f(1) = 0$ and

$$M_n(f) = \frac{1}{n}\sum_{m|n}\mu\left(\frac{n}{m}\right) m = \sum_{m|n}\mu\left(\frac{n}{m}\right)\left(\frac{n}{m}\right)^{-1} = \prod_{\substack{p\in\mathbb{P} \\ p|n}} (1-p^{-1}).$$

Note that this is the "truncated" Euler product. By Theorem 4.4.6, we have the absolute Euler product

$$\zeta_f(s) = \exp\left(\mathrm{Li}_0\left(\frac{1}{s}\right)\right) = \prod_{n=1}^{\infty}\left(1-s^{-n}\right)^{-\prod_{p|n}(1-p^{-1})}.$$

As we mentioned in Subsection 1.1.2, the validity of the name "absolute Euler product" has not been explained in previous work. In joint work with Y. Hirakawa, we found one of the reasons why the name is suitable using the Euler products of the congruent zeta function and the Ihara zeta function as follows.

As we mentioned in Subsection 1.1.2, it is well-known that the congruent zeta function of a smooth projective variety X over $\mathbb{F}_q$ has the following Euler product representation for $\mathrm{Re}(s) > \dim X$:

$$Z(X/\mathbb{F}_q, q^{-s}) = \prod_{x\in|X|}\left(1-\#k(x)^{-s}\right)^{-1} = \prod_{n=1}^{\infty}(1-q^{-sn})^{-\kappa(p,n;X)},$$

where $|X|$ is the set of closed points of X and $k(x)$ is the residue field of $x \in X$ and $\kappa(p,n;X)$ is the number of the closed points of X of degree n. Since $f_{X/\mathbb{F}_q} \in \mathcal{A}$ by Example 4.1.6, we have

$$Z(X/\mathbb{F}_q, q^{-s}) = q^{\chi_{\mathrm{top}}(X)s}\zeta_{f_{X/\mathbb{F}_q}}(q^s) = q^{(\chi_{\mathrm{top}}(X)-f_{X/\mathbb{F}_q}(1))s}\prod_{n=1}^{\infty}(1-q^{-sn})^{-M_n(f_{X/\mathbb{F}_q})}$$

by Corollary 4.1.8 and Theorem 4.4.6. Therefore, this absolute Euler product coincides with the Euler product by the following theorem.

Theorem 4.4.10. *Let X be a smooth projective variety over $\mathbb{F}_q$. Under the same notation in corollary 4.1.8, it holds that*

$$\chi_{\mathrm{top}}(X) = f_{X/\mathbb{F}_q}(1) = n_+ - n_-$$

and

$$\kappa(p,n;X) = M_n(f_{X/\mathbb{F}_q}) = \mathrm{Tr}\left(A^+_{X/\mathbb{F}_q}\right)^n - \mathrm{Tr}\left(A^-_{X/\mathbb{F}_q}\right)^n.$$

Proof. The equality $\chi_{\mathrm{top}}(X) = f_{X/\mathbb{F}_q}(1)$ follows from the proof of Corollary 4.1.8. We now show that $\kappa(p,n;X) = M_n(f_{X/\mathbb{F}_q})$. By the Lefschetz trace formula, it holds that

$$\alpha_m(f_{X/\mathbb{F}_q}) = \sum_{\lambda\in\Phi(X/\mathbb{F}_q)} m(\lambda)\lambda^m = \#X(\mathbb{F}_{q^m})$$

for any $m \in \mathbb{N}$. By the Möbius inversion formula, it holds that

$$\begin{aligned} nM_n(f_{X/\mathbb{F}_q}) &= \sum_{m|n}\mu\left(\frac{n}{m}\right)\#X(\mathbb{F}_{q^m}) = \sum_{m|n}\mu\left(\frac{n}{m}\right)\sum_{l|m} l\kappa(l,n;X) \\ &= \sum_{l|n} l\kappa(l,n;X)\sum_{\frac{m}{l}|\frac{n}{l}}\mu\left(\frac{n}{m}\right) = n\kappa(p,n;X) \end{aligned}$$

for any $n \in \mathbb{N}$. □

As we mentioned in Section 4.1, the Ihara zeta function of a finite connected graph G without degree 1 is defined as the infinite product running over primes in G, which corresponds to prime numbers in an Euler product. Hence, we may regard this infinite product as one of the Euler products. This product can be expressed by

$$Z_G(u) := \prod_{P\in\mathrm{Prim}(G)}\left(1-u^{l(P)}\right)^{-1} = \prod_{n=1}^{\infty}(1-u^{-n})^{-\kappa(n;G)},$$

where $\mathrm{Prim}(G)$ is the set of primes in G, the integer $l(P)$ is the length of a path P and the integer $\kappa(n;G)$ is the number of primes in G of length n. Since $f_G \in \mathcal{A}$ by Example 4.1.6, we have

$$Z_G(u) = u^{-\chi(G)}\zeta_{f_G}(u^{-1}) = u^{-\chi(G)+f_G(1)}\prod_{n=1}^{\infty}(1-u^{-sn})^{-M_n(f_G)}$$

by Corollary 4.1.8 and Theorem 4.4.6. Therefore, this absolute Euler product coincides with the above Euler product by the following theorem.

Theorem 4.4.11. *Let G be a finite connected graph without degree-1 vertices. Under the same notation in corollary 4.1.9, it holds that*

$$\chi(G) = f_G(1) = |V| - |E|$$

and

$$\kappa(n; G) = M_n(f_G) = \mathrm{Tr}\,(W_1)^n .$$

Proof. The equality $\chi(G) = f_G(1)$ follows from the proof of corollary 4.1.9. We now show that $\kappa(n; G) = M_n(f_G)$. It suffices to show that

$$\mathrm{Tr}(W_1)^m = \sum_{l|m} l\kappa(l; G)$$

due to the similar calculation in the proof of theorem 4.4.10. Let N_m be the number of closed paths of length m without backtracking or tails in G (cf. Terras [47, Definition 4.2]). Then, we have

$$\sum_{l|m} l\kappa(l; G) = N_m = \mathrm{Tr}(W_1)^m$$

since it holds that $N_m = \mathrm{Tr}(W_1)^m$ (see e.g. Terras [47, p. 30]). $\square$

By the above examples, the absolute Euler products of the congruent zeta function and the Ihara zeta function coincide with their respective Euler products. Therefore, it is reasonable to call the infinite product the "absolute Euler product" or the $\mathbb{F}_1$-theoretic Euler product from these points of view.

4.4.3 Region of absolute convergence of a certain absolute Euler product

Let $f \in \mathcal{A}$ and put

$$d(f) := \inf\{d > 0 \mid f \in \mathcal{A}_d\}.$$

In Theorem 4.4.6, the series

$$S_f(s) := \sum_{n=1}^{\infty} M_n(f) \log\left(1 - s^{-n}\right)$$

converges absolutely at least for $|s| > \max\{d(f), 1\}$. We want to extend the region where $S_f(s)$ converges absolutely as wide as possible. Thus, we wonder if the region $|s| > \max\{d(f), 1\}$ is best possible or not.

According to [49, Corollary 3.10] (Corollary 3.1.7), if f is a polynomial, it holds that $d(f) = \deg f$ and the region of absolute convergence of $S_f(s)$ is $\{s \in \mathbb{C} \mid |s| > \deg f\}$. Like this case, we want to obtain the region of absolute convergence even in the case of $f \in \mathcal{A}$ and describe it by using information of f. The following theorem is an answer to this problem.

Theorem 4.4.12. *Let Φ be a nonempty finite subset of $\mathbb{C}$ and put $d(\Phi) := \max_{\rho\in\Phi} |\rho|$. Set $\lambda = \lambda_\Phi := \max\{d(\Phi), 1\}$ and*

$$f_\Phi(t) := \sum_{\rho\in\Phi} c_\rho t^\rho \in \mathcal{A}_{d(\Phi)} \quad (c_\rho \in \mathbb{C}\setminus\{0\}).$$

Put $\Phi_{\max} := \{\rho\in\Phi \mid |\rho| = \lambda\} =: \{\rho_1,\dots,\rho_l\}$ and $\theta_k := \frac{\operatorname{Arg}\rho_k}{2\pi}$. Assume the following technical conditions by sorting the order of $\rho_1,\dots,\rho_l$ appropriately:

(1) There exists $0 \le r \le l$ such that $\theta_1,\dots,\theta_r \in \mathbb{Q}$ and $\theta_{r+1},\dots,\theta_l \notin \mathbb{Q}$.

(2) There exists $0 \le h \le \frac{l-r}{2}$ such that $\theta_{l-h+1} = -\theta_{l-2h+1},\dots,\theta_l = -\theta_{l-h}$.

(3) $1,\theta_{r+1},\dots,\theta_{l-h}$ are linearly independent over $\mathbb{Q}$.

(4) $c_{\rho_{l-h+1}},\dots,c_{\rho_l} \in \mathbb{R}$.

Then, the region of absolute convergence of $S_{f_\Phi}(s)$ is $\{s\in\mathbb{C} \mid |s| > \lambda_\Phi\}$.

$$\overbrace{\theta_1,\dots,\theta_r}^{\in\mathbb{Q}},\ \overbrace{\underbrace{\theta_{r+1},\dots,\theta_{l-2h+1},\dots,\theta_{l-h}}_{\&\ 1\text{ are linearly independent over }\mathbb{Q}},\ \underbrace{\theta_{l-h+1}}_{=-\theta_{l-2h+1}},\dots,\underbrace{\theta_l}_{=-\theta_{l-h}}}^{\notin\mathbb{Q}}$$

Figure 4.1: The conditions (1) to (3) of Theorem 4.4.12

Remark 4.4.13. The integer r is the number of θ_j's which are rational. The integer h is the number of θ_j's whose opposite signs exist in $\{\theta_{r+1},\dots,\theta_l\}$. If $r = l$, then all $\theta_1,\dots,\theta_l$ are rational. If $r = 0$, then all $\theta_1,\dots,\theta_l$ are irrational. If $h = 0$, then the conditions (2) and (4) are omitted.

Remark 4.4.14. We can treat the counting function of an absolute Riemann surface in this theorem. In regarding its absolute zeta function as "the finite version of the Riemann zeta function" as we mentioned in Example 1.1.5, the real numbers $1, \operatorname{Im}\rho_1,\dots,\operatorname{Im}\rho_l$ are expected to be linearly independent over $\mathbb{Q}$, which is called the Grand Simplicity Hypothesis (see e.g. Rubinstein and Sarnak [41, p. 176]). However, the claim that $1, \operatorname{Im}\rho_1,\dots,\operatorname{Im}\rho_l$ are linearly independent over $\mathbb{Q}$ is not equivalent to the condition that $2\pi, \operatorname{Arg}\rho_1,\dots,\operatorname{Arg}\rho_l$ are linearly independent over $\mathbb{Q}$. Indeed, if $\rho = re^{\frac{2\pi\sqrt{-1}}{6}} \in \Phi_{\max}$, then $\operatorname{Arg}\rho = \frac{\pi}{3}$ and $\operatorname{Im}\rho = \frac{\sqrt{3}}{2}$.

The following lemma is essential to prove Theorem 4.4.12. We prove this lemma in Subsection 4.4.4.

Lemma 4.4.15. *Put $e(\theta) := e^{2\pi\sqrt{-1}\theta}$. Let $l \ge 1$, $\theta_1,\dots,\theta_l \in \mathbb{R}$ and $c_1,\dots,c_l \in \mathbb{C}$. Assume the following technical conditions which are the same as those in Theorem 4.4.12:*

(1) There exists $0 \leq r \leq l$ *such that* $\theta_1, \ldots, \theta_r \in \mathbb{Q}$ *and* $\theta_{r+1}, \ldots, \theta_l \notin \mathbb{Q}$.

(2) There exists $0 \leq h \leq \frac{l-r}{2}$ *such that* $\theta_{l-h+1} = -\theta_{l-2h+1}, \ldots, \theta_l = -\theta_{l-h}$.

(3) $1, \theta_{r+1}, \ldots, \theta_{l-h}$ *are linearly independent over* $\mathbb{Q}$.

(4) $c_{l-h+1}, \ldots, c_l \in \mathbb{R}$.

Then, if $(c_1, \ldots, c_l) \neq (0, \ldots, 0)$*, then there exists* $N \subset \mathbb{N}$ *whose natural density is positive such that*

$$\liminf_{n \in N} \left| \sum_{j=1}^{l} c_j e(n\theta_j) \right| > 0.$$

Remark 4.4.16. The condition (3) is an essential assumption. For example, we consider the case when $l = 2$, $r = 0$, $h = 0$, and $c_1 = -c_2 \neq 0$. If θ_1 and θ_2 satisfy that $1 + \theta_1 - \theta_2 = 0$, then $c_1 e(n\theta_1) + c_2 e(n\theta_2) = (c_1 + c_2) e(n\theta_1) = 0$. Thus, this is a counterexample of Lemma 4.4.15 without the condition (3).

Admitting the above key lemma, we prove Theorem 4.4.12.

Proof of Theorem 4.4.12. By Theorem 4.4.6, the series $S_{f_\Phi}(s)$ converges absolutely for $|s| > \lambda$. Now, we show that $S_{f_\Phi}(s)$ does not converge absolutely for $|s| \leq \lambda$. If $\lambda = 1$, that is, $|\rho| \leq 1$ for each $\rho \in \Phi$, then the series $\log(1 - s^{-n})$ does not converge absolutely for $|s| \leq \lambda = 1$. Thus, we may assume that $\lambda > 1$. It holds that

$$\sum_{n=1}^{\infty} \left| M_n(f_\Phi) \log\left(1 - s^{-n}\right) \right| = \sum_{n=1}^{\infty} \left| \sum_{\rho \in \Phi} c_\rho \frac{n\kappa_\rho(n)}{\lambda^n} \right| \left| \frac{\lambda^n}{n} \log\left(1 - s^{-n}\right) \right|.$$

Since $\lambda > 1$, the set $\Phi_{\max} = \{\rho \in \Phi \mid |\rho| = \lambda\}$ is not empty. It holds that

$$\begin{aligned}
&\left| \sum_{\rho \in \Phi} c_\rho \frac{n\kappa_\rho(n)}{\lambda^n} \right| \\
&\geq \left| \sum_{\rho \in \Phi_{\max}} c_\rho \left(\frac{\rho}{\lambda} \right)^n \right| - \left| \sum_{\rho \in \Phi_{\max}} c_\rho \sum_{n \neq m \mid n} \frac{\mu\left(\frac{n}{m}\right) \rho^m}{\lambda^n} \right| - \left| \sum_{\rho \in \Phi \setminus \Phi_{\max}} c_\rho \sum_{m \mid n} \frac{\mu\left(\frac{n}{m}\right) \rho^m}{\lambda^n} \right| \\
&\geq \left| \sum_{\rho \in \Phi_{\max}} c_\rho e^{\sqrt{-1} n \arg \rho} \right| - \sum_{\rho \in \Phi_{\max}} |c_\rho| \sum_{n \neq m \mid n} \frac{|\rho|^m}{\lambda^n} - \sum_{\rho \in \Phi \setminus \Phi_{\max}} |c_\rho| \sum_{m \mid n} \frac{|\rho|^m}{\lambda^n}.
\end{aligned}$$

Here, we put

$$R_n := \sum_{\rho \in \Phi_{\max}} |c_\rho| \sum_{n \neq m \mid n} \frac{|\rho|^m}{\lambda^n} + \sum_{\rho \in \Phi \setminus \Phi_{\max}} |c_\rho| \sum_{m \mid n} \frac{|\rho|^m}{\lambda^n}.$$

By Lemma 4.4.15, there exists $N \subset \mathbb{N}$ whose natural density is positive such that

$$\liminf_{n \in N} \left| \sum_{\rho \in \Phi_{\max}} c_\rho e^{\sqrt{-1} n \arg \rho} \right| > 0.$$

Therefore, there exist $M > 0$, $n_0 \in \mathbb{N}$ such that for each $n \in N \cap [n_0, \infty)$

$$\left| \sum_{\rho \in \Phi_{\max}} c_\rho e^{\sqrt{-1} n \arg \rho} \right| > M.$$

Since $R_n \to 0$ as $n \to \infty$, there exists $n_1 \in \mathbb{N}$ such that $R_n < \frac{1}{2} M$ for each $n \geq n_1$. Hence, it holds that

$$\left| \sum_{\rho \in \Phi} c_\rho \frac{n \kappa_\rho(n)}{\lambda^n} \right| > \frac{1}{2} M$$

for each $n \in N \cap [\max\{n_0, n_1\}, \infty)$. Thus, we have

$$\sum_{n=1}^{\infty} \left| M_n(f_\Phi) \log\left(1 - s^{-n}\right) \right| \geq \frac{1}{2} M \sum_{n \in N \cap [\max\{n_0, n_1\}, \infty)} \left| \frac{\lambda^n}{n} \log\left(1 - s^{-n}\right) \right| = \infty$$

by the similar argument of Lemma 4.4.5. $\square$

Example 4.4.17. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g) \in [0, \infty)^g$. Then, the counting function $f_{\boldsymbol{\alpha}}$ of an absolute Riemann surface of genus g (cf. Example 1.1.5) coincides with

$$f_\Phi(t) = \sum_{\rho \in \Phi} a_\rho t^\rho = t - \sum_{k=1}^{g} \left(t^{\rho_k} + t^{\overline{\rho_k}} \right) + 1,$$

where $\rho_k := \frac{1}{2} + \sqrt{-1} \alpha_k$, $\Phi := \{0, 1, \rho_1, \ldots, \rho_g, \overline{\rho_1}, \ldots, \overline{\rho_g}\}$ and $a_0 = a_1 = 1$, $a_\rho = -1$ for $\rho \in \Phi \setminus \{0, 1\}$. By Example 4.4.7, it holds that

$$\zeta_{f_{\boldsymbol{\alpha}}}(s) = s^{-(2-2g)} \left(1 - \frac{1}{s}\right)^{-1} \prod_{n=1}^{\infty} \left(1 - s^{-n}\right)^{\sum_{k=1}^{g} \left(\kappa_{\rho_k}(n) + \kappa_{\overline{\rho_k}}(n)\right)}$$

for $|s| > \max\{d(\Phi), 1\}$.

Put $\theta_k := \frac{1}{2\pi} \operatorname{Arg} \rho_k$ $(1 \leq k \leq 2g)$. If necessary, we sort the order of $\rho_1, \ldots, \rho_g$ so that $\theta_1, \ldots, \theta_r \in \mathbb{Q}$ and $\theta_{r+1}, \ldots, \theta_g \notin \mathbb{Q}$ for some $0 \leq r \leq g$. Put $\rho'_1 := 0$, $\rho'_2 := 1$, $\rho'_{k+2} := \rho_k$ $(1 \leq k \leq r)$, $\rho'_{k+r+2} := \overline{\rho_k}$ $(1 \leq k \leq r)$, $\rho'_{k+2r+2} := \rho_{k+r}$ $(1 \leq k \leq g - r)$, $\rho'_{k+g+r+2} := \overline{\rho_{k+r}}$ $(1 \leq k \leq g - r)$, and $\theta'_k := \frac{1}{2\pi} \operatorname{Arg} \rho'_k$ $(1 \leq k \leq 2g + 2)$. Then, it holds that $\theta'_{k+g+r+2} = -\theta'_{k+2r+2}$ $(1 \leq k \leq g - r)$. By Theorem 4.4.12, if $1, \theta'_{2r+3}, \ldots, \theta'_{g+r+2}$ are linearly independent over $\mathbb{Q}$, then the region of absolute convergence of $S_{f_{\boldsymbol{\alpha}}}(s)$ is $\{s \in \mathbb{C} \mid |s| > d(\Phi)\}$, where $d(\Phi) := \max\{1, |\rho_1|, \ldots, |\rho_g|\}$.

4.4.4 Proof of the key lemma

We use the following lemma to prove Lemma 4.4.15. Hereinafter, we put $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and identify the interval $\left(-\frac{1}{2}, \frac{1}{2}\right]$ with $\mathbb{T}$ under the natural bijection $\varphi\colon \left(-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{T}$. We define $\varphi_l\colon \left(-\frac{1}{2}, \frac{1}{2}\right]^l \to \mathbb{T}^l$ by $\varphi_l(x) := (\varphi(x_1), \ldots, \varphi(x_l))$. Let $\iota\colon \mathbb{R} \to \mathbb{T}$ be the natural surjection and put $\psi := \varphi^{-1} \circ \iota$.

Lemma 4.4.18. *Let $g := (\theta_1, \ldots, \theta_l) \in \mathbb{R}^l$ and define $R_g\colon \mathbb{T}^l \to \mathbb{T}^l$ by $R_g(x) := x + g$. Fix $0 < \delta < \frac{1}{2}$ and put $V_\delta := \{x \in \mathbb{T}^l \mid |\varphi_l^{-1}(x)| < \delta\}$. For $x \in \mathbb{T}^l$, put $N^g_{V_\delta}(x) := \{n \in \mathbb{N} \mid R_g^n(x) \in V_\delta\}$. If $1, \theta_1, \ldots, \theta_l$ are linearly independent over $\mathbb{Q}$, then it holds that*

$$d(N^g_{V_\delta}(x)) = v_l \delta^l$$

for any $x \in \mathbb{T}^l$, where $d(N^g_{V_\delta}(x))$ is the natural density of $N^g_{V_\delta}(x)$ and $v_l := \frac{\pi^{\frac{l}{2}}}{\Gamma\left(\frac{l}{2}+1\right)}$.

Proof of Lemma 4.4.18. Fix $x_0 \in \mathbb{T}^l$. It holds that

$$\begin{aligned} d(N^g_{V_\delta}(x_0)) &= \lim_{m\to\infty} \frac{1}{m} \#\{n \mid 1 \le n \le m,\ \mathbf{1}_{V_\delta}(R_g^n(x_0)) = 1\} \\ &= \lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} \mathbf{1}_{V_\delta}(R_g^n(x_0)), \end{aligned}$$

where $\mathbf{1}_{V_\delta}$ is the indicator function of V_δ. Hereinafter, we show that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} \mathbf{1}_{V_\delta}(R_g^n(x_0)) = v_l \delta^l$$

in a similar way to [18, Lemma 4.17]. For $0 < \eta < \delta$, we define the continuous function $f_\eta^\pm\colon \left(-\frac{1}{2}, \frac{1}{2}\right]^l \to [0,1]$ by

$$f_\eta^-(x) := \begin{cases} 1 & \text{if } |x| \le \delta - \eta, \\ \frac{\delta - |x|}{\eta} & \text{if } \delta - \eta \le |x| \le \delta, \\ 0 & \text{if } |x| \ge \delta, \end{cases}, \quad f_\eta^+(x) := \begin{cases} 1 & \text{if } |x| \le \delta, \\ \frac{(\delta+\eta) - |x|}{\eta} & \text{if } \delta \le |x| \le \delta + \eta, \\ 0 & \text{if } |x| \ge \delta + \eta. \end{cases}$$

Note that $f_\eta^-(\varphi_l^{-1}(x)) \le \mathbf{1}_{V_\delta}(x) \le f_\eta^+(\varphi_l^{-1}(x))$. Hence, it holds that

$$\frac{1}{m} \sum_{n=1}^{m} f_\eta^-(\varphi_l^{-1}(R_g^n(x))) \le \frac{1}{m} \sum_{n=1}^{m} \mathbf{1}_{V_\delta}(R_g^n(x)) \le \frac{1}{m} \sum_{n=1}^{m} f_\eta^+(\varphi_l^{-1}(R_g^n(x)))$$

for any $x \in \mathbb{T}^l$. Now, since $1, \theta_1, \ldots, \theta_l$ are linearly independent over $\mathbb{Q}$ by the assumption, the irrational rotation R_g is uniquely ergodic according to Einsiedler and Ward [18, Corollary 4.15]. Therefore, by [18, Theorem 4.10 (3)], it holds that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f_\eta^\pm(\varphi_l^{-1}(R_g^j(x_0))) = \int_{\mathbb{T}^l} f_\eta^\pm(\varphi_l^{-1}(x))dx = \int_{\left(-\frac{1}{2},\frac{1}{2}\right]^l} f_\eta^\pm(x)dx.$$

We have

$$\int_{|x|\leq M} dx = \int_0^M r^{l-1}dr \cdot \int_0^\pi \sin^{l-2}\alpha_1 d\alpha_1 \cdot \cdots \cdot \int_0^\pi \sin\alpha_{l-2}d\alpha_{l-2} \cdot \int_0^{2\pi} d\alpha_{l-1}$$
$$= \frac{M^l}{l} \cdot \frac{\Gamma\left(\frac{l-1}{2}\right)}{\Gamma\left(\frac{l}{2}\right)} \frac{\Gamma\left(\frac{l-2}{2}\right)}{\Gamma\left(\frac{l-1}{2}\right)} \cdots \frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)} \left(\sqrt{\pi}\right)^{l-2} \cdot 2\pi = \frac{M^l}{l} \cdot \frac{2\pi^{\frac{l}{2}}}{\Gamma\left(\frac{l}{2}\right)} = v_l M^l$$

for any $M \in (0,\infty)$ by converting to polar coordinates. In a similar calculation, we have

$$\int_{|x|\leq M} |x|dx = \frac{M^{l+1}}{l+1} \cdot l v_l$$

for any $M \in (0,\infty)$. Therefore, it holds that

$$\begin{aligned}
\int_{\left(-\frac{1}{2},\frac{1}{2}\right]^l} f_\eta^-(x)dx &= \int_{|x|\leq\delta} f_\eta^-(x)dx \\
&= \int_{|x|\leq\delta-\eta} dx + \frac{\delta}{\eta}\int_{\delta-\eta\leq|x|\leq\delta} dx - \frac{1}{\eta}\int_{\delta-\eta\leq|x|\leq\delta} |x|dx \\
&= v_l(\delta-\eta)^l + \frac{\delta}{\eta}\left(v_l\delta^l - v_l(\delta-\eta)^l\right) - \frac{1}{\eta}\frac{\delta^{l+1}-(\delta-\eta)^{l+1}}{l+1} \cdot l v_l \\
&= \frac{v_l}{\eta}\left(\delta^{l+1} - (\delta-\eta)^{l+1}\right)\left(1 - \frac{l}{l+1}\right) = v_l\delta^l + O(\eta).
\end{aligned}$$

Similarly, we obtain

$$\int_{\left(-\frac{1}{2},\frac{1}{2}\right]^l} f_\eta^+(x)dx = \frac{v_l}{\eta}\left((\delta+\eta)^{l+1} - \delta^{l+1}\right)\left(1 - \frac{l}{l+1}\right) = v_l\delta^l + O(\eta).$$

Therefore, it holds that

$$\begin{aligned}
v_l\delta^l + O(\eta) = \int_{\left(-\frac{1}{2},\frac{1}{2}\right]^l} f_\eta^-(x)dx &\leq \liminf_{m\to\infty} \frac{1}{m}\sum_{n=1}^m \mathbf{1}_{V_\delta}(R_g^n(x_0)) \\
&\leq \limsup_{m\to\infty} \frac{1}{m}\sum_{n=1}^m \mathbf{1}_{V_\delta}(R_g^n(x_0)) \leq \int_{\left(-\frac{1}{2},\frac{1}{2}\right]^l} f_\eta^+(x)dx = v_l\delta^l + O(\eta).
\end{aligned}$$

As $\eta \to +0$, we have

$$\liminf_{m\to\infty} \frac{1}{m}\sum_{n=1}^m \mathbf{1}_{V_\delta}(R_g^n(x_0)) = \limsup_{m\to\infty} \frac{1}{m}\sum_{n=1}^m \mathbf{1}_{V_\delta}(R_g^n(x_0)) = v_l\delta^l.$$

Thus, we have $d(N_{V_\delta}^g) = v_l\delta^l$. □

Now, we show Lemma 4.4.15, which is the key to prove Theorem 4.4.12.

Proof of Lemma 4.4.15. Put

$$S_n := \sum_{j=1}^{l} c_j e(n\theta_j).$$

We may assume that all $\theta_1, \dots, \theta_l$ are distinct and $\theta_j \in (-\frac{1}{2}, \frac{1}{2}]$ for all $j \in \{1, \dots, l\}$.

Assume that $r = l$, i.e. all $\theta_1, \dots, \theta_l$ are rational. Put $\mathrm{Den}\left(\frac{a}{b}\right) := b \in \mathbb{N}$ for an irreducible fraction $\frac{a}{b} \in \mathbb{Q}$ and $m = \gcd\left(\mathrm{Den}(\theta_1), \dots, \mathrm{Den}(\theta_l)\right)$. Putting $N_k := \{n \in \mathbb{N} \mid n \equiv k \mod m\}$ for $k \in \{0, \dots, m-1\}$, it holds that

$$S_n = \sum_{j=1}^{l} c_j e(k\theta_j)$$

for each $k \in \{0, \dots, m-1\}$ and $n \in N_k$. We show that $c_1 = \cdots = c_l = 0$ by induction, assuming that

$$\sum_{j=1}^{l} c_j e(k\theta_j) = 0$$

for all $k \in \{0, \dots, m-1\}$. If $l = 1$, then we have $c_1 = 0$. Let $l \geq 2$. Since

$$e(\theta_l) \sum_{j=1}^{l} c_j e(k\theta_j) = 0, \quad \sum_{j=1}^{l} c_j e((1+k)\theta_j) = 0$$

for each $k \in \{0, \dots, m-1\}$, the subtraction of both equations leads to the equality

$$\sum_{j=1}^{l-1} c_j \left(e(\theta_l) - e(\theta_j)\right) e(k\theta_j) = 0.$$

Since k is arbitrary, by the hypothesis of induction, it holds that

$$c_j \left(e(\theta_l) - e(\theta_j)\right) = 0$$

for $j \in \{1, \dots, l-1\}$. Since $\theta_l \neq \theta_j$ and $\theta_j \in (-\frac{1}{2}, \frac{1}{2}]$ for all $j \in \{1, \dots, l-1\}$, we have $c_1 = \cdots = c_{l-1} = 0$. Thus, we have $c_l = 0$ since $c_l e(k\theta_l) = 0$ for each $k \in \{0, \dots, m-1\}$.

Assume that $0 \leq r \leq l-1$. Put $m := \gcd\left(\mathrm{Den}(\theta_1), \dots, \mathrm{Den}(\theta_r)\right)$ and $x_j := \frac{1}{2\pi} \operatorname{Arg} c_j$ for $j \in \{r+1, \dots, l-2h\}$. For $n \in m\mathbb{N}$, put $n = mk$ for some $k \in \mathbb{N}$. Then, we have

$$\begin{aligned} |S_n| &= \left| \sum_{j=1}^{r} c_j + \sum_{j=r+1}^{l-2h} c_j e(km\theta_j) + \sum_{j=l-2h+1}^{l-h} \left(c_j e(km\theta_j) + c_{j+h} e(-km\theta_j)\right) \right| \\ &= \left| \sum_{j=1}^{r} c_j + \sum_{j=r+1}^{l-2h} |c_j| e(km\theta_j + x_j) + \sum_{j=l-2h+1}^{l-h} \left(c_j e(km\theta_j) + c_{j+h} e(-km\theta_j)\right) \right|. \end{aligned}$$

Put $c_0 := \sum_{j=1}^r c_j$, $\theta'_j := \psi(m\theta_j)$ for $j \in \{r+1, \dots, l-h\}$ and $g' := (\theta'_{r+1}, \dots, \theta'_{l-h})$. Then, it is sufficient to show that if there exists $k_0 \in \{0, r+1, r+2, \dots, l\}$ such that $c_{k_0} \neq 0$, then there exists a set $N \subset \mathbb{N}$ whose natural density is positive such that

$$\begin{aligned}\liminf_{n \in mN} |S_n| &= \liminf_{k \in N} |S'_k| \\ &:= \liminf_{k \in N} \left| c_0 + \sum_{j=r+1}^{l-2h} |c_j| e(k\theta'_j + x_j) + \sum_{j=l-2h+1}^{l-h} \left(c_j e(k\theta'_j) + c_{j+h} e(-k\theta'_j) \right) \right| > 0.\end{aligned}$$

Note that this includes the case when $r = 0$ by putting $c_0 = 0$.

If $c_0 = 0$ and $k_0 \in \{r+1, \dots, l-2h\}$, then we may assume that $k_0 = r+1$ by sorting the order of the labels. Since $c_{l-h+1}, \dots, c_l \in \mathbb{R}$ by the assumption, we have

$$\begin{aligned}&\operatorname{Re}\left(c_j e(k\theta'_j) + c_{j+h} e(-k\theta'_j) \right) \\ &\quad = \operatorname{Re}\left((c_j + c_{j+h}) e(k\theta'_j) - c_{j+h} \left(e(k\theta'_j) - e(-k\theta'_j) \right) \right) \\ &\quad = |c_j + c_{j+h}| \cos 2\pi (k\theta'_j + x_j) + 2|c_{j+h}| \sin \arg c_{j+h} \sin 2\pi k\theta'_j \\ &\quad = |c_j + c_{j+h}| \cos 2\pi (k\theta'_j + x_j),\end{aligned}$$

where $x_j := \frac{1}{2\pi} \operatorname{Arg}(c_j + c_{j+h})$ for $j \in \{l-2h+1, \dots, l-h\}$. Hence, it holds that

$$|S'_k| \geq \left| \sum_{j=r+1}^{l-2h} |c_j| \cos 2\pi (k\theta'_j + x_j) + \sum_{j=l-2h+1}^{l-h} |c_j + c_{j+h}| \cos 2\pi (k\theta'_j + x_j) \right|.$$

Put $x := \left(\iota(x_{r+1}), \iota(x_{r+2} - \frac{1}{4}), \dots, \iota(x_{l-h} - \frac{1}{4}) \right) \in \mathbb{T}^{l-h-r}$. Since $1, \theta'_{r+1}, \dots, \theta'_{l-h}$ are linearly independent over $\mathbb{Q}$ by the assumption, the natural density of $N^{g'}_{V_\delta}(x)$ is positive for any $\delta > 0$ by Lemma 4.4.18. Therefore, we have $|R^k_{g'}(x)| < \delta$ for each $k \in N^{g'}_{V_\delta}(x)$. Since

$$|\psi\left(k\theta'_{r+1} + x_{r+1} \right)| < \delta, \quad \left| \psi\left(k\theta'_j + x_j - \frac{1}{4} \right) \right| < \delta \quad (j \in \{r+1, \dots, l-h\})$$

for each $k \in N^{g'}_{V_\delta}(x)$, it holds that

$$|\cos 2\pi (k\theta'_{r+1} + x_{r+1})| > \cos\delta, \quad |\cos 2\pi (k\theta'_j + x_j)| = \left| \sin 2\pi \left(k\theta'_j + x_j - \frac{1}{4} \right) \right| < \sin\delta$$

for each $j \in \{r+1, \dots, l-h\}$. Since $|c_{r+1}| = |c_{k_0}| > 0$, it holds that for a sufficiently small $\delta > 0$ and any $k \in N^{g'}_{V_\delta}(x)$,

$$\begin{aligned}|S'_k| &\geq |c_{r+1}| \left| \cos 2\pi (k\theta'_{r+1} + x_{r+1}) \right| - \sum_{j=r+2}^{l-2h} |c_j| \left| \cos 2\pi (k\theta'_j + x_j) \right| \\ &\qquad - \sum_{j=l-2h+1}^{l-h} |c_j + c_{j+h}| \left| \cos 2\pi k (k\theta'_j + x_j) \right| \\ &> |c_{r+1}| \cos\delta - \left(\sum_{j=r+2}^{l-2h} |c_j| + \sum_{j=l-2h+1}^{l-h} |c_j + c_{j+h}| \right) \sin\delta > 0.\end{aligned}$$

Therefore, when $c_0 = 0$ and $k_0 \in \{r+1, \ldots, l-2h\}$, we have

$$\liminf_{k \in N^{g'}_{V_\delta}(x)} |S'_k| > 0.$$

If $c_1 = 0$ and $k_0 \in \{l-2h+1, \ldots, l-h\}$, then we may assume that $k_0 = l-h$ by sorting the order of the labels, i.e. $c_{l-h} \neq 0$. Assume that $c_{l-h} \neq -c_l$. Then, we can show the desired inequality by a similar argument to the case when $k_0 \in \{r+1, \ldots, l-2h\}$ by taking $x \in \mathbb{T}^{l-h-r}$ satisfying that $\left|\psi\left(k\theta'_j + x_j - \frac{1}{4}\right)\right| < \delta$ for $j \in \{r+1, \ldots, l-h-1\}$ and $\left|\psi\left(k\theta'_{l-h} + x_{l-h}\right)\right| < \delta$. Assume that $c_{l-h} = -c_l$. Note that $c_{l-h} - c_l \neq 0$ since $c_{l-h} \neq 0$. Since $c_{l-h+1}, \ldots, c_l \in \mathbb{R}$ by the assumption, we have

$$\begin{aligned}
&\operatorname{Im}\left(c_j e(k\theta'_j) + c_{j+h} e(-k\theta'_j)\right) \\
&\quad = \operatorname{Im}\left((c_j - c_{j+h}) e(k\theta'_j) + c_{j+h}\left(e(k\theta'_j) + e(-k\theta'_j)\right)\right) \\
&\quad = |c_j - c_{j+h}| \sin 2\pi(k\theta'_j + x'_j) + 2|c_{j+h}| \sin \arg c_{j+h} \cos 2\pi k\theta'_j \\
&\quad = |c_j - c_{j+h}| \sin 2\pi(k\theta'_j + x'_j),
\end{aligned}$$

where $x'_j := \frac{1}{2\pi} \operatorname{Arg}(c_j - c_{j+h})$ for $j \in \{l-2h+1, \ldots, l-h\}$. Hence, it holds that

$$|S'_k| \geq \left| \sum_{j=r+1}^{l-2h} |c_j| \sin 2\pi(k\theta'_j + x_j) + \sum_{j=l-2h+1}^{l-h} |c_j - c_{j+h}| \sin 2\pi(k\theta'_j + x'_j) \right|.$$

Put $x := \left(\iota(x_{r+1}), \ldots, \iota(x_{l-2h}), \iota(x'_{l-2h+1}), \ldots, \iota(x'_{l-h-1}), \iota(x_{l-h} - \frac{1}{4})\right) \in \mathbb{T}^{l-h-r}$. Then, the natural density of $N^{g'}_{V_\delta}(x)$ is positive for any $\delta > 0$ by Lemma 4.4.18. Hence, we have $|R^n_{g'}(x)| < \delta$ for any $n \in N^{g'}_{V_\delta}(x)$. Since

$$|\psi\left(k\theta'_i + x_i\right)| < \delta, \quad |\psi\left(k\theta'_j + x'_j\right)| < \delta$$

for $i \in \{r+1, \ldots, l-2h\}$, $j \in \{l-2h+1, \ldots, l-h-1\}$ and

$$\left|\psi\left(k\theta'_{l-h} + x'_{l-h} - \frac{1}{4}\right)\right| < \delta,$$

it holds that

$$\begin{aligned}
&|\sin 2\pi(k\theta'_i + x_i)| < \sin\delta \qquad (i \in \{r+1, \ldots, l-2h\}), \\
&|\sin 2\pi(k\theta'_j + x'_j)| < \sin\delta \qquad (j \in \{l-2h+1, \ldots, l-h-1\}), \\
&|\sin 2\pi(k\theta'_{l-h} + x'_{l-h})| = \left|\cos 2\pi\left(k\theta'_{l-h} + x'_{l-h} - \frac{1}{4}\right)\right| > \cos\delta.
\end{aligned}$$

Since $c_{l-h} - c_l \neq 0$, it holds that for a sufficiently small $\delta > 0$ and any $n \in N^{g'}_{V_\delta}(x)$,

$$|S'_k| > |c_{l-h} - c_l| \cos\delta - \left(\sum_{j=r+1}^{l-2h} c_j - \sum_{j=l-2h+2}^{l-h-1} |c_j - c_{j+h}|\right) \sin\delta > 0.$$

Therefore, when $c_0 = 0$ and $k_0 \in \{l-2h+1, \dots, l-h\}$, we have

$$\liminf_{n \in N^{g'}_{V_\delta}(x)} |S'_k| > 0.$$

Assume that $c_0 \neq 0$. If $\mathrm{Re}(c_0) \neq 0$, then we have

$$|S'_k| \geq \left| \mathrm{Re}(c_0) + \sum_{j=r+1}^{l-2h} |c_j| \cos 2\pi(k\theta'_j + x_j) + \sum_{j=l-2h+1}^{l-h} |c_j + c_{j+h}| \cos 2\pi(k\theta'_j + x_j) \right|,$$

where $x_j := \frac{1}{2\pi} \mathrm{Arg}(c_j + c_{j+h})$ for $j \in \{l-2h+1, \dots, l-h\}$. By taking $x := \big(\iota(x_2 - \frac{1}{4}), \dots, \iota(x_{l-h} - \frac{1}{4})\big) \in \mathbb{T}^{l-h-r}$ satisfying that

$$\left| \psi\left(k\theta'_j + x_j - \frac{1}{4} \right) \right| < \delta$$

for $j \in \{r+1, \dots, l-h\}$, we can show the desired inequality by a similar argument to the case when $c_0 = 0$ and $k_0 \in \{r+1, \dots, l-2h\}$.

If $\mathrm{Im}(c_0) \neq 0$, then we have

$$|S'_k| \geq \left| \mathrm{Im}(c_0) + \sum_{j=r+1}^{l-2h} c_j \sin 2\pi(k\theta'_j + x_j) + \sum_{j=l-2h+1}^{l-h} |c_j - c_{j+h}| \sin 2\pi(k\theta'_j + x'_j) \right|,$$

where $x'_j := \frac{1}{2\pi} \mathrm{Arg}(c_j - c_{j+h})$ for $j \in \{l-2h+1, \dots, l-h\}$. By taking $x := \big(x_2, \dots, x_{l-2h}, x'_{l-2h+1}, \dots, x'_{l-h}\big) \in \mathbb{T}^{l-h-1}$ satisfying that

$$|\psi\left(k\theta'_i + x_i\right)| < \delta, \quad |\psi\left(k\theta'_j + x_j\right)| < \delta$$

for $i \in \{r+1, \dots, l-2h\}$, $j \in \{l-2h+1, \dots, l-h\}$, we can show the desired inequality by a similar argument to the case when $c_0 = 0$ and $k_0 \in \{r+1, \dots, l-2h\}$. $\square$

Chapter 5

Ceiling and floor Puiseux polynomials

In this chapter, we prove the third main result of this thesis. First, we introduce ceiling polynomials and give another interpretation of the result of Deitmar, Koyama and Kurokawa [14, Theorem 2.1]. Then, we give some examples of the ceiling and floor polynomials of specific schemes over $\mathbb{Z}[S^{-1}]$, where S is a subset of $\mathbb{P}$. Next, we extend ceiling and floor polynomials to ceiling and floor Puiseux polynomials and determine the ceiling and floor Puiseux polynomials of an elliptic curve defined over $\mathbb{Q}$, which leads to the pair of provisional definitions of its absolute zeta function.

Let S be a subset of $\mathbb{P}$. In what follows, we denote the set of prime powers p^m with $p \in \mathbb{P} \setminus S$ and $m \in \mathbb{N}$ by $\mathbb{P}_S^{\mathbb{N}}$.

5.1 Ceiling and floor polynomials

Let X be a monoid scheme of finite type. As we explained in Subsection 1.2.3, Deitmar, Koyama and Kurokawa [14] identified the absolute zeta function of the continuous function $N_{X_\mathbb{Z}}$ with the absolute zeta function of the polynomial $\mathfrak{C}_{X_\mathbb{Z}}$. In this section, we introduce the ceiling and floor polynomials of a scheme of finite type over $\mathbb{Z}[S^{-1}]$. After that, we characterise the polynomial $\mathfrak{C}_{X_\mathbb{Z}}$ as the ceiling polynomial of $X_\mathbb{Z}$.

5.1.1 Ceiling/Floor polynomials

Lemma 5.1.1 ([23, Lemma 2.4]). *Let $\mathcal{P}$ be an infinite subset of $\mathbb{N}$ and $\boldsymbol{A} = (A_n)_{n\in\mathcal{P}}$ be a sequence in $\mathbb{Z}$. Then, there exists at most one polynomial $f(t) \in \mathbb{R}[t]$ satisfying the following conditions:*

(1) The inequality $f(n) \geq A_n$ (resp. $f(n) \leq A_n$) holds for every $n \in \mathcal{P}$.

(2) There exist infinitely many $n \in \mathcal{P}$ such that $f(n) = A_n$.

Proof. Suppose that $f, g \in \mathbb{R}[t]$ satisfy both of the conditions. Then, since $f - g$ is a polynomial, we have the following three possibilities:

- There exists $N \in \mathbb{N}$ such that $f(n) - g(n) > 0$ for every $n > N$.
- There exists $N \in \mathbb{N}$ such that $f(n) - g(n) < 0$ for every $n > N$.
- $f(n) - g(n) = 0$ for every $n \in \mathbb{N}$, i.e. $f = g$ in $\mathbb{R}[t]$.

In the first case, since g (resp. f) satisfies the first condition, the inequality $f(n) > g(n) \geq A_n$ (resp. $A_n \geq f(n) > g(n)$) holds for every $n > N$, which contradicts that f (resp. g) satisfies the second condition. By changing the roles of f and g, we see that the second case is also impossible. Thus, we obtain the conclusion. □

Definition 5.1.2 ([23, Definition 2.5])**.** When the polynomial f in Lemma 5.1.1 exists, we call the unique polynomial f the *ceiling* (resp. *floor*) *polynomial* of $\boldsymbol{A}$.

Definition 5.1.3 ([23, Definition 2.6])**.** Let S be a proper subset of $\mathbb{P}$ and $\mathcal{X}$ be a scheme of finite type over $\mathbb{Z}[S^{-1}]$. We call the ceiling (resp. floor) polynomial of $(\#\mathcal{X}(\mathbb{F}_q))_{q\in\mathbb{P}_S^{\mathbb{N}}}$ the *ceiling* (resp. *floor*) *polynomial* of $\mathcal{X}$ and denote it by $\mathfrak{C}_{\mathcal{X}}$ (resp. $\mathfrak{F}_{\mathcal{X}}$).

According to Propositions 2.2.7 and 2.2.10, we obtain the ceiling (resp. floor) polynomial of the $\mathbb{Z}[S^{-1}]$-lift of a monoid scheme of finite type.

Theorem 5.1.4 ([23, Theorem 2.7])**.** *Let X be a monoid scheme of finite type and S be a finite subset of $\mathbb{P}$. Set $\mathcal{X} := X_{\mathbb{Z}[S^{-1}]}$,*

$$e_{x,j,S} := \begin{cases} 1 & \text{if } 2 \mid t_{x,j} \text{ and } 2 \in S, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad T_{x,S} := \prod_{j=1}^{l_x} 2^{e_{x,j,S}}.$$

Then, it holds that

$$\mathfrak{C}_{\mathcal{X}}(t) = \sum_{x\in X} T_x(t-1)^{r_x} \in \mathbb{Z}[t] \quad \text{and} \quad \mathfrak{F}_{\mathcal{X}}(t) = \sum_{x\in X} T_{x,S}(t-1)^{r_x} \in \mathbb{Z}[t].$$

In particular, $\mathfrak{C}_{\mathcal{X}}$ is independent of S. Moreover, it holds that

$$\zeta_{\mathfrak{C}_{\mathcal{X}}}(s) = \prod_{k=0}^{R_X} (s-k)^{\sum_{x\in X} T_x (-1)^{r_x-k+1}\binom{r_x}{k}},$$
$$\zeta_{\mathfrak{F}_{\mathcal{X}}}(s) = \prod_{k=0}^{R_X} (s-k)^{\sum_{x\in X} T_{x,S} (-1)^{r_x-k+1}\binom{r_x}{k}}.$$

Proof. First, we consider the polynomial $\mathfrak{C}_{\mathcal{X}}$. The first condition in Lemma 5.1.1 follows from the inequality $\gcd(n-1, t_{x,j}) \leq t_{x,j}$ for any $n \in \mathbb{N}$. We can check the second condition by applying Dirichlet's theorem on arithmetic progressions to the prime numbers p such that $p \equiv 1 \pmod{T_X}$. Thus, the polynomial $\sum_{x\in X} T_x(t-1)^{r_x}$ coincides with $\mathfrak{C}_{\mathcal{X}}$.

Next, we consider the polynomial $\mathfrak{F}_{\mathcal{X}}$. Let T'_X be the odd integer satisfying $T_X = 2^e T'_X$ for some $e \in \mathbb{N}_0$. The first condition follows from the inequality $\gcd(q-1, t_{x,j}) \geq 2^{e_{x,j,S}}$ for any $x \in X$, $j \in \{1,\ldots,l_x\}$ and $q \in \mathbb{P}_S^{\mathbb{N}}$. The second condition in the case where $2 \notin S$ follows from the fact that $2^{\varphi(T'_X)k+1} - 1 \equiv 1 \pmod{T'_X}$ for any $k \in \mathbb{N}$, where φ is Euler's totient function. In the case where $2 \in S$, we see that there are infinitely many $p \in \mathbb{P} \setminus S$ such that $p \equiv 2 \pmod{T'_X}$ and $p \equiv 3 \pmod 4$ by combining Dirichlet's theorem on arithmetic progression and the Chinese remainder theorem. We denote the set of such p's by P. For $p \in P$, it holds that $\gcd(p-1, T_X) = 2$ (resp. 1) when T_X is even (resp. odd), and hence $\gcd(p-1, t_{x,j}) = 2^{e_{x,j,S}}$ for any $x \in X$ and $j \in \{1,\ldots,l_x\}$. Thus, the second condition follows.

The equality on the absolute zeta function follows from Example 1.1.5 and the calculation of $\mathfrak{C}_{\mathcal{X}}$ and $\mathfrak{F}_{\mathcal{X}}$. □

Remark 5.1.5 ([23, Remark 2.8]). Let $X = (X, \mathcal{O}_X)$ be a monoid scheme of finite type. Then,

$$\sum_{x\in X} T_x(t-1)^{r_x} \in \mathbb{Z}[t] \quad \left(\text{resp. } \sum_{x\in X}(t-1)^{r_x} \in \mathbb{Z}[t]\right)$$

is the ceiling (resp. floor) polynomial of $(\#X(\mathbb{F}_{1^{n-1}}))_{n\in\mathbb{N}\cap[2,\infty)}$ by Proposition 2.2.10 and a similar argument of the proof of Theorem 5.1.4. In fact, the floor polynomial of $(\#X(\mathbb{F}_{1^{n-1}}))_{n\in\mathbb{N}\cap[2,\infty)}$ coincides with the polynomial $N(x)$ introduced by Deitmar in [12, Theorem 1] since it satisfies the condition therein and such a polynomial is unique.

Theorem 5.1.4 shows that $\zeta_{\mathfrak{C}_{\mathcal{X}}}(s)$ coincides with $\zeta_{N_{X_{\mathbb{Z}}}}(s)$ in Theorem 1.2.6, which Deitmar, Koyama and Kurokawa obtained in [14] by using the Fourier expansion. Thus, $\zeta_{N_{X_{\mathbb{Z}}}}(s)$ is an invariant of $X_{\mathbb{Z}[S^{-1}]}$ independent of S, and hence it is an invariant of its generic fiber $X_{\mathbb{Q}}$ (cf. Example 5.2.7).

5.1.2 Other examples of ceiling/floor polynomials

We give some examples of the ceiling (resp. floor) polynomials of other specific schemes over $\mathbb{Z}[S^{-1}]$, especially those of relative dimension 1.

Theorem 5.1.6 ([23, Theorem 2.9]). *Let $n \in \mathbb{N}$, $\mathcal{A}_n := \mathbb{A}^1_{\mathbb{Z}} \setminus \{0,1,\ldots,n-1\}$ and S be a finite subset of $\mathbb{P}$. Set $n_1 := \min\{n, \min(\mathbb{P}\setminus S)\}$. Then, it holds that*

$$\mathfrak{C}_{\mathcal{A}_{n,\mathbb{Z}[S^{-1}]}}(t) = t - n_1 \quad \textit{and} \quad \mathfrak{F}_{\mathcal{A}_{n,\mathbb{Z}[S^{-1}]}}(t) = t - n.$$

Proof. This follows from the fact that

$$\#\mathcal{A}_{n,\mathbb{Z}[S^{-1}]}(\mathbb{F}_q) = q - \#(\mathbb{F}_p \cap \{0,1,\ldots,n-1\}) = q - \min\{p,n\}$$

for each $q = p^m \in \mathbb{P}_S^{\mathbb{N}}$. □

Let $n \geq 2$. Replacing $\{0, 1, \ldots, n-1\}$ with $\{0\} \cup \mu_{n-1}$, where μ_{n-1} is the set of the $(n-1)$-th roots of unity, we obtain the following result.

Theorem 5.1.7 ([23, Theorem 2.10])**.** *Let $n \in \mathbb{N} \cap [2, \infty)$, $\mathcal{G}_n := \mathbb{A}^1_{\mathbb{Z}} \setminus (\{0\} \cup \mu_{n-1}) = \mathbb{G}_{m,\mathbb{Z}} \setminus \mu_{n-1}$ and S be a finite subset of $\mathbb{P}$. Set*

$$n_2 := \begin{cases} 3 & \text{if } 2 \nmid n \text{ and } 2 \in S, \\ 2 & \text{otherwise.} \end{cases}$$

Then, it holds that

$$\mathfrak{C}_{\mathcal{G}_{n,\mathbb{Z}[S^{-1}]}}(t) = t - n_2 \quad \text{and} \quad \mathfrak{F}_{\mathcal{G}_{n,\mathbb{Z}[S^{-1}]}}(t) = t - n.$$

Proof. This follows from Theorem 5.1.4 and the fact that μ_{n-1} is the $\mathbb{Z}$-lift of $\operatorname{spec} \mathbb{F}_{1^{n-1}}$. □

We give another example of ceiling (resp. floor) polynomials. Let $\mathcal{C}^\Delta$ be the Pell conic of discriminant $\Delta \neq 0$, defined as an affine curve over $\mathbb{Z}$ defined by

$$\begin{cases} x^2 - \frac{\Delta}{4} y^2 = 1 & \text{if } \Delta \equiv 0 \bmod 4, \\ x^2 + xy + \frac{1-\Delta}{4} y^2 = 1 & \text{if } \Delta \equiv 1 \bmod 4. \end{cases}$$

Then, the number of the $\mathbb{F}_q$-rational points of $\mathcal{C}^\Delta$ is given as follows.

Theorem 5.1.8 ([23, Theorem 2.11])**.** *Let $q = p^m \in \mathbb{P}^{\mathbb{N}}$. Then,*

$$\#\mathcal{C}^\Delta(\mathbb{F}_q) = \begin{cases} q - \left(\frac{\Delta}{p}\right)^m & \text{if } p \neq 2,\ p \nmid \Delta, \\ 2q & \text{if } p \neq 2,\ p \mid \Delta, \\ q - (-1)^{\frac{\Delta^2-1}{8} m} & \text{if } p = 2,\ 2 \nmid \Delta, \\ q & \text{if } p = 2,\ 2 \mid \Delta, \end{cases}$$

where $\left(\frac{\Delta}{p}\right)$ is the Legendre symbol. Moreover, let S_Δ be the set of prime numbers dividing Δ. For any finite subset S of $\mathbb{P}$, it holds that

$$\mathfrak{C}_{\mathcal{C}^\Delta_{\mathbb{Z}[S^{-1}]}}(t) = \begin{cases} 2t & \text{if } S_\Delta \setminus \{2\} \not\subset S, \\ t+1 & \text{if } \Delta \text{ is not a square and } S_\Delta \setminus \{2\} \subset S, \\ t-1 & \text{if } \Delta \text{ is a square and } S_\Delta \subset S, \\ t & \text{if } \Delta \text{ is an even square, } S_\Delta \setminus \{2\} \subset S \text{ and } 2 \notin S, \end{cases}$$

and

$$\mathfrak{F}_{\mathcal{C}^\Delta_{\mathbb{Z}[S^{-1}]}}(t) = t - 1.$$

Proof. Assume that $p \neq 2$ and $p \nmid \Delta$. If $\Delta \bmod p \in \mathbb{F}_q^{\times 2}$, then we have $\#\mathcal{C}^{\Delta}(\mathbb{F}_q) = q-1$ since it holds that

$$\mathcal{C}^{\Delta}(\mathbb{F}_q) \cong \mathbb{F}_q^{\times}; (x,y) \mapsto x + \frac{\sqrt{\Delta}}{2}y.$$

If $\Delta \bmod p \in \mathbb{F}_q^{\times} \setminus \mathbb{F}_q^{\times 2}$, then it holds that

$$\mathcal{C}^{\Delta}(\mathbb{F}_q) \cong \operatorname{Ker}\left(N_{\mathbb{F}_{q^2}/\mathbb{F}_q} \colon \mathbb{F}_{q^2}^{\times} \to \mathbb{F}_q^{\times}\right); (x,y) \mapsto x + \frac{\sqrt{\Delta}}{2}y$$

and the norm map $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}$ is surjective. Therefore, we have $\#\mathcal{C}^{\Delta}(\mathbb{F}_q) = \#\mathbb{F}_{q^2}^{\times}/\#\mathbb{F}_q^{\times} = q+1$. Thus, it holds that $\#\mathcal{C}^{\Delta}(\mathbb{F}_q) = q - \left(\frac{\Delta}{p}\right)^m$ if $p \neq 2$ and $p \nmid \Delta$.

Assume that $p \neq 2$ and $p \mid \Delta$. Then,

$$\#\mathcal{C}^{\Delta}(\mathbb{F}_q) = \#\{(x,y) \in \mathbb{F}_q \times \mathbb{F}_q \mid x^2 \equiv 4 \pmod{p}\} = 2q.$$

Assume that $p = 2$ and $p \nmid \Delta$. If $\Delta \equiv 1 \pmod 8$, then we have $\#\mathcal{C}^{\Delta}(\mathbb{F}_q) = q-1$ since

$$\mathcal{C}^{\Delta}(\mathbb{F}_q) \cong \mathbb{F}_q^{\times}; (x,y) \mapsto x.$$

If $\Delta \equiv 5 \pmod 8$ and m is even, then we have $\#\mathcal{C}^{\Delta}(\mathbb{F}_q) = q-1$ since

$$\mathcal{C}^{\Delta}(\mathbb{F}_q) \cong \mathbb{F}_q^{\times}; (x,y) \mapsto x + \zeta_3 y,$$

where $\zeta_3 \in \mathbb{F}_q$ denotes a primitive third root of unity. If $\Delta \equiv 5 \pmod 8$ and m is odd, then we have $\#\mathcal{C}^{\Delta}(\mathbb{F}_q) = q+1$ since

$$\mathcal{C}^{\Delta}(\mathbb{F}_q) \cong \operatorname{Ker} N_{\mathbb{F}_{q^2}/\mathbb{F}_q}; (x,y) \mapsto x + \zeta_3 y,$$

where $\zeta_3 \in \mathbb{F}_{q^2}$ denotes a primitive third root of unity.

Assume $p = 2$ and $p \mid \Delta$, then

$$\#\mathcal{C}^{\Delta}(\mathbb{F}_q) = \#\{(x,y) \in \mathbb{F}_q \times \mathbb{F}_q \mid x^2 \equiv 1 \pmod{2}\} = q.$$

The statements on the ceiling and floor polynomials of $\mathcal{C}^{\Delta}_{\mathbb{Z}[S^{-1}]}$ follow from the above calculation of $\#\mathcal{C}^{\Delta}(\mathbb{F}_q)$. □

Next, it is natural to study the ceiling (resp. floor) polynomial of a curve $\mathcal{C}$ of positive genus defined over $\mathbb{Z}[S^{-1}]$. According to Theorem 5.1.8, one can expect that the ceiling polynomial crucially depends on the bad reductions of $\mathcal{C}$ and becomes simpler if $\mathcal{C}$ is smooth over $\mathbb{Z}[S^{-1}]$. However, the following result shows that the ceiling polynomial does not exist even for an elliptic curve defined over $\mathbb{Z}[S^{-1}]$ whenever S is finite.

Proposition 5.1.9 ([23, Proposition 2.12])**.** *Let S be a finite subset of $\mathbb{P}$ and $\mathcal{E}$ be an elliptic curve defined over $\mathbb{Z}[S^{-1}]$. Then, there exists no ceiling or floor polynomial of $\mathcal{E}$.*

Proof. By Hasse's theorem, it holds that

$$\#\mathcal{E}(\mathbb{F}_p) < p + 1 + 2\sqrt{p}$$

for every $p \in \mathbb{P} \setminus S$. On the other hand, the Sato-Tate conjecture [3, 7] implies that for any $\varepsilon > 0$, there exist prime numbers $p \in \mathbb{P} \setminus S$ such that

$$\#\mathcal{E}(\mathbb{F}_p) > p + 1 + 2\sqrt{p}(1 - \varepsilon).$$

These facts imply that there exists no ceiling polynomial $\mathfrak{C}_{\mathcal{E}}$ of $\mathcal{E}$. Indeed, if such a polynomial $\mathfrak{C}_{\mathcal{E}}$ exists, then the Sato-Tate conjecture and the first condition in Lemma 5.1.1 imply that

$$\forall \alpha > 0,\ \exists N_0 \in \mathbb{N} \text{ s.t. } \forall p \in \mathbb{P} \setminus S\ (p > N_0 \Rightarrow \mathfrak{C}_{\mathcal{E}}(p) > p + \alpha).$$

However, since $\mathfrak{C}_{\mathcal{E}}$ is a polynomial, the above estimate is equivalent to the following formula:

$$\exists \delta > 0 \text{ s.t. } \exists N_1 \in \mathbb{N} \text{ s.t. } \forall p \in \mathbb{P} \setminus S\ (p > N_1 \Rightarrow \mathfrak{C}_{\mathcal{E}}(p) > (1 + \delta)p).$$

Since the inequality $(1 + \delta)p > p + 1 + 2\sqrt{p}$ holds for each $p \gg 1$, Hasse's theorem implies that

$$\exists N_2 \in \mathbb{N} \text{ s.t. } \forall p \in \mathbb{P} \setminus S\ (p > N_2 \Rightarrow \mathfrak{C}_{\mathcal{E}}(p) > \#\mathcal{E}(\mathbb{F}_p)),$$

which contradicts the second condition in Lemma 5.1.1.

The non-existence of the floor polynomial $\mathfrak{F}_{\mathcal{E}}$ of $\mathcal{E}$ follows from a similar argument. □

5.2 Ceiling and floor Puiseux polynomials

In this section, we introduce ceiling (resp. floor) Puiseux polynomials by replacing the polynomial condition in Lemma 5.1.1 by means of Puiseux polynomials. Then, after introducing the ceiling (resp. floor) Puiseux polynomial of a separated scheme of finite type over $\mathbb{Q}$, we identify the ceiling (resp. floor) Puiseux polynomial of an elliptic curve over $\mathbb{Q}$ as the Puiseux polynomial $t + 2t^{1/2} + 1$ (resp. $t - 2t^{1/2} + 1$).

5.2.1 Ceiling/Floor Puiseux polynomials

We begin with the definition of the ceiling (resp. floor) Puiseux polynomial of a general integer sequence.

Definition 5.2.1 ([23, Definition 3.1])**.** Let R be a commutative ring. We define $R[t^{1/\infty}]$ as the residue ring of the polynomial ring $R\,[t_n \mid n \in \mathbb{N}]$ in countably many indeterminates t_n by the ideal I generated modulo $t_{mn}^m - t_n$ for all $m, n \in \mathbb{N}$, and set $t^{1/n} := t_n \bmod I$. We call an element of $R[t^{1/\infty}]$ a *Puiseux polynomial* with coefficients in R.

Suppose that $R = \mathbb{R}$ (or its subring). Then, each Puiseux polynomial in $\mathbb{R}[t^{1/\infty}]$ defines a continuous function on $(0, \infty)$ to $\mathbb{R}$. In what follows, we identify each Puiseux polynomial with this function. Similarly to Lemma 5.1.1, we show the uniqueness of a certain Puiseux polynomial.

Lemma 5.2.2 ([23, Lemma 3.2])**.** *Let $\mathcal{P}$ be an infinite subset of $\mathbb{N}$ and $\boldsymbol{A} = (A_n)_{n\in\mathcal{P}}$ be a sequence in $\mathbb{Z}$. Then, there exists at most one Puiseux polynomial $f(t) \in \mathbb{R}[t^{1/\infty}]$ satisfying the following conditions:*

(1) The inequality $f(n) \geq A_n$ (resp. $f(n) \leq A_n$) holds for every $n \in \mathcal{P}$.

(2) There exist infinitely many $n \in \mathcal{P}$ such that the equality $\lfloor f(n) \rfloor = A_n$ (resp. $\lceil f(n) \rceil = A_n$) holds.

(3) $f(1) \in \mathbb{Z}$.

Proof. Suppose that $f, g \in \mathbb{R}[t^{1/\infty}]$ satisfy both of the conditions. Then, since $f - g$ is a Puiseux polynomial, it is a polynomial of $t^{1/m}$ for some $m \in \mathbb{N}$. Hence, we have the following three possibilities:

- There exists some $N \in \mathbb{N}$ such that $f(n) - g(n) \geq 1$ for every $n > N$.
- There exists some $N \in \mathbb{N}$ such that $f(n) - g(n) \leq -1$ for every $n > N$.
- $f - g$ is a constant in the open interval $(-1, 1)$.

In the first case, since g (resp. f) satisfies the first condition, the inequality $f(n) \geq g(n) + 1 \geq A_n + 1$ (resp. $g(n) \leq f(n) - 1 \leq A_n - 1$) holds for every $n > N$, which contradicts that f (resp. g) satisfies the second condition. By changing the roles of f and g, we see that the second case is also impossible. In the third case, it holds that $f = g$ since $f(1) - g(1) = 0$ by the third condition. Thus, we obtain the conclusion. □

Definition 5.2.3 ([23, Definition 3.3])**.** When the Puiseux polynomial f in Lemma 5.2.2 exists, we call the unique Puiseux polynomial f the *ceiling* (resp. *floor*) *Puiseux polynomial* of $\boldsymbol{A}$.

If there exists a polynomial with integral coefficients satisfying the conditions in Lemma 5.1.1, then it satisfies the conditions in Lemma 5.2.2. In this sense, the Puiseux polynomial in Lemma 5.2.2 is a generalisation of the polynomials with integral coefficients in Lemma 5.1.1, which contain polynomials having been studied in the context of absolute zeta functions (e.g. Soulé[46], Deitmar [12], Deitmar, Koyama and Kurokawa [14]).

As we mentioned after Theorem 5.1.8, we can expect a simpler ceiling Puiseux polynomial if the information on pathological prime numbers is excluded. Hence, it is fair to define a ceiling (resp. floor) Puiseux polynomial of an algebraic variety over $\mathbb{Q}$ (and more generally a separated scheme of finite type over $\mathbb{Q}$) as follows.

Definition 5.2.4 ([23, Definition 3.4]). Let X be a separated scheme of finite type over $\mathbb{Q}$. Assume that there exists a Puiseux polynomial f satisfying the following condition: for any separated scheme $\mathcal{X}$ of finite type over $\mathbb{Z}$ satisfying that $\mathcal{X}_{\mathbb{Q}} \cong X$, there exists a finite subset $S_{\mathcal{X}}$ of $\mathbb{P}$ such that for any finite subset S of $\mathbb{P}$ containing $S_{\mathcal{X}}$, the Puiseux polynomial f is the ceiling (resp. floor) Puiseux polynomial of $(\#\mathcal{X}(\mathbb{F}_q))_{q\in\mathbb{P}_S^{\mathbb{N}}}$. Then, we call f the *ceiling* (resp. *floor*) *Puiseux polynomial* of X and denote it by $\mathfrak{C}_X$ (resp. $\mathfrak{F}_X$).

The following facts are useful for verification of the uniqueness of the ceiling and floor Puiseux polynomials of X and their practical calculation.

Theorem 5.2.5 (cf. Serre [43, Theorems 4.12 and 4.13]). *Let $\mathcal{X}$ be a separated scheme of finite type over $\mathbb{Z}$ and l be a prime number. Then, there exists a finite subset Σ of $\mathbb{P}$ (independent of l) such that for every $p \in \mathbb{P} \setminus (\Sigma \cup \{l\})$ and every $m \in \mathbb{N}$, the following equality holds:*

$$\#\mathcal{X}(\mathbb{F}_{p^m}) = \sum_{i=0}^{2\dim \mathcal{X}_{\mathbb{Q}}} (-1)^i \operatorname{Tr}(\sigma_p^{-m} \mid H_c^i(\mathcal{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)),$$

where σ_p is the p-th power Frobenius automorphism in $\operatorname{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$, which acts on $H_c^i(\mathcal{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)$ via the specialization map $H_c^i(\mathcal{X}_{\overline{\mathbb{F}_p}}, \mathbb{Q}_l) \xrightarrow{\sim} H_c^i(\mathcal{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)$.

Corollary 5.2.6 ([23, Corollary 3.6]). *Let $\mathcal{X}, \mathcal{Y}$ be separated schemes of finite type over $\mathbb{Z}$ such that $\mathcal{X}_{\mathbb{Q}} \cong \mathcal{Y}_{\mathbb{Q}}$. Then, there exists a finite subset Σ' of $\mathbb{P}$ such that the following equality holds for every $q \in \mathbb{P}_{\Sigma'}^{\mathbb{N}}$:*

$$\#\mathcal{X}(\mathbb{F}_q) = \#\mathcal{Y}(\mathbb{F}_q).$$

In particular, in the setting of Definition 5.2.4, if f is the ceiling (resp. floor) Puiseux polynomial of $(\#\mathcal{X}(\mathbb{F}_q))_{q\in\mathbb{P}_S^{\mathbb{N}}}$ for some $\mathcal{X}$ and for some $S_{\mathcal{X}}$ with an arbitrary $S \supset S_{\mathcal{X}}$, then it coincides with the ceiling (resp. floor) Puiseux polynomial of X.

According to this corollary, it is sufficient to verify the condition in Definition 5.2.4 not for all $\mathcal{X}$ but for a single $\mathcal{X}$. Moreover, the ceiling and floor Puiseux polynomials for such an $\mathcal{X}$ are unique respectively if they exist. Using this fact, we obtain the ceiling and floor Puiseux polynomials for the generic fibres of specific schemes which appeared in Subsection 5.1.2 as follows.

Example 5.2.7 ([23, Example 3.7]). Let X be a monoid scheme of finite type such that $X_{\mathbb{Z}}$ is separated. Thus, it holds that

$$\mathfrak{C}_{X_{\mathbb{Q}}}(t) = \sum_{x\in X} T_x (t-1)^{r_x} \quad \text{and} \quad \mathfrak{F}_{X_{\mathbb{Q}}}(t) = \sum_{x\in X} T_{x,\{2\}} (t-1)^{r_x}$$

by Theorem 5.1.4 and Corollary 5.2.6. Indeed, it is sufficient to take $\mathcal{X} = X_{\mathbb{Z}}$ and $S_{\mathcal{X}} = \{2\}$. In particular, it holds that $\mathfrak{C}_{X_{\mathbb{Q}}} = \mathfrak{F}_{X_{\mathbb{Q}}}$ if and only if the torsion subgroup of $\mathcal{O}_{X,x}^{\times}$ is 2-torsion for all $x \in X$.

Example 5.2.8 ([23, Example 3.8]). Put $X = \mathcal{A}_{n,\mathbb{Q}}$. By Theorem 5.1.6 and Corollary 5.2.6, it holds that

$$\mathfrak{C}_{\mathcal{A}_{n,\mathbb{Q}}}(t) = \mathfrak{F}_{\mathcal{A}_{n,\mathbb{Q}}}(t) = t - n.$$

Indeed, it suffices to take $\mathcal{X} = \mathcal{A}_n$ and $S_{\mathcal{X}}$ as the set of prime numbers less than n.

Example 5.2.9 ([23, Example 3.9]). Put $X = \mathcal{G}_{n,\mathbb{Q}}$. By Theorem 5.1.7 and Corollary 5.2.6, it holds that

$$\mathfrak{C}_{\mathcal{G}_{n,\mathbb{Q}}}(t) = t - 2 \quad \text{and} \quad \mathfrak{F}_{\mathcal{G}_{n,\mathbb{Q}}}(t) = t - n.$$

Indeed, it is sufficient to take $\mathcal{X} = \mathcal{G}_n$ and $S_{\mathcal{X}} = \{2\}$. In particular, it holds that $\mathfrak{C}_{\mathcal{G}_{n,\mathbb{Q}}} = \mathfrak{F}_{\mathcal{G}_{n,\mathbb{Q}}}$ if and only if $n = 2$.

Example 5.2.10 ([23, Example 3.10]). Put $X = \mathcal{C}^{\Delta}_{\mathbb{Q}}$. By Theorem 5.1.8 and Corollary 5.2.6, it holds that

$$\mathfrak{C}_{\mathcal{C}^{\Delta}_{\mathbb{Q}}}(t) = \begin{cases} t-1 & \text{if } \Delta \text{ is a square,} \\ t+1 & \text{if } \Delta \text{ is not a square,} \end{cases} \quad \text{and} \quad \mathfrak{F}_{\mathcal{C}^{\Delta}_{\mathbb{Q}}}(t) = t - 1.$$

Indeed, it is sufficient to take $\mathcal{X} = \mathcal{C}^{\Delta}$ and $S_{\mathcal{X}}$ as the set of prime numbers dividing 2Δ. In particular, it holds that $\mathfrak{C}_{\mathcal{C}^{\Delta}_{\mathbb{Q}}} = \mathfrak{F}_{\mathcal{C}^{\Delta}_{\mathbb{Q}}}$ if and only if Δ is square, which is equivalent to $\mathcal{C}^{\Delta}_{\mathbb{Q}} \cong \mathbb{G}_{m,\mathbb{Q}}$. Note that even if Δ is not a square, the scalar extension (base change) $\mathcal{C}^{\Delta}_{\mathbb{Q}} \otimes \mathbb{Q}(\sqrt{\Delta})$ can be identified with the $\mathbb{Q}(\sqrt{\Delta})$-lift of the monoid scheme $\mathbb{G}_{m,\mathbb{F}_1}$.

5.2.2 Ceiling/Floor Puiseux polynomial of a projective curve and its maximal/minimal reduction

Let C be a smooth proper curve over $\mathbb{Q}$ which is geometrically irreducible of genus $g > 0$. Then, by the spreading out principle (see Poonen [40, Theorem 3.2.1]), there exist a finite subset S_C of $\mathbb{P}$ and a smooth proper scheme $\mathcal{C}$ of finite type over $\mathbb{Z}[S_C^{-1}]$ such that $\mathcal{C}_{\mathbb{Q}} \cong C$.

For $q = p^m \in \mathbb{P}^{\mathbb{N}}_{S_C}$, the Hasse-Weil bound (see Serre [43, §4.7.2.2]) implies that

$$q - 2g\sqrt{q} + 1 \leq \#\mathcal{C}(\mathbb{F}_q) \leq q + 2g\sqrt{q} + 1.$$

The closed fiber $\mathcal{C}_{\mathbb{F}_p}$ of $\mathcal{C}$ is called $\mathbb{F}_q$-*maximal* (resp. $\mathbb{F}_q$-*minimal*) if $\#\mathcal{C}(\mathbb{F}_q)$ attains the Hasse-Weil upper (resp. lower) bound, i.e.

$$\#\mathcal{C}(\mathbb{F}_q) = q + 2g\sqrt{q} + 1 \quad (\text{resp. } \#\mathcal{C}(\mathbb{F}_q) = q - 2g\sqrt{q} + 1).$$

In view of the ceiling (resp. floor) Puiseux polynomial, we are interested in the distribution of the prime powers q for which $\mathcal{C}_{\mathbb{F}_p}$ is $\mathbb{F}_q$-maximal (resp. $\mathbb{F}_q$-minimal). By the definition of the ceiling (resp. floor) Puiseux polynomial of C, we obtain the following proposition.

Proposition 5.2.11 ([23, Proposition 3.11]). *Assume that there exist infinitely many prime numbers $p \in \mathbb{P} \setminus S_C$ for which $\mathcal{C}_{\mathbb{F}_p}$ is $\mathbb{F}_{p^m}$-maximal (resp. $\mathbb{F}_{p^m}$-minimal) for some $m \in \mathbb{N}$. Then, it holds that*

$$\mathfrak{C}_C(t) = t + 2gt^{1/2} + 1 \quad \left(\textit{resp. } \mathfrak{F}_C(t) = t - 2gt^{1/2} + 1\right).$$

5.2.3 Ceiling/Floor Puiseux polynomial of an elliptic curve

Let E be an elliptic curve defined over $\mathbb{Q}$. Like Subsection 5.2.2, there exist a finite subset S_E of $\mathbb{P}$ and an elliptic curve $\mathcal{E}$ over $\mathbb{Z}[S_E^{-1}]$ such that $\mathcal{E}_{\mathbb{Q}} \cong E$. Then, the following fact is known concerning a supersingular elliptic curve.

Lemma 5.2.12 (see e.g. Silverman [44, p. 155]). *Suppose that $p \in \mathbb{P} \setminus (S_E \cup \{2, 3\})$. Then, the following conditions are equivalent:*

(1) $\mathcal{E}_{\mathbb{F}_p}$ is supersingular, i.e. $\#\mathcal{E}(\mathbb{F}_p) = p + 1$.

(2) $\mathcal{E}_{\mathbb{F}_p}$ is $\mathbb{F}_{p^{4k-2}}$-maximal and $\mathbb{F}_{p^{4k}}$-minimal for any $k \in \mathbb{N}$, i.e. $\#\mathcal{E}(\mathbb{F}_{p^{4k-2}}) = p^{4k-2} + 2p^{2k-1} + 1$ and $\#\mathcal{E}(\mathbb{F}_{p^{4k}}) = p^{4k} - 2p^{2k} + 1$.

(3) $\mathcal{E}_{\mathbb{F}_p}$ is $\mathbb{F}_{p^2}$-maximal.

(4) It holds that

$$Z(\mathcal{E}_{\mathbb{F}_p}, T) := \exp\left(\sum_{m=1}^{\infty} \frac{\#\mathcal{E}(\mathbb{F}_{p^m})}{m} T^m\right) = \frac{1 + pT^2}{(1 - T)(1 - pT)}.$$

Proof. Let α be an eigenvalue of the p-th power Frobenius endomorphism on the Tate module of E. Then, it holds that

$$\#\mathcal{E}(\mathbb{F}_{p^m}) = 1 - \left(\alpha^m + \frac{p^m}{\alpha^m}\right) + p^m \tag{5.1}$$

for any $m \in \mathbb{N}$ (see e.g. Silverman [44, Theorem 2.3.1]). In particular, by specialising it to $m = 1$, the equivalence (1) $\Leftrightarrow$ $\alpha^2 = -p$ follows. The equation $\alpha^2 = -p$ is equivalent to (2) and (3), respectively. Moreover, the equivalence (1) $\Leftrightarrow$ (4) follows since

$$\begin{aligned} Z(\mathcal{E}_{\mathbb{F}_p}, T) &= \exp\left(\sum_{m=1}^{\infty} \left(1 - \left(\alpha^m + \frac{p^m}{\alpha^m}\right) + p^m\right) \frac{T^m}{m}\right) \\ &= \frac{(1 - \alpha T)(1 - \frac{p}{\alpha} T)}{(1 - T)(1 - pT)} = \frac{1 + (\#\mathcal{E}(\mathbb{F}_p) - p - 1)\, T + pT^2}{(1 - T)(1 - pT)}. \end{aligned}$$

□

Proposition 5.2.11 and Lemma 5.2.12 (1) $\Leftrightarrow$ (2) lead us to the natural question whether there exist infinitely many prime numbers p such that $\mathcal{E}_{\mathbb{F}_p}$ is supersingular. The answer is known to be affirmative due to Elkies as follows.

Theorem 5.2.13 (Elkies [19]). *Let E be an elliptic curve over $\mathbb{Q}$. Then, there exist infinitely many prime numbers at which E has good supersingular reduction.*

Remark 5.2.14 ([23, Remark 3.14]). In fact, Elkies [20] obtained a similar result for every elliptic curve over an arbitrary number field F (of finite degree) which has at least one field embedding $F \subset \mathbb{R}$.

As the consequence of Theorem 5.2.13 and Lemma 5.2.12 (1) $\Leftrightarrow$ (2), we see that there exist infinitely many prime numbers $p \in \mathbb{P} \setminus S_E$ for each of which $\mathcal{E}_{\mathbb{F}_p}$ is $\mathbb{F}_{p^m}$-maximal for some $m \in \mathbb{N}$. Therefore, we can determine the ceiling (resp. floor) Puiseux polynomial of an elliptic curve defined over $\mathbb{Q}$ as follows.

Corollary 5.2.15 ([23, Corollary 3.15]). *Let E be any elliptic curve over $\mathbb{Q}$. Then, it holds that*

$$\mathfrak{C}_E(t) = t + 2t^{1/2} + 1 \quad \textit{and} \quad \mathfrak{F}_E(t) = t - 2t^{1/2} + 1.$$

Moreover, the absolute zeta functions of $\mathfrak{C}_E$ and $\mathfrak{F}_E$ are

$$\zeta_{\mathfrak{C}_E}(s) = \frac{1}{s\left(s-\frac{1}{2}\right)^2(s-1)} \qquad \textit{and} \qquad \zeta_{\mathfrak{F}_E}(s) = \frac{\left(s-\frac{1}{2}\right)^2}{s(s-1)}.$$

Remark 5.2.16 ([23, Remark 1.7]). According to Corollary 5.2.15, it holds that

$$\zeta_{\mathfrak{C}_E}(s) = \left(\frac{1}{s\left(s-\frac{1}{2}\right)}\right)^{\widetilde{\otimes}2} \quad \text{and} \quad \zeta_{\mathfrak{F}_E}(s) = \left(\frac{s}{s-\frac{1}{2}}\right)^{\widetilde{\otimes}2},$$

where $\widetilde{\otimes}$ denotes the tensor product that we replace $m(\rho_1, \ldots, \rho_r)$ to $-m(\rho_1, \ldots, \rho_r)$ in the definition of the Kurokawa tensor product in Definition 3.1.3. These are compatible with the factorizations $\mathfrak{C}_E(t) = (t^{1/2}+1)^2$ and $\mathfrak{F}_E(t) = (t^{1/2}-1)^2$.

Remark 5.2.17 ([23, Remark 3.16]). If X is a monoid scheme of finite type whose $\mathbb{Z}$-lift is a smooth projective variety, then Deitmar, Koyama and Kurokawa deduced the equality

$$\#X(\mathbb{F}_1) = N_{X_{\mathbb{Z}}}(1) = \chi_{\text{top}}(X_{\mathbb{Z}}(\mathbb{C}))$$

from the Weil conjecture for $X_{\mathbb{F}_p}$ (cf. the proof of [14, Theorem 2.1]). In fact, we could formally obtain the similar equality

$$\text{“}\#\mathcal{E}(\mathbb{F}_1)\text{”} = 0 = \chi_{\text{top}}(\mathcal{E}(\mathbb{C}))$$

if we substituted $m = 0$ in Equation (5.1) in the proof of Lemma 5.2.12, which is the consequence of the Weil conjecture for $\mathcal{E}_{\mathbb{F}_p}$. Moreover, the Puiseux polynomial $\mathfrak{F}_E$ satisfies that

$$\mathfrak{F}_E(1) = \chi_{\mathrm{top}}(E(\mathbb{C})) = \chi_{\mathrm{top}}(S^1 \times S^1).$$

These observations are all consistent with the philosophy we mentioned in Remark 1.1.8. On the other hand, the Puiseux polynomial $\mathfrak{C}_E$ is not consistent with it. In this view, it is fair to say that $\zeta_{\mathfrak{F}_E}$ is better than $\zeta_{\mathfrak{C}_E}$.

Remark 5.2.18 ([23, Remark 3.17])**.** According to Charles [6], for any pair of elliptic curves E_1, E_2 over a number filed K, there are infinitely many prime ideals of K at which the reductions of E_1 and E_2 are geometrically isogenous. Corollary 5.2.15 might suggest that "the reductions modulo 1" of all elliptic curves over K are "geometrically isogenous over $\mathbb{F}_1$" in some sense. On the other hand, if $K = \mathbb{Q}$, then Corollary 5.2.15 shows that both $\mathfrak{C}_E$ and $\mathfrak{F}_E$ are determined purely in terms of the Betti numbers of the topological 2-dimensional torus $S^1 \times S^1$. In particular, they are independent of the isogeny class of E. This might even suggest that all elliptic curves over $\mathbb{Q}$ are "isogenous over $\mathbb{F}_1$" at least in view of Tate's isogeny theorem over $\mathbb{F}_p$ (see e.g. Silverman [44, III.7.7]).

Appendix A

Ceiling/Floor Puiseux polynomial of an elliptic curve in the case of $\mathcal{P} = \mathbb{P} \setminus S$

Let S be a finite subset of $\mathbb{P}$. In this appendix, we discuss the ceiling and floor Puiseux polynomials of the sequence $(\#\mathcal{E}(\mathbb{F}_p))_{p\in\mathbb{P}\setminus S}$ instead of $(\#\mathcal{E}(\mathbb{F}_q))_{q\in\mathbb{P}_S^{\mathbb{N}}}$ in Section 5.2. As a result, in the case of elliptic curves defined over $\mathbb{Q}$ with complex multiplication, we obtain the same Puiseux polynomial as its ceiling and floor Puiseux polynomial.

Definition A.0.1 ([23, Definition A.1])**.** Let X be a separated scheme of finite type over $\mathbb{Q}$. Assume that there exists a Puiseux polynomial f satisfying the following condition: for any separated scheme $\mathcal{X}$ of finite type over $\mathbb{Z}$ satisfying that $\mathcal{X}_{\mathbb{Q}} \cong X$, there exists a finite subset $S_{\mathcal{X}}$ of $\mathbb{P}$ such that for any finite subset S of $\mathbb{P}$ containing $S_{\mathcal{X}}$, the Puiseux polynomial f is the ceiling (resp. floor) Puiseux polynomial of $(\#\mathcal{X}(\mathbb{F}_p))_{p\in\mathbb{P}\setminus S}$. Then, we call f the *prime ceiling* (resp. *floor*) *Puiseux polynomial* of X and denote it by $\mathfrak{C}'_X$ (resp. $\mathfrak{F}'_X$).

Remark A.0.2 ([23, Remark A.2])**.** Comparing it with Definition 5.2.4, the first condition in Lemma 5.2.2 gets weaker and the second one gets stronger for $\boldsymbol{A} = (\#\mathcal{X}(\mathbb{F}_p))_{p\in\mathbb{P}\setminus S}$ than for $\boldsymbol{A} = (\#\mathcal{X}(\mathbb{F}_q))_{q\in\mathbb{P}_S^{\mathbb{N}}}$.

Let E be an elliptic curve defined over $\mathbb{Q}$. As mentioned in Subsection 5.2.3, there exist a finite subset S_E of $\mathbb{P}$ and an elliptic curve $\mathcal{E}$ over $\mathbb{Z}[S_E^{-1}]$ such that $\mathcal{E}_{\mathbb{Q}} \cong E$. Then, for $p \in \mathbb{P} \setminus S_E$, the Hasse bound implies that

$$p + 1 - 2\sqrt{p} < \#\mathcal{E}(\mathbb{F}_p) < p + 1 + 2\sqrt{p}.$$

Then, p is called a *champion* (resp. *trailing*) prime if the equality

$$\#\mathcal{E}(\mathbb{F}_p) = p + 1 + \lfloor 2\sqrt{p} \rfloor \quad (\text{resp. } \#\mathcal{E}(\mathbb{F}_p) = p + 1 - \lceil 2\sqrt{p} \rceil)$$

holds (cf. James and Pollack [25]). Let π_E^+ (resp. π_E^-) be the set of champion (resp. trailing) prime numbers for E and $\pi_E^{\pm}(x) := \pi_E^{\pm} \cap (0, x]$ for every $x \in (0, \infty)$. Then, the following is obvious:

Proposition A.0.3 (cf. Proposition 5.2.11)**.** *Assume that $\#\pi_E^{\pm} = \infty$, then it holds that $\mathfrak{C}'_E = \mathfrak{C}_E$ and $\mathfrak{F}'_E = \mathfrak{F}_E$.*

For a CM elliptic curve over $\mathbb{Q}$, the following fact on $\pi_E^{\pm}(x)$ is known.

Theorem A.0.4 (James and Pollack [25, Theorem 1])**.** *Suppose that E has complex multiplication over $\overline{\mathbb{Q}}$. Then, the following asymptotic relation holds:*

$$\pi_E^{\pm}(x) \sim \frac{2}{3\pi} \cdot \frac{x^{3/4}}{\log x} \quad (x \to \infty).$$

In particular, it holds that $\#\pi_E^{\pm} = \infty$.

According to Theorem A.0.4, the prime ceiling (resp. floor) Puiseux polynomial of a CM elliptic curve coincides with the Puiseux polynomial in Proposition A.0.3. On the other hand, for an elliptic curve defined over $\mathbb{Q}$ without complex multiplication, it is conjectured in [26, Conjecture 2.3] that

$$\pi_E^{\pm}(x) \sim c_E \cdot \frac{x^{1/4}}{\log x} \quad (x \to \infty),$$

where $c_E \in (0, \infty)$ is a constant. Currently, the above estimate of $\pi_E^{\pm}(x)$ in the case where E is a non-CM elliptic curve is verified only under some assumptions such as the Generalised Riemann Hypothesis (cf. David, Gafni, Malik, Prabhu and Turnage-Butterbaugh [10]).

Appendix B

Another proofs of Lemma 4.4.5 and Theorem 4.4.12

In this chapter, we give another proof of Lemma 4.4.5 using ergodic theory. Moreover, we show another version of Theorem 4.4.12 whose conditions are simpler.

B.1 Another proof of Lemma 4.4.5

Lemma B.1.1 (Lemma 4.4.5). *It holds that*

$$\log(1-\rho u) = S_{t^\rho}(u) := \sum_{n=1}^{\infty} \kappa_\rho(n) \log\left(1-u^n\right) \ \in \mathbb{Q}[\![\rho, u]\!].$$

Moreover, for $\rho \in \mathbb{C}$, the region of absolute convergence of the series in the right-hand side is

$$\begin{cases} \left\{ u \in \mathbb{D} \;\middle|\; |u| < \frac{1}{|\rho|} \right\} & \text{if } \rho \neq 0, 1, \\ \mathbb{C} & \text{if } \rho = 0, \\ \mathbb{D} & \text{if } \rho = 1. \end{cases}$$

Another proof of Lemma 4.4.5. We assume that $\rho \neq 0, 1$. We show that the series $S_{t^\rho}(u)$ does not converge absolutely for $|u| \geq \min\{\frac{1}{|\rho|}, 1\}$. If $|u| \geq 1$, then $S_{t^\rho}(u)$ does not converge absolutely. Hence, we may assume that $|\rho| > 1$ and it is sufficient to prove that $S_{t^\rho}(u)$ does not converge absolutely for $\frac{1}{|\rho|} \leq |u| < 1$.

Now, we show that $S_{t^\rho}(u)$ does not converges absolutely for $\frac{1}{|\rho|} \leq |u| < 1$. Assume that $\frac{\operatorname{Arg} u}{2\pi} \in \mathbb{Q}$. Then, we put $\frac{\operatorname{Arg} u}{2\pi} = \frac{k_1}{k_2}$ ($k_1, k_2 \in \mathbb{Z}$, $k_2 \neq 0$) and $N_{\mathrm{rat}} := \{k_2 m \mid m \in \mathbb{N}\}$. Then, since $u^n \in [\frac{1}{|\rho|}, 1)$ for any $n \in N_{\mathrm{rat}}$, it holds that

$$\sum_{n=1}^{\infty} \left|\kappa_\rho(n) \log\left(1-u^n\right)\right| \geq \sum_{n \in N_{\mathrm{rat}}} \left|\kappa_\rho(n) \log\left(1-u^n\right)\right| = \sum_{n \in N_{\mathrm{rat}}} |\kappa_\rho(n)| \log \frac{1}{1-u^n}.$$

Put $N_\rho := \left\{ n \in \mathbb{N} \;\middle|\; n \geq \max\left\{ \frac{5}{\log|\rho|} \log \frac{2}{|\rho|-1}, 3 \right\} \right\}$. Since $\frac{n}{5} < n - \lfloor \frac{n}{2} \rfloor - 1$ for each $n \in N_\rho$, we have

$$\left| \sum_{n \neq d \mid n} \mu\left(\frac{n}{d}\right) \rho^d \right| \leq \sum_{n \neq d \mid n} |\rho|^d \leq \sum_{d=1}^{\lfloor \frac{n}{2} \rfloor} |\rho|^d = \frac{|\rho|^{\lfloor \frac{n}{2} \rfloor + 1} - |\rho|}{|\rho| - 1} < \frac{|\rho|^{\lfloor \frac{n}{2} \rfloor + 1}}{|\rho| - 1} < \frac{|\rho|^n}{2}.$$

Thus, for any $n \in N_\rho$, it holds that

$$|n\kappa_\rho(n)| \geq |\rho|^n - \left| \sum_{n \neq d \mid n} \mu\left(\frac{n}{d}\right) \rho^d \right| > |\rho|^n - \frac{|\rho|^n}{2} = \frac{|\rho|^n}{2}.$$

Since $\left(1 - \frac{1}{x}\right)^{-x} > e$ for any $x > 1$ and the natural density of N_{rat} is equal to $\frac{1}{k_2} > 0$, it holds that

$$\begin{aligned} \sum_{n \in N_{\mathrm{rat}}} |\kappa_\rho(n)| \log \frac{1}{1-u^n} &\geq \sum_{n \in N_{\mathrm{rat}}} |\kappa_\rho(n)| \log \frac{1}{1-|\rho|^{-n}} > \sum_{n \in N_{\mathrm{rat}} \cap N_\rho} \frac{|\rho|^n}{2n} \log \frac{1}{1-|\rho|^{-n}} \\ &= \sum_{n \in N_{\mathrm{rat}} \cap N_\rho} \frac{1}{2n} \log \left(1 - \frac{1}{|\rho|^n}\right)^{-|\rho|^n} > \frac{1}{2} \sum_{n \in N_{\mathrm{rat}} \cap N_\rho} \frac{1}{n} = \infty. \end{aligned}$$

Assume that $\frac{\operatorname{Arg} u}{2\pi} \notin \mathbb{Q}$. We put $\varphi_n := \operatorname{Arcsin} |u|^n$, $\psi_n := \operatorname{Arcsin} \frac{1}{2|\rho|^n}$,

$$\begin{aligned} \theta_n^\pm &:= \operatorname{Arccos} \left(\frac{1}{2|u|^n} \left(\left(\sqrt{1 - \frac{1}{4|\rho|^{2n}}} \pm \sqrt{|u|^{2n} - \frac{1}{4|\rho|^{2n}}} \right)^2 - |u|^{2n} - 1 \right) \right) \\ &= \operatorname{Arccos} \left(-\frac{1}{4(|u||\rho|^2)^n} \pm \sqrt{\left(1 - \frac{1}{4|\rho|^{2n}}\right)\left(1 - \frac{1}{4(|u||\rho|)^{2n}}\right)} \right), \end{aligned}$$

and $\Theta_n := [\theta_n^+, \theta_n^-] \cup [2\pi - \theta_n^-, 2\pi - \theta_n^+]$ as described in Figure B.1. Note that

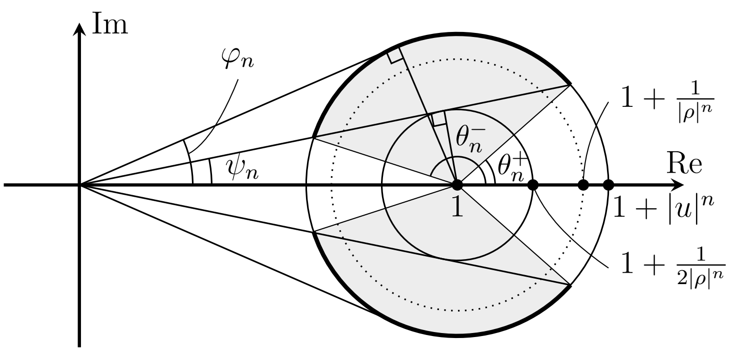


Figure B.1: The definitions of φ_n, ψ_n, $\theta_n^\pm$ in the complex plane

$|\operatorname{Arg}(1-u^n)| \leq \varphi_n$ when $n \in \Theta_n$. Since θ_n^+ is strictly monotonically decreasing and $\theta_n^- > \frac{\pi}{2}$ by a simple calculation, we have $\theta_n^+ \leq \frac{\pi}{3}$ and $\theta_n^- \geq \frac{\pi}{2}$ for any $n \in \mathbb{N}$. Putting $\Theta := \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$, we have $\Theta_n \supset \Theta$ for any $n \in \mathbb{N}$. Since $|\operatorname{Arg}(1-u^n)| \geq \psi_n \geq \frac{1}{2|\rho|^n}$ for n satisfying that $\operatorname{Arg} u^n \in \Theta \subset \Theta_n$,

$$\begin{aligned}\sum_{n=1}^{\infty} |\kappa_\rho(n) \log(1-u^n)| &\geq \sum_{n \in N_\rho} |\kappa_\rho(n)| \, |\log(1-u^n)| > \sum_{n \in N_\rho} \frac{|\rho|^n}{2n} |\operatorname{Arg}(1-u^n)| \\ &\geq \sum_{n \in N_\rho \cap N} \frac{|\rho|^n}{2n} |\operatorname{Arg}(1-u^n)| \geq \sum_{n \in N_\rho \cap N} \frac{|\rho|^n}{2n} \frac{1}{2|\rho|^n} = \frac{1}{4} \sum_{n \in N_\rho \cap N} \frac{1}{n},\end{aligned}$$

where $N := \{n \in \mathbb{N} \mid \operatorname{Arg} u^n \in \Theta\}$. Therefore, it is sufficient to prove that the natural density of N is positive.

Put $\theta := \frac{\operatorname{Arg} u}{2\pi}$ and $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Let $\iota\colon \mathbb{R} \to \mathbb{T}$ be the natural isomorphism defined by $\iota(x) := x \bmod \mathbb{Z}$ and $R_\theta\colon \mathbb{T} \to \mathbb{T}$ be the map satisfying that $R_\theta(x) = \iota(x+\theta)$. Put $\overline{\Theta} := \iota\left(\left[\frac{1}{6}, \frac{1}{4}\right]\right)$. Then, we have the equality $N = \{n \in \mathbb{N} \mid R_\theta^n(0) \in \overline{\Theta}\}$. Since $\theta \notin \mathbb{Q}$, the continuous function R_θ on $\mathbb{T}$ is uniquely ergodic [18, Example 1.3 and Example 4.11]. Since $\overline{\Theta}$ is an interval in $\mathbb{T}$, for any $x \in \mathbb{T}$

$$\lim_{M \to \infty} \frac{1}{M} \#\{n \in \mathbb{N}_0 \cap [0, M) \mid R_\theta^n(x) \in \overline{\Theta}\} = m_{\mathbb{T}}(\overline{\Theta}) = \frac{1}{12},$$

where $m_{\mathbb{T}}$ is a Lebesgue measure on $\mathbb{T}$ [18, Example 4.18 and Lemma 4.17] (also see Einsiedler and Ward [18, Example 1.3]). Substituting $x = 0$ gives the inequality

$$\lim_{M \to \infty} \frac{\#(N \cap [1, M])}{M} = \lim_{M \to \infty} \frac{1}{M} \#\{n \in \mathbb{N}_0 \cap [0, M) \mid R_\theta^n(0) \in \overline{\Theta}_\infty\} = \frac{1}{12} > 0.$$

Thus, the natural density of N is positive. □

B.2 Another version of Theorem 4.4.12

While the conditions in Theorem 4.4.12 are so complicated, the conditions in the following similar theorem are simpler. Note that, however, Theorem B.2.1 cannot treat the counting function of an absolute Riemann surface, since it violates the second condition.

Theorem B.2.1. *Let Φ be a nonempty finite subset of $\mathbb{C}$ and put $d(\Phi) := \max_{\rho \in \Phi} |\rho|$. Set $\lambda = \lambda_\Phi := \max\{d(\Phi), 1\}$ and*

$$f_\Phi(t) := \sum_{\rho \in \Phi} c_\rho t^\rho \in \mathcal{A}_{d(\Phi)} \quad (c_\rho \in \mathbb{C} \setminus \{0\}).$$

Put $\Phi_{\max} := \{\rho \in \Phi \mid |\rho| = \lambda\} =: \{\rho_1, \ldots, \rho_l\}$ and $\theta_k := \frac{\operatorname{Arg} \rho_k}{2\pi}$. Assume the following technical conditions:

(1) It holds that

$$\sum_{\rho\in\Phi_{\max}} c_\rho \neq 0,$$

(2) $1, \theta_1, \ldots, \theta_l$ *are linearly independent over* $\mathbb{Q}$.

Then, the region of absolute convergence of $S_{f_\Phi}(s)$ *is* $\{s \in \mathbb{C} \mid |s| > \lambda_\Phi\}$.

Proof. We use the same notations in Section 4.4. By Theorem 4.4.6, the series $S_{f_\Phi}(s)$ converges absolutely for $|s| > \lambda$. Now, we show that $S_{f_\Phi}(s)$ does not converge absolutely for $|s| \leq \lambda$. If $\lambda = 1$, that is, $|\rho| \leq 1$ for each $\rho \in \Phi$, then the series $\log(1 - s^{-n})$ does not converge absolutely for $|s| \leq \lambda = 1$. Thus, we may assume that $\lambda > 1$. It holds that

$$\sum_{n=1}^{\infty} \left|M_n(f)\log\left(1 - s^{-n}\right)\right| = \sum_{n=1}^{\infty}\left|\sum_{\rho\in\Phi} c_\rho \frac{\kappa_\rho(n)}{\kappa_\lambda(n)}\right|\left|\kappa_\lambda(n)\log\left(1 - s^{-n}\right)\right|$$

Put $M := \frac{1}{2}\left|\sum_{\rho\in\Phi_{\max}} c_\rho\right|$ and fix $0 < \varepsilon < \frac{1}{2}$ satisfying that

$$\varepsilon < \frac{2M}{4\#\Phi_{\max}\max|c_\rho| + M}.$$

By the assumption, it holds that $M > 0$. Set

$$N_\varepsilon := \left\{n \in \mathbb{N} \;\middle|\; \forall\rho \in \Phi_{\max},\ \left|e^{\sqrt{-1}n\arg\rho} - 1\right| < \varepsilon\right\},$$

$$N_\lambda := \left\{n \in \mathbb{N} \;\middle|\; (\log\lambda)^{-1}\log\frac{2}{\varepsilon(\lambda - 1)} < n - \left\lfloor\frac{n}{2}\right\rfloor - 1\right\}.$$

Since

$$\kappa_\rho(n) = \frac{1}{n}\sum_{m|n}\mu\left(\frac{n}{m}\right)\lambda^m e^{\sqrt{-1}m\arg\rho} = \kappa_\lambda(n) + \frac{1}{n}\sum_{m|n}\mu\left(\frac{n}{m}\right)\lambda^m\left(e^{\sqrt{-1}m\arg\rho} - 1\right),$$

putting $R_n := \sum_{m|n}\mu\left(\frac{n}{m}\right)\lambda^m\left(e^{\sqrt{-1}m\arg\rho} - 1\right)$, we have

$$\left|\sum_{\rho\in\Phi_{\max}} c_\rho\frac{\kappa_\rho(n)}{\kappa_\lambda(n)}\right| = \left|\sum_{\rho\in\Phi_{\max}} c_\rho + \sum_{\rho\in\Phi_{\max}}\frac{c_\rho R_n}{n\kappa_\lambda(n)}\right| \geq \left|\left|\sum_{\rho\in\Phi_{\max}} c_\rho\right| - \left|\sum_{\rho\in\Phi_{\max}}\frac{c_\rho R_n}{n\kappa_\lambda(n)}\right|\right|.$$

Then, it holds that

$$\frac{|R_n|}{\lambda^n} \leq \frac{1}{\lambda^n}\sum_{m|n}\lambda^m\left|e^{\sqrt{-1}m\arg\rho} - 1\right| = \left|e^{\sqrt{-1}n\arg\rho} - 1\right| + \frac{1}{\lambda^n}\sum_{n\neq m|n}\lambda^m\left|e^{\sqrt{-1}m\arg\rho} - 1\right|$$

$$< \varepsilon + \frac{2}{\lambda^n}\sum_{m=1}^{\lfloor\frac{n}{2}\rfloor}\lambda^m = \varepsilon + \frac{2\left(\lambda^{\lfloor\frac{n}{2}\rfloor+1} - \lambda\right)}{\lambda^n(\lambda - 1)} = \varepsilon + \frac{2}{\lambda^{n-\lfloor\frac{n}{2}\rfloor-1}(\lambda - 1)}$$

for each $n \in N_\varepsilon$. Moreover, since

$$\frac{2}{\lambda^{n-\lfloor \frac{n}{2} \rfloor -1}(\lambda-1)} < \varepsilon$$

for $n \in N_\lambda$, we have $|R_n| < 2\varepsilon\lambda^n$ for each $n \in N_\varepsilon \cap N_\lambda$. Since

$$n\kappa_\lambda(n) > \lambda^n - \frac{\lambda^{\lfloor \frac{n}{2} \rfloor}+1}{\lambda-1} > \left(1-\frac{\varepsilon}{2}\right)\lambda^n$$

for any $n \in N_\lambda$, it holds that

$$\begin{aligned}\left|\sum_{\rho\in\Phi_{\max}} \frac{c_\rho R_n}{n\kappa_\lambda(n)}\right| &\leq \sum_{\rho\in\Phi_{\max}} \left|\frac{c_\rho R_n}{n\kappa_\lambda(n)}\right| < \sum_{\rho\in\Phi_{\max}} \frac{2\varepsilon\lambda^n|c_\rho|}{\left(1-\frac{\varepsilon}{2}\right)\lambda^n} = \sum_{\rho\in\Phi_{\max}} \frac{4\varepsilon|c_\rho|}{2-\varepsilon} \\ &< \sum_{\rho\in\Phi_{\max}} \frac{M|c_\rho|}{\#\Phi_{\max}\max|c_\rho|} = \frac{M}{\#\Phi_{\max}} \sum_{\rho\in\Phi_{\max}} \frac{|c_\rho|}{\max|c_\rho|} \leq M\end{aligned}$$

for each $n \in N_\varepsilon \cap N_\lambda$. Therefore, the desired inequality

$$\left|\sum_{\rho\in\Phi_{\max}} c_\rho \frac{\kappa_\rho(n)}{\kappa_\lambda(n)}\right| \geq \left|\sum_{\rho\in\Phi_{\max}} c_\rho\right| - \left|\sum_{\rho\in\Phi_{\max}} \frac{c_\rho R_n}{n\kappa_\lambda(n)}\right| \geq 2M - M = M$$

follows for any $n \in N_\varepsilon \cap N_\lambda$.

Since

$$\frac{1}{\lambda^n}\left|\sum_{n\neq m|n} \mu\left(\frac{n}{m}\right)\lambda^m\right| \leq \frac{1}{\lambda^n}\cdot\frac{\lambda^{\lfloor \frac{n}{2} \rfloor+1}-1}{\lambda-1} < \frac{1}{\lambda^{n-\lfloor \frac{n}{2} \rfloor -1}(\lambda-1)} < \frac{1}{\lambda-1},$$

it holds that

$$\begin{aligned}\left|\frac{\kappa_\rho(n)}{\kappa_\lambda(n)}\right| = \left|\frac{\sum_{m|n}\mu\left(\frac{n}{m}\right)\rho^m}{\sum_{m|n}\mu\left(\frac{n}{m}\right)\lambda^m}\right| &\leq \frac{\frac{1}{\lambda^n}\sum_{m|n}|\rho|^m}{\left|1+\frac{1}{\lambda^n}\sum_{n\neq m|n}\mu\left(\frac{n}{m}\right)\lambda^m\right|} \\ &\leq \frac{\frac{|\rho|}{|\rho|-1}\left(\frac{|\rho|}{\lambda}\right)^n}{1-\frac{1}{\lambda-1}} \longrightarrow 0 \quad (n\to\infty)\end{aligned}$$

for $\rho \in \Phi \setminus \Phi_{\max}$. Hence, for each $\rho \in \Phi \setminus \Phi_{\max}$, there exists $n_\rho \in \mathbb{N}$ such that,

$$\left|\frac{\kappa_\rho(n)}{\kappa_\lambda(n)}\right| < \frac{M}{\#\Phi|c_\rho|}$$

for each $n \geq n_\rho$. For any $n \in \mathbb{N}$, if $n \geq \max\{n_\rho \mid \rho \in \Phi\setminus\Phi_{\max}\}$, then it holds that

$$\left|\sum_{\rho\in\Phi\setminus\Phi_{\max}} c_\rho\frac{\kappa_\rho(n)}{\kappa_\lambda(n)}\right| \leq \sum_{\rho\in\Phi\setminus\Phi_{\max}} |c_\rho|\left|\frac{\kappa_\rho(n)}{\kappa_\lambda(n)}\right| < \sum_{\rho\in\Phi\setminus\Phi_{\max}} \frac{M}{\#\Phi} = \frac{(\#\Phi-\#\Phi_{\max})M}{\#\Phi}.$$

Therefore, by putting $N := \{n \in N_\varepsilon \cap N_\lambda \mid n \geq \max\{n_\rho \mid \rho \in \Phi \setminus \Phi_{\max}\}\}$, we have

$$\begin{aligned}\left|\sum_{\rho\in\Phi} c_\rho \frac{\kappa_\rho(n)}{\kappa_\lambda(n)}\right| &= \left|\sum_{\rho\in\Phi_{\max}} c_\rho \frac{\kappa_\rho(n)}{\kappa_\lambda(n)} + \sum_{\rho\in\Phi\setminus\Phi_{\max}} c_\rho \frac{\kappa_\rho(n)}{\kappa_\lambda(n)}\right| \\ &\geq \left|\sum_{\rho\in\Phi_{\max}} c_\rho \frac{\kappa_\rho(n)}{\kappa_\lambda(n)}\right| - \left|\sum_{\rho\in\Phi\setminus\Phi_{\max}} c_\rho \frac{\kappa_\rho(n)}{\kappa_\lambda(n)}\right| \\ &> M - \frac{(\#\Phi - \#\Phi_{\max})M}{\#\Phi} = \frac{\#\Phi_{\max}}{\#\Phi} M\end{aligned}$$

for any $n \in N$. Therefore, it holds that

$$\sum_{n=1}^{\infty} \left|M_n(f)\log\left(1 - s^{-n}\right)\right| > \frac{\#\Phi_{\max}}{\#\Phi} M \sum_{n\in N} \left|\kappa_\lambda(n)\log\left(1 - s^{-n}\right)\right|.$$

Set $\delta = \frac{\varepsilon}{2\pi}$ and $g = (\theta_1, \ldots, \theta_l)$. Let V_δ and $N^g_{V_\delta}(x)$ be the sets in Lemma 4.4.18. Then, we have $N^g_{V_\delta}(0) \subset N_\varepsilon$. Indeed, if $n \in N^g_{V_\delta}(0)$, then it holds that

$$\sum_{\rho\in\Phi_{\max}} \left(\frac{\operatorname{Arg}\rho^n}{2\pi}\right)^2 < \left(\frac{\varepsilon}{2\pi}\right)^2$$

by taking the arguments of any $\rho \in \Phi_{\max}$ in $-\pi < \operatorname{Arg}\rho \leq \pi$. Then, since we have $|\operatorname{Arg}\rho^n| < \varepsilon$ for each $\rho \in \Phi_{\max}$, it holds that

$$\left|e^{\sqrt{-1}n\operatorname{Arg}\rho} - 1\right| < \sqrt{2 - 2\cos\varepsilon} < \varepsilon$$

for any $\rho \in \Phi_{\max}$. Thus, if $n \in N^g_{V_\delta}(0)$, then $n \in N_\varepsilon$.

Since the natural density of $N^g_{V_\delta}(0) \subset N_\varepsilon$ is positive by Lemma 4.4.18, the natural density of $N' := N^g_{V_\delta}(0) \cap N$ is positive. By Lemma 4.4.5, the series $\sum_{n\in N'} |\kappa_\lambda(n)\log(1 - s^{-n})|$ diverges for $|s| \leq \lambda$. Therefore, the series $S_{f_\Phi}(s)$ does not converges absolutely for $|s| \leq \lambda$. □

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